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András Némethi

Normal Surface Singularities

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to my family

Balázs
János
Barna

and

Krisztina

Preface

The aim of this book is to provide an introduction to the theory of complex normal surface singularities. However, alongside the presentation of certain general elements of this topic, I also wanted to focus on certain recent developments in the theory, with remarkable connections and possibilities for further developments. The presentation targets both the topological and analytical aspects. From the topological point of view, we study invariants of the link, and their connection with the resolution graph, and we make connections with invariants provided by low dimensional topology (like the Seigerg–Witten invariant). From the analytical point of view, we study invariants of the local ring via sheaf cohomologies of line bundles of the resolution, multivariable Hilbert series, and holomorphic forms. An important bridge between them is formed by the topological and analytic lattice cohomologies. The main message is the deep symbiosis of these topological–analytical structures.

The book contains a large number of examples based on the treatment of several key families of singularities (e.g., rational, elliptic, Kulikov, weighted homogeneous, suspension, superisolated, quotient, splice quotient, and Newton nondegenerate). They support the understanding of the theoretical parts.

The book is intended for readers who already have a certain amount of knowledge of algebraic geometry, multivariable complex analysis, sheaf theory, and algebraic topology. However, we have tried to be self-contained, to make the material more legible and accessible. Also, in order to help younger readers, or those researchers who wish to learn techniques from this book, we provide fairly detailed proofs and arguments in most of the cases. In this way, we hope that we will partially meet a need regarding a book on complex normal surface singularities.

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I am also extremely indebted to all my collaborators. Our joint efforts crystallized into a deeper understanding of the subject and without doubt contributed to my vision of surface singularities.

During the preparation of the book, I was unstintingly supported by Gert-Martin Greuel. Without his constant encouragements, this book would never have been (started and) finished. I am very grateful to him.

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Budapest, Hungary András Némethi
2022

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Chapter 1
Introduction

1.1 General Introduction

Singularity theory aims to study the singular points of algebraic/analytic varieties. It was born together with the classical algebraic geometry, but step by step became an independent discipline within algebraic and complex geometry. Since the 1980s we witness its extraordinary development producing a large number of key results, research programs, and remarkable connections with other fields, like topology or differential equations. It has created its own specific language and tools.

Historically, in parallel with the study of projective curves and their singularities, the study of global complex surfaces—together with their singularities—became a central object as well. In the global nonsingular case, after the 'classical' classification of Enriques and the Italian school, a 'modern' classification was provided in the 1960s by Kodaira (and independently, by the seminar of Shafarevich) based on sheaves, cohomologies and characteristic classes with special emphasis on the relationships of the analytic structures with invariants of the underlying smooth 4–manifolds. Additionally, crucial open questions were formulated targeting ties of topological nature: e.g. the purely topological characterization of rational surfaces (as an addendum of Castelnuovo's criterion) or of K3 surfaces (asked by Kodaira).

Later, by crucial classification projects in mainstream mathematics (e.g. the Mori program in algebraic geometry targeting classification of varieties, or developments in low-dimensional topology) had significant influences on surface theory and generated a series of new open questions. The discovery of Donaldson and Seiberg–Witten theories gave a powerful impetus of such comparison efforts targeting the symbiosis of the smooth and the algebraic/analytic structures.

All these were inherited by local surface singularity theory too, involving 3-dimensional topology (of the local links). The lack of classification results in 3 and 4 dimensional topology put even more in the highlight the possible analytic/algebraic connections (with the hope that the classification of these rigid structures might

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provide new perspectives). The analytic/algebraic structures served as inspiration and testing ground for contact and symplectic geometry, too.

This is the guiding principle of the present book too: what are the ties between analytic and topological invariants of a local complex normal surface singularity? Historically this program was started by Mumford, Artin and Laufer: Mumford realized the link as a plumbing and computed its fundamental group, Artin and Laufer characterized topologically the rational and minimally elliptic germs (respectively), and computed several analytic invariants for them from the resolution graph.

Let us exemplify a few pairs of analytic/topological objects, which play a central role in this book from the point of view of the above comparison.

On the analytic side our fundamental objects are the dimensions of the sheaf cohomologies of line bundles on a resolution (including e.g. the geometric genus) and the multivariable Poincaré series of the divisorial filtration associated with a resolution. This is enhanced with a family of linear subspace arrangements, indexed by the first Chern classes of the line bundles. On the topological side, the link, as an oriented 3-manifold, carries the Casson invariant whenever the link is an integral homology sphere. In the rational homology sphere case, it carries the Seiberg–Witten invariants. The Seiberg–Witten invariant will be compared with the ranks of cohomologies of line bundles (formulated by the Casson Invariant Conjecture of Neumann and Wahl, or by the Seiberg–Witten Invariant Conjecture of Nicolaescu and the author). Moreover, a topological multivariable Poincaré series (a 'zeta' function, associated with the dual graph) will be compared with its analytic counterpart provided by the divisorial filtration (as extensions of Campillo–Delgado–Gusein-Zade identity). Furthermore, certain topological linear subspace arrangements will mirror their analytic counterparts, too. The parallelism will be emphasized by several surgery and additivity formulae of similar shape present in both sides.

All these are complemented by the corresponding lattice cohomologies on both sides. The Euler characteristic of the topological lattice cohomology is the Seiberg–Witten invariant, while of the analytic lattice cohomology is the equvariant geometric genus. The two cohomology theories will be compared as well.

Regarding the topological invariants, the author was influenced by the Heegaard Floer theory of 3-manifolds. However, the techniques developed by the author to create a bridge between singularities and low dimensional topology differ from those used in Heegaard Floer theory. The effort to create such a bridge bore fruit and culminated in the topological lattice cohomology. It is defined combinatorially from the graph. However, its definition and several of its properties resemble sheaf cohomology long exact sequences. Indeed, behind certain definitions and techniques in lattice cohomology theory one experiences certain generalizations of ideas of Laufer regarding computation sequences, used in sheaf cohomological arguments. In the new context these sequences appear as discrete 'homotopy deformation retractions'. Our presentation emphasises this continuity with Laufer's work.

The appearance of the new invariants presented in the book suggested to us the desirability of rethinking the Mumford–Artin–Laufer program, and in fact, of the whole theory of surface singularities.

1.2 Why a New Book?

In any developing areas of mathematics it is highly desired to have a few books, which cover not only the basic knowledge of the area, but also the important connections with other fields, and also the main guiding problems, conjectures and leading principles of the actual research. Such a source helps not only the young researchers to learn the basic material, but it is very helpful for the whole community to have a good grip on the hot topics and questions of the research. This is even more necessary, when different mathematical areas are combined; in this way a good source creates a common language and helps the connected areas to have a fast access to the needed techniques.

In *algebraic geometry* a great number of excellent books were published in the last period, mostly motivated and influenced by the modern developments, as the powerful Minimal Model Program. In several of these studies singular spaces are inevitably studied, e.g., the global geometry of a birational equivalence class is more transparent in a canonical singular representative. However, the singularities which appear in this way are 'mild' (and, in fact, their 'mildness' is a key ingredient of the theory). In particular, by the very essence of the program, the class of singularities is limited by the needs and motivation of the MMP. Already in the surface case the difference is considerable—they include basically the (log)terminal and (log)canonical singularities only. But, several other important questions of surface singularities have their natural place beyond these restrictions.

There is an impressive list of excellent books, published in the last decades, covering different areas of local *singularity theory* as well. Some important topics covered by them include the following (the list of subjects and of the cited monographs is not exhaustive):

- special families, like isolated hypersurface and complete intersections, simple or quotient singularities, weighted homogeneous, plane curve, or even non-isolated hypersurface singularities [19–21, 85, 87, 105, 147, 148, 169, 241, 346, 395, 405, 408, 430, 622, 644, 715, 745],
- deformation theory of isolated singularities, smoothing, local moduli spaces, equisingular deformations [240, 263, 357, 664],
- construction of resolutions (general and low-dimensional cases) [2, 127, 128, 260, 330],
- classification of map germs and critical points, stability questions [159, 212, 441],
- connection with intersection cohomology, constructible and perverse sheaves, mixed Hodge structures and modules [343, 413, 563, 619],

- links of singularities, their exotic smooth structures [275],
- relation with global singularity theory, generalization of characteristic classes, Thom polynomials, enumerative geometry [317].

Though these books and monographs definitely include important information regarding surface singularities too, the need for a more focused source (showing the power and the potentials of surface singularities) is really visible.

Regarding the theory of *projective surfaces* one has several excellent books. The extensive and comprehensive book of Friedman and Morgan [206] targets the smooth invariants (Donaldson polynomials) of (global non-singular) complex surfaces, showing that their smooth classification is deeply connected to their algebro-geometric structures. The books of Bădescu [38], Beauville [46] and Barth, Peters and Van de Ven [41] target the classification of surfaces. They include certain discussions of local singularities too (needed in the proof of global classification statements), but they do not go beyond the cyclic quotient, rational, or minimally elliptic cases. The book of Ishii [288] contains a wonderful presentation of the algebraic aspects (with a focus on the cases motivated by the MMP) but it provides no topological connection. The book of Okuma [528] is an important introduction to the plurigenera of surface singularities, but it contains no topological connections. On the other hand, the book of Eisenbud and Neumann [178] is a fundamental topological treatment of the integral homology sphere plumbed/spliced 3-manifolds, but without further connection with the analytic theory of surface singularities. The presentations of Singularity Seminars from important singularity centers from the 1970s and the 1980s (edited e.g. by Lê, Demazure, Pinkham, Teissier) reflect the research of that period. We end this list with the excellent book of Laufer [358], an influential source of knowledge, techniques and inspiration regarding analytic properties of resolution of surface singularities, however, since its publication (1971) several new directions and results have appeared, which should be reported and organized.

Having all this in mind, the author had to take certain decisions about the structure of the present book. Though the primary goal was to give a presentation of the new analytic/topological interferences in the theory of complex normal surface singularities, the author realized that this cannot be done properly without (at least a partial) foundation. Even more, besides the central theorems the presentation should contain the detailed proofs and the main auxiliary constructions. In this way the reader will be essentially helped to create a good intuition and to acquire the necessary techniques. This addition is one of the reasons for the length of the book.

Naturally, in the presentation of the basic material we had to be very selective. The needed choices were dictated by the main goals of the book. In this way we added what is really required to understand the main messages, and to attain sufficient experience in the needed techniques. Furthermore, our goal was to make the book fairly self–contained as well.

1.3 Examples and Constructions

The book contains many examples, they constitute an important part of the presentation. They are the compass for the definitions, the testing trial for theorems and the guide for the intuition.

Again, we had to take some decisions regarding the balance between the number of examples and supplementary comments versus the length of the manuscript. The decision probably was done in favour of the examples. In this way we decided to build in the text (and run them simultaneously with the theory) several key families of singularities. They include the following: cyclic quotients (or Hirzebruch–Jung germs), quotients, weighted homogeneous singularities (in particular, Brieskorn–Hamm complete intersections), cusps, quotient-cusps, splice quotients, special hypersurface singularities (as superisolated, suspensions, and Newton non-degenerate), rational (and almost rational), elliptic, and Kulikov singularities. In our presentation of the examples we provide all the necessary machinery to handle their (both analytic and topological) invariants. In this way the reader will have sufficient information to verify the statements and the formulae exemplified by them. We briefly review some statements regarding the Milnor package of hypersurface singularities as well, but (having so many wonderful books in this subject) we restricted ourself to the minimum.

Similarly, we present several key constructions and the related techniques as well. Some of them are vital, like the package related with a resolution. Some of them are connected with the above families (e.g. the splice quotient construction of Neumann and Wahl). Some others support the theory. For example, the (ramified and o–ramified) coverings, determined either by analytic or by topological data, are needed e.g. in the definition of the analytic/topological canonical coverings, or in the discussion of $\mathbb{Q}$-Gorenstein germs. The universal abelian covering is also a central object together with the natural line bundles associated with a resolution.

In some discussions sometimes we mention certain invariants, even though their detailed presentation takes place further back. We hope that such cases will not create serious obstructions in the comprehension of the material.

1.4 Prologue of the Chapters

Though *Contents* suggests briefly the covered material, here we provide some additional information and guiding motivations about each chapter as well. It can be advantageous to read this introductory presentation before—and in parallel to—the study of the corresponding parts of the book.

Below (X, o) is a complex normal surface singularity.

Chapter 2: Resolution of Surface Singularities The total space $\widetilde{X}$ of a resolution of a singularity (X, o) is an important source of invariants. It is smooth, hence it carries all the 'classical' invariants of smooth spaces (and several of them are

resolution independent). It also carries the combinatorial structure of the dual resolution graph, which, as it will turn out, is equivalent with the topological type of (X, o).

In Chap. 2 we introduce the notions of modifications and resolutions together with their exceptional curves, and we prove Zariski's Main Theorem regarding the connectivity of the exceptional locus. The exceptional curve combinatorially can be codified either in the dual resolution graph, or in the lattice $L = H_2(\widetilde{X}, \mathbb{Z})$ endowed with the intersection form. According to a theorem of Du Val, this form is negative definite. We introduce the dual lattice $L' = H_2(\widetilde{X}, \partial\widetilde{X}, \mathbb{Z}) \simeq H^2(\widetilde{X}, \mathbb{Z})$ and Lipman's cone (of anti–nef cycles), and we list some of their basic properties. In the presence of a Weil divisor $(D, o) \subset (X, o)$ we speak about the embedded resolutions.

Strings are graphs of Hirzebruch–Jung (or, cyclic quotient) singularities. In 2.3 we provide a rather complete presentation of their properties (including five analytic appearances). They are the most important blocks, which appear in several geometrical constructions (e.g. in the proof of the existence of the resolution as well). The arithmetical properties of strings are compared with continued fractions and Dedekind sums.

Finally, we prove the existence and uniqueness of the minimal resolution of any surface singularity via Jungian strategy.

Chapter 3: The Link We introduce the link L_X of (X, o), an oriented connected 3-manifold, and we state that (X, o) is a real cone over L_X. In this way L_X determines the topological type of (X, o). We prove that the link is the plumbed 3-manifold associated with any resolution graph considered as plumbing graph. The ambiguity of the graph is discussed via the plumbing calculus. In fact, we consider the calculus of all oriented plumbed 3-manifolds (i.e., not only those which are associated with negative definite connected graphs). This extension will be necessary later (e.g. when we discuss the steps of the calculus dictated by splice decompositions).

In 3.6 we list certain basic classification results of 3-manifolds; the Jaco–Shalen–Johannson decomposition will show up later several times. We deduce several consequences of the general theorems for the link. E.g., if the fundamental group π_1 is infinite then the link is an Eilenberg–Mac Lane space with a torsion free π_1. We provide the presentation of π_1 with special emphasise on the cases when π_1 is finite (reproving Brieskorn's classification), on the Seifert case (showing the connection with the crystalographic groups), and on the rational homology sphere case (when a certain presentation of π_1 has deficiency zero). We prove Mumford's theorem, which says that $\pi_1 = 0$ if and only if (X, o) is analytically smooth.

Chapter 4: Coverings We prove an algorithm, which determines the resolution graph of the cyclic covering (ramified along $\{f = 0\}$) associated with a germ $f : (X, o) \to (\mathbb{C}, 0)$. In the book several examples are based on coverings (e.g. suspension singularities). Thanks to this algorithm the reader will be able to determine the corresponding graphs. But the study of the coverings has more theoretical reasons as well. Several constructions depend essentially on the existence

and properties of o–ramified abelian coverings (this means that the covering over $X \setminus \{o\}$ is unramified). The point is that the above algorithm extends to such cases too.

If the link is a rational homology sphere, then there are two key o–ramified cases: the universal abelian covering and a cyclic covering associated with a character $\alpha \in \widehat{H} = \mathrm{Hom}_{\mathbb{Z}}(H, \mathbb{C}^*)$, where $H = H_1(L_X, \mathbb{Z}) = L'/L$.

The universal abelian covering $(X_a, 0) \to (X, o)$ (with its Galois action H) is the tool to extend any geometrical construction and invariant (analytic or topological) from the level of (X, o) to its *equivariant* version. Indeed, one takes the corresponding invariant at the level of (X_a, o) with the action of H. In topology many invariants, like the torsion or the Seiberg–Witten invariant, are indexed by the set Spin^c of the spin^c structures. Since for L_X, $\mathrm{Spin}^c(L_X)$ is an H-torsor, they are in fact H-indexed invariants. Hence, it is natural to compare them with objects associated with the universal abelian covering. In some cases such an H-enhancement can also be done via the enlargement of the lattice L by L', a trace of the above covering.

In the case when the link is a rational homology sphere ($\mathbb{Q}HS^3$), cyclic coverings associated with characters have the following reinterpretation. Since H and $\widehat{H}$ can be identified (via Γ), any class $[l'] \in L'/L$ ($l' \in L'$) provides a o–covering. E.g., in this way we have the 'topological canonical covering' associated with the class of the *canonical cycle* in L'/L. Note that the construction is not obstructed by any analytic property. In particular, in this particular dimension, this is an improvement of the standard procedure of the (algebraic/analytic) literature, when one defines the canonical covering only under the assumption that the canonical *divisor* is $\mathbb{Q}$-Cartier. In fact, in Chap. 6, cf. 6.1.A, we will also define for any $\mathbb{Q}$-Cartier divisor a cyclic o–ramified covering. If the canonical divisor of $\widetilde{X}$ is $\mathbb{Q}$-Cartier, then its covering is called the 'analytic canonical covering'. If $b_1(L_X) = 0$ then the coverings associated with characters of H and the coverings associated with $\mathbb{Q}$-Cartier divisors can be identified (and the 'canonical' ones too).

In certain cases by taking the 'right' covering we obtain a nicer singularity than the original one (e.g. the topological canonical covering of a topological type with rational homology sphere link is numerically Gorenstein, the analytic canonical covering of a $\mathbb{Q}$-Gorenstein germ is Gorenstein, the universal abelian covering of a splice quotient singularity is a complete intersection, the universal covering of a quotient is smooth, a 'right' covering of a weighted homogeneous germ is a cone).

Chapter 5: Examples As we already mentioned, in the book several special families will continuously support the theory. In this chapter we introduce some of them. Again, we focus on the analytic/topological consonances.

Section 5.1 First, we discuss the **weighted homogeneous** germs. Such a germ is isomorphic with the germ at a normal singular point of an affine variety with a good $\mathbb{C}^*$-action. Their links admit an S^1–action, they are Seifert 3–manifolds. We provide their topological and analytical classifications. When the link is a $\mathbb{Q}HS^3$ (i.e. $b_1(L_X) = 0$), the analytic classification (in the presence of the $\mathbb{C}^*$-action) is

not very far from the topological classifications (of Seifert 3-manifolds). Hence, one can really hope that the discrete analytic invariants can be read from the Seifert invariants. E.g., the output of the Dolgachev–Pinkham–Demazure construction (as an automorphic factor) provides such a topological formula (in the non-equivariant case) for the Poincaré series and the geometric genus. Further expressions and properties (even at the equivariant level) can be derived using an additional construction: the realization of the resolution space $\widetilde{X}$ by analytic plumbing via special (monoidal) gluings.

In the case $b_1(L_X) = 0$ we also prove a theorem of Neumann: the universal abelian covering is a Brieskorn–Hamm complete intersection. This statement led Neumann and Wahl to the invention of splice-quotients.

Section 5.2 The analytic types of hypersurface **superisolated singularities** with $\mathbb{Q}HS^3$ link reflect faithfully the set of projective rational cuspidal curves C in $\mathbb{CP}^2$. Their links are special surgery 3-manifolds.

Via superisolated singularities we transform a global algebro-geometric question into the theory of local surface singularities. The question (still open, central in the classification of singular projective curves) is the following: Given a set of topological types of irreducible local plane curve singularities and $d \in \mathbb{Z}_{>0}$, is there a projective rational curve C of degree d with local singularities listed by this set? Or, said in a different way: How do the local knot invariants of the singular points $(C, p_i) \subset (\mathbb{CP}, p_i)$ of C obstruct the realizability of a projective C? Surprisingly, this question is closely related with one of the main motives of the book, namely with the Seiberg–Witten Invariant Conjecture (SWIC), which compares the geometric genus of the associated superisolated germ with the Seiberg–Witten invariant of the link. For superisolated singularities this leads to the 'Semigroup Distribution Property', which formulates an obstruction of the realizability of C in terms of the semigroups of the local singularities.

Furthermore, these singularities produce the most important counterexamples for the SWIC (and for some other 'too optimistic conjectures' as well).

Sections 5.3 and 5.4 Following Neumann and Wahl, a **splice quotient singularity** is constructed from a (convenient) potential resolution graph Γ as an H-quotient of an isolated complete intersection, such that the equations are of semi-quasihomogeneous type (though each equation might have different weights). The variables correspond to the end vertices, while the equations correspond to the nodes. The point is that the constructed germ has a resolution with dual graph exactly Γ (and the quotient covering is exactly the universal abelian covering of the factor). It turns out that the multivariable Poincaré series (associated with the divisorial filtration of this resolution) can be read combinatorially from Γ. It is an equivariant Molien-type rational function $Z(\mathbf{t})$. Similarly, several other basic analytic invariants are readable from Γ.

This motivates to define and consider the rational function $Z(\mathbf{t})$ for *any* resolution graph Γ. It turns out that it is one of the main source of *topological candidates* for analytic invariants of 'nice' singularities.

The equations of the universal abelian covering of a splice quotient germ can be written elegantly from the associated 'splice diagram', introduced as a replacement of the graph Γ. Since the splice diagrams constitute an important topological tool, we will present their theory in an independent subsection. In the case of integral homology spheres they are complete invariants, the plumbing graphs and splice diagrams determine each other. In some parts of their presentation we follow the book of Eisenbud and Neumann. Later, in the computation of topological invariants of integral homology spheres, the splice decomposition (corresponding to the minimal JSJ–decomposition) will be the basic inductive tool. (The arithmetical properties of the JSJ–decomposition can be much harder followed in the combinatorics of the plumbing graph.)

One cannot associate splice equations to any Γ, the graph must satisfy certain arithmetical condition. Neumann and Wahl formulated them as the 'semigroup' and 'congruence' conditions. In a later version, Okuma replaced them by the equivalent 'monomial' condition. In our discussions we use Okuma's formalism.

Section 5.5 In the case of a **hypersurface with Newton non-degenerate principal part** we combine two combinatorial objects. The first one is the Newton diagram $N\Gamma$, related with the (equisingularity type of the) equation (providing almost all the discrete analytic invariants and also the embedded topological type). The second one is the resolution graph Γ (equivalent with the abstract topological type of the link). Using toric resolution arguments one recovers Γ from $N\Gamma$ (we call this Oka's algorithm). On the other hand, in [76] we have proved that if $b_1(L_X) = 0$, then $N\Gamma$ can be recovered from Γ, hence the link can even provide the equation (up to equisingularity). This shows that (at least conceptually) those discrete analytic invariants which can read from $N\Gamma$ can be read topologically from the link as well. (But finding the concrete expressions can be a challenge.)

In the second part we discuss a generalization, the case of (local analytic Newton non-degenerate) Weil divisors $(X, 0)$ in toric affine varieties. We provide combinatorial criterions which guarantee certain analytic properties of (X, o), like having an isolated singularity, or being normal or Gorenstein or Cartier in $(Y, 0)$. We provide combinatorial formulae for the delta invariant (in the curve case) and for the geometric genus (in the surface case).

Chapter 6: Invariants Associated with a Resolution This is a basic foundation of invariants of normal surface singularities with a focus on the geometry of line bundles on $\widetilde{X}$. These line bundles are classified by $\mathrm{Pic}(\widetilde{X})$. We also consider $\mathrm{Cl}(X, o)$, the group of Weil divisors of (X, o) modulo Cartier divisors. One has the Chern class morphism $c_1 : \mathrm{Pic}(\widetilde{X}) \to L'$ (as part of the exponential exact sequence). If $b_1(L_X) = 0$, then this morphism naturally splits: for any $l' \in L'$ there is a well-defined bundle $\mathscr{O}_{\widetilde{X}}(l')$ with $c_1(\mathscr{O}_{\widetilde{X}}(l')) = l'$ and such that some power of it has the form (in the classical sense) $\mathscr{O}_{\widetilde{X}}(l)$ for some integral cycle

$l \in L$. In other words, the bundles $\mathscr{O}_{\widetilde{X}}(l')$ are exactly those whose projection to $\mathrm{Cl}(X,o) \simeq \mathrm{Pic}(\widetilde{X})/L$ have finite order (i.e. in $\mathrm{Cl}(X,o)$ they are $\mathbb{Q}$–Cartier). They behave compatibly with coverings: the local algebra of a Galois abelian covering has an eigenspace decomposition identifiable with sections of certain natural line bundles. They constitute the equivariant setup at the level of line bundles (and at the level of their cohomology groups).

The line bundle $\Omega^2_{\widetilde{X}} = \mathscr{O}_{\widetilde{X}}(K_{\widetilde{X}})$ is analysed with a special interest. We discuss numerically Gorenstein, Gorenstein, $\mathbb{Q}$–Gorenstein structures and their relationships with (canonical) coverings and natural line bundles. We also focus on the topological numerical invariant $c_1(K_{\widetilde{X}})^2 + |\mathscr{V}|$ ($\mathscr{V}$=vertices of Γ), present in several index formulae.

There are several powerful tools, which handle the cohomology of line bundles. They include e.g. vanishing theorems and statements targeting base components and base points. Besides the 'standard' vanishing theorems of the literature (Grauert–Riemenschneider vanishing, valid for any germ, or the Lipman vanishing, valid for rational ones) we will prove the local version of the Kawamata–Viehweg Vanishing Theorem. Moreover, in our surgery/additivity formulae we will need a different type of theorem as well. (In applications, it is used when we wish to separate 'bad' or 'special' exceptional curves). In the assumption of this 'Unbalances Vanishing Theorem' the required positivity is distributed in a special way along the exceptional locus.

The study of divisors of different sections is guided by the 'analytic monoid' associated with the analytic structure. The *non-equivarint analytic monoid* $\mathscr{S}_{an}$ consists of the divisors of the analytic function (restricted on the exceptional locus). They sit naturally in the *non-equivariant topological monoid* $\mathscr{S}$, the (integral) Lipman's cone. For rational (and under some condition, for minimally elliptic) singularities they agree, $\mathscr{S}_{an} = \mathscr{S}$. However, in general, the determination of $\mathscr{S}_{an}$ is very difficult. We spend considerable time with their study and exemplification. A possible application is the determination of the multiplicity of (X,o), or of the Hilbert–Samuel function (associated with the filtration of the maximal ideal). In fact, $\mathscr{S}_{an}$ is some kind of discrete simplification of the local algebra $\mathscr{O}_{X,o}$, hence its influence is present everywhere. The equivariant version of the pair $\mathscr{S}_{an} \subset \mathscr{S}$, defined via L' and via the universal abelian covering, is the pair of monoids $\mathscr{S}'_{an} \subset \mathscr{S}'$.

The geometric genus $p_g = h^1(\mathscr{O}_{\widetilde{X}})$, and their equivariant versions (and several other cohomology groups), can be analysed via Laufer duality, that is, via global differential forms, with possible poles along the exceptional curves. The space of such differential forms enriches the theory with several new ideas and techniques. E.g. the linear subspace arrangement of forms with prescribed poles is a powerful invariant, it even produces an inductive additivity formulae for cohomology groups. Integration of forms (i.e. the Abel map of surface singularities) explains the variation of divisors, bundles, or existence of sections (with given topological data).

We believe that the following problems are crucial, and they code essential information regarding the possible analytic structures supported on a fixed topological

type: (a) assign combinatorially an integer $\mathrm{MAX}(\Gamma)$ to any resolution graph Γ, such that $p_g \leq \mathrm{MAX}(\Gamma)$ for any analytic type supported by Γ, and furthermore, for a certain analytic structure one has equality; (b) the same problem asking for the optimal minimum $p_g \geq \mathrm{MIN}(\Gamma)$; (c) find all the values between $\mathrm{MIN}(\Gamma)$ and $\mathrm{MAX}(\Gamma)$, which can be realized as the geometric genus of some analytic structure.

Let us assume that $b_1(L_X) = 0$. Then the answer to (b) was found recently: $\mathrm{MIN}(\Gamma)$ is the 'arithmetical genus' and it is realized by the generic analytic structure [453]. The answer to $\mathrm{MAX}(\Gamma)$ is unknown (here one searches for the existence of 'very special' analytic structures). Nevertheless, we will introduce a 'candidate', Path^i, which is an upper bound of p_g. Although, in general it is not optimal, for certain topological types the equality $\mathrm{Path}^i = p_g$ is realized for some (well-known) special analytic structures. This happens e.g. for weighted homogeneous, rational, maximally elliptic, superisolated, Newton non-degenerate analytic types. (I.e., their geometric genus is maximal among the geometric genera provided by all analytic types supported on the corresponding topological type, and it is given by Path^i).

We prove other p_g-estimates too, needed in the proof of Durfee's Conjecture.

Beautiful identities appear when we compare resolution invariants with smoothing invariants. Though in this book we do not plan to discuss the theory of smoothings, we still decided to mention briefly some famous connections: certain formulae of Laufer, Durfee and Wahl, together with some famous conjectures regarding hypersurfaces. Also, in order to help the reader in concrete computations along the book, we listed some well-known formula for Milnor number, signature and geometric genus, valid for Thom–Sebastiani or suspension type hypersurfaces.

In the investigation of topological invariants we will need the classification (and 'parametrization') of the spin^c structures of $\widetilde{X}$ and L_X. This is done in 6.10. As we already have mentioned, if $b_1(L_X) = 0$ then $\mathrm{Spin}^c(L_X)$ is an H-torsor, hence it fits perfectly into the equivariant setup.

Chapter 7: The Artin–Laufer Program Initially, the 'Artin–Laufer program', or 'Mumford–Artin–Laufer' program, covered the collection of those types of results, which aimed to determine analytic invariants from the topology of the link (for complex normal surface singularities). The first starting models were the following. Mumford in [448] determined the fundamental group of the link and proved that if this group is trivial then the germ is analytically smooth. Artin in [27, 28] proved that rational singularities can be recognized topologically, and he determined their Hilbert–Samuel function from the combinatorics of the resolution graph. Then Laufer proved similar statements for minimally elliptic singularities (the simplest non-rational germs) [363]. The program was continued by the author in [473] for Gorenstein elliptic singularities with rational homology sphere links. This was followed by several results e.g. by topological description of the geometric genus (in certain cases), and the connection of the geometric genus with the Seiberg–Witten invariant of the link. In this spirit several 'pairs of invariants' were developed (that is, certain analytic invariants together with their 'topological candidates'); e.g. the

analytic and topological Poincaré series, or, analytic and topological linear subspace arrangements (several of them will be discussed in this book, see also [486]).

In particular, the 'Artin–Laufer program' step by step outgrew (from the collection of results about certain concrete invariants, like the geometric genus, or Hilbert–Samuel function) to a large collection of methods and constructions, which stay at the basis of the establishment of even more general and comprehensive projects regarding the analytic-topological comparisons.

This development can be seen even at the level of some technical tools, e.g. via different generalizations of Laufer's 'computational sequences'. A substantial generalization constitutes one of the basic machineries in lattice cohomology (which mimics some kind of combinatorial-homotopy).

The first part of Chap. 7 develops the theory of rational and elliptic singularities (in the spirit of Artin and Laufer). In the rational case we determine $h^1(\widetilde{X}, \mathscr{L})$ for any line bundle $\mathscr{L} \in \mathrm{Pic}(\widetilde{X})$, we prove $\mathscr{S}'_{an} = \mathscr{S}'$, we compute the Hilbert–Samuel function, and we establish p_g and multiplicity formulae even for the universal abelian covering. For elliptic singularities we target basically the same type of results (sometimes under some minor restrictions). Moreover, we analyse with detail the Kulikov elliptic and minimally elliptic germs as well.

For an a arbitrary elliptic germ its geometry is guided by the combinatorics of the elliptic sequence (it is like a combinatorial 'skeleton'). In the numerically Gorenstein case we use the elliptic sequence considered by Laufer and Yau, however in the non numerically Gorenstein case—besides the sequence of Laufer–Yau—we introduce a new elliptic sequence too, which is more compatible with the (non-integral) canonical cycle. A typical result what we establish: Path^i equals the length of the elliptic sequence, and it is also realized as the geometric genus of a special/existing analytic structure (characterized by the fact that its restriction to the maximal numerically Gorenstein subgraph should be Gorenstein).

In the last part of the Chapter we develop the theory of generalized computation sequences and their applications. The theory will be substantially exploited in the next chapters, when we discuss lattice cohomology.

The starting objects in the lattice cohomology are the weighted cubes. We tile $\mathbb{R}^{|\mathscr{V}|}$ with q-dimensional cubes ($0 \le q \le |\mathscr{V}|$) whose vertices are in L ($L \subset L \otimes \mathbb{R} = \mathbb{R}^{|\mathscr{V}|}$). We fix a weight function on L, which can be e.g. the Riemann–Roch expression $\chi_k(l) = -(l, l+k)/2$, where k is a characteristic element in L'. Then for any $n \in \mathbb{Z}$ we consider $S_{k,n}$, the union of all cubes whose vertices have χ_k-weight $\le n$. (Since χ_k is positive definite, $S_{k,n} = \emptyset$ for $n \ll 0$.) The topology of the collection $\{S_{k,n}\}_n$ contains a very deep information about the compatibility of the position of the lattice points with the level-sets of the weight function. Though a level set in $L \otimes \mathbb{R}$ bounds a convex—hence contractible—domain, each $S_{k,n}$ might have a complicated topology, and surprisingly, it reveals very essential information about Γ (deeper than what we naively might expect from a 'simple graph'). For example, Γ is rational if and only if all the (non-empty) spaces $S_{k,n}$ are contractible. For an elliptic graph each connected component of $S_{k,n}$ is contractible. Furthermore,

for $k = K$, $S_{K,n} = \emptyset$ for $n < 0$, it is contractible for $n > 0$, and for $n = 0$ the number of components is exactly the length of the elliptic sequence.

In the homotopy identifications of $S_{k,n}$ we use deformation retractions guided by computation sequences (that is, collection of elements $\{l_i\}_{I=0}^{N}$, $l_i \in L$ of type $l_{i+1} = l_i \pm E_{v(i)}$). Then χ_k-monotonity of type $\chi_k(l_{i+1}) \leq \chi_k(l_i)$ or $\chi_k(l_{i+1}) \geq \chi_k(l_i)$ produces homotopy at the level of certain cubes.

It turns out that for some vertices v (depending on the global arithmetical properties of Γ) retraction in direction E_v can be done naturally, they are not obstructed. E.g., for a rational graph no direction is obstructed (hence one can prove that $S_{k,n}$ is contractible). In general, we say that $\overline{\mathcal{V}} \subset \mathcal{V}$ is a set of 'bad' vertices if replacing their Euler decorations $\{e_v\}_{v\in\overline{\mathcal{V}}}$ with a sufficiently negative integers we get a rational graph. (They index those directions E_v, $v \in \overline{\mathcal{V}}$, which are or might be 'obstructed'.) In particular, if $|\overline{\mathcal{V}}| = \overline{s}$, then each $S_{k,n}$ has the homotopy type of $\leq \overline{s}$ dimensional cubical complex. E.g., if $\overline{s} = 1$ (in this case Γ is called 'almost rational', AR), one can construct an infinite 1-dimensional simplicial complex (path) γ such that $S_{k,n} \cap \gamma \hookrightarrow S_{k,n}$ is a homotopy equivalence for every n and k. The elements of γ are constructed by a special algorithm (generalization of that algorithm of Laufer, which finds the fundamental cycle). They are 'universal' cycles, some of them can be identified by some universal properties as well (e.g., fundamental cycle, canonical cycle, members of the elliptic sequence). The local minima and maxima of the weight function along γ tell very deep information about the topological type, while the long cohomology exact sequences associated with sheaf-morphisms of type $\mathcal{F}|_{l_i+E_{v(i)}} \to \mathcal{F}|_{l_i}$ carry important information about the analytic type.

In this chapter we develop the needed technical machinery (for arbitrary $\overline{s}$).

Chapter 8: Multivariable Divisorial Filtration In general, when we wish to understand the structure of a ring, it is helpful to filter it by a (decreasing) filtration $\{\mathcal{F}_i\}_i$ and study each $\mathcal{F}_i/\mathcal{F}_{i+1}$, together with the Poincaré series $P(t) = \sum_i \dim(\mathcal{F}_i/\mathcal{F}_{i+1})\, t^i$. In commutative algebra it is common to use the filtration given by powers of a certain ideal.

In this chapter, once we fix a resolution $\widetilde{X} \to X$ with exceptional curve E, we consider the multivariable divisorial filtration $\{\mathcal{F}(l)\}_{l\in L}$ of $\mathcal{O}_{X,o}$, indexed by L, the integral divisors supported on E. Let us assume that $b_1(L_X) = 0$, hence the universal abelian covering (X_a, o) is well-defined. Then this filtration extends to an equivariant filtration $\{\mathcal{F}(l')\}_{l'\in L'}$ of $\mathcal{O}_{X_a,o}$, indexed by L'. Then the equivariant multivariable Hilbert series $H(\mathbf{t})$ is defined as $\sum_{l'\in L'} \dim(\mathcal{O}_{X_a,o}/\mathcal{F}(l'))_{[l']}\, \mathbf{t}^{l'}$ (where $(\cdot)_{[l']}$ denotes the $\theta([l'])$-eigenspace, where $\theta : H \to \widehat{H}$ is a natural identification). One also defines the equivariat multivariable Poincaré series, $P(\mathbf{t}) = -H(\mathbf{t}) \cdot \prod_v (1 - t_v^{-1})$. Both $H(\mathbf{t})$ and $P(\mathbf{t})$ have an H-decomposition, $H(\mathbf{t}) = \sum_{h\in H} H_h(\mathbf{t})$, $P(\mathbf{t}) = \sum_{h\in H} P_h(\mathbf{t})$, such that H_0 and P_0 are the Hilbert and Poincaré series associated with the filtration $\{\mathcal{F}(l)\}_{l\in L}$ of the algebra of invariants $\mathcal{O}_{X,o} = (\mathcal{O}_{X_a,o})^H$.

It turns out that H and P determine each other, and they also determine all the numerical invariants of type $\dim \frac{H^0(\mathscr{O}_{\widetilde{X}}(-l'))}{H^0(\mathscr{O}_{\widetilde{X}}(-l'-l))}$ ($l' \in L'$, $l \in L_{>0}$) and $h^1(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(-l'))$.

In some cases (e.g. cyclic quotient) $P(\mathbf{t})$ can be computed directly and characterized topologically from Γ. This leads to consider a combinatorial series $Z(\mathbf{t})$ (already mentioned), defined for any Γ. The identity $P(\mathbf{t}) = Z(\mathbf{t})$ holds e.g. for rational singularities (proved by Campillo, Delgado and Gusein-Zade), minimally elliptic or weighted homogeneous singularities. However, when we fix a topological type, $P(\mathbf{t})$ might vary with its analytic type. Hence it is natural to ask: (a) what is a possible list of series, whose members are realized as $P(\mathbf{t})$, when we vary the analytic types, and (b) what are the principles how the analytic structure makes its choice for $P(\mathbf{t})$? (These are very hard questions, we will not provide answers for them.) A simpler question, focusing on the candidate $Z(\mathbf{t})$ merely, is the following: for what analytic structures does the identity $P(\mathbf{t}) = Z(\mathbf{t})$ hold? It turns out that $P(\mathbf{t}) = Z(\mathbf{t})$ holds if and only if (X, o) is splice quotient. In Sect. 8.5 we will spend considerable time with the study of the invariant of the splice quotients. In fact, our investigation starts with germs, which satisfy the 'end curve condition'. We prove for them $P(\mathbf{t}) = Z(\mathbf{t})$, we determine topologically and explicitly $\mathscr{S}'_{an}$, we determine the base points of the maximal ideal of $\mathscr{O}_{X,o}$, and we compute the multiplicity of (X, o) as well. Then, we prove that such germs are in fact the splice quotients, this is formulated by the 'End Curve Theorem', firstly proved by Neumann and Wahl.

Then we return to the general case of singularities (with $b_1(L_X) = 0$).

The series $Z(\mathbf{t})$ has many manifestations, we will provide five characterizations of it. One of them is based on a family of linear subspace arrangements (see below), another one on the space of effective Cartier divisors supported on E, yet another one is based on the weighted cubes. All the manifestations carry important geometrical interpretations and consequences.

The pair $P \leftrightarrow Z$, $P(\mathbf{t}) = \sum_{l'} \mathfrak{p}(l')\mathbf{t}^{l'}$, $Z(\mathbf{t}) = \sum_{l'} \mathfrak{z}(l')\mathbf{t}^{l'}$ is just the starting of a list of *pairs* of analytic-topological invariants. Let us mention another one. For each $l' \in L'$ we construct from Γ a linear subspace arrangement $\{T(l')_v\}_{v \in \mathscr{V}} \subset T(l')$ such that $\chi_{top}(\mathbb{P}(T(l') \setminus \cup_v T(l')_v)) = \mathfrak{z}(l')$. Moreover, each analytic type supported on Γ determines in each $T(l')$ a linear subspace $A(l')$, hence also a linear subspace arrangement $\{A(l')_v\}_{v \in \mathscr{V}} \subset A(l')$, defined by $A(l')_v := A(l') \cap T(l')_v$. It turns out that $\chi_{top}(\mathbb{P}(A(l') \setminus \cup_v A(l')_v)) = \mathfrak{p}(l')$. This gives the possibility to extend $P(\mathbf{t})$ and $Z(\mathbf{t})$ in several ways, e.g. the motivic version replaces the topological Euler characteristic by the corresponding class in the Grothendieck ring of varieties.

For any subset $\mathscr{I} \subset \mathscr{V}$ we can reduce the variables (in both P and Z) by taking $t_v = 1$ for every $v \notin \mathscr{I}$. One obtains the series $P_{h,\mathscr{I}}$ and $Z_{h,\mathscr{I}}$ (for any $h \in H$) in variables $\{t_v\}_{v \in \mathscr{I}}$. For example, when $\mathscr{I} = \{v_0\}$, $P_{0,\{v_0\}}(t_{v_0})$ equals the Poincaré series of the divisorial filtration associated with E_{v_0}. The case $|\mathscr{I}| = 1$ is exemplified via weighted homogeneous and superisolated germs (when we recover certain 'classical series' associated with the specific geometry). In the superisolated unicuspidal case the difference $Z_{0,\mathscr{I}}(t) - P_{0,\mathscr{I}}(t)$ is a symmetric polynomial and its vanishing obstructs the realizability of the projective curve C.

In 8.7, by a regularization procedure, we associate to a one-variable formal power series $F(t) = \sum_{i \geq 0} a_i t^i$ a new invariant, called the '*periodic constant* $\mathrm{pc}(F)$*of* F'. Suppose that there exists a positive integer p and a polynomial $\mathfrak{P}_p(t)$ such that $\sum_{0 \leq i < pn} a_i = \mathfrak{P}_p(n)$ for every $n \in \mathbb{Z}_{>0}$. Then we set $\mathrm{pc}(F) := \mathfrak{P}_p(0)$. (In certain concrete cases, the periodic constant measures the difference between the Hilbert function and its Hilbert polynomial.) In the book we will discuss several additivity formulae (both analytic and topological), where the correction term is given by a periodic constant associated with some series. The first such formula appears in Sect. 8.7.A, it is Okuma's additivity formula valid for the geometric genus. The setup is the following. Consider a normal surface singularity (X, o) and fix one of its resolutions $\phi : \widetilde{X} \to X$ with exceptional curve E and graph Γ. We fix a vertex $v \in \mathcal{V}$. Let $\cup_{j \in J} \Gamma_j$ be the connected components of $\Gamma \setminus \{v\}$, and let X_j be the singularities associated with the subgraphs Γ_j. Then (under some mild assumptions) $p_g(X, o) - \sum_j p_g(X_j, o_j) = \mathrm{pc}(P_{0,\{v\}})$.

Later we will generalize this in both analytic and topological directions by considering the periodic constant of multivariable series as well. This periodic constant will serve as correction term in generalized additivity formulae. In the $\mathcal{I} = \mathcal{V}$ case the Γ_j terms disappear, and we get direct (non-relative) formulae. E.g., the periodic constant of $\{P_h(\mathbf{t})\}_h$ is the equivariant geometric genus, while the periodic constant of $\{Z_h(\mathbf{t})\}_h$ is the Seiberg–Witten invariant of the link.

Chapter 9: Topological Invariants—The Seiberg–Witten Invariant In this part we assume that the link is a rational homology sphere. The first part introduces three topological invariants of the link: the Casson–Walker invariant, the Reidemeister–Turaev torsion and the Seiberg–Witten invariant. The Casson invariant assigns an integer to a connected oriented integral homology 3-sphere. Its generalization, the Casson–Walker invariant, assigns a rational number to an oriented rational homology 3-sphere. The original definition is based on a Heegaard splitting and on the behaviour of the induced SU_2-representation varieties.

Torsion was introduced in the context of 3-dimensional manifolds by Reidemeister in 1935 in order to give a classification of the 3-dimensional lens spaces up to combinatorial equivalence and homeomorphism [588]. It was extended to arbitrary dimensions by Franz [205]. Later it was studied by many authors (de Rham, Whitehead, Milnor, see e.g. [429]).

Even its original version had many powerful applications (e.g. the simple homotopy classification) and subtle connections with other invariants (e.g. with the first elementary ideal and the Alexander polynomial of $\pi_1(M)$, or with the linking form on $\mathrm{Tors}\, H_1(M)$). But, recent developments of low dimensional topology (certain reinterpretations and generalizations) transformed the torsion into one of the most important combinatorial invariants of 3-dimensional manifolds. The most important generalizations give rise to the sign-refined torsion and to the torsion associated with Euler structures, both developed by Turaev.

Turaev proved (for $b_1(M) \neq 0$) that the refined maximal abelian torsion is equivalent with the Seiberg–Witten invariant (at least up to a sign), hence it gives a possibility to compute combinatorially the Seiberg–Witten invariants. (The proof

uses the connection between the Alexander polynomial and the Seiberg–Witten invariant, established by Meng and Taubes, where they also use the theory of torsion.)

For those readers working in singularity theory, who might not be familiar with these recent developments in low dimensional topology, let us comment a little bit more on the Seiberg–Witten invariant from a historical point of view.

Gauge theory, originated in mathematical physics, had an enormous influence in the study of smooth three and four dimensional manifolds. In the 1980s Donaldson used anti–self-dual Yang–Mills ('instanton') equations to create new invariants of 4-manifolds. The theory produced a series of extraordinary consequences and applications. In 1994 Witten in [754], based on the (far simpler) 'monopole' equations, introduced a new set of invariants, which reproduced and considerably improved the earlier results for 4-manifolds.

In dimension three, the first ('gauge theoretical') homology theory goes back to the work of Floer (late 1980s), who using the anti–self-dual equation on the cylinder worked out invariants of integral homology spheres (and later, invariants of integral homology $S^1 \times S^2$'s). The Euler characteristic of this mod 8 graded theory was identified with the Casson invariant by Taubes.

After the discovery of the Seiberg–Witten equations and invariants associated with 4–manifolds, it was immediately applied for 3-manifolds too (via $M \times \mathbb{R}$). Since then we witness an intensive search for different (other types of) realizations of this invariant, (starting from different topological/geometrical properties of 3-manifolds). One of them is the Reidemeister–Turaev torsion. As we already said, a result of Meng–Taubes–Turaev identifies this torsion with the Seiberg–Witten invariant, whenever $H_1(M, \mathbb{Q}) \neq 0$. In the case $H_1(M, \mathbb{Q}) = 0$ Nicolaescu proved that the Seiberg–Witten invariant is the difference of the torsion and the Casson–Walker invariant [520]. In the present book this identity will be used as starting definition.

In fact, the Seiberg–Witten theory produces a homology theory (similarly as the Floer theory), called the Seiberg–Witten–Floer homology, whose Eluer characteristic is the Seiberg–Witten invariant. This initiated an intensive search for other homology theories of 3-manifolds with similar properties. The approach of Ozsváth and Szabó (bypassing some of the analytic problems with a novel topological set-up) around 2000 led to the Heegaard Floer homology of any closed, oriented, connected three-manifold (with the wished Euler characteristic property). Soon after, Kronheimer and Mrowka came out in 2004 with another construction, leading them to Monopole Floer homology. This theory is based on the Seiberg–Witten equations (and on their solution space), and as it has been verified later on, it produces an invariant isomorphic to the Heegaard Floer groups. Another approach to the Seiberg–Witten version of Floer homology is presented by Marcolli and Wang. A symplectic geometric construction was found by Hutchings, which is now called Embedded Contact homology, yet another definition of the same groups. The 'modified' Euler characterisic of the 'lattice cohomology' of the author, valid for negative definite plumbed rational homology spheres, is again the Seiberg–Witten invariant, cf. Chap. 11.

In our presentations we will not follow the original definitions of the Casson–Walker, Reidemeister–Turaev and the Seiberg–Witten invariants, but we will adopt definitions valid for negative definite plumbed 3-manifolds. These are combinatorial expressions in terms of Γ. [In fact, $\mathfrak{sw}(M)$ in our sense is $\mathfrak{sw}(-M)$ in the sense of Heegaard Floer $+$–theory HF^+, cf. 11.7.2. Our choice in 9.6.1 is motivated by the fact that in this way the Euler characteristic of the topological lattice cohomology $\mathbb{H}^*(M)$ will be $\mathfrak{sw}_\sigma$ (with our sign), and also we will have a sign–correction free comparison with the analytic theory, cf. 9.6.]

Both the torsion and the Seiberg–Witten invariant are indexed by H (or, by $\mathrm{Spin}^c(L_X)$). Our adopted definitions make it possible to combine them with methods from singularity theory and to obtain new powerful statements for these invariants (valid for singularity links). In particular, we will establish additivity formulae for them (in a similar decomposition context as in the above p_g-additivity formula). E.g., for the Seiberg–Witten invariant the correction term will be the periodic constant of $Z_{h,\{v\}}(t)$. Using such additivity formulae we can identify different combinations of invariants by induction.

We wish to emphasize that our additivity (or surgery) formulae are of different nature than those from low dimensional topology based on 'exact triangles'. In those cases one compares the invariants of three manifolds, in our case we compare the invariants of two manifolds (the one associated with Γ and another one associated with $\cup_{j\in J}\Gamma_j$) via a correction term, which is usually a periodic constant. The author does not know what the analogue of the periodic constant is (and of the series, which generate them) in the general context of 3-manifolds.

Next, we wish to understand how these invariants are reflected by the analytic structure of the singularity. The first such amazing connection is formulated by the Casson Invariant Conjecture (CIC) of Neumann and Wahl in the context of an isolated complete intersection surface singularity whose link is an integral homology sphere. Note that in such a case we have a well-defined (and unique) Milnor fiber F, and $\chi_{top}(F)$, $\sigma(F)$(=signature(F)) and $p_g(X,o)$ determine each other (up to the link). (Hence if for one of them one can prove that it is link-invariant, then all of them are so.) Then the CIC says that $\sigma(F)/8$ is the Casson invariant of the link. We present its verification for several cases (the conjecture is still open in general).

Since there are several important singularities, whose links are not integral homology spheres (but we have a big amount of knowledge about them), it is natural to ask for a generalization of the CIC to the case when the link is a rational homology sphere. This is the Seiberg–Witten Invariant Conjecture (SWIC), formulated by Nicolaescu and the author, which compares the (normalized) Seiberg–Witten invariant of the link with the equivariant geometric genera (both indexed by H). We verify it for several cases (rational, weighted homogeneous, or more generally for splice quotients). But we also show that for certain superisolated singularities it fails.

If $P(\mathbf{t}) = Z(\mathbf{t})$ then the SWIC holds. If the choice of an analytic structure for P is 'very different' than Z, then the SWIC fails. In particular, the failure of the SWIC gives hints for behaviours of the different analytic structures. The superisolated

case is a 'lucky' counterexample, since we can understand (later, in terms of lattice cohomology) how these germs make their choice for the geometric genus. (We will comment these facts again in the chapter of the lattice cohomology).

Chapter 10: Ehrhart Theory and the Seiberg–Witten Invariant Let us consider again the multivariable series $Z(\mathbf{t}) = \sum_{l' \in L'} \mathfrak{z}(l')\mathbf{t}^{l'}$ with H–decomposition $Z(\mathbf{t}) = \sum_{h \in H} Z_h(\mathbf{t})$, $Z_h(\mathbf{t}) = \sum_{[l']=h} \mathfrak{z}(l')\mathbf{t}^{l'}$, associated with the graph Γ.

It turns out that several invariants of the topological type can be related with the *counting function* of the coefficients of $Z(\mathbf{t})$, namely with the function $l' \mapsto Q_h(l') := \sum_{\tilde{l}' \not\geq l'} \mathfrak{z}_h(\tilde{l}')$. (Here it is worth to mention that at the analytic side, the similarly defined counting function of the coefficients of $P(\mathbf{t})$ is exactly the Hilbert series $H(\mathbf{t})$. This means that Q_h is the topological pair of H_h.)

Since Z_h is supported on $\mathscr{S}'$, this sum is indexed by $\{\tilde{l}' \in L' : [\tilde{l}'] = h,\ \tilde{l}' \in \mathscr{S}',\ \tilde{l}' \not\geq l'\}$. This consists of those lattice points of L', which have class h (are 'collored' by $h \in H$), and are contained in a (non-convex) finite polytope. Some of the walls/facets of this polytope (those given by $\tilde{l}' \not\geq l'$) are 'moving' with l'. The counting function along this 'equivariant, rational vector-dilated polytope' creates the bridge with the multivariable equivariant Ehrhart theory.

The 'truncation' $\sum_{\tilde{l}' \not\geq l'}$ is motivated by concrete formulae imposed by singularity theory (topology of the link). E.g., the formula from 9.6.21 relates the counting function (defined by this truncation) with the Seiberg–Witten invariant of the link. This truncation imposes the inequalities, which determine the dilated polytope.

In fact, the connection with the Ehrhart theory is presented in a more general context. We start with a pair of lattices $L \subset L'$ of the same rank, a fixed basis $\{E_v\}_v$ of L, and with a multivariable rational function $f(\mathbf{t})$, whose denominator $\widetilde{f}(\mathbf{t})$ has the form $\prod_{i=1}^{n}(1 - \mathbf{t}^{a_i})$, where $a_i \in L'$ and all the (rational) E_v-coordinates are strictly positive. Then the summation set of the counting function of $\widetilde{f}(\mathbf{t})$ (with truncation $\sum_{\tilde{l}' \not\geq l'}$, $l', \widetilde{l}' \in L'$) is again a dilated polytope (with lattice points colored by $H = L'/L$). This counting function is a piecewise quasipolynomial. This means that $L \otimes \mathbb{R}$ decomposes into several (cone-type) chambers such that in each chamber the counting function of $\widetilde{f}$ is a quasipolynomial. The chambers are determined by a general algorithm from $\{a_i\}_i$ and $\{-E_v\}_v$, as part of the general theory of 'vector partitions', developed by Sturmfels, Brion, Vergne and others. The counting function of f is determined from the counting function of $\widetilde{f}$ as a sum depending on the coefficients of the numerator. It is a quasipolynomial in a shifted cone pushed deeply inside the chambers of $\widetilde{f}$.

In our application, regarding $f(\mathbf{t}) = Z(\mathbf{t})$ we are basically interested in those chambers, which intersect the Lipman's cone $\mathscr{S}'$. The reason is that from the quasipolynomial associated with such a chamber we can determine the Seiberg–Witten invariant of the link. The general procedure is the following. If the counting function of a series (like $f(\mathbf{t})$, or $Z(\mathbf{t})$) evaluated 'deeply' in a cone (chamber) behaves as a quasipolynomial, then we define the 'multivariable periodic constant of the series in that cone' as the evaluation of this quasipolynomial at zero. In the case of a plumbing graph and the series $Z(\mathbf{t})$, and for any chamber which

intersects the Lipman's cone, the multivariable periodic constant is the Seiberg–Witten invariant (where the H-decomposition corresponds to the spin^c structures). Note that this procedure, in general, is a hardly computable regularization procedure of the summation of the coefficients.

Using the general theory of Ehrhart and of vector partitions, applied for $Q_h(l') := \sum_{\tilde{l}' \not\geq l'} \mathfrak{z}_h(\tilde{l}')$, the chamber decomposition of $Z(\mathbf{t})$ is not 'optimal': Lipman's cone is cut in several chambers. Therefore, we reorganize our summation/truncation (hence our dilated polytopes as well) in such a way that in the new context the new chambers are larger, and in particular the Lipman's cone will be one of the chambers.

More precisely, we replace $\sum_{\tilde{l}' \not\geq l'} \mathfrak{z}_h(\tilde{l}')$ by $\sum_{\mathscr{I} \subset \mathscr{V}} (-1)^{|\mathscr{I}|+1} q_{h,\mathscr{I}}(l')$, where $q_{h,\mathscr{I}}(l')$ equals $\sum_{\tilde{l}'|_{\mathscr{I}} \prec l'|_{\mathscr{I}}} \mathfrak{z}_h(\tilde{l}')$ and the summation runs over those $\tilde{l}'$ for which all the $\mathscr{I}$–coordinates of $\tilde{l}' - l'$ are strictly negative. The new counting functions $l' \mapsto q_{h,\mathscr{I}}(l')$ are called 'modified counting functions'. They are indexed by subsets $\mathscr{I} \subset \mathscr{V}$ and have several additional pleasant properties as well. E.g., the associated dilated polytopes are convex, their inclusion properties allow us to reduce the variables, and the chambers are larger. This leads to important additivity formulae of the corresponding (modified) quasipolynomials and of the topological invariants (e.g. of the Seiberg–Witten invariant). In such formulae, one compares invariants associated with Γ and with $\Gamma \setminus \mathscr{I}$, where $\mathscr{I}$ is any subset of vertices. Again, the correction terms are multivariable periodic constants. Furthermore, this technique allows us to reduce the Γ-dependence of the invariants to the subset of 'bad' vertices.

In the above presentation, the Ehrhart theory might appear as a nice reorganization tool. However, using the symmetry of $Z(\mathbf{t})$, and the duality provided by the equivariant Ehrhart–Macdonald–Stanley reciprocity law, we obtain really unexpected consequences as well.

Firstly, we are able to decompose $Z(\mathbf{t})$ in a unique way (according to a conveniently chosen decomposition of its support) as $Z^+(\mathbf{t}) + Z^-(\mathbf{t})$, where $Z^+(\mathbf{t})$ is a finite sum—it is called the 'polynomial part' of $Z(\mathbf{t})$, and $Z^-(\mathbf{t})$ is a rational function with negative degree in all the variables. The main point is that $Z^+(\mathbf{t})|_{t_v=1,\forall\, v}$ is exactly the multivariable periodic constant of $Z_h(\mathbf{t})$. Hence, in this way, we recover the periodic constant as a finite sum of the coefficients of $Z_h(\mathbf{t})$ instead of the hard (mostly theoretical) regularization procedure. In particular, the Seiberg–Witten invariant itself appears as a convenient (precisely determined) finite sum of the coefficients of Z_h. Therefore, the polynomial part Z^+ is, in fact, a multivariable polynomial extension of the Seiberg–Witten invariant.

In fact, via reinterpretation of the polytopes used in the Ehrhart part, the Seiberg–Witten invariant appears as the alternating sum of number of lattice points in certain (well-characterized) polytopes.

Chapter 11: Lattice Cohomology The chapter has three main parts. They focus on the *topological, combinatorial and analytic* lattice cohomologies. The topological and analytic lattice cohomologies are associated with a surface singularity with a

$\mathbb{Q}HS^3$ link (though the analytic one can be extended to arbitrary dimension). The combinatorial lattice cohomology is purely combinatorial.

The common setup is the following. Firstly, we need a free $\mathbb{Z}$-module with a fixed basis, $\mathbb{Z}^s$. Here, the linear structure is unessential, the really essential object is the induced cubical decomposition of $\mathbb{Z}^s \otimes_{\mathbb{Z}} \mathbb{R}$, determined by the lattice points $\mathbb{Z}^s$ and the basis (coding the combinatorial proximity). In fact, in most cases we can replace $\mathbb{Z}^s \otimes_{\mathbb{Z}} \mathbb{R}$ by a rectangle $R(l_1, l_2) = \{x \in \mathbb{Z}^s \otimes_{\mathbb{Z}} \mathbb{R} : l_1 \leq x \leq l_2\}$.

Then, we also need a system of weights of the cubes. In most of the cases they are induced by a function $w_0 : \mathbb{Z}^s \to \mathbb{Z}$, and the weight of a cube $\square$ is $\max\{w_0(l) : l \text{ is a vertex of } \square\}$.

The output of the construction is $\mathbb{H}^*(w) = \oplus_{q \geq 0} \mathbb{H}^q(w)$, where each $\mathbb{H}^q(w)$ is a graded $\mathbb{Z}[U]$-module. It measures the compatibility of the lattice points (and their neighboring properties) with the level sets of w_0. In 11.1 we provide two constructions. The first one is algebraic, based on the construction of a chain complex. The second approach determines a tower of spaces $\{S_n\}_{n \in \mathbb{Z}}$, $\cdots \subset S_n \subset S_{n+1} \subset \cdots$, such that $\mathbb{H}^q(w) = \oplus_{n \in \mathbb{Z}} H^q(S_n, \mathbb{Z})$, and the U-operator is induced by the cohomological restrictions $H^q(S_{n+1}) \to H^q(S_n)$. In fact, $S_n := \cup\{\square : w(\square) \leq n\}$. Via reduced cohomologies and the decomposition $H^*(S_n, \mathbb{Z}) = \mathbb{Z} \oplus \widetilde{H}^*(S^n, \mathbb{Z})$, we get a $\mathbb{Z}[U]$-module direct sum decomposition $\mathbb{H}^*(w) = \mathscr{T}^+_{\min w_0} \oplus \mathbb{H}^*_{red}(w)$, where $\mathscr{T}^+_{\min w_0}$ is a tower of $\mathbb{Z}$'s in degrees $\geq \min w_0$, and $\mathbb{H}^*_{red}(w)$ has a finite $\mathbb{Z}$–rank. We define the Euler characteristic of $\mathbb{H}^*(w)$ by $eu(\mathbb{H}^*) := -\min\ w_0 + \sum_q (-1)^q \mathrm{rank}_{\mathbb{Z}} \mathbb{H}^q_{red}(w)$.

It is convenient to define $eu(\mathbb{H}^0) := -\min\ w_0 + \mathrm{rank}_{\mathbb{Z}} \mathbb{H}^0_{red}(w)$ as well.

In our first **topological** context the weight function w_0 is defined as follows. We fix a negative definite connected plumbing graph Γ with $M(\Gamma)$ a $\mathbb{Q}HS^3$. The lattice $\mathbb{Z}^s$ is the lattice L (with a natural basis associated with the vertices), and we define $w_0(l)$ as $\chi_k(l) = -(l, l+k)/2$, where k is a characteristic element of L. Convenient choices k_r of characteristic elements provide the topological lattice cohomology $\{\mathbb{H}^*_{top}(\Gamma, k_r)\}_{k_r}$, indexed by the spin^c structures $\sigma[k_r] \in \mathrm{Spin}^c(M(\Gamma))$. It turns out that it is independent of the choice of the graph Γ, it depends only on the 3-manifold $M(\Gamma)$ and its spin^c structure $\sigma[k_r]$. Furthermore, its Euler characteristic is the (normalized) Seiberg–Witten invariant of $M(\Gamma)$:

$$eu(\mathbb{H}^*_{top}(\Gamma, k_r)) = \mathfrak{sw}_{\sigma[k_r]}(M(\Gamma)) - (k_r^2 + |\mathscr{V}|)/8.$$

From singularity point of view, it characterizes e.g. the rational and elliptic graphs.

The $q = 0$ case can be improved: instead of $\oplus_n H^0(S_n, \mathbb{Z})$, the information can be coded in a sharper way via 'graded roots'. (They are graded trees, such that their grading endows them with a special shape, as the roots of a plant.) The graded root determines $\mathbb{H}^0$, but it provides deeper information, and a more intuitive picture.

In fact, we show that the homotopy type of the whole tower of spaces $\{S_n\}_n$ is independent of the choice of Γ, hence it is a much stronger invariant than the lattice cohomology. It can be reinterpreted as the '*lattice homotopy type*' of $M(\Gamma)$.

In 11.3 we determine the graded roots of *almost rational* (AR) graphs. (For AR graphs, and the structure of the spaces S_n see also this introduction, Chap. 7.) E.g., star–shaped graphs and resolution graphs of superisolated germs associated with rational unicuspidal curves are AR. (In the second case, the link is a surgery 3–manifold $S^3_{-d}(K)$.) For them we prove the vanishing $\mathbb{H}^{\geq 1}(\Gamma, k) = 0$. In particular, $\mathbb{H}^*$ is determined from the graded root. On the other hand, the root is explicitly computed by a concatenated computation sequence (path), provided by generalized Laufer computation sequences. In the star–shaped case, the output fits perfectly with the analytic computation of the Poincaré series and geometric genus of weighted homogeneous singularities. In the case of surgery 3-manifolds (superisolated case) we determine the graded root in terms of the semigroup of the cusp.

Such results are interesting from purely topological point of view as well. Indeed, for AR graphs Γ the author in [480] proved that $\mathbb{H}^*_{top}(M(\Gamma))$ equals the Heegaard Floer cohomology of $M(\Gamma)$, hence in this way we provide new formulae for these cohomologies for such plumbed 3-manifolds. For more information regarding the connection with the Heegaard Floer theory see 11.7.

The topological lattice cohomology plays a role in the analytic study of the germs as well. E.g., we verify the SWIC for rational unicuspidal superisolated singularities. It turns out that the validity of the SWIC is equivalent with a special property of the semigroup of the cusp, we call it 'Semigroup Distribution Property' (SDP). This was verified in several cases via algebraic geometry/singularity theory in [198], the complete proof was given in [68] via Heegaard Floer Knot theory.

In 11.3.C we suggest a deeper geometric reason, which stands behind the validity of the SWIC: We expect the stability of the (canonical) graded roots along K^2_{min}-constant deformations of Gorenstein surface singularities.

The most important tool for the computation of the topological lattice cohomology is the (Topological) Reduction Theorem. If $\overline{\mathcal{V}} \subset \mathcal{V}$ is a set of 'bad' vertices (this can be e.g. the set of nodes, or even a smaller subset of vertices) then a conveniently defined new weight function on $\mathbb{Z}^{|\overline{\mathcal{V}}|}$ provides the same lattice cohomology. The new weight is determined by universal cycles determined by generalized Laufer type algorithms (see Chap. 7). In the AR case ($|\overline{\mathcal{V}}| = 1$) we provide complete proofs, however for the proof of the general case we refer to [350].

Using the Reduction Theorem we determine the lattice cohomology of surgery 3-manifolds $S^3_{-d}(\#_i K_i)$ ($d > 0$), in particular, of the links of all superisolated germs. The main combinatorial input is the set of semigroups of the cusps (algebraic knots K_i). If the number of cusps is ν, then $\mathbb{H}^{\geq \nu} = 0$, however, all other groups might be non-trivial. We also show that from $\mathbb{H}^*$ one can recover the number of cusps as well.

Using topological lattice cohomology we can understand better the validity or failure of the SWIC for superisolated singularities. It turns out that the generalization (for more cusps and for their semigroups) of the Semigroup Distribution Property (valid by Heegaard Floer Knot theory, cf. [68]) is equivalent with $eu\left(\mathbb{H}^0_{top}(S^3_{-d}(\#_i K_i), K)\right) = p_g$. (Here K stays for the canonical characteristic element, or for the Chern class of the canonical bundle.) The SWIC requires

$eu\left(\mathbb{H}^*_{top}(S^3_{-d}(\#_i K_i), K)\right) = p_g$. Hence, the possible failure of the SWIC for superisolated germs is provided by the non-vanishing of the higher lattice cohomology groups. (It is worth to mention that if we compare the lattice cohomology with the Heegaard Floer cohomology, then in the second one the q-grading of $\{\mathbb{H}^q\}_{q\geq 0}$ is invisible. This is a very important additional property of the lattice cohomology.)

Next, for any increasing path γ in L we can consider the restriction of the weight function to the lattice points and segments of γ. In this way we obtain a lattice cohomology $\mathbb{H}^0(\gamma)$ and its Euler characteristic $eu(\mathbb{H}^0_{top}(\gamma)) = -\min w|_\gamma + \operatorname{rank}_{\mathbb{Z}}\mathbb{H}^0_{top,red}(\gamma)$. This is the *topological path lattice cohomology* of γ.

It turns out that $\min_\gamma eu(\mathbb{H}^0_{top}(\gamma)) = \mathrm{Path}^i$ (where Path^i is the invariants discussed in Chap. 6). We show that for superisolated germs (with an arbitrary number of cusps, even if the SWIC fails) the identity $p_g = \min_\gamma eu(\mathbb{H}^0_{top}(\gamma))$ holds. That is, the analytic type of a superisolated germ, for its p_g, 'choices' this topological numerical invariant from the package of lattice cohomology. For these surgery 3–manifolds $\min_\gamma eu(\mathbb{H}^0_{top}(\gamma)) = eu(\mathbb{H}^0_{top})$ holds as well.

In 11.6.A we discuss germs with Newton non-degenerate principal part using topological lattice cohomology. For them we prove that (surprisingly) they simultaneously satisfy both identities $p_g = \min_\gamma eu(\mathbb{H}^0(\gamma))$ and $p_g = eu(\mathbb{H}^*(M(\Gamma), K))$. This in particular means that their weight table $\{\chi(l)\}_{l\in L}$ is very peculiar. (This property can be compared with a similar property of the *combinatorial lattice cohomology* associated with certain combinatorial weight functions, see below.)

The next part of Chap. 11 (Sect. 11.8) is a preparatory part for the analytic lattice cohomology, though it has an independent value as well: we define the **combinatorial lattice cohomology**. We consider a weight function $l \mapsto w_0(l) = h(l)+h^\circ(l)-h^\circ(0)$ for l in some rectangle $R(0,c)$, where $h(0) = 0$, h is increasing, h° is decreasing. Our main interest is when they satisfy certain additional properties. One of them is the *stability property* of h. This is a consequence of the *matroid rank inequality*, usually satisfied by the Hilbert function associated with 'nice' filtrations. The second requirement is that the pair (h, h°) satisfies the *Combinatorial Duality Property* (CDP) (in applications usually implied by some geometric duality).

The point is that if we consider the lattice cohomology associated with w_0 having these properties then $eu(\mathbb{H}^*(w_0)) = eu(\mathbb{H}^0(\gamma, w_0))$ for any increasing path γ connecting 0 with c, and this common values is $h^\circ(0) - h^\circ(c)$.

The range of applications is rather extensive. E.g., having a (convenient) Newton diagram in $\mathbb{R}^n_{\geq 0}$, using the weights of the facets we can define a *combinatorial lattice cohomology associated with a Newton diagram* whose Euler characteristic is the number of strict positive lattice points not above the diagram.

Another case provides the **analytic lattice cohomology** of surface singularities. Since this theory is very new and in rapid development, we decided to present only certain limited part of it. In particular, we describe only the module associated with the canonical spin^c structure (this is the non–equivariant case associated with the local ring $\mathcal{O}_{X,o}$). See [9] for the general case and [10, 11] for other dimensions.

In this case, we fix a resolution, and h is the restriction of the Hilbert function to $R(0,c)$, where $c \geq Z_K$. Its pair is $h^\circ(l) = p_g - h^1(\mathcal{O}_l)$. One shows that the

cohomology $\mathbb{H}^*_{an,0}(X, o)$ is independent of the choice of c and of the resolution, it is called the *analytic lattice cohomology* (associated with the canonical spin^c structure). On the other hand, it might depend essentially on the choice of the analytic structure of (X, o) (on a fixed topological type). This can be seen already on the case of elliptic topological types (with different analytic realizations), or by analysing the analytic lattice cohomology associated with the generic analytic structure.

Furthermore, the pair (h, h°) satisfies the stability and CDP properties, hence $eu(\mathbb{H}^*_{an}(X, o)) = p_g$, that is, it is a categorification of the geometric genus.

The module $\mathbb{H}^*_{an,0}(X, o)$ admits a graded $\mathbb{Z}[U]$-linear morphism to the topological lattice cohomology $\mathbb{H}^*_{top}(M, K)$ (associated with the canonical spin^c structure). This map is an isomorphism for weighted homogeneous germs, or for superisolated unicuspidal germs (to prove this we need to establish the 'Analytic Reduction Theorem' too). However, they differ in general (since $\mathbb{H}^*_{an}(X, o)$ mirror the different analytic structures). Hence, the comparison of the Seiberg–Witten invariant of the link and the geometric genus of the germ is 'lifted' to the level of their categorifications: to the topological and analytical lattice cohomologies.

We formulate several open problems and conjectures regarding this cohomology pair. Furthermore, we expect that the analytic lattice cohomology will play a crucial role in the analytic study of the germs as well: e.g. we conjecture the stability of $\mathbb{H}^*_{an}$ along p_g-constant deformations.

Chapter 12: Appendix—Complex Analytic Spaces Here we collect some general material, which might help the reader during the reading of the book.

1.5 What Is New?

The book gathers in one place considerable material of the literature, the presentation of some 'classical' parts follows the original sources, but some of them are reorganized, complemented and supported by new statements and examples. Here is a (non complete) list of statements, examples and proofs, which has not been published yet:

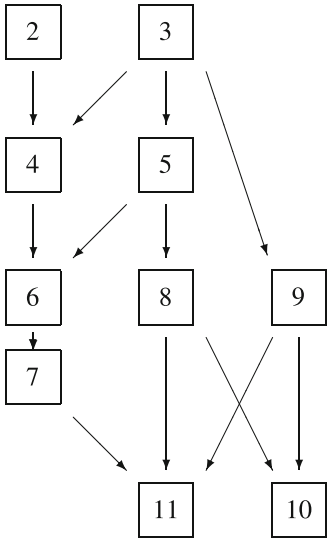
- Chapter 3: the 'optimal deficiency presentation of $\pi_1(L_X)$' in 3.7.8 is new, and the proof of Mumford's Theorem 3.7.10 is slightly modified.
- Chapter 4: parts 4.1.39, 4.2.4, 4.2.5, 4.2.7.
- Chapter 5: the construction of the 'special analytic plumbing' 5.1.36 is new, as well as the fact that it admits a weighted homogeneous analytic structure, as stated in 5.1.37.
- Chapter 6: the proof of Tomari-Watanabe's theorem 6.1.11; the definition of the analytic/topological canonical coverings 6.1.19 together with several statements in 6.3.B; the local version of the Kawamata–Viehweg Vanishing Theorem 6.4.3 together with several consequences, and different versions of the 'Unbalances Vanishing Theorem' 6.4.12; certain properties of $\mathscr{S}'_{an}$, formulae

from 6.8.18, 6.8.19, examples 6.8.22, 6.8.23, 6.8.24, and lemma 6.8.38 regarding Path^i.

- Chapter 7: parts of 7.2.40 and 7.2.42 regarding elliptic sequences and Corollary 7.2.66; the proof of Proposition 7.3.5 (which was rather sketchy in [480]) is now complete; in 7.3.A several statements regarding the homotopy type of the spaces $S_{k,n}$: e.g. 7.3.12, 7.3.14, 7.3.31, 7.3.35, 7.3.37; and the organization of the 'bad' vertices 7.3.21.
- Chapter 8: Proposition 8.6.11, and several generalizations of Okuma's additivity formula in 8.7.A
- Chapter 9: we provide new proofs for certain Casson invariant formulae in 9.1, and for the verification in certain cases of the Casson Invariant Conjecture in 9.2.
- Chapter 10: the two technical lemmas from 10.3.B are new, they will be fully used in the proof of several statements regarding the modified counting function.
- Chapter 11: Proposition 11.5.21 is new (it shows that from the lattice cohomology one can recover the number of link-components of the surgery 3-manifold). The material of the analytic lattice cohomology is a newly born theory.

1.6 Organization of the Sections

Although there are many interactions between the sections, in a first reading focused on specific parts one can leave out some of them. The next diagram shows how different sections are built on each other (considering the most essential connections). One might skip even certain parts which contain examples (e.g. Chap. 5, 7.1, 7.2 or 11.5).



1.7 What Is Not Covered

In the choice of the material of the book we had to be very selective. In particular, several very important areas of surface singularity theory are not present, or are mentioned only partially. Here is (again, a non-complete) list of such areas. Some of them are already covered in very good monograph (we provide some references, where the interested reader will find the needed information to start).

- deformation of singularities [141, 233, 240, 664],
- smoothing of singularities, rational homology ball smoothing, the new conjectures of Wahl on the Milnor and Tjurina numbers and $h^1(S_X)$ [397, 667, 730, 732]
- connection of the weighted homogeneous singularities with the geometry of Lie groups [153, 155, 431, 622],
- weighted superisolated (Lê–Yomdin) singularities [26]
- Lipschitz geometry of singularities [510]
- reflexive modules, maximal Cohen–Macaulay modules, McKay correspondence [156, 188, 216, 295, 592, 757]
- polar geometry, Nash transform [215, 380, 381, 677, 681, 682]
- the Abel map of normal surface singularities [452, 453, 455, 456]
- exact sequences of lattice cohomology [231, 483]
- curve singularities embedded into surfaces, relative Poincaré series [98, 116, 117, 481]
- more connection with topology, symplectic fillings, contact geometry, Floer teory, Heegaard Floer theory.

In **References** we collected a considerable list of titles, which are related with the treated material (though not all of them are specifically cited in the body of the book). In this way we try to help the reader to find sufficient support for the covered topics.

Chapter 2
Resolution of Surface Singularities

2.1 Modifications and Resolutions

We introduce the notions of modification and resolution. For resolutions $\widetilde{X}$ of a surface singularity we also define the exceptional curve, the lattice $L = H_2(\widetilde{X}, \mathbb{Z})$ (with its intersection form) and its dual L', the finite discriminant group $H = L'/L$, and the Lipman's cone. We prove several of their properties, e.g., the negative definiteness of the intersection from. The existence of a resolution, and the uniqueness of the minimal one, is also proved. ♣

Definition 2.1.1 (Local Modifications, Resolutions) Consider the germ (X, o) of a complex analytic set with singular points $(\Sigma(X), o)$.

Let $\phi : \widetilde{X} \to X$ be a proper analytic map, where $\widetilde{X}$ is normal and X is a sufficiently small representative of (X, o). We also set $E := \phi^{-1}(\Sigma(X))$.

We say that ϕ is a local *partial resolution* (or *modification*) of (X, o) if $\widetilde{X} \setminus E$ is dense in $\widetilde{X}$ and ϕ is an isomorphism over the complement of an analytic set $A \subset X$, which contains $\Sigma(X)$, but does not contain any of the irreducible components of X.

Given two modifications $\phi_i : \widetilde{X}_i \to X_i$ $(i = 1, 2)$ of (X, o), we say that ϕ_1 *dominates* ϕ_2 if after replacing both X_i by some smaller representative X, there exists an analytic map $\psi : \widetilde{X}_1 \to \widetilde{X}_2$ such that $\phi_2 \circ \psi = \phi_1$.

A modification ϕ is called *resolution* if $\widetilde{X}$ is a smooth analytic manifold.

A resolution ϕ is called *strong* if the restriction $\phi : \phi^{-1}(X \setminus \Sigma(X)) \to X \setminus \Sigma(X)$ is an isomorphism and $\phi^{-1}(\Sigma(X))$ is a normal crossing divisor in $\widetilde{X}$. (In the literature such a ϕ is also called 'log–resolution'.)

A strong resolution is called *good* if all the irreducible components of $\phi^{-1}(\Sigma(X))$ (with reduced structure) are smooth (that is, if they have no self-intersections).

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A resolution is called *minimal* if it does not dominate (with ψ non-isomorphism) any other resolution. One defines similarly the *minimal strong* or *minimal good resolutions*.

Definition 2.1.2 (Modifications, Resolutions [55, 225]) Let X be a complex analytic set with singular points $\Sigma(X)$. A proper analytic map $\phi : \widetilde{X} \to X$ is a modification (resolution) if for any $o \in \Sigma(X)$ a convenient restriction is a local modification (resolution) of the germ (X, o).

Example 2.1.3 (See 12.2.6) The normalization $n : \widehat{X} \to X$ is a modification. In this case n is finite, in particular $n^{-1}(o)$ is a finite set $\{\widehat{o}_i\}_i$. If (X, o) has dimension one, then the normalization is a good resolution. If (X, o) has dimension two, then each $(\widehat{X}, \widehat{o}_i)$ is normal, hence irreducible with an isolated singularity.

By the universal property of the normalization (cf. 12.2.6) any resolution dominates the normalization. Hence, any resolution procedure might start with the normalization after which one has to resolve normal singular germs.

Remark 2.1.4 ([225]) If $\phi : \widetilde{X} \to X$ is a modification with X normal, then the minimal set $A \subset X$, with ϕ isomorphism over $X \setminus A$, has codimension ≥ 2.

Example 2.1.5 Several modifications appear as incidence varieties.

(a) **(Blow up of a point**, see e.g. [632, II.4.1]) Let $(X, o) \subset (\mathbb{C}^n, o)$ be a germ with an isolated singularity $\Sigma(X) = \{o\}$, and let $\mathbb{P}^{n-1}$ stay for the parameter space of lines ℓ in $\mathbb{C}^n$ with $\ell \ni o$. Then $\phi' : B := \{(x, \ell) \in \mathbb{C}^n \times \mathbb{P}^{n-1} : x \in \ell\} \to \mathbb{C}^n$, $\phi'(x, \ell) = x$, is a modification of $\mathbb{C}^n$, which is an isomorphism over $\mathbb{C}^n \setminus o$. It is called the *blow up* of $\mathbb{C}^n$ at o. Next, consider the subspace $\{(x, \ell) \in X \times \mathbb{P}^{n-1} : x \in \ell,\ x \neq 0\}$ and let $\widetilde{X}$ be its closure in B. $\widetilde{X}$ is called the *strict transform* of X along ϕ'. The map $\phi : \widetilde{X} \to X$, induced by ϕ', is called the blow up of X at o. If (X, o) is smooth then $\phi^{-1}(o)$ is isomorphic to $\mathbb{P}^{\dim X - 1}$. In general, $\phi^{-1}(o)$ considered with its reduces structure is the reduced projectivized tangent cone of X at $o \in X$.

(b) **(Affine cones)** Let $C \subset \mathbb{P}^{n-1}$ be an irreducible smooth projective curve in $\mathbb{P}^{n-1}$, and let $X \subset \mathbb{C}^n$ be the (complex) affine cone over it with vertex o being the origin. Though X has an isolated singularity, it is not automatically normal at o. The embedding $C \subset \mathbb{P}^{n-1}$ is called *projectively normal* if (X, o) is normal. E.g., the embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^3$, $[u : v] \mapsto [u^4 : u^3v : uv^3, v^4]$ is not projectively normal. The normalization of its cone is the cone over the embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^4$, $[u : v] \mapsto [u^4 : u^3v : u^2v^2 : uv^3 : v^4]$.

Independently on whether (X, o) is normal or not, if we blow up $o \in X$ (as in part (a)), we obtain a resolution of (X, o) with $\widetilde{X}$ the total space of the line bundle $\mathcal{O}_{\mathbb{P}^{n-1}}(-1)|_C$, such that its zero section C is $\phi^{-1}(o)$ (considered with its reduced structure).

(c) **(Atiyah flop)** Set $(X, o) := (\{xt = zy\}, o) \subset (\mathbb{C}^4, o)$. Consider the family of planes $H_{[\alpha:\beta]} := \{\beta x = \alpha z,\ \beta y = \alpha t\}$ parametrized by $[\alpha : \beta] \in \mathbb{P}^1$.

Then the incidence variety $\widetilde{X} := \{(p, [\alpha : \beta]) \in X \times \mathbb{P}^1 : p \in H_{[\alpha:\beta]}\}$ together with the projection $\phi : \widetilde{X} \hookrightarrow X \times \mathbb{P}^1 \to X$ is a resolution of X with $\phi^{-1}(o) = \mathbb{P}^1$. Hence, $\phi^{-1}(o)$ might have codimension > 1. Symmetrically, taking $H'_{[\alpha,\beta]} := \{\beta x = \alpha y,\ \beta z = \alpha t\}$ one obtains another resolution. Both of them are minimal, however they are not isomorphic: the first one resolves (makes regular) the map $X \dashrightarrow \mathbb{P}^1$, $(x, y, z, t) \mapsto [x : z] = [y : t]$, the second one resolves $(x, y, z, t) \mapsto [x : y] = [z : t]$. Both of them are dominated by the resolution obtained by blow up $o \in X$ with $E = \mathbb{P}^1 \times \mathbb{P}^1$.

In particular, this (X, o) does not admit a unique minimal resolution, cf. 2.4.7.

(d) **(Degeneration loci of matrices)** For $k \geq n \geq 2$ set $X := \{M \in \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n) : \mathrm{rank}(M) \leq n - 1\}$. Then $\Sigma(X) = \{M \in X : \mathrm{rank}(M) \leq n - 2\}$. Let $\mathbb{P}^{n-1}$ be the space of hyperplanes H in $\mathbb{C}^n$, and consider $\widetilde{X} := \{(M, H) \in X \times \mathbb{P}^{n-1} : \mathrm{im}(M) \subset H\}$. Then $\widetilde{X} = \gamma^{\oplus k}$, where γ is the tautological hyperplane bundle over $\mathbb{P}^{n-1}$, hence $\widetilde{X}$ is smooth. Morever, $\widetilde{X} \to X$ is an isomorphism over the complement of $\Sigma(X)$, hence it is a resolution. The codimension of $\phi^{-1}(\Sigma(X))$ in $\widetilde{X}$ is $\geq k - n + 2$ [395, p. 54].

2.1.6 The Exceptional Set of a Modification [358] Assume that (X, o) is a germ of a *normal* surface singularity and fix a modification $\phi : \widetilde{X} \to X$, which is *not* an isomorphism. Then ϕ is an isomorphism over $X \setminus \{o\}$, cf. 2.1.4.

Lemma 2.1.7 (Zariski's Main Theorem) *(See [781], [258, p. 280] for the algebraic and [225, 228] for the analytic case)*

$E = \phi^{-1}(o)$ is connected, compact and one-dimensional.

Proof This follows from the connectivity properties of normal points from 12.2.6. Indeed, there exists a neighborhood X of o such that $X \setminus o$ is connected. This means that $\phi^{-1}(X) \setminus E$ is connected. Since $\phi^{-1}(X)$ is an arbitrarily small neighbourhood of E, E itself should be connected. E is compact since ϕ is proper. E cannot be a point, since then ϕ would be finite, and the identity map of (X, o) (as the normalization of (X, o)) would dominate ϕ (cf. 12.2.6), hence ϕ would be an isomorphism. □

Definition 2.1.8 Let (X, o) be a normal surface singularity and ϕ a modification.

(a) The analytic (reduced) curve E is called the *exceptional set (or curve)* of ϕ. Usually we write $\{E_v\}_{v=1}^s$ for the irreducible components of E and $g_v = g(E_v)$ denotes the geometric genus of (the normalization of) E_v.

(b) Assume that ϕ is a resolution. Then the intersection matrix I of ϕ consists of the intersection numbers $(E_v, E_u)_{v,u}$ in $\widetilde{X}$. Furthermore, let $f : (X, o) \to (\mathbb{C}, 0)$ be the germ of a holomorphic function. Then the divisor $\mathrm{div}(f \circ \phi)$ on $\widetilde{X}$ decomposes into $\mathrm{div}_E(f \circ \phi) + S(f \circ \phi)$, abbreviated as $\mathrm{div}_E(f) + S(f)$, where $\mathrm{div}_E(f)$ is the part supported on E, while $S(f)$ is the *strict transform of the divisor of* f. (This means that $S(f)$ is supported on the closure of $\phi^{-1}(\{f = 0\} \setminus \{o\})$. The irreducible components of $S(f)$ are non-compact.)

Example 2.1.9 Let C, X and $\widetilde{X}$ be as in 2.1.5(b). If d is the degree of $C \subset \mathbb{P}^{n-1}$, and we take a generic linear function f through o, then $\mathrm{div}_E(f) = E$ and $S(f)$ consists of d smooth components intersecting E transversally, hence $(E, S(f)) = d$. Therefore $E^2 = -d$ (since $(E, \mathrm{div}(f \circ \phi)) = 0$).

If $C \simeq \mathbb{P}^1$ and $d = 1$, then the cone is a plane, hence (X, o) is smooth. Thus, by blowing up o we get a modification with an exceptional curve $E \simeq \mathbb{P}^1$ and $E^2 = -1$.

In general, if C is a curve on a smooth surface $\widetilde{X}$ with $C \simeq \mathbb{P}^1$ and $C^2 = -1$ then we say that C *is a* (-1)*-curve on* $\widetilde{X}$. In 2.4.4 *(1)* (Castelnuovo's Contractibility Criterion) we will show that *any* (-1)-curve appears as a blow up of a smooth point.

Remark 2.1.10 Assume that $\widetilde{X}$ is a smooth surface and C is an irreducible curve on it with $(C, C) < 0$. Let us denote by $g(C)$ its geometric genus and by $\delta(C)$ the sum of the delta invariants of its points. Then by the *adjunction formula* $(K_{\widetilde{X}}, C) + (C, C) = -2 + 2g(C) + 2\delta(C) \geq -2$, cf. 12.4.8. Therefore, C is a (-1)-curve if and only if $(K_{\widetilde{X}}, C) < 0$. (Here $K_{\widetilde{X}}$ denotes a canonical divisor of $\widetilde{X}$. In the above formula it can be replaced by $K := c_1(\Omega^2_{\widetilde{X}})$ too, cf. 6.1 and 12.4.7.)

Remark 2.1.11

(a) The normality of (X, o) implies the following fact, see e.g. 12.2.6. Assume that $\tilde{f} : \widetilde{X} \to \mathbb{C}$ is holomorphic. Then, by Zariski's Main Theorem 2.1.7, $\tilde{f}|_E$ is constant. Hence there exists a continuous and bounded map $f : X \to \mathbb{C}$, holomorphic on $X \setminus \{o\}$, such that $\tilde{f} = f \circ \phi$. Then, by normality of X, f is holomorphic. In other words, $\phi_*(\mathscr{O}_{\widetilde{X}}) = \mathscr{O}_X$.

(b) If (X, o) is normal, then $X \setminus \{o\}$ is smooth. Above, in the definition of the resolution, X was an open representative. However, (in topological discussions) we can assume additionally that X is contractible to $o \in X$ and it is closed with a compact and C^∞ boundary (cf. Theorem 3.1.2). In particular, $\widetilde{X}$ has the homotopy type of E and it also has a C^∞ boundary $\partial\widetilde{X}$.

The first proof of the next result is attributed to Du Val [167], see also [27, 358, 448].

Proposition 2.1.12 *Let (X, o) be a normal surface singularity and ϕ a resolution. Then the intersection matrix $(E_v, E_u)_{v,u=1}^s$ is negative definite.*

Proof Take a holomorphic function $f : (X, o) \to (\mathbb{C}, 0)$ and its divisor $D := \mathrm{div}_E(f)$. Since $D + S(f)$ is principal divisor (or, its class is zero in $H_2(\widetilde{X}, \partial\widetilde{X}, \mathbb{Z})$), we have $(D + S(f), E_v) = 0$ for any v. Since $(S(f), E_v) \geq 0$ for all v, and the inequality is strict for at least one v (since $S(f)$ intersects E),

$$(D, E_v) \leq 0 \ \text{ for all } v, \ \text{ and } \ (D, E_v) < 0 \ \text{ for at least one } v. \tag{2.1.1}$$

Since the multiplicities of the irreducible curves E_v in D are the vanishing orders of $f \circ \phi$, D is effective and its support $|D|$ is the whole E.

We wish to prove that if $Z = \sum_v r_v E_v$ ($r_v \in \mathbb{Q}$) and $Z^2 \geq 0$ then $Z = 0$. Write Z as $Z_1 - Z_2$ with both Z_i effective and having no common component. Then $0 \leq Z^2 = Z_1^2 + Z_2^2 - 2(Z_1, Z_2) \leq Z_1^2 + Z_2^2$. Hence it is enough to prove the statement for each Z_v, that is, we can assume that Z is effective. If the support $|Z|$ of Z is smaller than E, then we restrict D to $|Z|$; note that the property (2.1.1) will be preserved. Now, the proof is an induction on the number of E_v's in $|Z|$. Let $a \in \mathbb{Q}_{\geq 0}$ be the largest number with $aD \leq Z$. Then $Z' := Z - aD$ is effective, with smaller support than Z. Moreover, $Z'^2 = (Z - aD, Z') \geq (Z, Z') = (Z, Z - aD) \geq Z^2 \geq 0$. By induction $Z' = 0$ hence $Z = aD$. Since $D^2 < 0$ by (2.1.1), we get $a = 0$, hence $Z = 0$. □

Remark 2.1.13 The converse of Proposition 2.1.12 is also true. By a famous theorem of Grauert [223], any connected collection of (compact) curves on a smooth surface with negative definite intersection form can *analytically* be contracted to an analytic normal singular point, hence it appears as the exceptional curve of a resolution of a certain analytic normal surface singularity. For more see 2.5.

Remark 2.1.14 A similar property characterizes families of curves $f : W \to D$, where W is a smooth surface, D is a (sufficiently small) disc centered at 0 and f is a surjective *proper* holomorphic map. We assume that all the fibers over $D \setminus 0$ are smooth, connected, of genus g. Let $(\ ,\ )_W$ denote the intersection form on W.

Assume that the central fiber has irreducible components $\{C_v\}_v$ (they are not necessarily smooth) and the divisor of f is the effective cycle $\mathrm{div}(f) = \sum_v n_v C_v$. Note that $H_2(W, \mathbb{Z})$ is freely generated by the classes of $\{C_v\}_v$. Then the following facts hold (the statement is usually called *Zariski's Lemma*, see e.g. [41, III.8]): $(C_v, \mathrm{div}(f))_W = (C_v, f^{-1}(t))_W = 0$ (where t is a generic element of D), hence $\mathrm{div}(f)^2 = 0$. Moreover, the intersection form $(\ ,\ )_W$ is negative semi–definite (that is, for any $C = \sum_n r_v C_v$, $r_v \in \mathbb{Q}$, one has $C^2 \leq 0$), and if $C^2 = 0$ then $C = r \cdot \mathrm{div}(f)$ for some $r \in \mathbb{Q}$. The proof is a small modification of the proof of Proposition 2.1.12.

We refer to such an f as a *holomorphic family of genus g curves*.

2.1.15 The Lattice Associated with a Resolution Let (X, o) be a complex normal surface singularity and let $\phi : \widetilde{X} \to X$ be a (not necessarily good) resolution. Here we take X sufficiently small and contractible (as in Theorem 3.1.2).

Set $L := H_2(\widetilde{X}, \mathbb{Z})$. Since $\widetilde{X}$ has the homotopy type of E, L is freely generated by the classes of $\{E_v\}_v$ (still denoted by the same symbol E_v), and it carries the intersection form I. Define also $L' := H_2(\widetilde{X}, \partial\widetilde{X}, \mathbb{Z})$. It is dual to L. If for each $v \in \mathcal{V}$ one takes a transversal disc D_v to E_v (at a generic point of E_v), then their classes form a basis of L'. The duality map $L \otimes L' \to \mathbb{Z}$ in these bases is $(E_v, D_w) = \delta_{vw}$. Moreover, the homological map $L \to L'$ in the bases $\{E_v\}$ and $\{D_v\}$ is exactly the matrix I. Since I is non-degenerate, $L \to L'$ is injective. We write $H := L'/L$. Clearly, $|H| = |\mathrm{coker}(I)| = |\det(I)|$. (For a topological interpretation of L'/L see Proposition 3.4.2.)

We extend the intersection form I of L to $L \otimes \mathbb{Q}$. By the perfect pairing between L and L', L' is identified with $\mathrm{Hom}(L, \mathbb{Z})$. On the other hand, $\mathrm{Hom}(L, \mathbb{Z})$ is also identified with those elements l' of $L \otimes \mathbb{Q}$ for which $(l', l) \in \mathbb{Z}$ for any $l \in L$. In the sequel we will think about L' in this way, as a sublattice of $L \otimes \mathbb{Q}$, and as an overlattice of L, endowed with the (rational) intersection form I.

Effective classes $l = \sum r_v E_v \in L'$ with all $r_v \in \mathbb{Q}_{\geq 0}$ are denoted by $L'_{\geq 0}$, and $L_{\geq 0} := L'_{\geq 0} \cap L$. There is a natural partial ordering in $L \otimes \mathbb{Q}$ associated with the bases $\{E_v\}_v$: we say that $l_1 \geq l_2$ if $l_1 - l_2$ is effective. We write $l_1 > l_2$ if $l_1 \geq l_2$ and $l_1 \neq l_2$. The cycle $\min\{l_1, l_2\}$ is the largest l with $l_1, l_2 \geq l$. If $l' = \sum_v r_v E_v$ is a rational cycle, its *support* $|l'|$ is $\cup_{v\,:\,r_v \neq 0} E_v$. Moreover, we set $\lfloor l' \rfloor := \sum_v \lfloor r_v \rfloor E_v$, and $\{l'\} := l' - \lfloor l' \rfloor$.

2.1.16 The Pontrjagin Dual of H We denote the Pontrjagin dual $\mathrm{Hom}(H, S^1)$ of H by $\widehat{H}$. Let $\theta : H \to \widehat{H}$ be the isomorphism $[l'] \mapsto e^{2\pi i (l', \cdot)}$ of H with $\widehat{H}$.

2.1.17 Lipman's Cones Associated with the Resolution [389] We prefer to replace the classes $[D_v] \in H_2(\widetilde{X}, \partial \widetilde{X}, \mathbb{Z})$, reinterpreted in L', by their 'opposites', denoted by E_v^*. That is, $E_v^* \in L' \subset L \otimes \mathbb{Q}$ satisfies $(E_v^*, E_u) = -1$ for $v = u$, and 0 otherwise.

In particular, the vectors E_v^*, written in the base $\{E_v\}_v$, are exactly the columns of the matrix $-I^{-1}$, and $(I^{-1})_{vu} = (E_v^*, E_u^*)$. Let

$$\mathscr{S}_{\mathbb{Q}} := \{l' \in L \otimes \mathbb{Q} \,:\, (l', E_v) \leq 0 \text{ for all } v \in \mathscr{V}\}$$

be the anti-nef rational cone, $\mathscr{S}' := \mathscr{S}_{\mathbb{Q}} \cap L'$ and $\mathscr{S} := \mathscr{S}_{\mathbb{Q}} \cap L$. $\mathscr{S}'$ is generated over $\mathbb{Z}_{\geq 0}$ by the elements E_v^*.

The definition of the cone $\mathscr{S}$ is motivated by the following fact, cf. (2.1.1):

Lemma 2.1.18 *Let $f : (X, o) \to (\mathbb{C}, 0)$ be a holomorphic function, and ϕ a good resolution of (X, o). Then $\mathrm{div}_E(f) \in \mathscr{S} \setminus \{0\}$.*

The divisor $\mathrm{div}_E(f) = \sum_{v \in \mathscr{V}} m_v E_v$ obviously satisfies $m_v > 0$ for all v. This is a general fact of all the elements of $\mathscr{S}_{\mathbb{Q}}$ by the next corollary. In particular, $\mathscr{S}_{\mathbb{Q}}$ is in the first quadrant. (This motivates the sign modification in the definition of E_v^*.)

Corollary 2.1.19

(a) If $l = \sum_v r_v E_v \in \mathscr{S}_{\mathbb{Q}} \setminus \{0\}$ then $r_v > 0$ for all $v \in \mathscr{V}$.
In particular, all the entries of E_v^* are strictly positive.
(b) For any fixed $l' \in L'$ the set $\{\tilde{l}' \in \mathscr{S}',\ \tilde{l}' \not\geq l'\}$ is finite.

Proof Write $l = l_1 - l_2$ with both l_1 and l_2 effective, such that $|l_1|$ and $|l_2|$ have no common E_v-components. Write $l_2 = \sum_v r_v E_v$ with $r_v \geq 0$. Fix $E_v \subset |l_2|$. Since $(l_1 - l_2, E_v) \leq 0$ and $(l_1, E_v) \geq 0$ one gets $(l_2, E_v) \geq 0$. Hence $(l_2, l_2) = (l_2, \sum_v r_v E_v) \geq 0$. Since I is negative definite, $l_2 = 0$. Hence $l \geq 0$. Since E is connected, the support of l should be E, otherwise there would exists a neighboring v of $|l|$ with $(l, E_v) > 0$. □

Since $\{l \in L \otimes_{\mathbb{Z}} \mathbb{R} \ : \ (l, E_v) \leq 0 \text{ for all } v\}$ is a rational cone, by Gordan's Lemma [88, 2.A.2.9] one has:

Corollary 2.1.20 *$\mathscr{S}'$ and $\mathscr{S}$ are finitely generated (affine, normal) monoids.*

The next statement guarantees the existence of a resolution.

Theorem 2.1.21 *Let (X, o) be a normal surface singularity germ. Then the following facts hold.*

1. *A good resolution exists.*
2. *There is a unique minimal resolution and a unique minimal good (or strong) resolution.*
3. *A resolution is minimal if and only if none of the curves E_v is a (-1)-curve.*
4. *A good resolution is minimal good if and only if none of the curves E_v is a (-1)-curve with $(E - E_v, E_v) \leq 2$ (that is, any (-1)-curve must intersect at least three other components).*

For comments, references and the proof see Sect. 2.4.

2.1.22 Embedded Resolutions Fix $(X, 0)$ and a local Weil divisor $(D, o) \subset (X, o)$.

We say that a strong resolution $\phi : \widetilde{X} \to X$ is an *embedded resolution of the pair* (X, D), if the reduced inverse image $\phi^{-1}(D)$ of D is a normal crossing divisor in $\widetilde{X}$. The embedded resolution is *good* if all the irreducible components of $\phi^{-1}(D)$ (with reduced structure) are smooth.

If $(X, o) \approx (\mathbb{C}^2, 0)$ is a smooth germ, that is, $(D, o) \subset (X, o)$ is a *plane curve singularity*, then a classical result guarantees the existence of an embedded resolution. It can be obtained by iterated blow ups of (infinitely near) smooth points, which are not normal crossing points of (the inverse images of) D. See e.g. [258, p. 391] or [85, 745]. Therefore, if (X, o) is a two-dimensional germ, then one might resolve first X via some $\phi : \widetilde{X} \to X$ (using Theorem 2.1.21), then one proceeds with an embedded resolution of $(\widetilde{X}, \phi^{-1}(D))$ (with $\widetilde{X}$ smooth) by resolving the plane curve singularities of (the reduced) $\phi^{-1}(D)$. Hence, Theorem 2.1.21 proves the existence of embedded resolution of (X, D) too, provided that $\dim(X) = 2$.

In fact, the pair (X, D) admits even a *minimal good embedded resolution*, whose existence follows similarly as the existence of the minimal good resolution of X, cf. Theorem 2.1.21 parts 2 and 4. In this case, ϕ is minimal good embedded resolution if and only if there is no smooth irreducible exceptional curve E_v with $g(E_v) = 0$, $E_v^2 = -1$ and $(\phi^{-1}(D) - E_v, E_v) \leq 2$, where $\phi^{-1}(D)$ is the reduced set-theoretical inverse image.

2.2 The Embedded Resolution Graph

In this section we define the resolution and the embedded resolution graphs, and we list some of their properties. Arithmetical properties of strings will be discussed via continued fractions and Dedekind sums. ♣

2.2.1 Let (X, o) be a normal surface singularity and $f : (X, o) \to (\mathbb{C}, 0)$ be the germ of an analytic function. We write $(V(f), o)$ for $(f^{-1}(0), o)$ (with reduced structure). Let $\phi : \widetilde{X} \to X$ be a *good embedded resolution* of the pair $(X, V(f))$. Denote by E the exceptional curve of ϕ with irreducible decomposition $\{E_v\}_{v\in\mathcal{V}}$, and let $\cup_{a\in\mathcal{A}} S_a$ be the irreducible components of the reduced strict transform $S(f)$.

2.2.2 The Dual Embedded Resolution Graph $\Gamma(X, f)$ Associated with the Resolution ϕ [178, 277, 358, 390] We construct a graph $\Gamma(X, f)$ as follows. Its *vertices* $\mathcal{W} = \mathcal{V} \sqcup \mathcal{A}$ consist of *non-arrowhead vertices* $\mathcal{V}$ corresponding to the irreducible exceptional components, and arrowhead vertices $\mathcal{A}$ corresponding to the irreducible components of the support of the strict transform. If two irreducible divisors corresponding to $w_1, w_2 \in \mathcal{W}$ have k intersection points then we connect w_1 and w_2 by k edges in $\Gamma(X, f)$.

The graph $\Gamma(X, f)$ is decorated as follows. Any vertex $v \in \mathcal{V}$ is decorated with the self-intersection $e_v := E_v^2$ and genus g_v of E_v (denoted as $[g_v]$). Furthermore, the third decoration is the *multiplicity* m_w of f (denoted as (m_w)), defined for any $w \in \mathcal{W}$ as the vanishing order of $f \circ \phi$ along the irreducible component corresponding to w. (For example, if f defines an isolated singularity, then $m_a = 1$ for any $a \in \mathcal{A}$.)

In general we assume that $\mathcal{V} \neq \emptyset$ except one case, when $(X, o) = (\mathbb{C}^2, 0)$ and f in some local coordinates is $x^a y^b$, hence it defines a normal crossing. In this 'unblow up' satiation $\Gamma(X, f)$ is the 'double arrow' $(a) \longleftrightarrow (b)$.

$\Gamma(X, f)$ is called the *(dual) embedded resolution graph of the pair* (X, f) associated with ϕ. If (X, o) is clear from the context then we denote it by Γ_f.

If $g_v = 0$ then usually we omit the decoration $[g_v]$. Note also that e_v is also the Euler number of the normal bundle of the embedding $E_v \subset \widetilde{X}$.

For any $v \in \mathcal{V}$ denote by $\mathcal{W}_v$ the set of vertices $w \in \mathcal{W}$, which are connected by an edge with v.

2.2.3 The Resolution Graph Γ_X of (X, o) If we start with a space germ (X, o), without any function on it, then any good resolution ϕ provides in a similar way the *dual resolution graph Γ_X of (X, o) associated with ϕ*. In this case $\mathcal{A} = \emptyset$, and any $v \in \mathcal{V}$ has two decorations: e_v and $[g_v]$. From any embedded resolution graph Γ_f one obtains the graph Γ_X by deleting the arrowheads (and their connecting edges) and multiplicities.

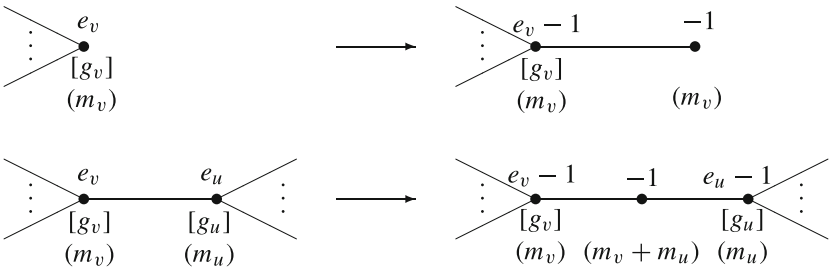
The valency (number of adjacent edges) of a vertex v is denoted by κ_v.

Note that the intersection matrix I can be read from the decorated graph.

Definition 2.2.4 If Γ is a graph with negative definite intersection matrix I, then the determinant of Γ is defined as $\det(\Gamma) := \det(-I)$. Furthermore, by convention, the determinant of the empty graph is $\det(\emptyset) = 1$.

2.2.5 First Properties of the Resolution Graphs

1. The graphs Γ_f and Γ_X are connected graphs by Lemma 2.1.7.
2. The resolution is not unique, e.g. one can blow up a point of the exceptional divisor of a resolution. Accordingly, the graphs Γ_f and Γ_X depend on the choice of ϕ. Theorems 2.1.21 and 2.4.4 imply that the dual graphs associated with different resolutions are connected by a sequence of blow ups and blow downs of (-1)-curves. The next diagram shows the corresponding modification of the graph via blow up of a smooth point on E_v or of the intersection point $E_v \cap E_u$. The decoration modifications are based on the fact that if we blow up a smooth point of a curve C, then the self-intersection of its strict transform is one less than the self-intersection of C (see also 12.4.9). These operations in the language of the graph are called *blow up the vertex* v or *the edge* (v, u).

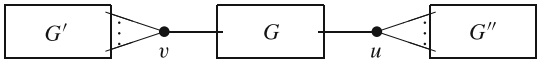


3. Let $I := (I_{vu} = (E_v, E_u))_{v,u \in \mathscr{V}}$ be the intersection matrix associated with ϕ. Since $\operatorname{div}(f \circ \phi) = \sum_{v \in \mathscr{V}} m_v E_v + \sum_{a \in \mathscr{A}} m_a S_a$ is a principal divisor, its intersection with any E_u is zero. This reads as

$$\sum_{v \in \mathscr{V}} m_v I_{vu} + \sum_{a \in \mathscr{A} \cap \mathscr{W}_u} m_a = 0 \text{ for any } u \in \mathscr{V}. \tag{2.2.1}$$

   Since I is non-degenerate, cf. Proposition 2.1.12, the multiplicities $\{m_v\}_{v \in \mathscr{V}}$ can be recovered from I and the multiplicities $\{m_a\}_{a \in \mathscr{A}}$. Furthermore, since $m_v > 0$ for any $v \in \mathscr{V}$, the set of multiplicities (and the shape of Γ) determine the Euler numbers $\{e_v\}_{v \in \mathscr{V}}$ too. This has a rather important technical advantage: a multiplicity can always be determined by a local computation (in contrast with the Euler number, which is a global characteristic class, and usually it is harder to identify).
4. By Corollary 2.1.19 all the entries of the inverse matrix I^{-1} are strictly negative. This also implies that if a (rational) divisor $D = \sum_{v \in \mathscr{V}} m_v E_v + \sum_{a \in \mathscr{A}} m_a S_a$ (with $\mathscr{A} \neq \emptyset$) has the property that $(D, E_u) = 0$ for all $u \in \mathscr{V}$, and $m_a > 0$ for all $a \in \mathscr{A}$, then automatically $m_v > 0$ for all $v \in \mathscr{V}$ too.

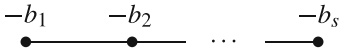
5. Assume that Γ_X is a tree. Then for any pair $v, u \in \mathcal{V}$, the entry $(-I^{-1})_{vu}$ multiplied by $\det(\Gamma_X)$ equals the determinant of the subgraph Γ_{vu}, which is obtained from Γ_X by deleting the 'shortest path' in Γ_X connecting the vertices v and u and all the adjacent edges. (For a proof use the 0-entries in I; for the $\det(\Gamma_X) = 1$ case see [178].)
6. ([77]) Assume that Γ is a tree, and it has a decomposition as follows:



The subgraphs G', G and G'' can be empty. If G is empty then v and u is connected by a single edge. The vertices v and u are not allowed to be the same. $G \cup G' \cup v$ and $G \cup G'' \cup u$ contain the edges adjacent to v and u too, respectively. G_{uv} is obtained from G by deleting the minimal path (and adjacent edges) connecting u and v. Then one has:

$$\begin{aligned} \det(\Gamma_X) \cdot \det(G) = \ & \det(G \cup G' \cup v) \cdot \det(G \cup G'' \cup u) \\ & - \det(G') \cdot \det(G'') \cdot \det(G_{uv})^2. \end{aligned} \tag{2.2.2}$$

Definition 2.2.6 A vertex of a graph with positive genus decoration, or adjacent to at least three edges, is called a *node*. A 'linear' (sub)graph of type



(with all genus-decorations zero) is called a *string*.

Strings can be characterized by continued fractions, see Sect. 2.2.A.

2.2.7 Examples In the body of the book several resolution algorithms will be presented for special families of surface singularities together with their graphs. See

- Examples 3.3.10 and 5.3.20 for the embedded resolution graph of irreducible plane curve singularities;
- Sect. 2.3 for Hirzebruch–Jung (cyclic quotient) surface singularities;
- Theorem 4.1.17 and Example 5.3.21 for suspension hypersurface singularities $\{f(x, y) + z^n = 0\}$;
- Section 5.1 for weighted homogeneous singularities;
- Example 5.1.17 for Brieskorn–Hamm complete intersections;
- Section 5.2 for superisolated (hypersurface) singularities;
- 5.5.23 for hypersurfaces with non-degenerate Newton principal part and for Weil divisors with non-degenerate Newton principal part in 3-dimensional affine toric singularities.
- Graphs of rational and elliptic singularities are characterized in 7.1 and 7.2.
- Graphs of cusps (that is, cyclic graphs, cf. 7.2.19) are analysed in 3.5.B.

2.2.A *Continued Fraction and Dedekind Sums*

Certain arithmetical properties of string graphs will be present over and over in the computation of graph invariants. Here we start the list with the relevant connection with continued fractions.

Definition 2.2.8 To any two relative prime positive numbers n and q we associate the following (Hirzebruch, or negative) continued fraction:

$$\frac{n}{q} = [[b_1, b_2, \ldots, b_s]] := b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots - \cfrac{1}{b_s}}}, \quad b_1 \geq 1,\ b_2, \ldots, b_s \geq 2. \tag{2.2.3}$$

The entries $(b_1, \ldots, b_s)$ characterize a string graph with decorations $-b_1, \ldots, -b_s$.

Associated with the continued fraction, it is helpful to consider its 'partial' fractions as well. For $1 \leq i \leq j \leq s$ set $n_{ij}/q_{ij} = [[b_i, \ldots, b_j]]$, with $n_{ij} > 0$ and $\gcd(n_{ij}, q_{ij}) = 1$. We also set $n_{i,i-1} := 1$, $n_{i,j} = 0$ for $j < i - 1$. Since $n_{ij}/q_{ij} = b_i - q_{i+1,j}/n_{i+1,j}$, one gets that $q_{ij} = n_{i+1,j}$ and $n_{ij} = b_i n_{i+1,j} - n_{i+2,j}$ for $i \leq j$. This has its 'symmetric' version too, namely $n_{ij} = b_j n_{i,j-1} - n_{i,j-2}$ for $i \leq j$. Furthermore,

$$n \cdot n_{i+1,j-1} = n_{i+1,s} \cdot n_{1,j-1} - n_{j+1,s} \cdot n_{1,i-1} \quad (i < j). \tag{2.2.4}$$

Note that $n_{1,s} = n$ and $n_{2,s} = q$. Symmetrically, $n_{1,s-1} = q'$, where q' is the unique integer $0 < q' < n$ with $qq' \equiv 1 \pmod{n}$. Indeed, by (2.2.4) (for $i = 1$ and $j = s$)

$$qq' - 1 = n\tau, \text{ where } \tau := n_{2,s-1}. \tag{2.2.5}$$

For several arithmetical properties of continued fractions see e.g. [578].

2.2.9 Continued fractions are tightly related with Dedekind sums as well. For any pair n and q besides the continued fraction (2.2.3) and the string Definition 2.2.6 one also defines the *Dedekind sum*

$$\mathbf{s}(q, n) = \sum_{l=0}^{n-1} \left(\!\!\left(\frac{l}{n}\right)\!\!\right)\left(\!\!\left(\frac{ql}{n}\right)\!\!\right),$$

where $((x))$ is the *Dedekind symbol*:

$$((x)) = \begin{cases} \{x\} - 1/2 & \text{if } x \in \mathbb{R} \setminus \mathbb{Z} \\ 0 & \text{if } x \in \mathbb{Z}. \end{cases}$$

Here $\{\cdot\}$ is the 'fractional part'. Note that $\sum_{l=0}^{n-1} \left(\!\left(\frac{l}{n}\right)\!\right) = 0$.

The Dedekind sum $\mathbf{s}(q, n)$ satisfies the *Dedekind reciprocity law* [244, 583]

$$\mathbf{s}(q, n) + \mathbf{s}(n, q) = \frac{n^2 + q^2 + 1}{12nq} - \frac{1}{4}. \tag{2.2.6}$$

Note that $n = qb_1 - n_{3,s}$, hence $\mathbf{s}(n, q) = \mathbf{s}(-n_{3,s}, q) = -\mathbf{s}(n_{3,s}, n_{2,s})$, and $\mathbf{s}(n_{3,s}, n_{2,s})$ is associated with $[[b_2, \ldots, b_s]]$ as $\mathbf{s}(q, n)$ is associated with $[[b_1, \ldots, b_s]]$. This fact, induction and (2.2.6) give the following identity, which connects the entries of the continued fraction with the corresponding Dedekind sum:

$$12 \cdot \mathbf{s}(q, n) = \frac{q + q'}{n} + \sum_{i=1}^{s} (b_i - 3). \tag{2.2.7}$$

The Dedekind symbol and Dedekind sum have the following trigonometric expression, cf. [244].

$$\left(\!\left(\frac{l}{n}\right)\!\right) = \frac{1}{2n} \cdot \sum_{\xi^n = 1 \neq \xi} \frac{1 + \xi}{1 - \xi} \cdot \xi^l,$$

$$\mathbf{s}(q, n) = -\frac{1}{4n} \cdot \sum_{\xi^n = 1 \neq \xi} \frac{1 + \xi}{1 - \xi} \cdot \frac{1 + \xi^q}{1 - \xi^q} = -\frac{1}{n} \cdot \sum_{\xi^n = 1 \neq \xi} \frac{1}{(1 - \xi)(1 - \xi^q)} + \frac{n - 1}{4n}.$$

Dedekind sums have several generalizations. The first one is a slight modification [244, 582, 774]: for integers $a,\ b,\ c$ $(a \neq 0)$ one writes

$$\mathbf{s}(b, c; a) := \sum_{l=0}^{a-1} \left(\!\left(\frac{lb}{a}\right)\!\right)\left(\!\left(\frac{lc}{a}\right)\!\right). \tag{2.2.8}$$

Since $\mathbf{s}(b, c; a) = (a, b, c) \cdot \mathbf{s}(b/(a, b), c/(a, b, c); a/(a, b))$ we can assume that $(a, b) = 1$, and in this case for $b'b \equiv 1 \pmod{a}$ one has $\mathbf{s}(b, c; a) = \mathbf{s}(b'c, a)$. One also has the generalized reciprocity law (whenever $(a, b, c) = 1$):

$$\mathbf{s}(b, c; a) + \mathbf{s}(c, a; b) + \mathbf{s}(a, b; c) = -\frac{1}{4} + \frac{a^2(b, c)^2 + b^2(a, c)^2 + c^2(b, a)^2}{12abc}.$$

Another *generalized Dedekind sum* (cf. [583]) what we will need is the following

$$\mathbf{s}(q,n;\frac{a}{n},0)=\sum_{l=0}^{n-1}\Big(\Big(\frac{l}{n}\Big)\Big)\Big(\Big(\frac{ql+a}{n}\Big)\Big), \tag{2.2.9}$$

where again n and q are relative prime, and $a\in\mathbb{Z}$. By a computation, for any $a\in\mathbb{Z}$,

$$\mathbf{s}(q,n;\frac{a}{n},0)-\mathbf{s}(q,n)=-\frac{1}{2}\Big(\Big(\frac{aq'}{n}\Big)\Big)+\sum_{j=1}^{a}\Big(\Big(\frac{jq'}{n}\Big)\Big). \tag{2.2.10}$$

A parallel computation shows

$$\frac{1}{n}\cdot\sum_{\xi^n=1\neq\xi}\frac{1}{(1-\xi)(1-\xi^q)}-\frac{1}{n}\cdot\sum_{\xi^n=1\neq\xi}\frac{\xi^{-a}}{(1-\xi)(1-\xi^q)}=\frac{1}{2}\Big\{\frac{a}{n}\Big\}+\sum_{j=1}^{a}\Big(\Big(\frac{jq'}{n}\Big)\Big).$$

Hence (see also [51])

$$\mathbf{s}(q,n;\frac{a}{n},0)= -\frac{1}{n}\cdot\sum_{\xi^n=1\neq\xi}\frac{\xi^{-a}}{(1-\xi)(1-\xi^q)}+\frac{n-1}{4n}-\frac{1}{2}\Big\{\frac{a}{n}\Big\}-\frac{1}{2}\Big(\Big(\frac{aq'}{n}\Big)\Big). \tag{2.2.11}$$

2.3 Example: Hirzebruch–Jung Singularities

We discuss five equivalent characterizations of Hirzebruch–Jung (cyclic quotient) singularities with several additional complements. ♣

Theorem 2.3.1 ([41, 269, 358, 590, 591]) *For a normal surface singularity, the following conditions are equivalent. If (X,o) satisfies any of them, then it is called Hirzebruch–Jung or cyclic quotient singularity.*

(a) *(X,o) is isomorphic with one of the 'model spaces' $\{X_{n,q}\}_{n,q}$, where $X_{n,q}$ is the normalization of $(\{xy^{n-q}=z^n\},0)$, where $0<q<n$, $(n,q)=1$.*

(b) *There is an analytic covering $p:(X,o)\to(\mathbb{C}^2,0)$ such that the reduced branch locus of p is $\{uv=0\}$ in some local coordinates (u,v) of $(\mathbb{C}^2,0)$.*

(c) *The resolution graph Γ_X is a string (with $g_v=0$ for any $v\in\mathcal{V}$).*

(d) *(X,o) is the quotient singularity $(\mathbb{C}^2,0)/\mathbb{Z}_n$ of the cyclic group $\mathbb{Z}_n=\{\xi\in\mathbb{C}:\xi^n=1\}$ of order n, where the action is $\xi*(z_1,z_2)=(\xi z_1,\xi^q z_2)$ for some $0<q<n$ with $(q,n)=1$.*

(e) *(X,o) is the germ at o of $\operatorname{Spec}\mathbb{C}[\sigma^\vee\cap\mathbb{Z}^2]$, where $\sigma^\vee$ is the real cone generated in $\mathbb{R}^2$ by the vectors $(1,0)$ and (q,n).*

Proof $(a \Rightarrow b)$ follows from the composition $(X_{n,q}, o) \to (\{xy^{n-q} = z^n\}, 0) \to (\mathbb{C}^2, 0)$, where the second map is induced by $(x, y, z) \mapsto (u, v) = (x, y)$ with branch locus $\Delta = \{uv = 0\}$. If in $\Pi = \pi_1(\mathbb{C}^2 \setminus \Delta) = \mathbb{Z}^2$ the base elements $(1, 0)$ and $(0, 1)$ are represented by the class of the oriented elementary loops around the u and v axes, and $\xi := \exp(2\pi i/n)$, then the monodromy representation $\rho : \Pi \to \mathbb{Z}_n$ of the covering is $(1, 0) \mapsto \xi$ and $(0, 1) \mapsto \xi^{-q}$. Hence $\ker \rho = \mathbb{Z}\langle(n, 0), (q, 1)\rangle \subset \Pi$.

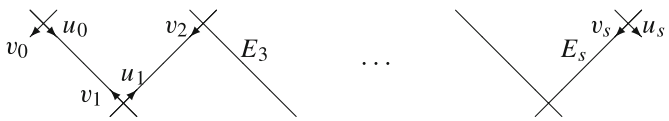
Now, consider an arbitrary covering p as in *(b)*. We show that p has a factorization $p_2 \circ p_1 : (X, o) \to (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$, where p_1 is isomorphic to $(X_{n,q}, o) \to (\mathbb{C}^2, 0)$ considered in the previous paragraph and p_2 is of type $(u, v) \mapsto (u^{m_1}, v^{m_2})$.

For this, set $K := \pi_1(X \setminus p^{-1}(\Delta)) \subset \Pi$. Any subgroup K of rank two in $\mathbb{Z}^2$ has a decomposition into the form

$$K = \mathbb{Z}\langle(nm_1, 0), (qm_1, m_2)\rangle \subset \mathbb{Z}m_1 \oplus \mathbb{Z}m_2 \subset \Pi \tag{2.3.1}$$

for some m_1, m_2, n, q positive integers, $q < n$ and $(n, q) = 1$. These inclusions corresponds to $p_2 \circ p_1$: both p_1 and p_2 can be identified by Stein's Theorem, cf. 12.3.3, once we identify their branch locus and the corresponding monodromy representations of the regular coverings. This shows $(b \Rightarrow a)$.

In order to prove $(a \Rightarrow c)$, we construct the resolution of $X_{n,q}$. Set $n/q = [[b_1, \ldots, b_s]]$ with $b_i \geq 2$. We will show that $\Gamma_{X_{n,q}}$ is the string with s vertices and self-intersection numbers $-b_1, \ldots, -b_s$. Indeed, consider $s + 1$ copies $U_0, \ldots, U_s$ of $\mathbb{C}^2$ with affine coordinates (u_i, v_i). For each $1 \leq i \leq s$ we glue $U_{i-1} \setminus \{u_{i-1} = 0\}$ with $U_i \setminus \{v_i = 0\}$ by $v_i = u_{i-1}^{-1}$ and $u_i = u_{i-1}^{b_i} v_{i-1}$. We call this space $\widetilde{X}$.



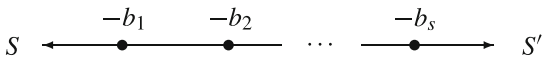
Consider the integers n_{ij} $(1 \leq i, j \leq s)$ as in Definition 2.2.8. Using their inductive properties from Definition 2.2.8, one shows by induction on i that $n_{i+1,s} + (n - q)n_{1,i-1}$ is divisible by n, hence equals nm_i for some positive integer m_i.

Then we define three functions f, g and h on $\widetilde{X}$ via their restrictions on the open sets U_i by:

$$\begin{aligned} f &= u_0^n v_0^q = \cdots = u_i^{n_{i+1,s}} v_i^{n_{i+2,s}} = \cdots = u_{s-1}^{b_s} v_{s-1} = u_s v_s^0, \\ g &= u_0^0 v_0 = u_1 v_1^{b_1} = \cdots = u_i^{n_{1,i-1}} v_i^{n_{1,i}} = \cdots = u_s^{n_{1,s-1}} v_s^n, \\ h &= u_0 v_0 = u_1 v_1^{b_1 - 1} = \cdots = u_i^{m_i} v_i^{m_{i+1}} = \cdots . \end{aligned} \tag{2.3.2}$$

Then $fg^{n-q} = h^n$, hence (f, g, h) defines a map $\widetilde{X} \to \{xy^{n-q} = z^n\}$, which (since $\widetilde{X}$ is smooth) factorizes through $\widetilde{X} \to X_{n,q}$. Analyzing the maps on the

affine charts U_i one shows that it is the minimal resolution with exceptional curve $\sum_{i=1}^{s} E_i$, where $E_i := \{v_{i-1} = 0\} \cup \{u_i = 0\}$. The strict transform of f and g are $S := \{u_0 = 0\}$ and $S' := \{v_s = 0\}$ respectively. Hence the dual graph of all three functions f, g, h is the following:



where the multiplicities of f, g, h are $n, 0, 1$ along S, and $0, n, n-q$ along S', while $n_{i+1,s}, n_{1,i-1}, m_i$ along each E_i respectively.

$(c \Rightarrow a)$. Assume that the resolution graph of (X, o) associated with the resolution ϕ is a string with decorations $-b_1, \ldots, -b_s$. In the proof we will use the following fact: *there exists a small neighborhood $\widetilde{X}$ of the exceptional curve E such that* $\mathrm{Pic}^0(\widetilde{X}) = 0$. This will be proved for more general (namely, rational) graphs in 7.1.10 and 7.1.14. Once we accept this fact, the proof runs as follows. Insert two transversal (non-compact) curves S and S' in order to get the dual graph as in the above picture. Consider the three divisors $F = nS + \sum_{i=1}^{s} n_{i+1,s} E_i$, $G = nS' + \sum_{i=1}^{s} n_{1,i-1} E_i$ and $H = S + (n-q)S' + \sum_{i=1}^{s} m_i E_i$, where the coefficients n_{ij} and m_{ij} are the same as in the proof $(a \Rightarrow c)$. By their construction, any of them intersected by any E_i gives zero. Hence the line bundles associated with these divisors are topologically trivial. But since $\mathrm{Pic}^0(\widetilde{X}) = 0$, they are even analytically trivial. This means that these divisors are linearly equivalent to zero. Being positive divisors, they are divisors of holomorphic functions, say f, g, h. Moreover, the divisor of $i := fg^{n-q}/h^n$ is trivial, hence i is an invertible function. Hence, if we replace f by f/i, they define a holomorphic map $\widetilde{X} \to \{xy^{n-q} = z^n\}$, which factorizes through $\psi : \widetilde{X} \to X_{n,q}$ (where n and q are as in the proof of $(a \Rightarrow c)$).

Since the E_i-coefficients of f, g, h are positive, ψ contracts E. Hence ψ factorizes through a continuous map $\alpha : X \to X_{n,q}$, which is holomorphic since X is normal.

We wish to show that the degree of α is one. Indeed, take the projection $c : X_{n,q} \to \mathbb{C}^2$ induced by the coordinates (x, y) (cf. the proof $(a \Rightarrow b)$), and $r = (f, g) : \widetilde{X} \to \mathbb{C}^2$. Then $r = c \circ \psi$. Clearly, c has degree n. But this is true for the restriction r' of r to $\widetilde{X} \setminus E$ too. Indeed, in some local coordinates (u_0, v_0) of the intersection point $S \cap E_1$ one has $S = \{u_0 = 0\}$, $E_1 = \{v_0 = 0\}$, $f = u_0^n v_0^q$ and $g = v_0$. Hence S is sent to the divisor $V(x) = \{x = 0\}$ of $\mathbb{C}^2$ isomorphically. Therefore, $\deg r' \cdot V(x) = r_* r^* V(x) = r_*(F) = r_*(nS) = nV(x)$. Hence both c and r' have the same degree n and $\psi : \widetilde{X} \setminus E \to X_{n,q} \setminus 0$ is biholomorphic. Therefore, $\alpha : X \setminus o \to X_{n,q} \setminus o$ is an isomorphism too, which extends to an isomorphism over the singular points because the spaces are normal.

$(a \Leftrightarrow d)$. Write $\mathbb{Z}_n = G$. Consider the invariant monomials $x := z_2^n$, $y := z_1^n$ and $z := z_1^{n-q} z_2$. Note that $xy^{n-q} = z^n$. The inclusions of algebras $\mathbb{C}[x, y, z] \subset \mathbb{C}[z_1, z_2]^G \subset \mathbb{C}[z_1, z_2]$ (and the fact that $\mathbb{C}^2/G$ is normal, cf. 12.3.5), provide maps $\mathbb{C}^2 \xrightarrow{q} \mathbb{C}^2/G \xrightarrow{p} X_{n,q}$. Note that both maps q and $p \circ q$ have degree n, hence p has degree one. Since $\mathbb{C}^2/G$ and $X_{n,q}$ are normal, p is an isomorphism.

$(a \Leftrightarrow e)$. The proof uses some notions of affine toric varieties, for terminology see e.g. [208, 522]; for this equivalence especially [208, 2.2] and [522, 1.6]. For a short introduction see also 5.5.B here.

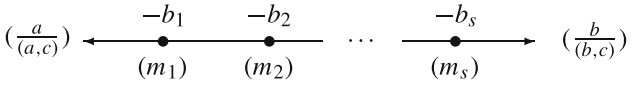
Consider the terminology of the proof $(a \Leftrightarrow c)$. We consider $U_0 = \operatorname{Spec} \mathbb{C}[v_0, u_0]$. Equation (2.3.2) shows that the functions f, g, h in this chart is given by $f = v_0^q u_0^n$, $g = v_0$ and $h = v_0 u_0$. Let $\mathfrak{S}$ be the semigroup in $\mathbb{Z}^2$ generated by the corresponding exponents, namely by $(q, n), (1, 0), (1, 1)$. Then $\operatorname{Spec} \mathbb{C}[\mathfrak{S}] = \{xy^{n-q} = z^n\}$ and (f, g, h) induces a map $X_{n,q} \to \operatorname{Spec} \mathbb{C}[\mathfrak{S}]$, which is isomorphism on a dense open set. The normalization $\widehat{\mathfrak{S}}$ of the semigroup $\mathfrak{S}$ consists of all the lattice points $\sigma^\vee \cap \mathbb{Z}^2$ of the cone $\sigma^\vee$. Indeed, $nl \in \mathfrak{S}$ for any $l \in \sigma^\vee \cap \mathbb{Z}^2$. Hence the normalization of $\operatorname{Spec} \mathbb{C}[\mathfrak{S}]$ is $\operatorname{Spec} \mathbb{C}[\widehat{\mathfrak{S}}]$ providing the isomorphism $X_{n,q} \to \operatorname{Spec} \mathbb{C}[\widehat{\mathfrak{S}}]$. □

2.3.2 Addendum to Part *(b)* of Theorem 2.3.1 Usually, Hirzebruch–Jung singularities appear via characterization *(b)*. From the proof $(b \Rightarrow a)$ we also read the following facts. If a normal singularity (X, o) admits an analytic covering $p : (X, o) \to (\mathbb{C}^2, 0)$ whose reduced branch locus is smooth then (X, o) is smooth. If the branch locus is the union of the two axes, usually we have the 'uncanonical' situation, and the proof shows a way how one can rewrite it in a canonical form. Here we will emphasize a situation which appears in the case of cyclic coverings 4.1.C or in the Jung construction 2.4.2. Assume that (X, o) is the normalization of $(\{x^a y^b = z^c\}, 0) \subset (\mathbb{C}^3, 0)$, where a, b, c are positive integers, and p is the projection to the (x, y)-plane. Then (X, o) is locally irreducible if and only if $\gcd(a, b, c) = 1$. In this case, the embedded minimal resolution graph of (X, o) and of the function $z : (X, o) \to (\mathbb{C}, 0)$ induced by the z-coordinate projection can be obtained as follows. (Below we write (a, c) for $\gcd(a, c)$.)

First, consider the unique $0 \le \lambda < c/(a, c)$ and $m_1 \in \mathbb{N}$ with:

$$b + \lambda \cdot \frac{a}{(a, c)} = m_1 \cdot \frac{c}{(a, c)}. \tag{2.3.3}$$

If $\lambda \neq 0$, consider the continued fraction: $\frac{c/(a,c)}{\lambda} = [[b_1, \ldots, b_s]]$, with $b_1, \ldots, b_s \ge 2$. Then the embedded resolution graph of z is the next string:



The first non-arrowhead vertex has multiplicity m_1 given by (2.3.3); all the others $m_2, \ldots, m_s$ can be computed by (2.2.1). Moreover, $X = X_{n,q}$, where $n = c/(a, c)(b, c)$ and $q = \lambda/(b, c)$ (see also [41, pages 83–84]).

If $\lambda = 0$, then (X, o) is smooth, and the embedded resolution graph of z becomes a double arrow (without non-arrowhead vertices) with the same decorations as above.

2.3.3 Addendum to Part *(d)* of Theorem 2.3.1

(i) In parallel to 2.3.2 there is a similar extension of part *(d)*, too. Consider a linear action of an arbitrary cyclic group $\mathbb{Z}_c$ on $\mathbb{C}^2$. This, in some coordinates, can be written as $\xi * (z_1, z_2) = (\xi^b z_1, \xi^{-a} z_2)$, for some positive integers a, b. If $d = \gcd(a, b, c) > 1$, then the action factorizes to an action of $\mathbb{Z}_{c/d}$, hence we can assume that $d = 1$. Take the invariants $x = z_1^c$, $y = z_2^c$ and $z = z_1^a z_2^b$, then these, as maps, define an isomorphism $\mathbb{C}^2/\mathbb{Z}_c \to \widehat{\{x^a y^b = z^c\}}$. Hence X is the Hirzebruch–Jung singularity described in Addendum 2.3.2.

(ii) Next, consider a linear action of a *finite abelian* group G on $\mathbb{C}^2$. We claim that $\mathbb{C}^2/G$ is again a cyclic quotient. Indeed, we can assume that G acts diagonally, hence $G \subset \mathbb{Z}_n \times \mathbb{Z}_n$ (for some n). If $pr_1 : \mathbb{Z}_n \times \mathbb{Z}_n \to \mathbb{Z}_n$ denotes the first projection then the nonzero elements of $\ker\left(pr_1|_G : G \to \mathbb{Z}_n\right)$ are pseudo reflections. Since by 12.3.7 G is small, $pr_1|_G$ is a monomorphism. Hence $G \subset \mathbb{Z}_n$ and it is cyclic.

(iii) In particular, using the linearization result 12.3.6, we obtain that if a finite abelian group acts on the regular germ $(\mathbb{C}^2, o)$, then $(\mathbb{C}^2/G, o)$ is a cyclic quotient.

2.3.4 Addendum to Parts *(a)* and *(c)* of Theorem 2.3.1 Note that the graph of part *(c)* is a string with decoration $-b_1, \ldots, -b_s$. Since this graph 'is the same' as its left-right mirror image, the string with decoration $-b_s, \ldots, -b_1$ should provide the same singularity. This symmetry at the level of continued fractions corresponds to

$$n/q = [[b_1, \ldots, b_s]] \quad n/q' = [[b_s, \ldots, b_1]], \tag{2.3.4}$$

where $0 < q' < n$, $qq' \equiv 1$ modulo n (for the proof use (2.2.5)). Since this is the only ambiguity at the level of graphs, we obtain:

$$X_{n,q} \text{ and } X_{m,p} \text{ are isomorphic if and only if } m = n \text{ and } p \in \{q, q'\}. \tag{2.3.5}$$

This isomorphism can be seen at the level of the $\mathbb{Z}_n$ action as well: if we choose the primitive root $\eta = \xi^{q'}$ instead of ξ, then $(\eta, \eta^q) = (\xi^{q'}, \xi^{q'q}) = (\xi^{q'}, \xi)$.

Moreover, $(a \Rightarrow c)$ also shows that once we fix the resolution graph of a cyclic quotient singularity, the analytic type of the singularity is uniquely determined; that is, the topological type carries only one analytic structure. Singularities with this rather special property are called *taut*, cf. [361].

2.3.5 Addendum to Parts *(d)* and *(e)* of Theorem 2.3.1 The monoids in $\mathbb{C}[z_1, z_2]^{\mathbb{Z}_n}$ correspond to the lattice points of $\sigma^\vee \cap \mathbb{Z}^2$ providing the isomorphism $\mathbb{C}[z_1, z_2]^{\mathbb{Z}_n} \simeq \mathbb{C}[\sigma^\vee \cap \mathbb{Z}^2]$. This can be seen as follows (cf. e.g. with [522]).

Consider the convex closure of the lattice points of $\sigma^\vee \cap \mathbb{Z}^2$, and the sequential lattice points $\{(p_j, q_j)\}_{0 \le j \le r+1}$ on the part of its boundary connecting $(1, 0)$ with (q, n). We number them in such a way that $(p_0, q_0) = (1, 0)$, $(p_1, q_1) = (1, 1)$,

$(p_{r+1}, q_{r+1}) = (q, n)$. We rewrite the vector $V_j = (p_j, q_j)$ as (m_j^1, m_j^0), and we identify it with the monomial $u_0^{m_j^0} v_0^{m_j^1}$ in the $\mathbb{C}[u_0, v_0]$ plane (cf. the proof of $(a \Rightarrow c)$). Hence m_j^0 serves as the multiplicity of S and m_j^1 as the multiplicity of E_1. These can be continued uniquely to a multiplicity system $\{m_j^i\}_{i=0}^{s+1}$ via the identities $m_j^{i-1} + m_j^{i+1} = b_i m_j^i$ providing multiplicities for $S, E_1, \ldots, E_s, S'$. Set $r_j := m_j^{s+1}$, the multiplicity of S'. Then the monomials $g_j := z_1^{r_j} z_2^{q_j}$ form a minimal set of generators of $\mathbb{C}[z_1, z_2]^{\mathbb{Z}_n}$. (The string with multiplicity system $\{m_j^i\}_i$ is the embedded resolution graph of g_j.) Clearly the pairs $\{(p_j, q_j)\}_{j=0}^{r+1}$ and $\{(r_j, q_j)\}_{j=0}^{r+1}$ determine each other; $(p_0, q_0) = (1, 0)$ provides $(r_0, q_0) = (n, 0)$ and $g_0 = z_1^n$, $(p_1, q_1) = (1, 1)$ provides $(n-q, 1)$ and $g_1 = z_1^{n-q} z_2$, while $(p_{r+1}, q_{r+1}) = (q, n)$ gives $(0, n)$ and $g_{r+1} = z_2^n$.

The point is that for each $j = 1, \ldots, r$ there exists $a_j \in \mathbb{Z}_{>0}$ such that $V_{j-1} + V_{j+1} = a_j V_j$. Furthermore, the entries $\{a_j\}_{j=1}^r$ determine exactly the continued fraction of $n/(n-q)$, that is $n/(n-q) = [[a_1, \ldots, a_r]]$ (for the geometric meaning of the duality connecting n/q and $n/(n-q)$, hence $[[b_1, \ldots, b_s]]$ and $[[a_1, \ldots, a_r]]$, see Example 3.1.6.)

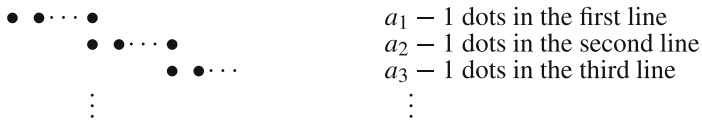
There is a much shorter combinatorial way to compare the dual entries of the continued fractions of n/q and $n/(n-q)$ formulated in *Riemenschneider's point diagram* [590]. Assume that

$$n/(n-q) = [[a_1, \ldots, a_r]] = [[(2)^{m_1-1}, n_1, (2)^{m_2-1}, n_2, \ldots, n_t, (2)^{m_{t+1}-1}]],$$

where $(2)^\ell$ means the constant sequence with ℓ terms all equal to 2, and $n_j \geq 3$ for all $1 \leq j \leq t$. Then $n/q = [[b_1, \ldots, b_s]]$ is

$$n/q = [[m_1 + 1, (2)^{n_1-3}, m_2 + 2, (2)^{n_2-3}, \ldots, m_t + 2, (2)^{n_t-3}, m_{t+1} + 1]].$$

The corresponding *point diagram* providing this 'transformation' is the following:



while the number of dots in the columns are $b_1 - 1, b_2 - 1, \ldots$ respectively.

In particular:

$$r = 1 + \sum_{1 \leq i \leq s} (b_i - 2) = -1 + \sum_{1 \leq i \leq t+1} m_i \quad \text{and} \quad s = 1 + \sum_{1 \leq j \leq t} (a_j - 2).$$

Besides the identities $V_{j-1}+V_{j+1}=a_jV_j$ the vectors V_j in $\sigma^\vee\cap\mathbb{Z}^2$ satisfies also

$$V_{j-1}+V_{k+1}=V_j+\sum_{j\le l\le k}(a_l-2)V_l+V_k\quad(1\le j\le k\le r)$$

and all the semigroup relations connecting $\{V_j\}_j$ are generated by these identities. These automatically extend to a similar set of identities connecting the multiplicities $\{m^i_j\}_j$, hence providing relations in $\mathbb{C}[z_1,z_2]^{\mathbb{Z}_n}$ (cf. [54, 110, 662]):

$$g_{j-1}g_{k+1}=g_j\cdot(\prod_{j\le l\le k}g_l^{a_l-2})\cdot g_k\quad(1\le j\le k\le r).$$

These are the (toric) equations of $X_{n,q}$ in $\mathbb{C}[g_0,\dots,g_{r+1}]$, realized as a toric variety $\mathrm{Spec}\mathbb{C}[\sigma^\vee\cap\mathbb{Z}^2]$. It is worth to emphasise again that the entries $\{m^i_j\}_{i,j}$ for fixed j satisfy $m^{i-1}_j+m^{i+1}_j=b_im^i_j$, and for fixed i the identities $m^i_{j-1}+m^i_{j+1}=a_jm^i_j$.

In order to have a complete picture of the duality, let us consider the dual cone σ of $\sigma^\vee$ in $\mathbb{R}^2$, that is the cone generated by $(0,1)$ and $(n,-q)$. Similarly as above, we can determine the minimal set of generators of $\sigma\cap\mathbb{Z}^2$ by taking the lattice point on the boundary broken segment connecting $(0,1)$ and $(n,-q)$ of the convex closure of all lattice points of $\sigma\cap\mathbb{Z}^2$. It turns out that we get $s+2$ vectors $\{W_i\}_{i=0}^{s+1}$, and they satisfy $W_{i-1}+W_{i+1}=b_iW_i$ for all $1\le i\le s$.

In particular, the cone σ is responsible for the resolution graph of $X_{n,q}$ (hence for its topology, cf. next chapter), while $\sigma^\vee$ for the equations of $X_{n,q}$.

If we rotate σ by $\pi/2$, we can replace it by the cone σ' generated by $(-1,0)$ and (q,n) (hence it becomes an adjacent cone of $\sigma^\vee$), then even more combinatorial duality properties can be read from the pair $(\sigma',\sigma^\vee)$, cf. [578].

Example 2.3.6 Fix positive integers $\{a_k\}_{k=1}^\kappa$, $\{b_k\}_{k=1}^\kappa$, $\{c_k\}_{k=1}^\kappa$ (for some $\kappa\ge 1$). Define X as the normalization of $\{(x,y,z_1,\dots,z_\kappa)\ :\ x^{a_k}y^{b_k}=z_k^{c_k}\ \text{ for all } 1\le k\le\kappa\}$.

The branch locus of the projection to the (x,y)-plane is contained in $\{xy=0\}$, hence each locally irreducible component of X has a Hirzebruch–Jung singularity.

Consider the monodromy map $\rho:\mathbb{Z}^2\to\mathbb{Z}_{c_1}\times\cdots\times\mathbb{Z}_{c_\kappa}$ of the regular covering above $\mathbb{C}^2\setminus\{xy=0\}$ (cf. the proof of part (b)⇒(a) of Theorem 2.3.1). The image of ρ is generated by $\rho((1,0))=(e^{2\pi ia_1/c_1},\dots,e^{2\pi ia_\kappa/c_\kappa})$ and $\rho((0,1))=(e^{2\pi ib_1/c_1},\dots,e^{2\pi ib_\kappa/c_\kappa})$. Therefore, the number of irreducible components of (X,o) is the order of $\prod_k\mathbb{Z}_{c_k}/\mathrm{im}\,\rho$. Since the group $\prod_k\mathbb{Z}_{c_k}$ acts transitively on these components, all the components are isomorphic to each other.

Next, we determine the lattice $K:=\ker\rho\subset\mathbb{Z}^2$. Using the above generators, one gets that $(m,n)\in K$ if and only if (†) $ma_k+nb_k\equiv 0\ (\mathrm{mod}\,c_k)$ for all $1\le k\le\kappa$.

We search for solutions of (†). First assume that $n=0$. Then m should be divisible by $M:=\mathrm{lcm}_k\{c_k/\gcd(a_k,c_k)\}$, hence $K\cap(\mathbb{Z}\times\{0\})=\mathbb{Z}\langle(M,0)\rangle$. Next, note that $\mathcal{N}:=\{n\ :\ (m,n)\in K\ \text{for some}\ m\}$ is a subgroup of $\mathbb{Z}$. Let n_0 be the smallest positive integer from $\mathcal{N}$. Then there exists a unique integer

Λ with $0 \leq \Lambda < M$ such that $(\Lambda, n_0) \in K$. Moreover, K is generated over $\mathbb{Z}$ by $(M, 0)$ and (Λ, n_0). Therefore, via a similar decomposition as in (2.3.1), an irreducible component of (X, o) has type $X_{n,q}$, where $n = M/\gcd(M, \Lambda)$ and $q = \Lambda/\gcd(M, \Lambda)$.

2.4 Existence and Uniqueness of the Minimal Resolution

2.4.1 The Existence of a Resolution At the beginning of the twentieth century the Italian geometers proposed several approaches to resolve complex surface singularities, see e.g. [784, I.6] for the early history. The most significant one, the method of Albanese [13], can be made precise, and it had serious impact and several (even higher dimensional) generalizations. For a detailed presentation of this and several other methods see also [330, 390]. Albanese's method proves that every d-dimensional projective variety X is birational to another variety whose points have multiplicity $\leq d!$. In short the algorithm runs as follows. First, we start with a convenient embedding of X into $\mathbb{P}^n$. Then we choose a singular point $p \in X$ with high multiplicity, and projects $X \setminus p$ to $\mathbb{P}^{n-1}$ from p, and takes its closure. Repeating this step finitely many times one gets the wished birational transform. In dimension two, in this way we obtain a surface whose singularities have multiplicity 2, hence locally they are of hypersurface suspension type $\{z^2 = f(x, y)\}$ (see Example 6.7.13), and they can be resolved basically via resolution theory of plane curve singularities (see 4.1.18).

The first completely rigourous proof for resolution of analytic surface singularities was given by Walker [740] patching together the local resolutions of Jung [293]. Based on Jung's work, another proof was given by Hirzebruch in [269] (who additionally also realized the power and elegance of continued fractions in the resolution procedure of surface singularities). These methods can be carried out algebraically as well. Later Zariski provided several algebraic proofs over fields of characteristic zero. In [779] he showed that if one blows up the singular set and normalizes the resulting surface, this procedure repeated finitely many times provides a resolution. In [780] he obtained the resolution by successively blowing up points and smooth non-singular curves of maximal multiplicity. The existence of resolution in arbitrary dimensions (in the analytic case and in the algebraic case over fields of characteristic zero) was established by Hironaka in his celebrated article [265]. For new proofs and recent developments, see [260, 330] and the references therein (e.g. [58, 79, 185, 718, 755]).

Below, we will present the 'oldest' algorithm, given by Jung in 1908 [293], discussed also in [41, 330, 358, 390]. It gives not only a qualitative proof of the existence of the resolution, but also a rather clear recipe how this resolution can be constructed. Moreover, it can be related with several geometric constructions as well, e.g. (cyclic) coverings, see [358, §III] or [475, 501, 502].

2.4.2 The Jungian Strategy Jung's method reduces the construction of a resolution in dimension d to the existence and properties of a strong embedded resolution of $(d-1)$-dimensional divisors embedded in $(\mathbb{C}^d, 0)$. In order to run the program, one needs to understand the resolution of finite covers of $(\mathbb{C}^d, 0)$ with normal crossing branch loci, and one has to glue these local resolutions together. In dimension two, for a normal surface singularity (X, o), it can be summarized in the diagram:

$$\begin{array}{ccccccc} \widetilde{X} & \longrightarrow & \widehat{X}' & \longrightarrow & X' & \longrightarrow & (X,o) \\ & & & \searrow^{\widehat{\pi}'} & \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ & & & & (\widetilde{Y}, B') & \xrightarrow{\phi_B} & (\mathbb{C}^2,0) \supset (B,0) \end{array}$$

where:

(a) $\pi : (X, o) \to (\mathbb{C}^2, 0)$ is an analytic covering (guaranteed by Noether's Finiteness Lemma) with branch locus $(B, 0) \subset (\mathbb{C}^2, 0)$;
(b) ϕ_B is an embedded resolution of $(B, 0) \subset (\mathbb{C}^2, 0)$. In particular, $\widetilde{Y}$ is smooth and $B' = \phi_B^{-1}(B)$ is a divisor with normal crossing singularities;
(c) $\pi' : X' \to \widetilde{Y}$ is the pullback of π via ϕ_B, and $\widehat{X}'$ is the normalization of X'. Then $\widehat{X}'$ has only normal singularities, and the branch locus of the projection $\widehat{\pi}'$ has normal crossing singularities only. This property characterizes exactly the Hirzebruch–Jung singularities, cf. Theorem 2.3.1. In particular, $\widehat{X}'$ has only finitely many singular points, all of them of Hirzebruch–Jung type.
(d) $\widetilde{X} \to \widehat{X}'$ is the resolution of the Hirzebruch–Jung singularities of $\widehat{X}'$, cf. Sect. 2.3.

Hence the knowledge of the embedded resolution of plane curve singularities and of the resolution of Hirzebruch–Jung singularities provides the wished resolution.

The resolution $\phi : \widetilde{X} \to X$ can be made good by composing with a good embedded resolution of the pair ($\widetilde{X}$, exceptional curve of ϕ).

2.4.3 The Existence of Minimal Resolutions and Its Uniqueness The minimality and uniqueness (under minimality assumption) of a resolution are based on the following facts regarding (-1)-curves on smooth surfaces.

Theorem 2.4.4

*(1) **(Castelnuovo's Contractibility Criterion)** Let S be a smooth surface and $C \subset S$ a (-1)-curve on it. Then C can be contracted to a smooth point: there exists a smooth surface S_C and a modification $\kappa : S \to S_C$ which maps C to a point o_C and its restriction $S \setminus C \to S_C \setminus \{o_C\}$ is an isomorphism.*
(2) Assume that $\phi : S \to T$ is a modification, where both S and T are smooth surfaces. Assume that ϕ is an isomorphism over $T \setminus \{o_T\}$ for some point $o_T \in T$, but ϕ is not an isomorphism over T. Then at least one of the irreducible components of $\phi^{-1}(o_T)$ is a (-1)-curve.
(3) Assume that $\phi : S \to T$ is a modification with both S and T smooth surfaces, and all rational curves on T have negative self-intersections. If C is a (-1)-

curve on S then ϕ maps C either to a (-1)-curve or to a point. In the second case, ϕ factorizes through κ: that is, there exists $\phi_C : S_C \to T$ such that $\phi_C \circ \kappa = \phi$.

Proof

(1) The first part has a similar proof as the proof of $(c \Rightarrow a)$ in Theorem 2.3.1. In the present case the starting graph is $A \overset{-1}{\longleftarrow\!\bullet\!\longrightarrow} A'$. Consider the divisors $F = A + C$ and $G = C + A'$. Similarly as in the proof of 2.3.1, one obtains that both are divisors of holomorphic functions, say of f and g, defined in a sufficiently small neighborhood U of C. Since $A \cap A' = \emptyset$, $(C, A) = (C, A') = 1$ and $C^2 = -1$, we get by topological reasons, that for any $(c, d) \neq (0, 0)$, but in a sufficiently small neighborhood V of $(0, 0)$ in $\mathbb{C}^2$, the curves $f^{-1}(c)$ and $g^{-1}(d)$ (in U) intersect each other in exactly one point transversely. (Indeed, the graph, as a plumbing graph, cf. 3.3, represents S^3, and A and A' represent two link components with linking number one.) In other words, the holomorphic map $(f, g) : (f, g)^{-1}(V) \setminus C \to V \setminus (0, 0)$ is biholomorphic. Since F and G contain C with positive coefficients, (f, g) contracts C. Finally, the contraction globalizes: S_C can be obtained from gluing $S \setminus C$ and V via $(f, g) : (f, g)^{-1}(V) \setminus C \to V \setminus (0, 0)$.

See also [41, p. 78], and [587, p. 103] or [258, 330, 632] for algebraic proofs. (The above existence of holomorphic functions with given topology/divisor will be explained and generalized for all resolutions of rational singularities in 7.1.13.)

(2) Let x and y be local coordinates at o_T. Then $\phi^*(dx \wedge dy)$ is a holomorphic section of Ω^2_S in a neighborhood of $E = \phi^{-1}(o_T)$ with divisor K supported on E. In fact, K is supported all along E. (Indeed, at a generic point of E, with convenient local coordinates (u, v) one has $\phi^*(x) = u^a$, $\phi^*(y) = u^b\varphi(u, v)$, $E = \{u = 0\}$, where a, b are positive since ϕ contracts E. Then $\phi^*(dx \wedge dy)$ vanishes along $u = 0$.) K being a positive cycle, $K^2 < 0$ by Proposition 2.1.12, hence there exists at least one component E_i of E with $(K, E_i) < 0$. Then E_i is a (-1)-curve by Remark 2.1.10.

(3) Assume that ϕ does not contract C. Set $C' := \phi(C)$. Then $\phi|C : C \to C'$ is finite. Since ϕ over a dense set is an isomorphism, $\phi|C$ is a homeomorphism, hence $\phi_*(C) = C'$, and the multiplicity of C in $\phi^*(C')$ is one. Hence, by projection formula, $C'^2 = (\phi_*(C), C') = (C, \phi^*(C')) = C^2 + (C, \phi^*(C') - C)$. Since $C'^2 < 0$, and $(C, \phi^*(C') - C) \geq 0$, and $C^2 = -1$, we must have $C'^2 = -1$ and $(C, \phi^*(C') - C) = 0$. The last condition guarantees that ϕ does not modify any point along C', hence C and C' are isomorphic. Hence C' is a (-1)-curve.

If ϕ contracts C then ϕ factorizes to a continuous map $\phi_C : S_C \to T$. Since continuous maps are weakly holomorphic, and S_C is smooth, ϕ_C is in fact holomorphic.

□

2.4.5 The Proof of the Existence of a Unique Minimal Resolution

Let $\phi : \widetilde{X} \to X$ be a resolution of (X, o) (guaranteed by 2.4.1). Contract all (-1)-curves via Castelnuovo's Criterion successively. What we get is a minimal resolution by Theorem 2.4.4*(2)*. Next, we show the uniqueness of such a resolution.

Assume that we have two minimal resolutions $\phi_i : \widetilde{X}_i \to X$, $i = 1, 2$. Let Π be the closure in $\widetilde{X}_1 \times \widetilde{X}_2$ of $\{(x_1, x_2) \in \widetilde{X}_1 \times \widetilde{X}_2 : \phi_1(x_1) = \phi_2(x_2)\}$. We normalize it, and if it is not smooth then we even resolve it by 2.4.1. We obtain a smooth space $\widetilde{X}$ and natural holomorphic maps $\psi_i : \widetilde{X} \to \widetilde{X}_i$ with $\phi_1 \circ \psi_1 = \phi_2 \circ \psi_2$. If ψ_i is not an isomorphism, being birational (and $\widetilde{X}_i$ smooth), the domain of definition of its birational inverse is the complement of certain points. Hence, by Theorem 2.4.4, parts *(2)* and *(3)*, if at least one of the maps ψ_i is not an isomorphism, then there exists a (-1)-curve C in $\widetilde{X}$ which is contracted by both ψ_i. Moreover, both ψ_i factor through $\psi_{i,C} : \widetilde{X}_C \to \widetilde{X}_i$. We continue this procedure till both maps ψ_i became isomorphisms. Then $\psi_1 \circ \psi_2^{-1}$ establishes the needed isomorphism.

This proof shows part 3 of Theorem 2.1.21 as well.

The existence and uniqueness of the minimal good (strong) resolution follows from the existence and uniqueness of the good (strong) embedded resolution of plane curve singularities (or it can be proved by similar arguments as above).

Remark 2.4.6 The existence of a unique minimal resolution of a normal surface singularity can be compared with the following combinatorial fact. Using blow downs of rational (-1)-vertices with at most two adjacent edges, cf. 2.2.5.2, any (negative definite) plumbing graph can be blown down to a graph which contains no such vertices. The point is that any graph admits a *unique* such minimal graph.

Since any graph has an analytic realization, and for this fixed analytic structure any blow down of the graph has an analytic realization too, this combinatorial statement follows from the previous analytic one. For a combinatorial proof see [505].

Remark 2.4.7 Uniqueness of the *minimal* resolution exists only in dimension one and two, cf. 2.1.5.

2.5 Analytic Realizations: Theorems of Grauert and Winters

We review two deep classical theorems. The first one is related with the analytic realization of a connected curve configuration with negative definite intersection form as an exceptional curve (Grauert theorem [223]). The second one is related with the realization of a negative semi–definite intersection form provided by a central fiber of a holomorphic family of curves over a disc (Winters theorem [753]). Additionally, we also recall a statement regarding the realizability of certain analytic germs defined on normal surface singularities from [567], a consequence of Winters theorem. ♣

2.5.1 The next theorem guarantees that any connected negative definite graph can be realized as the dual resolution graph of a normal surface singularity.

Theorem 2.5.2 (Grauert [223], See Also [358]) *Let S be a smooth complex manifold of dimension two, and $\{C_v\}_{v\in\mathcal{V}}$ a collection of irreducible curves in S such that $\cup_v C_v$ is connected and the intersection matrix $(C_v, C_u)_{v,u}$ is negative definite. Then there exists a normal surface $\bar{S}$ and an analytic contraction $\phi : S \to \bar{S}$ such that $\phi(\cup_v C_v)$ is a point $p \in \bar{S}$ and ϕ is an isomorphism over $\bar{S} \setminus \{p\}$.*

For any negative definite connected graph Γ (whose intersection form satisfies $(E_v, E_u) \geq 0$ for $v \neq u$) one can construct S and a normal crossing divisor $\cup_v C_v$ such that the intersection matrix of Γ agrees with $(C_v, C_u)_{v,u}$. In particular, after contracting $\cup_v C_v$, Γ appears as the dual resolution graph of the resolution of $(\bar{S}, p)$ given by the restriction of ϕ. (This is the 'converse statement' of Proposition 2.1.12.)

We emphasize that if we allow in the graph negative edges too (that is, the entry (E_v, E_u) in the intersection matrix is negative for an edge (v, u))—as in the case of general plumbing graphs, cf. 3.3.13—then even if Γ is negative definite it cannot be a resolution graph. (In Example 3.7.17 we exemplify such negative definite graph which cannot be resolution graphs even if we modify them by plumbing calculus from 3.3.13.)

2.5.3 In the projective algebraic context the analogue of the Grauert theorem is not true: for a connected configuration of projective curves sitting on a projective smooth surface $\widetilde{X}$, such that $(C_v, C_u)_{v,u}$ is negative definite, does not necessarily exist an algebraic variety X and projective map $\widetilde{X} \to X$ which contracts $\cup_v C_v$.

An example of Hironaka is the following (for details see [258, p. 417]). Take an elliptic smooth curve $C_3 \subset \mathbb{CP}^2$ (of degree 3). Blow up 10 *generic* points $\{P_i\}_i$ on it, let $\widetilde{X}$ be the resulting surface. The strict transform C of C_3 is a smooth elliptic curve with self–intersection -1. The statement is that this cannot be blown down algebraically.

Indeed, assume that this contraction $\widetilde{X} \to X$ exists, where C is contracted into o. Let D be a curve in X which does not contain o. Let $\widetilde{D}$ its inverse image in $\widetilde{X}$, and D_d its image in $\mathbb{CP}^2$. It is a plane curve, say of degree d, which intersects C_3 only in the points $\{P_i\}_i$, with intersection multiplicities $\{m_i\}_i$. Then the divisor of the intersection is $\sum_i m_i P_i$ (of total degree $3d$). Consider the group low of C_3 such that its neutral element is an inflexion point O, whose tangent line is L. Since $D_d \sim dL$ as divisors of $\mathbb{CP}^2$, in the group low of C_3 we have $\oplus_i P_i^{\oplus m_i} = \oplus_{3d} O = O$. But for generic choice of the points this identity cannot hold.

See also 7.2.101 for another analytic contraction, which is not projective.

On the other hand, some configurations admit algebraic projective contractions. For example, Artin in [27] proved that if the dual graph of the curve configuration $\cup_v C_v$ is rational (in the sense of Chap. 7, that is, after analytic contraction the analytic normal singularity will have geometric genus zero), then the curve configuration can be algebraically contracted.

Despite the above difference between analytic and algebraic cases, we wish to observe the following 'positive' comparison statement. If (X, o) is an analytic normal surface singularity germ (in fact, any analytic isolated germ), by a theorem of Artin [29, 3.8], there exists an affine algebraic variety Y and a point $y \in Y$ such that (Y, y) and (X, o) have isomorphic formal completions. Moreover, by a result of Hironaka [266], (Y, y) and (X, o) are analytically isomorphic. In particular, for any analytic germ (X, o) there exists an affine algebraic variety Y and $y \in Y$ such that (X, o) is isomorphic to the associated analytic germ (Y, y).

In particular, any analytic structure supported on a fixed topological type of an analytic normal surface singularity comes from a certain algebraic situation in the above sense of Artin–Hironaka. This also shows that if $\phi : \widetilde{X} \to X$ is an analytic resolution of (X, o) (X is a small Stein representative), then there exists an algebraic $\widetilde{Y}$, a projective contraction $\phi_a : \widetilde{Y} \to Y$ such that $\widetilde{X}$ can be analytically identified with a convenient strictly Levi pseudoconvex neighbourhood of the exceptional divisor of ϕ_a, and the analytic restriction of ϕ_a to this neighborhood is analytically isomorphic with ϕ.

2.5.4 Regarding **negative semi–definite intersection forms** we have the following. (It should be compared with 2.1.14.) The context is the following.

Let $\mathbb{Z}\langle E_v\rangle_{v=1}^s$ be a free lattice with an integral negative semi–definite intersection form $(E_v, E_u)_{v,u}$ such that $(E_v, E_u) \geq 0$ for any $v \neq u$. We assume that for any fixed u the restriction of the form to the sublattice $\mathbb{Z}\langle E_v\rangle_{v\neq u}$ (of rank $s-1$) is negative definite, and that the kernel of $(\ ,\ )$ is generated by $\sum_v m_v E_v$ with $m_v > 0$ for all v (and $\gcd_v\{m_v\} = 1$). We also assume that for any v, u there exists a sequence $v = v_0, v_1, \ldots, v_t = u$ such that $(E_{v_i}, E_{v_{i+1}}) > 0$ for any $0 \leq i < t$.

Theorem 2.5.5 (Winters [753]) *Let us consider an integral negative semi-definite matrix as above. We also fix certain non-negative integers g_v for any v. Then there exists a holomorphic family of curves $f : W \to D$ (cf. 2.1.14) such that $f^{-1}(0)$ is a connected curve with all irreducible components $\{C_v\}_v$ smooth, the genus of C_v is g_v, and all the intersections are transversal, and the intersection matrix $(C_v, C_u)_{v,u}$ agrees with $(E_v, E_u)_{v,u}$. Furthermore, the divisor of f in W is $\sum_v m_v C_v$.*

Remark 2.5.6 There is a conceptual difference between the two theorems.

In Grauert Theorem we can start with a smooth analytic surface and a collection of embedded analytic curves, and by the theorem we can contract the curves to a normal point. In particular, the analytic structure of the germ $(\bar{S}, p)$ inherits the analytic structure of $(S, \cup_v C_v)$ we started with.

On the other hand, in Winters theorem we start with a combinatorial (topological) data, and the theorems provides an analytic realization: the existence of f is linked with certain analytic realization of the ambient space W. More precisely, if we would like to start with a fixed smooth analytic surface W and a collection of embedded curves $\{C_v\}_v$ satisfying all the topological requirements, the existence of the holomorphic f (with divisor $\sum_v m_v C_v$ on this W) is not guaranteed.

2.5.7 Winters Theorem has the following consequence.

Theorem 2.5.8 ([567], For Integral Homology Sphere Link See [178]) *Let us fix a connected negative definite graph Γ with intersection matrix $(E_v, E_u)_{v,u}$, and also a nonzero element of the Lipman's cone $l = \sum_v m_v E_v \in \mathscr{S} \setminus \{0\}$ (hence, with all $m_v \in \mathbb{Z}_{>0}$). Then there exists a normal surface singularity germ (X, o) and a holomorphic function $f : (X, o) \to (\mathbb{C}, 0)$, and a resolution ϕ of (X, o) such that the dual resolution graph agrees with Γ, and* $\mathrm{div}_E(f) = l$.

Again, the theorem guarantees the existence of the pair (X, f) simultaneously: f exists only for certain analytic types of the ambient space. If one starts with a fixed analytic type (X, o) (with a resolution graph Γ), the existence of a holomorphic f (defined on this analytic (X, o)) is not guaranteed. In particular, any $l \in \mathscr{S} \setminus \{0\}$ can be realized as $\mathrm{div}_E(f)$ *whenever we choose* (X, o) *properly*. For more see 6.7.

Theorem 2.5.8 follows from Winters Theorem as follows. The graph Γ and the cycle l can be completed to a larger negative semi–definite graph Γ' and a cycle l' such that $l'|_\Gamma = l$ (and the assumptions of Winters Theorem are satisfied)—this is a combinatorial procedure. Then we consider the pair (W, f) given by Winters theorem, and we set $\widetilde{X}$ a small neighbourhood of $\cup_{v \in \Gamma} C_v$ in W. This is the resolution space of (X, o). The expected function is given by the restriction of f to $\widetilde{X}$.

Chapter 3
The Link

3.1 The Local Conic Structure of Isolated Analytic Germs. The Link

We define the link of an isolated singularity and we state its first properties. The most important one says that it is a complete topological invariant: it characterizes topologically the local germ. This is guaranteed by the real cone structure of a complex analytic set near any given point. ♣

3.1.1 Let X be a complex analytic set, $o \in X$ an isolated singular point, and U a neighborhood of o such that $U \setminus \{o\}$ is smooth. We fix a *real analytic* function $\rho : U \to [0, \infty)$ with $\rho^{-1}(0) = \{o\}$. Such a function always exists: take e.g. any embedding $(X, o) \subset (\mathbb{C}^n, 0)$ and the restriction of the distance function $\rho(z) = |z|^2$. More generally, take $h_1, \ldots, h_m \in \mathcal{O}(U)$ with $\{x \in U : h_1(x) = \cdots = h_m(x) = 0\} = \{o\}$. Then $\rho = \sum_i |h_i|^2$ also works.

It is convenient to write X_S for $\rho^{-1}(S)$ for different subsets S of $[0, \infty)$.

The next theorem characterizes the local homeomorphism type of (X, o) showing its *conic structure*. In the general case the conic structure of semi-analytic sets was proved by Łojasiewicz [393]. The next set of results for germs of complex algebraic hypersurfaces with isolated singularities was established by Milnor [430], while the generalization to non-isolated hypersurface singularities was done by Burghelea and Verona [93]. For the neighbourhood structure of an arbitrary algebraic set see also [164]. Proofs valid for analytic sets with isolated singularities can be found in [251, 377, 395].

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Theorem 3.1.2

(a) There exists a sufficiently small $\epsilon_0 > 0$ with the property that for any $0 < \epsilon \leq \epsilon_0$ the inverse image $X_{\{\epsilon\}} := \rho^{-1}(\epsilon)$ is a C^∞ manifold. Its C^∞ type is independent of the choice of ϵ and ρ.

(b) The homeomorphism type of the pair $(X_{[0,\epsilon]}, X_{\{\epsilon\}})$ is independent of the choice of ϵ, and it is the same as the homeomorphism type of the pair $(\text{real cone}(X_{\{\epsilon\}}), X_{\{\epsilon\}})$, where the vertex of the cone corresponds to o.

(c) The homeomorphism type of $(X_{[0,\epsilon]}, X_{\{\epsilon\}})$ is independent of the choice of ρ too, hence it depends only on the germ of the analytic set (X, o).

As $X_{[0,\epsilon]} \setminus \{o\}$ is a C^∞ manifold with a canonical orientation (induced by the complex structure), its boundary $X_{\{\epsilon\}}$ inherits a canonical orientation too. If the complex dimension of $X \setminus \{o\}$ is d then the real dimension of $X_{\{\epsilon\}}$ is $2d - 1$.

Definition 3.1.3 The oriented diffeomorphism type of $X_{\{\epsilon\}}$ is called the *link of X at o*. We denote it by $L(X, o)$, or just by L_X if there is no danger of confusion.

Example 3.1.4

(a) If (X, o) is smooth and of dimension n then $L_X = S^{2n-1}$.

(b) Let $C \subset \mathbb{P}^{n-1}$ be an irreducible smooth projective variety in $\mathbb{P}^{n-1}$. Let $X \subset \mathbb{C}^n$ be the affine cone over it; cf. 2.1.5(b) and 2.1.9. Then $L(X, o)$ is an oriented S^1-bundle over C whose Euler class is $c_1(\mathscr{O}_{\mathbb{P}^{n-1}}(-1)|_C)$. In particular, if C is a curve, then the Euler number of the S^1-bundle is $-\deg(C)$.

(c) Assume that X is a normal affine surface, which admits a good $\mathbb{C}^*$ action (cf. 5.1). Then $L(X, o)$ is a Seifert 3-manifold (see 5.1.12).

(d) Usually, the link is not connected and is not even equidimensional. (Take for X e.g. the union of the (z_1, z_2)-plane and the z_3-axis of $\mathbb{C}^3$.)

Lemma 3.1.5

(a) Consider a holomorphic map $f : \bar{X} \to X$ such that the restriction of f to $f^{-1}(X \setminus \{o\})$ is an isomorphism and $f^{-1}(o) = \{\bar{o}_1, \ldots, \bar{o}_k\}$. Then $L(X, o)$ is the disjoint union of the manifolds $\{L(\bar{X}, \bar{o}_j)\}_{j=1}^k$.

(b) If (X, o) is locally irreducible then $L(X, o)$ is connected. Hence, in general, the number of connected components of $L(X, o)$ equals the number of local irreducible components of (X, o) at o.

Proof *(a)* If ρ, as in 3.1.1, defines $L(X, o)$, then $\rho \circ f$ defines the links of $\cup_j (\bar{X}, \bar{o}_j)$, and the diffeomorphism is induced by f. For *(b)* use 12.2.6 and the fact that the normalization of a locally irreducible germ with an isolated singularity is a homeomorphism. □

Example 3.1.6 Consider the situation of Theorem 2.3.1*(d)*. Set $S^3 = \{|z_1|^2 + |z_2|^2 = \epsilon\}$. Then the $\mathbb{Z}_n$-action preserves S^3, where it *acts freely*. Hence the link $L(X_{n,q}, o)$ is the lens space $L(n, q) = S^3/\mathbb{Z}_n$. Moreover, (2.3.5) is compatible with

the fact that $L(n,q)$ and $L(m,p)$ are *orientation preserving diffeomorphic* if and only if $m=n$ and $p\in\{q,q'\}$.

Let us denote by $-L(n,q)$ the manifold $L(n,q)$ with opposite orientation. It is known that $-L(n,q)$ is again a lens space, namely $L(n,n-q)$. (This can be verified e.g. by plumbing calculus from 3.3.13, proving that $-\Gamma(n,q)$ is equivalent to $\Gamma(n,n-q)$, cf. 3.3.15). For more combinatorial properties of this duality see 2.3.5.

Remark 3.1.7 Replacing complex analytic sets with real analytic ones (or real algebraic ones), the link of an isolated singularity still can be defined. Though not any manifold M might appear as such link (e.g. necessarily $\chi(M)$ should be even [671]), the restrictions are not so severe. Indeed, Nash proved in [460] that every compact differentiable manifold M can be realized as a real algebraic variety in some $\mathbb{RP}^N$, hence, by taking real cone over M in $\mathbb{R}^{N+1}$, one sees that the double cover of M (via $S^N\to\mathbb{RP}^N$) is realized as the link of a real algebraic isolated singularity.

On the other hand, links of *complex analytic sets* are more restrictive: e.g. the link of a complex cone over a smooth algebraic projective variety $C\subset\mathbb{CP}^N$ is an S^1-bundle over C. More generally, a 3-dimensional link is always a graph manifold (plumbed S^1-bundles)! This last fact has the following advantage: since graph manifolds (in particular, the topological types of isolated complex surface singularities) are classified, one has a better chance to understand the effects of the topological (=link) invariants on the analytic structure.

3.2 Embedded Links. Milnor Fibration

We review the definition and basic properties of the nearby fibration and Milnor fibration associated with a germ of analytic function f defined on an isolated singularity. If f itself defines an isolated singularity then the Milnor fibration extends to an open book decomposition. In dimension two we generalize the open book decomposition to the case when f is not isolated via the open book decomposition of multilinks. The basic terminologies and properties related with the homological package are also introduced and discussed. ♣

3.2.1 The Link of Pairs Let (X,o) be an irreducible analytic set with an isolated singularity at $o\in X$, and assume that (D,o) is an analytic subset of (X,o) such that $D\setminus\{o\}$ is smooth, equidimensional and nonempty. Then Theorem 3.1.2 can be repeated for the pair (X,D). In particular, for a real analytic function $\rho:U\to[0,\infty)$ with $\rho^{-1}(0)=\{o\}$, the level sets $(X_{\{\epsilon\}},X_{\{\epsilon\}}\cap D)$ determine a pair of oriented smooth manifolds, which is independent of the choice of ϵ and ρ. In particular, we get a canonical embedding $L(D,o)\subset L(X,o)$.

Therefore, if $f:(X,o)\to(\mathbb{C},0)$ is a holomorphic function such that the reduced zero set $V(f)$ is smooth off o, then one gets the oriented diffeomorphism

type of the embedding $L_{V(f)} \subset L_X$, where $L_{V(f)}$ has real codimension two. The normal bundle of $L_{V(f)} \subset L_X$ has a complex line bundle structure trivialized by $\mathrm{grad}(f)$.

Definition 3.2.2 The embedding $L_{V(f)} \subset L_X$ is called the embedded link of f, and denoted by $L(f, o)$, or L_f. (The notation $L_{V(f)}$ will stay for the 'abstract' link of $V(f)$, when L_X and the embedding are disregarded.)

Not any embedding $L(D, o) \subset L(X, o)$, associated with a reduced Weil divisor, can be realized (up to an isotopy of $L(X, o)$) as L_f for a certain f. This is obstructed even topologically, e.g. the homology class of $L(D, o)$ should be zero in $H_*(L(X, o), \mathbb{Z})$, cf. Theorem 3.2.6. The analytic obstruction is coded in the local divisor class group, see 6.1.

If (X, o) is smooth, then $(V(f), o) \subset (X, o)$ is called a *hypersurface singularity*.

Proposition 3.2.3 ([372, 430]) *Fix an isolated singularity (X, o), a real analytic function ρ and a holomorphic germ $f : (X, o) \to (\mathbb{C}, 0)$. Then there exist a sufficiently small $\epsilon > 0$ and $0 < \delta \ll \epsilon$, such that*

(a) $X_{[0,\epsilon]} \cap f^{-1}(t)$ is smooth and intersects the link $X_{\{\epsilon\}}$ transversally for any $t \in D_\eta \setminus \{0\}$, where $D_\eta := \{t : |t| \leq \eta\}$;
(b) the restriction of f on $(X_{[0,\epsilon]} \cap f^{-1}(\partial D_\eta), X_{\{\epsilon\}} \cap f^{-1}(\partial D_\eta))$ is a locally trivial C^∞ fibration of pair of spaces over ∂D_η ;
(c) if $V(f)$ is an isolated singularity, then the fibration $X_{\{\epsilon\}} \cap f^{-1}(\partial D_\eta) \to \partial D_\eta$ extends to a trivial fibration $X_{\{\epsilon\}} \cap f^{-1}(D_\eta) \to D_\eta$ given by the restriction of f.

Definition 3.2.4 The above fibration is called the 'nearby' fibration associated with the germ f. The space $F_t := X_{[0,\epsilon]} \cap f^{-1}(t)$ for $t \in D_\eta \setminus \{0\}$ is called the *Milnor fiber* of f at o; it is a smooth oriented manifold with boundary $\partial F_t = X_{\{\epsilon\}} \cap f^{-1}(t)$.

If f defines an isolated singularity, then 3.2.3*(c)* guarantees that ∂F_t is diffeomorphic to the link $L_{V(f)}$. Moreover, 3.2.3*(c)* also shows that the restriction of the nearby fibration to the boundary is the product projection $\partial D_\eta \times \partial F \to \partial D_\eta$.

By the next result of Milnor [430], whenever f defines an isolated singularity, the nearby fibration of f can be realized inside of the link L_X as well, and it can be identified with an *open book decomposition* of L_X.

Definition 3.2.5 An open book decomposition of a closed manifold M consists of a codimension 2 submanifold L, embedded in M with trivial normal bundle, together with a smooth fiber bundle decomposition of its complement $p : M \setminus L \to S^1$. One also requires a trivialization of the a tubular neighborhood of L into the form $L \times D$ (D a complex disc) such that the restriction of p to $L \times (D \setminus 0)$ is the map $(x, y) \mapsto y/|y|$.

The existence of an open book decomposition of an arbitrary oriented closed odd-dimensional manifold M was establish for $\dim(M) = 3$ by Alexander [14], for $\dim(M) = 5$ by A'Campo [4], for higher dimensions see [675, 752], for discussions

and other references see [403]. (For connection with contact geometry see e.g. [191].)

The submanifold L is called the *binding* of the open book, while the fibers of p are the *pages*. The differential dp induces natural co-orientations on the binding and on the pages of the open book. Thus, any fixed orientation of M induces a natural orientation on L. If L itself is oriented a priori, then we say that the open book is *compatible with the orientations of* M *and* L if the two orientations of L coincide.

Two open books (M, L, p) and (M', L', p') are isomorphic if there exists a diffeomorphism $\phi : (M, L) \to (M', L')$ which preserves the pages and their co-orientations.

Theorem 3.2.6 *[113, 377, 430] Fix an isolated singularity (X, o) and a holomorphic germ $f : (X, o) \to (\mathbb{C}, 0)$. Then there exists a sufficiently small $\epsilon > 0$, such that $\arg(f) = f/|f| : X_{\{\epsilon\}} \setminus X_{\{\epsilon\}} \cap V(f) \to S^1$ is a locally trivial C^∞ fibration. Its isotopy type does not depend on the choice of ϵ and ρ, and it is isomorphic with the nearby fibration of f. Furthermore, if f defines an isolated singularity, then the above fibration $f/|f|$ extends to an open book decomposition $(X_{\{\epsilon\}}, X_{\{\epsilon\}} \cap V(f), f/|f|)$ which is compatible with the natural orientations of $X_{\{\epsilon\}}$ and $X_{\{\epsilon\}} \cap V(f)$.*

Definition 3.2.7 The fibration $\arg(f) : L_X \setminus L_{V(f)} \to S^1$ of Theorem 3.2.6 is called the *Milnor fibration of* f, respectively the *Milnor open book decomposition of* f.

Remark 3.2.8

(a) Let M be a smooth manifold and L be an oriented smooth submanifold of real codimension 2. Assume that F is an oriented manifold with boundary of real dimension $\dim(M) - 1$, $F \subset M$, such that its interior F° embeds in M and its oriented boundary ∂F equals L. Then F is called a *Seifert 'surface'* of L.

Note that the pages of an open book decomposition (M, L, p) are Seifert surfaces.

(b) If f defines a non-isolated singularity, then the Milnor fibration near $V(f)$ is more complicated. In the next paragraphs we describe it in the case $\dim(M) = 3$.

3.2.9 Seifert Surfaces of Multilinks Let M be a smooth 3-manifold and L be an oriented smooth submanifold of dimension 1 with trivial normal bundle. Furthermore, we associate with each component $\{L_a\}_{a \in \mathscr{A}}$ of L an integer $m_a \in \mathbb{Z}_{\geq 0}$. We say that $\mathbf{m}\, L = \{m_a L_a\}_{a \in \mathscr{A}}$ is a *multilink in* M.

Definition 3.2.10

(a) A *Seifert surface of the multilink* $\mathbf{m}\, L$ is an embedded oriented open 2-manifold $F^\circ \in M \setminus L$, such that its closure F satisfies: (i) $F \setminus F^\circ \subset L$, and (ii) F intersects a tubular neighborhood $T(L_a) \subset M$ of L_a as follows.

Assume first that $m_a \neq 0$. Identify $T(L_a)$ with a disc-bundle over L_a with projection $p_a : T(L_a) \to L_a$. Then, for any $x \in L_a$, the subset $F \cap p_a^{-1}(x)$ of the disc $p_a^{-1}(x)$ can be identified with the subset $\bigstar := \{z \in \mathbb{C} : |z| \leq 1,\ z^{m_a} \in \mathbb{R}_{\geq 0}\}$ of the disc $\{z \in \mathbb{C} : |z| \leq 1\}$. The orientation of F is such that $F \cap \partial T(L_a)$ in $T(L_a)$ homologically is $m_a L_a$. (Though $T(L_a)$ is a trivial bundle, we do not require that $F \cap T(L_a)$ has a product structure $\bigstar \times L_a$.)

If $m_a = 0$, then $F \cap T(L_a)$ consists of several discs transversal to L_a (of type $p_a^{-1}(x)$). In particular, $F \cap T(L_a)$ is an embedded smooth manifold. The orientation of F is such that at each intersection point with L_a the intersection multiplicity is 1.

(b) An open book decomposition of M with binding the multilink $\mathbf{m}\, L$ is a smooth fibration $p : M \setminus L \to S^1$ such that the closure of each page $p^{-1}(\theta)$ is a Seifert surface of the multilink.

This definition is motivated by the following fact. (For the proof combine Theorem 3.2.6 with the fact that the equation of f near a generic point of $V(f)_a$ is $(z, z') \mapsto z^{m_a}$.)

Lemma 3.2.11

(a) Fix an isolated surface singularity (X, o) and let $f : (X, o) \to (\mathbb{C}, 0)$ be the germ of a holomorphic function. Consider the irreducible components $\{V(f)_a\}_{a \in \mathscr{A}}$ of the reduced zero set of f, and let $m_a \in \mathbb{Z}_{>0}$ be the vanishing order of f along $V(f)_a$. Then the Milnor fibration $\arg(f) : L_X \setminus L_{V(f)} \to S^1$ extends to an open book decomposition of the multilink $\mathbf{m}\, L_{V(f)} = \{m_a L_{V(f)_a}\}_{a \in \mathscr{A}}$.

(b) Moreover, consider additionally a Weil divisor $(D, o) \subset (X, o)$ with irreducible components $\{D_{a'}\}_{a' \in \mathscr{A}'}$ such that $D \cap V(f) = \{o\}$. Then all the pages of the above filtration $\arg(f)$ intersect D transversally, with intersection multiplicity $+1$ at all the intersection points, hence it can be interpreted as an open book decomposition of the multilink $\mathbf{m}\, L_{V(f)} \cup \mathbf{0}\, L_D$, where the multiplicity of any $D_{a'}$ is zero.

3.2.12 The Geometric Monodromy and Some of Its Properties Any C^∞ fibration over S^1 with fiber F determines a *geometric monodromy*, a diffeomorphism h_{geo} of F, well defined up to isotopy and conjugation. In the context of Lemma 3.2.11*(b)*, h_{geo} is a diffeomorphism of the pair $(F, F \cap D)$. Moreover, if $m_a = 1$ for all $a \in \mathscr{A}$, then one can assume that the restriction of h_{geo} to a neighborhood of ∂F is the identity. (For this, first one fixes a trivialization of the extension $X_{\{\epsilon\}} \cap f^{-1}(D_\eta) \to D_\eta$, cf. Proposition 3.2.3*(c)*.) Conversely, any oriented manifold with boundary $(F, \partial F)$, and orientation preserving diffeomorphism h of F, which is identity in a neighborhood of ∂F, determines an open book decomposition of the mapping torus $M = F \times [0, 1]/\sim$, where $(h(x), 0) \sim (x, 1)$ and $(x, t) \sim (x, t')$ for $x \in \partial F$ and $t, t' \in [0, 1]$. The isotopy and conjugacy class of h determines M up to an isomorphism.

If (M, L, p) is an open book decomposition, where $\dim(M) = 3$ and F is connected, then the page F is a $K(\pi_1(F), 1)$ space with free fundamental group. Hence, via the homotopy exact sequence of the fibration, $M \setminus L$ is a $K(\pi, 1)$ space too. The exact sequence $1 \to \pi_1(F) \to \pi_1(M \setminus L) \xrightarrow{p_*} \pi_1(S_1) = \mathbb{Z} \to 0$ determines an action of the (positive) generator of $\mathbb{Z}$ on $\pi_1(F)$, which, in fact, is the action induced by h_{geo}.

If, additionally, M is an integral homology sphere and $|\mathscr{A}| = 1$, then p_* is the abelianization (cf. 3.4.6), hence $\pi_1(F)$ is the commutator subgroup $[\pi_1, \pi_1]$ of $\pi_1(M \setminus L)$. In particular, $[\pi_1, \pi_1]$ is free. If $L_X = S^3$ then the converse is also true: by a theorem of Neuwirth and Stallings [518, 648], if $L \subset S^3$ is a (tame) knot, and the commutator subgroup of $\pi_1(S^3 \setminus L)$ is free (or, 'only' finitely generated), then $S^3 \setminus L$ is the total space of a fiber bundle over S^1 with a connected orientable fiber.

3.2.A The Homological Package of the Milnor Fibration

We give a nutshell presentation of the standard terminology related with Milnor fibration and of its basic properties and invariants (which frequently will be used in the sequel). For more see e.g. [20, 21, 395, 430] or [471]. ♣

3.2.13 The Homology of F Fix an isolated singularity (X, o) of dimension $n + 1$, and let $f : (X, o) \to (\mathbb{C}, 0)$ be a holomorphic germ as in 3.2.3 and 3.2.4. Since the nearby Milnor fiber F is a Stein space of (complex) dimension n, by a theorem of Andreotti–Frankel [17] it has the homotopy type of a CW complex of (real) dimension n. Hence, its homology $H_q(F, \mathbb{Z})$ vanishes for $q > n$ and $H_n(F, \mathbb{Z})$ is free.

If (X, o) is smooth and f defines an isolated singularity, then by Milnor [430] F is $(n-1)$ connected, hence its homotopy type is a bouquet of spheres $F \sim \vee_\mu S^n$. Here μ is the Milnor number, it can also be computed as the codimension in $\mathscr{O}_{X,o}$ of the Jacobian ideal generated by the partial derivatives of f. More generally, if (X, o) is a complete intersection, and f defines an isolated singularity then by a result of Hamm [252] $F \sim \vee_\mu S^n$ again (see also [395]). Furthermore, the link L_f is $n-2$ connected. In particular, if $n = 2$ we get the connectivity of the link.

Above, the assumption that $V(f)$ has an isolated singularity is essential. E.g., if (X, o) is smooth but the dimension of the singular locus of $V(f)$ is k, then F is 'merely' $(n-1-k)$ connected [304], and this bound is sharp. In this case about the (homotopy) type of F it is hard to say anything (however for some special cases see e.g. [375, 376, 470, 636, 639, 640, 686, 775, 776] and the references therein).

Since F is an oriented $2n$–dimensional smooth oriented manifold, its middle homology carries an $(-1)^n$–symmetric intersection form. If n is even we write μ_+, μ_0, μ_- for the Sylvester invariants of this quadratic form; hence $\mu = \mu_+ + \mu_0 + \mu_-$. We define the signature of f as $\sigma(f) := \mu_+ - \mu_-$, the signature of this form.

Assume next that f defines an isolated complete intersection singularity. Then the only interesting part in the long homology exact sequence of the pair $(F, \partial F)$ is

$$0 \to H_n(\partial F)/H_{n+1}(F, \partial F) \to H_n(F) \xrightarrow{i} H_n(F, \partial F) \to \tilde{H}_{n-1}(\partial F) \to 0. \tag{3.2.1}$$

Here $H_{n+1}(F, \partial F) = 0$ if $n > 1$. However for $n = 1$ it equals $\mathbb{Z}$ and $H_1(\partial F)/H_2(F, \partial F) \simeq \mathbb{Z}^{r-1}$, where r is the number of components of ∂F.

Consider the perfect pairing (Lefschetz duality) $\langle\ ,\ \rangle : H_n(F, \partial F) \otimes H_n(F) \to \mathbb{Z}$. By this, $H_n(F, \partial F)$ is the Lefschetz dual of $H_n(F)$, and i identifies with the natural morphism $b : H_n(F) \to H_n(F)^*$ given by the intersection form. This provides the description of the homologies of the link in terms of the intersection form of F.

3.2.14 The Monodromy Let (X, o) and f be arbitrary. The geometric monodromy induces (algebraic) monodromies $h_q : H_q(F) \to H_q(F)$ with characteristic polynomials $\Delta_q(t) := \det(I - th_q)$. The monodromy zeta function is a 'multiplicative' generalization of the Euler characteristic of F: $\zeta(t) := \prod_q \Delta_q(t)^{(-1)^q}$.

Since h_n preserves the intersection form, the complex intersection form decomposes into forms $\oplus_\lambda (\ ,\ )_\lambda$ according to the generalized eigenspace decomposition $H_n(F, \mathbb{C}) = \oplus_\lambda H_n(F, \mathbb{C})_\lambda$, with $(a, b)_\lambda = (-1)^n \overline{(b, a)_\lambda}$.

If I is any complex hermitian form ($\bar{I}^t = I$) its signature $\sigma(I)$ is defined similarly as in the real symmetric case. If I satisfies $\bar{I}^t = -I$ then set $\sigma(I) := \sigma(\sqrt{-1}I)$. In particular, we have the equivariant signatures $\sigma_\lambda(f) := \sigma((\ ,\ )_\lambda)$ for any eigenvalue λ. Clearly $\sum_\lambda \sigma_\lambda(f) = \sigma(f)$ and $\sigma_{\bar{\lambda}}(f) = (-1)^n \sigma_\lambda(f)$.

By the celebrated *Monodromy Theorem* (see, e.g. [6, 7, 84, 347, 373]), all the eigenvalues of h_q are roots of unity, and the maximal size of the Jordan block is $q + 1$. (The proof for $n = q = 1$ is given in 4.1.D.)

The Wang exact homology sequence of the Milnor fibration $L_X \setminus L_{V(f)} \to S^1$ connects the monodromy with the homology of $L_X \setminus L_{V(f)}$. Via Lefschetz duality this reads as

$$0 \to H^n(L_X, \partial F) \to H_n(F) \xrightarrow{h_n - I} H_n(F) \to H^{n-1}(L_X, \partial F) \to H_{n-1}(F).$$

For a detailed study of the homological monodromy when $n = 1$ see 4.1.D. *If L_X is an integral homology sphere* and $\tilde{H}_{n-1}(F) = 0$, then we get

$$0 \to H_n(\partial F)/H_{n+1}(F, \partial F) \to H_n(F) \xrightarrow{h_n - I} H_n(F) \to \tilde{H}_{n-1}(\partial F) \to 0. \tag{3.2.2}$$

This provides the description of the homologies of the link in terms of $h_n - I$.

3.2.15 The Variation Map (See also [160, 303].) If f defines an isolated singularity, then one can choose the geometric monodromy in such a way that

$h_{geo}|_{\partial F} = id_{\partial F}$. The correspondence $[c] \mapsto [h_{geo}(c) - c]$ determines a well-defined map $V : H_n(F, \partial F, \mathbb{Z}) \to H_n(F, \mathbb{Z})$, called the *variation map*. Automatically, $V \circ i = h - 1$ (where $h = h_n$). If L_X is an integral homology sphere, then in fact V is an isomorphism. Furthermore, in this case, under the (Lefschetz) identification $H_n(F, \partial F, \mathbb{Z}) = H_n(F, \mathbb{Z})^*$,

$$h = -(-1)^n V(\theta^{-1} \circ V^*)^{-1}, \quad b = -V^{-1} - (-1)^n(\theta^{-1} \circ V^*)^{-1},$$

where $\theta : H_n(F, \mathbb{Z}) \to H_n(F, \mathbb{Z})^{**}$ is the natural isomorphism. Note also that via the identity $V \circ i = h - 1$ one identifies the exact sequences (3.2.1) and (3.2.2).

Local trivialization of $\arg(f)$ over a small interval $e^{2\pi i[0,\epsilon]}$ $(0 < \epsilon \ll 1)$ provides an identification of $F = \arg(f)^{-1}(1)$ with $F^+ = \arg(f)^{-1}(e^{2\pi i\epsilon})$, denoted by h^+_{geo}. Then, whenever L_X is an integral homology sphere (hence the linking number ℓn is well-defined), one defines the Seifert form $S : H_n(F) \otimes H_n(F) \to \mathbb{Z}$ by $S(a, b) := \ell n(a, h^+_{geo}(b))$. Then, S and V can be identified via $S(a, b) = \langle V^{-1}(a), b\rangle$.

Under the assumption that L_X is a simply connected integral homology sphere, Durfee [160] and Kato [303] proved independently (based on results of Levine) that the set of isomorphism classes of open book decompositions in L_X is equivalent via the variation map with the set of conjugacy classes of unimodular bilinear forms over $\mathbb{Z}$, whenever $n \geq 3$. The bijection cannot be extended to the cases $n \leq 2$, see e.g. [157, 158] for $n = 1$ and [22] for $n = 2$.

3.3 Plumbed 3-Manifolds: Plumbing Graphs

In this section we describe the construction of oriented plumbed three and four manifolds associated with plumbing graphs. Then we discuss two cases of the plumbing calculus: the case of connected graphs with negative definite intersection form (which provide links of normal surface singularities) and the more general case of graphs of oriented plumbed 3-manifolds. ♣

3.3.1 To any normal surface singularity (X, o) we associated its link L_X and its resolution graph Γ_X (well-defined up to blow up/down of (-1)-curves). In this section we explain how they determine each other.

First, we show how L_X is recovered from Γ_X via the *plumbing construction*, by considering Γ_X as a *plumbing graph*. For more details, see [270, 277, 448, 505, 578].

3.3.2 Plumbing Graphs A *plumbing graph* (of a closed oriented plumbed 3-manifold) is a graph whose vertices carry two integral decorations. One of them is the *'Euler number'* e_v, which can be any integer, while the other one is the *'genus'*, a non-negative integer, written as $[g_v]$ (and it is omitted if it is zero).

3.3.3 The Plumbing Construction Fix a *connected* plumbing graph Γ. As usual, we denote the set of vertices and edges by $\mathscr{V}$ and $\mathscr{E}$.

The *oriented plumbed 3-manifold* $M(\Gamma)$ associated with Γ is constructed using the set of S^1–bundles $\{\pi_v : B_v \to S_v\}_{v\in\mathscr{V}}$, whose total space B_v has a fixed orientation. They are indexed by the set of vertices $\mathscr{V}$ of the plumbing graph, so that the base-space S_v of B_v is a closed orientable real surface of genus g_v, and the Euler number of the bundle B_v is the Euler number decoration of the vertex v on the graph. Then one glues these bundles corresponding to the edges of Γ as follows. First, for each v one chooses an orientation of S_v and of the fibers compatible with the orientation of B_v. Then, for each edge adjacent to v one fixes a point $p \in S_v$, an orientation preserving trivialization $D_p \times S^1 \to \pi_v^{-1}(D_p)$ above a small closed disc $D_p \ni p$, and one deletes its interior $D_p^\circ \times S^1$. Here, S^1 is the unit circle with its natural orientation. Then, any edge (v, w) of Γ determines $\partial D_p \times S^1$ in B_v and $\partial D_q \times S^1$ in B_w, both diffeomorphic to $S^1 \times S^1$. They are glued by the map $(x, y) \mapsto (y, x)$.

In this construction in the plumbing graph we can allow loops as well (edges with identical end-vertices). If the vertex v supports several loops then in the plumbing construction for each loop we choose a pair of points $p, q \in S_v$ and we make a 'self–gluing' of $\partial D_p \times S^1$ and $\partial D_q \times S^1$ in B_v (in the bundle with Euler number e_v).

The orientations of B_v provide a unique well-defined orientation on M, (and this is independent of the choices of the base/fiber orientations in the construction: we can change all the orientations of the of base/fiber spaces simultaneously, but this will not effect the orientation of M).

Next, assume that Γ is not connected. If $\{\Gamma_i\}_i$ are its connected components, then one defines $M(\Gamma)$ as the connected sum $\#_i M(\Gamma_i)$.

Furthermore, we allow the empty graph too: it corresponds to $M(\emptyset) = S^3$.

3.3.4 Oriented Links in Oriented Closed Plumbed 3-Manifolds Any plumbing graph as in 3.3.2 can be complemented with a set of *arrowhead vertices*; the other 'old' vertices will be called *non–arrowheads*. Arrowhead vertices have no 'Euler number' or 'genus' decorations, each of them is connected by a unique edge to some non-arrowhead vertex. We write $\mathscr{A}$ for the set of arrowheads, and $\mathscr{V}$ for the set of non-arrowheads, and $\mathscr{W} = \mathscr{A} \sqcup \mathscr{V}$.

If we disregard the arrowheads and their supporting edges, by 3.3.3 we obtain an oriented 3–manifold $M(\Gamma)$. The arrowheads represent an oriented link L (1-dimensional submanifold) in $M(\Gamma)$: any arrowhead supported by a non-arrowhead v codifies a generic S^1-fiber of B_v. The collection of all such fibers carries a natural orientation well defined up to a simultaneous change of orientation of all the components. (In the construction 3.3.3 we fix an orientation of the base spaces and fibers in order to glue these pieces. These fiber-orientations are 'remembered'.) Usually we fix a choice, hence we orient compatibly all the fibers.

There is an exception to the above description when the graph consists of a double arrow. In this case $M(\Gamma) = S^3$, and the arrows represent two oriented Hopf link–components.

3.3.5 Oriented Plumbed 4-Manifolds The above construction provides even an oriented 4–manifold $P(\Gamma)$ with boundary such that $\partial P(\Gamma) = M(\Gamma)$. Indeed, if one replaces the S^1–bundles with the corresponding disc-bundles $\mathscr{D}_v$ and one glues them by a similar procedure (and one smoothes the corners) one gets $P(\Gamma)$. In this case, $\pi_v^{-1}(D_p)$ and $\pi_w^{-1}(D_q)$ are bidiscs, which will be identified via $(x, y) \mapsto (y, x)$. Again, in the presence of a loop, we take for each loop two points $p, q \in S_v$ and we self-glue the bidiscs $\pi_v^{-1}(D_p)$ and $\pi_w^{-1}(D_q)$.

The arrowheads will correspond to oriented 2-discs $(D^2, \partial D^2) \subset (P, \partial P)$: corresponding to an arrowhead supported on $v \in \mathscr{V}$ on takes a generic fiber of $\mathscr{D}_v$ chosen similarly as the link-components of L. The boundary of their union is L.

Each vertex $w \in \mathscr{W}$ determines a 2–cycle C_w in $P = P(\Gamma)$. If $a \in \mathscr{A}$ is an arrowhead supported by $v \in \mathscr{V}$ then C_a is an oriented generic disc–fiber of $\mathscr{D}_v$; it is a relative cycle of $(P, \partial P)$. If v is a non–arrowhead vertex then C_v is the oriented 'core' (i.e. the zero section) of $\mathscr{D}_v$, hence it is an absolute cycle. $H_2(P, \mathbb{Z})$ is freely generated by the classes of $\{C_v\}_{v\in\mathscr{V}}$. The intersection matrix $I = \{(C_v, C_u)\}_{v,u\in\mathscr{V}}$ codifies the Euler numbers and the combinatorics of the edges connecting non–arrowheads. Indeed, if we have no loops, then $I_{v,u}$ is e_v if $v = u$, and the number of edges connecting v and u if $v \neq u$. However, if v supports k_v loops, then $I_{v,v} = e_v + 2k_v$.

3.3.6 Multiplicity Systems Plumbing graphs with arrowheads might carry an extra set of decorations, the *'multiplicities'*, denoted by (m_w), for every $w \in \mathscr{W}$.

By convention, a system of multiplicities $\{m_w\}_{w\in\mathscr{W}}$ represent a relative 2-cycle $C(\underline{m}) := \sum_{w\in\mathscr{W}} m_w C_w$ in P whose homology class in $H_2(P, \partial P, \mathbb{Z})$ is zero. Therefore, since $(C(\underline{m}), C_v) = 0$ for all $v \in \mathscr{V}$, it satisfies a similar set of relations as the system of multiplicities of an embedded resolution graph (2.2.1).

A geometric motivation for the above definition is the following. Assume that (M, L, p) is an open book decomposition of the multilink $\{m_a L_a\}_{a\in\mathscr{A}}$. Then p automatically induces a set of multiplicities, called the *multiplicity system of* p. For this first we define for each $w \in \mathscr{W}$ an oriented loop γ_w in $P \setminus L$. For $v \in \mathscr{V}$ let γ_v be an oriented generic S^1-fiber of B_v. For $a \in \mathscr{A}$ we define γ_a as the oriented meridian of L_a. Then $m_a = p_*([\gamma_a]) \in H_1(S^1, \mathbb{Z}) = \mathbb{Z}$ for any $a \in \mathscr{A}$, and if we set $m_v := p_*([\gamma_v]) \in \mathbb{Z}$ for any $v \in \mathscr{V}$ too, then $\{m_w\}_{w\in\mathscr{W}}$ is a multiplicity system. (Indeed, $C(\underline{m})$ is homologous with a page F pushed in the interior of P keeping $L = \partial F$ fixed, and F in the relative homology represents zero since $F \subset \partial P$).

3.3.7 Example: The Link L_X. Assume that (X, o) is a normal surface singularity, $f : (X, o) \to (\mathbb{C}, 0)$ holomorphic and $\rho : (X, o) \to [0, \infty)$ as in 3.1.1. We fix a *good* resolution $\phi : \widetilde{X} \to X$, which determines its dual embedded resolution graph Γ_f. (Note that Γ_f has no loops.)

Proposition 3.3.8 *For* $0 < \epsilon \ll 1$ *one has the oriented diffeomorphisms* $\phi^{-1}(X_{[0,\epsilon]}) \simeq P(\Gamma_f)$ *and* $(L_X, L_{V(f)}) \simeq (M(\Gamma_f), L)$. *The diffeomorphism identifies the cores* C_v *with the irreducible exceptional divisors; this induces a canonical choice of orientations in the base spaces and fibers in the plumbing construction (hence of the corresponding link components as well). The multiplicities of* Γ_f *equal the multiplicities associated with the Milnor open book* $(L_X, L_{V(f)}, \arg f)$ *(as in 3.3.6).*

Proof The C^∞ tubular neighborhood of the smooth embedded curve $E_w \subset \widetilde{X}$ is determined by its genus and the Euler number of its normal bundle. □

Remark 3.3.9 By Proposition 3.3.8 the dual resolution graph Γ_X of (X, o) determines the link L_X, the dual graph Γ_f of f determines the pair $(L_X, L_{V(f)})$ (hence the binding of the open book decomposition $(L_X, L_{V(f)}, \arg f)$). Nevertheless, in general, from Γ_f *one cannot recover* the projection $\arg f$ of the open book decomposition, see Sect. 3.4.

Example 3.3.10 (Plane Curve Singularities [85, 178, 240, 430, 745]) If (X, o) is smooth then the embedded resolution of the pair $(V(f), o) \subset (X, o)$ is obtained by a successive sequence of blow ups at smooth points, and the graph Γ_f is a tree with all $g_v = 0$. Hence, a graph is the embedded resolution graph of a plane curve singularity if and only if after deleting all its arrowheads and multiplicities it can be blow down to the empty graph. In this case the embedded resolution graph determines completely (besides the embedded link $L_{V(f)} \subset S^3$) the Milnor fibration too (cf. 3.4.10). In fact, Γ_f determines the link $L_{V(f)} \subset S^3$ not only up to a diffeomophism, but even up to an isotopy. We say that it is a complete local topological invariant of f.

If f is locally irreducible, another complete invariant is the set of Puiseux pairs $\{(n_i, m_i)\}_{i=1}^s$, where $m_i \geq 2$, $\gcd(n_i, m_i) = 1$, $1 < n_1/m_1 < n_2/(m_1 m_2) < \cdots$, defined as follows. Let $n : (\mathbb{C}, 0) \to (V(f), o)$ be the normalization, $t \mapsto (x(t), y(t))$. After change of local variables, n has the form $x(t) = t^m$, where m is the multiplicity of f, and $y(t)$ is some convergent power series $\sum_{\ell > m} a_\ell t^\ell$. This symbolically can be written as $y = \sum_{\ell > m} a_\ell x^{\ell/m}$. Then there exist pairs $\{(n_i, m_i)\}_{i=1}^s$ as above such that $\frac{\ell}{m} \in \frac{1}{m_1 \cdots m_{k-1}} \cdot \mathbb{Z}$ whenever $\frac{\ell}{m} < \frac{n_k}{m_1 \cdots m_k}$ with $a_\ell \neq 0$.

The set of Puiseux pairs is equivalent to the set of Newton pairs $\{(p_i, q_i)\}_{i=1}^s$ defined by $p_i = m_i$ for $i \geq 1$ and $q_1 = n_1$, $q_i = n_i - n_{i-1} m_i$ for $i > 1$.

Consider the 'model' singularity f_N associated with equivalent parametrizations

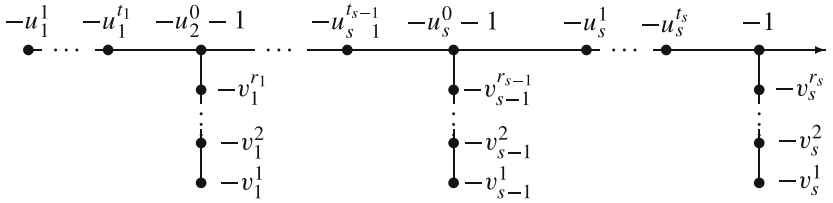
$$y = x^{q_1/p_1}(1 + x^{q_2/p_1 p_2}(1 + \cdots + (1 + x^{q_s/p_1 \cdots p_s}) \cdots)) = x^{n_1/m_1} + \cdots + x^{n_s/m_1 \cdots m_s}.$$

Then $(S^3, V(f))$ and $(S^3, V(f_N))$ are isotopic. Furthermore, the links of two models are isotopic if and only if the set of Newton pairs coincide.

The set of Newton pairs and the embedded resolution graph determine each other as follows. Consider the continued fractions

$$p_i/q_i = [[u_i^0, u_i^1, \ldots, u_i^{t_i}]] \qquad q_i/p_i = [[v_i^0, v_i^1, \ldots, v_i^{r_i}]],$$

with $u_i^0, v_i^0 \geq 1$ and $u_i^l, v_i^l \geq 2$ for $l \geq 1$. Then the graph Γ_f (without multiplicity decorations) is the following [178, 468]:



If $(C, o) = (V(f), o)$ has more irreducible components, say $\{(C_k, o)\}_{k=1}^r$, then its embedded link is determined by the Newton pairs of each component and by the intersection multiplicities $(C_k, C_l)_o$ at the origin $(k \neq l)$. For Γ_f and more details see [85, 140, 178].

There are several other complete invariants, which determine the embedded topological type of a plane curve singularity, e.g. the multiplicity sequence [78, 778], splice diagram [178], or in the irreducible case the semigroup (for their definitions and some discussion see 5.2.2 and 5.3.20 below). For more see [85, 178].

3.3.A The Plumbing Calculus

3.3.11 The plumbing construction associates a manifold $M(\Gamma)$ with any plumbing graph Γ. But different plumbing graphs might produce the same 3–manifold. Therefore, one introduces a set of operations acting on plumbing graphs, which, in the optimal situation, satisfy the following properties. (We concentrate here on operations, which preserve the orientation.)

1. (**Stability of the Calculus**) Applying any of the operations, or their inverses, to a plumbing graph Γ does not change the oriented diffeomorphism type of $M(\Gamma)$.
2. (**Sufficiency of the Calculus**) If Γ_1 and Γ_2 are two plumbing graphs and $M(\Gamma_1)$ and $M(\Gamma_2)$ are diffeomorphic by an orientation preserving diffeomorphism, then Γ_1 and Γ_2 are related by a sequence of operations or/and their inverses.

Usually, in specific geometric situations, we do not use all the possible plumbing graphs. Different geometric situations determine special, restricted families of graphs and 3-manifolds whose classification and description we target in that

specific situation. According to this, we also consider different sets of allowed operations connecting these graphs. In this book we will consider two such families.

Firstly, we make a choice dictated by the complex analytic theory (or complex algebraic geometry), motivated by the properties of the resolution graphs of normal surface singularities. More precisely, we consider the family of those plumbing graphs which are *connected and have negative definite intersection matrix*.

For this class, the plumbing calculus consists of two operations (and their inverses): blow up/down a vertex or an edge. They are presented in 2.2.5.2. (In the presence of arrowheads the pictures should be modified accordingly.) The point behind this combinatorial discussion is that these graph operations, in fact, are realized by complex analytic modifications. Furthermore, one has the following fact.

Theorem 3.3.12 *The blow up/down operation satisfies the stability and sufficiency properties of connected graphs with negative definite intersection form.*

The *stability* part of the calculus is immediate: its proof basically copies the analytic fact that blow up of a point on a smooth surface realizes an isomorphism over the complement of the point. Furthermore, one verifies that under blow up/down the negative definiteness of the intersection matrix is preserved.

On the other hand, the proof of the sufficiency of the calculus is hard and it is based on deep results of 3-manifold theory. It involves the classification of oriented plumbed 3-manifolds. This family is precisely the family of graph-manifolds; they were classified, with minor exceptions, by Waldhausen [734–739]. This classification was completed and rewritten in terms of plumbing graphs by Neumann in [505]. Some of the statements from [505] regarding graph-manifolds and singularity links will be listed in 3.6 after we review some basic theorems on the classification of 3-manifolds.

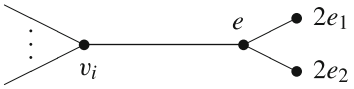
3.3.13 Extended Plumbing Graphs and Calculus for Oriented 3-Manifolds [505] Our second family is motivated by several proofs and constructions in the body of the book involving certain topological/C^∞ invariants (see e.g. the procedure how one recovers the plumbing graphs from a splice diagram in 5.3.10). It is the family of oriented plumbed 3-manifolds. In this case we allow non-connected graphs too, and we have to extend even the set of connected graphs. Namely, in the new situation the edges will have signs $+$ or $\ominus$. The $(+)$-edge represents a gluing via $(x, y) \mapsto (y, x)$ as in 3.3.3, while a $\ominus$-edge represents a gluing via $(x, y) \mapsto (y^{-1}, x^{-1})$. The $+$ sign usually will be not marked. We invite the reader to verify that the corresponding 'extended' plumbing construction still provides an oriented 3-manifold.

The intersection matrix I associated with Γ is defined similarly as in 3.3.5, but now we have to be careful on the edge-signs as well. For $v \neq u$ the entry $I_{v,u}$ is the number of edges connecting v with u summed with their signs, and $(I_{v,v} - e_v)/2$ equals the number of loops supported by v summed with their signs.

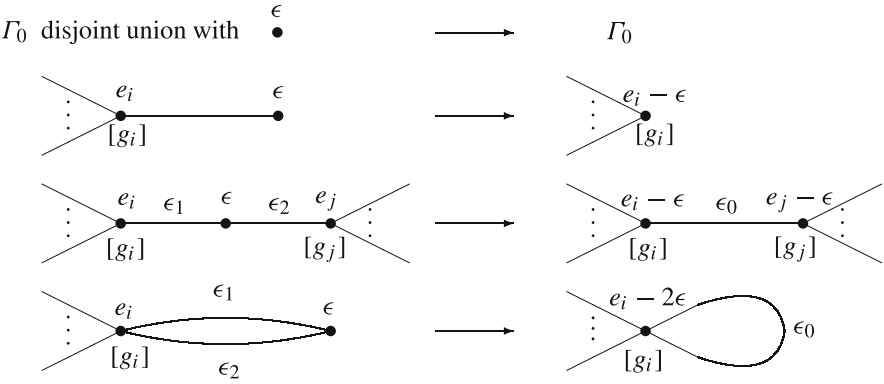
For oriented 3-manifolds the set of operations, which satisfy both the stability and the sufficiency of the calculus are the following:

[O1] Reverse the signs on all edges other than loops adjacent to any fixed vertex.

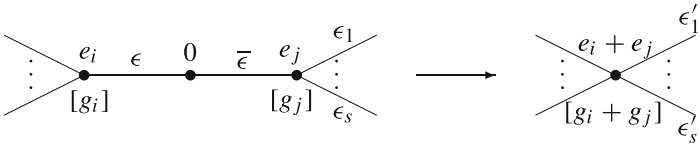
Furthermore, if the vertex v_i supports a subgraph as below, with $e_1 = \pm 1$, $e_2 = \pm 1$ and $e = (e_1 + e_2)/2$, then reverse the sign on all edges adjacent to v_i.



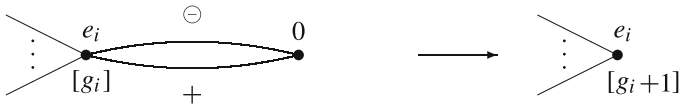
[O2] (Blow Down) Below $\epsilon = \pm 1$ and the edge signs $\epsilon_0,\ \epsilon_1,\ \epsilon_2$ are related by $\epsilon_0 = -\epsilon\epsilon_1\epsilon_2$.



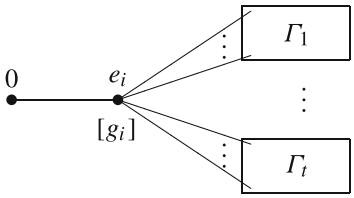
[O3] (0-Chain Absorption) The edge signs ϵ_i' $(i = 1, \ldots, s)$ are related by $\epsilon_i' = -\epsilon\overline{\epsilon}\epsilon_i$ provided that the edge sign in question is not on a loop, and $\epsilon_i' = \epsilon_i$, if it is on a loop.



[O4] (Oriented Handle Absorption)

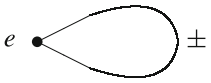


[O5] (Splitting) If Γ has the form



where each Γ_j is connected and, for each $j \in \{1, \dots, t\}$, Γ_j is connected to the vertex v_i by k_j edges, then replace Γ by the disjoint union of $\Gamma_1, \dots, \Gamma_t$, and $2g_i + \sum_j (k_j - 1)$ copies of $\overset{0}{\bullet}$.

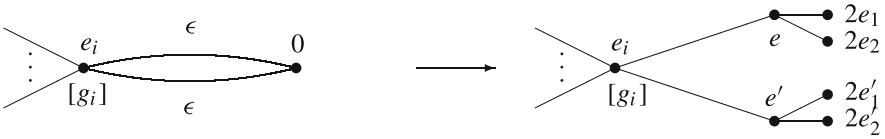
[O6] (Seifert Graph Exchange) Replace



where $e \in \{-1, 0, +1\}$, by a star–shaped graph with all genera zero.

There are six cases, the pair $(e, \pm)$ can be $(-1, +)$, $(0, +)$, $(1, +)$, $(-1, \ominus)$, $(0, \ominus)$ and $(1, \ominus)$. The corresponding star-shaped graphs are in Kodaira's notation (for the graph see 7.2.A) II, III, IV, IV^*, III^*, II^*.

[O7] (One Unoriented Handle Absorption Followed by Two $\mathbb{RP}^2$-Extrusions)



where $\epsilon = \pm 1$, $e_1, e_2, e'_1, e'_2 \in \{-1, +1\}$ and $e = (e_1 + e_2)/2$, $e' = (e'_1 + e'_2)/2$.

Remark 3.3.14 For extended graphs we still use the convention $\det(\Gamma) := \det(-I)$, cf. 2.2.4. However, though blowing down a (-1) vertex preserves $\det(\Gamma)$, in this generalized case not any operation preserves $\det(\Gamma)$. E.g., blow down of a $(+1)$-vertex or the 0-chain absorption reverse the sign of $\det(\Gamma)$.

3.3.15 Changing the Orientation If Γ is an orientable plumbing graph (that is, with $e_v \in \mathbb{Z}$ and $g_v \geq 0$) then let $-\Gamma$ be the same graph with the signs of all Euler and edge decorations reversed. Then $M(-\Gamma) = -M(\Gamma)$ [505, 3.3], that is, $-\Gamma$ provides the same manifold as Γ but with opposite orientation.

3.3.16 For the plumbing calculus of general (unoriented) graph manifolds see [505]. In this case some additional operations appear, e.g., the unoriented handle

absorption and $\mathbb{RP}^2$-absorption (and their inverses, the corresponding extrusions). The second operation in [O1] and operation [O7] above constitute their 'traces'.

3.4 Homological Properties of the Link

In this subsection all the homologies will have integral coefficients.

Let (X, o) be a complex normal surface singularity and let $\phi : \widetilde{X} \to X$ be a good resolution. We will use the notations Γ, $E = \cup_{v\in\mathcal{V}} E_v$, $\{D_v\}_v$, L, L', I as in 2.1.15.

In the following $\widetilde{X}$ will denote any 'small neighbourhood' of E topologically echivalent with $\phi^{-1}(\rho^{-1}([0, \epsilon]))$, where ρ is as in 3.1.1 (and 3.1.2) and $0 < \epsilon \ll 1$. According to Proposition 3.3.8, in all the topological discussions below, the pair $(\widetilde{X}, \partial\widetilde{X} = L_X)$ can be replaced by the plumbed pair $(P, \partial P)$ associated with Γ.

3.4.1 The Homology of the Link Since $\widetilde{X}$ has the homotopy type of E, and $H_1(\widetilde{X}, \partial\widetilde{X}) = H_3(\widetilde{X})^* = H_3(E)^* = 0$, and $i : L \to L'$ is injective (see 2.1.15), one has the exact sequence of $(\widetilde{X}, \partial\widetilde{X})$

$$0 \to H_2(\widetilde{X}) \xrightarrow{i} H_2(\widetilde{X}, \partial\widetilde{X}) \to H_1(L_X) \to H_1(E) \to 0. \tag{3.4.1}$$

Set $g(\Gamma) := \sum_{v\in\mathcal{V}} g_v$ and let $c(\Gamma)$ be the number of independent cycles in Γ.

Proposition 3.4.2 ([277, 448, 614, 615]) $L'/L = \operatorname{coker}(I) = \operatorname{Tors}(H_1(L_X, \mathbb{Z}))$, *and*

$$H_1(L_X, \mathbb{Z}) = \operatorname{coker}(I) \oplus H_1(E, \mathbb{Z}) = \operatorname{coker}(I) \oplus \mathbb{Z}^{2g(\Gamma)+c(\Gamma)}.$$

Hence, L_X is a rational homology sphere if and only if Γ is a tree with all $g_v = 0$, and L_X is an integral homology sphere when additionally $\det(-I) = 1$.

Remark 3.4.3 Since $-E_u = \sum_v I_{uv}E_v^*$, L'/L is generated by the classes $\{[E_v^*]\}_{v\in\mathcal{V}}$ with relations $\sum_v I_{uv}[E_v^*] = 0$. If Γ_X is a tree then these identities and induction show that L'/L is already generated by the classes $\{[E_v^*]\}_{v:\kappa_v=1}$ of end-vertices. In fact, if v_0 is such an end, then L'/L is already generated by $\{[E_v^*]\}_{v:\kappa_v=1,\, v\neq v_0}$. (Use induction as in the proof of 3.7.8.)

Remark 3.4.4 Usually, for an arbitrary manifold, the *cohomology ring* contains more information than the (co)homology groups. But this is not the case for $H^*(L_X, \mathbb{Q})$: the cup product $H^1(L_X, \mathbb{Q})^{\otimes 2} \xrightarrow{\cup} H^2(L_X, \mathbb{Q})$ is trivial [672]. Indeed, in the exact sequence

$$H^1(\widetilde{X}) \xrightarrow{r^1} H^1(\partial\widetilde{X}) \longrightarrow H^2(\widetilde{X}, \partial\widetilde{X}) \xrightarrow{i^*} H^2(\widetilde{X}) \xrightarrow{r^2} H^2(\partial\widetilde{X}) \tag{3.4.2}$$

with $\mathbb{Q}$-coefficients, i^* is an isomorphism. Hence $r^2 = 0$ and r^1 is onto. Therefore, if $a = r^1(a')$, $b = r^1(b')$ then $a \cup b = r^2(a' \cup b') = 0$.

3.4.5 The Homology of the Pair $(L_X, L_{V(f)})$ Fix an analytic germ $f : (X, o) \to (\mathbb{C}, 0)$. Our next goal is the computation of $H_1(L_X \setminus L_{V(f)})$.

Let $\mathbb{Z}^{|\mathscr{W}|}$ be the free abelian group generated by $\{\langle w\rangle\}_{w\in\mathscr{W}}$. Define the group $H_{\mathscr{A}}$ as the quotient of $\mathbb{Z}^{|\mathscr{W}|}$ factorized by the subgroup generated by:

$$\sum_{v\in\mathscr{V}} I_{uv}\langle v\rangle + \sum_{a\in\mathscr{A}_u}\langle a\rangle \quad (\text{for all } u \in \mathscr{V}),$$

where for any $u \in \mathscr{V}$ we denote by $\mathscr{A}_u$ the set of arrowheads supported by u.

Proposition 3.4.6 *[277, 505, 614, 615] One has the following commutative diagram with exact rows and columns:*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathbb{Z}^{|\mathscr{A}|} & \xrightarrow{=} & \mathbb{Z}^{|\mathscr{A}|} & & & & \\
 & & \downarrow k & & \downarrow & & & & \\
0 & \to & H_{\mathscr{A}} & \xrightarrow{j} & H_1(L_X \setminus L_{V(f)}) & \xrightarrow{q} & H_1(E) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow = & & \\
0 & \to & \operatorname{coker}(I) & \to & H_1(L_X) & \xrightarrow{q} & H_1(E) & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Above k is the composed map $\mathbb{Z}^{|\mathscr{A}|} \hookrightarrow \mathbb{Z}^{|\mathscr{W}|} \to H_{\mathscr{A}}$, and $j([\langle w\rangle]) = [\gamma_w]$, where the classes $\{[\gamma_w]\}_{w\in\mathscr{W}}$ are defined in 3.3.6.

Proof The exact sequence (3.4.1) is modified as follows. Consider the homology exact sequence of the pair $(\widetilde{X}, L_X \setminus L_{V(f)})$. Note that $H_1(\widetilde{X}, L_X \setminus L_{V(f)}) = H_3(\widetilde{X}, L_{V(f)})^* = 0$, and $H_2(\widetilde{X}, L_X \setminus L_{V(f)})$ is $\mathbb{Z}^{|\mathscr{W}|}$, freely generated by the relative homology classes of discs of type $\{D_v\}_{v\in\mathscr{V}}$ (as in 2.1.15), and of type $\{D_a\}_{a\in\mathscr{A}}$, where for each $a \in \mathscr{A}$, D_a is a local disc in L_X, transversal to L_a, intersecting L_a with intersection multiplicity 1. Note also that $H_2(\widetilde{X}) \simeq \mathbb{Z}^{|\mathscr{V}|}$. Moreover, in

$$\cdots \to H_2(\widetilde{X}) \xrightarrow{i'} H_2(\widetilde{X}, L_X \setminus L_{V(f)}) \to H_1(L_X \setminus L_{V(f)}) \to H_1(E) \to 0 \tag{3.4.3}$$

$\operatorname{coker}(i')$ is exactly $H_{\mathscr{A}}$. Furthermore, $H_{\mathscr{A}}/\mathbb{Z}^{|\mathscr{A}|} \simeq H_2(\widetilde{X}, L_X)/\operatorname{im}(i) \simeq \operatorname{coker}(I)$, cf. (3.4.1). Finally note that the circle γ_w $(w \in \mathscr{W})$ is the boundary of the disc D_w. □

3.4.7 Our next goal is to connect the morphism $\arg_* : H_1(L_X \setminus L_{V(f)}) \to H_1(S^1)$ induced by $\arg(f)$ with the multiplicity system of f.

Define $\mathbf{m} : H_{\mathscr{A}} \to \mathbb{Z}$ by $\mathbf{m}([\langle w \rangle]) = m_w$. This is well defined, and, in fact, $\mathbf{m}$ is the restriction of $\arg_*$ to $H_{\mathscr{A}}$, cf. 3.3.6. Hence $\arg_* \circ j = \mathbf{m}$ inserts into the diagram:

$$\begin{array}{ccccccccc} 0 & \to & H_{\mathscr{A}} & \xrightarrow{j} & H_1(L_X \setminus L_{V(f)}) & \xrightarrow{q} & H_1(E) & \to & 0. \\ & & & \searrow^{\mathbf{m}} & \downarrow{\scriptstyle \arg_*} & & & & \\ & & & & \mathbb{Z} & & & & \end{array}$$

Regarding $\arg_*$ we emphasize two facts.

3.4.8 Fact I [475] As the above diagram shows, in general $\mathbf{m}$ does not determine $\arg_*$, hence it does not determine the Milnor fibration. Examples show that both conditions $g(\Gamma) > 0$ or $c(\Gamma) > 0$ can appear as obstructions. For example, if one takes $(X, o) = (\{x^2 + y^7 - z^{14} = 0\}, 0) \subset (\mathbb{C}^3, 0)$ and the two functions $f_1 = z^2$ and $f_2 = z^2 - y$ on it, then $\Gamma(X, f_1) = \Gamma(X, f_2)$, hence $\mathbf{m}(f_1) = \mathbf{m}(f_2)$ (with $\operatorname{coker}(\mathbf{m}) = \mathbb{Z}_2$). Nevertheless, in the first function $\operatorname{coker}(\arg_*) = \mathbb{Z}_2$, while in the second case it is trivial. Here both functions define non-isolated singularities, however this can be modified by their 'series of singularities'. E.g., on the very same (X, o), the two functions $z^2 + y^k$ and $z^2 - y + y^k$ ($k \geq 2$) have the same minimal good embedded resolution graphs but their $\arg_*$'s are different. Here $c(\Gamma) = 0$ but $g(\Gamma) > 0$.

Examples with $c(\Gamma) > 0$ and $g(\Gamma) = 0$ are given by $(X, o) = (\{z^2 + (x^2 - y^3)(x^3 - y^2) = 0\}, 0)$ and the pair of functions $f_1 = x^2$ and $f_2 = x^2 - y^3$ (non-isolated case), and $f_1 = x^2 + y^k$, $f_2 = x^2 - y^3 + y^k$ ($k \geq 4$) in the isolated case. For details see [475].

Note that a page has k connected components, where $\mathbb{Z}_k = \operatorname{coker}(\arg_*)$. In general one has the divisibilities $k \,|\, \gcd\{m_w : w \in \mathscr{W}\} \,|\, \gcd\{m_a : a \in \mathscr{A}\}$, and it might happen that these integers are different, and k cannot be determined from $\mathbf{m}$.

3.4.9 Fact II $\arg_*$ determines uniquely the Milnor fibration.

Theorem 3.4.10 ([59, 108, 737], [178, page 34–35]) *Fix an oriented 3-manifold M and an oriented link L in it. Then the following facts hold.*

(a) If two smooth locally trivial fibrations $p_i : M \setminus L \to S^1$ ($i = 1, 2$) induce the same morphism $(p_1)_* = (p_2)_* : H_1(M \setminus L, \mathbb{Z}) \to \mathbb{Z}$, then they are isotopic.

(b) Fix a multilink structure $\mathbf{m}L$ of L. If two open book decompositions of M with binding $\mathbf{m}L$ and projections p_i ($i = 1, 2$) induce the same morphism $H_1(M \setminus L, \mathbb{Z}) \to \mathbb{Z}$ (which automatically extends $[\gamma_a] \mapsto m_a$ for all $a \in \mathscr{A}$), then they are isotopic.

(c) In particular, if L_X is a rational homology sphere, then $\mathbf{m}L$, the multilink binding determines completely the open book decomposition.

In fact, in this case, any page of the multilink $\mathbf{m}L$ *is a Seifert surface with minimal genus, and conversely, any Seifert surface of the multilink with minimal genus is isotopic with the page of the open book decomposition.*

3.5 Examples of Special Graphs

3.5.A Star-Shaped Graphs

3.5.1 A connected graph Γ is called *star-shaped* if it has a *central vertex* v_0, and $\Gamma \setminus v_0$ consists of $\nu \geq 0$ strings (legs). Each string is connected to v_0 by an edge at one of the end-vertices of the string. In some cases, for a fixed Γ, the choice of the central vertex is not unique; e.g. if Γ itself is a string then any vertex can be central.

We use the following notations: v_0 has Euler number $-b_0$ and genus $g \geq 0$. The Euler numbers of the vertices v_{ji} of the j^{th} string ($1 \leq j \leq \nu$) are $-b_{j1}, \ldots, -b_{js_j}$, with $b_{ji} \geq 2$, determined by the continued fraction $\alpha_j/\omega_j = [[b_{j1}, \ldots, b_{js_j}]]$, where $\gcd(\alpha_j, \omega_j) = 1,\ 0 < \omega_j < \alpha_j$. For each j, v_0 is connected with v_{j1} by one edge.

In such a case the plumbed 3-manifold $M(\Gamma)$ is a *Seifert fibered 3-manifold*, which means that $M(\Gamma)$ is foliated by circles such that any circle has a compact orientable saturated neighbourhood [289, 291, 505, 511, 624]. The Seifert structure, the Seifert S^1-action, and its relation with singularities with good $\mathbb{C}^*$-action will be discussed in Sect. 5.1. $M(\Gamma)$ and the foliation is characterized by the collection $(b_0, g; \{(\alpha_j, \omega_j)\}_j)$, called the *Seifert invariants*, for details see again 5.1. Here we only fix some notation and list some combinatorial invariants of Γ, L or $M(\Gamma)$ and we prepare the terminology for the basic classification theorems of plumbed 3-manifolds and for the discussion of their fundamental groups.

If either $g > 0$ or $\nu \geq 3$ then the choice of the central vertex is unique (compare also with 5.1.14). In the sequel we assume this fact. (The remaining case of strings was fully studied in Sect. 2.3.)

The *virtual (or orbifold) Euler number* e is defined by

$$e := -b_0 + \sum_j \omega_j/\alpha_j. \tag{3.5.1}$$

The meaning of the word 'virtual' will be explained after Theorem 5.1.23.

Note that $e < 0$ if and only if the intersection matrix $I = I(\Gamma)$ is negative definite.

By plumbing calculus, $-\Gamma$ also can be represented as a star-shaped graph with decorations in the above 'normal form': the Seifert invariants of $-\Gamma$ are $(\nu - b_0, g; \{(\alpha_j, \alpha_j - \omega_j)\}_j)$ (compare also with 3.1.6), hence $e(-\Gamma) = -e(\Gamma)$

too. Therefore, if $I(\Gamma)$ is negative definite then $-\Gamma$ cannot be represented (even up to the plumbing calculus) by a negative definite plumbing graph.

3.5.2 Lattice Identities [478] We will assume that Γ determines a negative definite lattice L, the base elements $\{E_v\}$ will be indexed as the vertices.

The following identities will be used several times. For their proof use $(I^{-1})_{vu} = (E_v^*, E_u^*)$ and the last two parts of 2.2.5, cf. [478, (11.1)].

$$\begin{aligned} (E_0^*, E_0^*) &= e^{-1}, \\ (E_0^*, E_{js_j}^*) &= (e\alpha_j)^{-1}, \\ (E_{is_i}^*, E_{js_j}^*) &= \begin{cases} (e\alpha_i\alpha_j)^{-1} & \text{if } i \neq j, \\ (e\alpha_i^2)^{-1} - \omega_i'/\alpha_i & \text{if } i = j, \end{cases} \end{aligned} \tag{3.5.2}$$

where $0 < \omega_j' < \alpha_j$ and $\omega_j\omega_j' \equiv 1 \pmod{\alpha_j}$, and $1 \le i, j \le \nu$.

Remark 3.5.3 The analogue of the identity (2.2.7), valid for quotient singularities, in the case of star-shaped graphs transforms into the following identity:

$$\begin{aligned} &\sum_{v\in\mathcal{V}} (2\chi(E_v) - \kappa_v)(I^{-1})_{vv} + \sum_{v\in\mathcal{V}} (E_v^2 + 3) = \\ &\qquad \frac{1}{e}(2 - 2g - \nu + \sum_{j=1}^{\nu} \frac{1}{\alpha_j^2}) + e + 3 - 12 \cdot \sum_{j=1}^{\nu} \mathbf{s}(\omega_j, \alpha_j). \end{aligned} \tag{3.5.3}$$

(Use the above lattice identities for I_{vv}^{-1} and (2.2.7)). For topological interpretation, see 6.3.10–6.3.12.

3.5.4 The Group $H = \mathrm{Tors}(H_1(M(\Gamma), \mathbb{Z})) = L'/L$ [506]

Let h_j denote the class $[E_{js_j}^*]$ $(j = 1, \dots, \nu)$ and h_0 the class $[E_0^*]$ in H. Since (by similar notations as in 2.2.8) $[E_{ji}^*] = [n_{i+1,s_j}^j E_{js_j}^*]$ in H, $\{h_j\}_{j=1}^{\nu}$ and h_0 generate H. The relations are the following:

$$b_0h_0 = \textstyle\sum_{j=1}^{\nu} \omega_j h_j, \quad \text{and} \quad \alpha_j h_j = h_0 \ \ (j = 1, \dots, \nu). \tag{3.5.4}$$

(Use repeatedly $[\sum_u I_{vu}E_u^*] = 0$.) Let $\mathfrak{o}$ be the order of h_0 in H. Then (cf. [506])

$$|H| = \alpha_1 \cdots \alpha_\nu |e|, \quad \mathfrak{o} = \alpha|e|, \tag{3.5.5}$$

where

$$\alpha := \mathrm{lcm}\{\alpha_1, \dots, \alpha_\nu\}.$$

For the first relation compute the determinant of the matrix of relations; for the second one use the exact sequence $0 \to G \hookrightarrow \mathbb{Z}_{\alpha_1} \times \cdots \times \mathbb{Z}_{\alpha_\nu} \to H/\langle h_0 \rangle \to 0$, where G is the cyclic subgroup of $\mathbb{Z}_{\alpha_1} \times \cdots \times \mathbb{Z}_{\alpha_\nu}$ generated by the ν-tuple $(\bar{\omega}_1, \ldots, \bar{\omega}_\nu)$, whose entries are the classes of ω_j's; hence $G \simeq \mathbb{Z}_\alpha$.

Similarly, from an exact sequence of type $0 \to \langle(\bar{\omega}_2, \ldots, \bar{\omega}_\nu)\rangle \hookrightarrow \mathbb{Z}_{\alpha_2} \times \cdots \times \mathbb{Z}_{\alpha_\nu} \to H/\langle h_1 \rangle \to 0$, we get that the order of h_j in H is $\mathrm{ord}(h_j) = \alpha_j \alpha_j^\vee |e|$, where $\alpha_j^\vee := \mathrm{lcm}\{\alpha_i \,;\, i \neq j\}$.

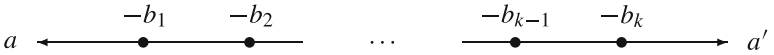
3.5.B Cyclic Graphs

3.5.5 A connected plumbing graph is cyclic if $g_v = 0$ for all v, and each vertex has exactly two adjacent edges, and all the edge decorations are $+$. We denote the Euler numbers of the ordered vertices by $-b_1, \ldots, -b_k$ (up to cyclic permutation), $k \geq 2$. We consider also the graph with one vertex with genus zero and Euler number $-b_1$, which supports a $(+)$-loop (this is the $k = 1$ case). In this $k = 1$ case the intersection matrix has entry $-b_1 + 2$. For all cases $k \geq 1$ we assume that each $b_i \geq 1$ and the intersection matrix is negative definite. If each $b_i \geq 2$ then I is negative definite exactly when $\sum_i (b_i - 2) > 0$. If we regard Γ as a resolution graph with exceptional curve E (cf. 2.1.13), then $\pi_1(E) = \mathbb{Z}$ and we fix one of its generators. If $k \geq 2$ then fixing a generator of $\pi_1(E)$ is equivalent to fixing the cyclic order of the vertices of the graph. Usually we prefer a minimal graph, however, in order to keep the next discussion uniform, in the $k = 1$ it is convenient to blow up the loop and introduce a new (-1)-vertex. (Or, the next parts should be slightly adopted to the $k = 1$ case.)

In several parts of this subsection we follow [333], for more see also [273].

By the plumbing construction, $M = M(\Gamma)$ consists of k copies of $(\mathbb{P}^1 \setminus \{2 \text{ discs}\}) \times S^1 \simeq [0, 1] \times T$ (where T denotes the torus $S^1 \times S^1$). The segments $[0, 1]$ are glued together into another S^1 (whose fundamental group can be identified with $\pi_1(E)$), and M becomes a torus bundle over this S^1. This is what we will explain next.

We fix an edge, say (v_k, v_1), which by the plumbing construction determines a torus T in M, cf. 3.3.3. Then M cut along T is represented by



We fix a base point in M, and we complete the free loops γ_* (defined as in 3.3.6) by connecting paths to get elements of $\pi_1(M)$; we denote them by the same symbols $\gamma_0 = \gamma_a, \gamma_1, \ldots, \gamma_k, \gamma_{k+1} = \gamma_{a'}$. By a convenient choice of the connecting paths we may assume that they commute with each other and $\gamma_i^{b_i} = \gamma_{i-1}\gamma_{i+1}$ for $1 \leq i \leq k$. (See e.g. the proof of 3.7.2.) In particular, γ_0 and γ_1 generate freely $\mathbb{Z}^2 \simeq \pi_1(T)$, and all γ_i can be expressed by them. It is convenient to organize this as follows.

For any collection of integers $a_1, \dots, a_n$ we define the matrix

$$A(a_1, \dots, a_n) := \begin{pmatrix} a_1 & 1 \\ -1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

Introduce the column vectors $\{v_i\}_{i=0}^{k+1}$ of $\mathbb{Z}^2$ by $v_0 = \binom{0}{1}$ and $A(b_1, \dots, b_i) = (v_{i+1}, v_i)$ for $1 \le i \le k$. E.g., $v_1 = \binom{1}{0}$, $v_2 = \binom{b_1}{-1}$. For any $v = \binom{r}{s}$ we set $\underline{\gamma}^v := \gamma_1^r \gamma_0^s \in \pi_1(T)$. Then, by induction, $\gamma_i = \underline{\gamma}^{v_i}$ for any $0 \le i \le k+1$.

The matrix $A = A(b_1, \dots, b_k) \in \mathrm{SL}_2(\mathbb{Z})$ is called the monodromy operator associated with the choice of the torus T and the bases $\{v_0, v_1\}$ of $\mathbb{Z}^2$, and the representation $\gamma_0 \mapsto v_0$, $\gamma_1 \mapsto v_1$ of $\pi_1(T) \to \mathbb{Z}^2$. Note that $Av_0 = v_k$ and $Av_1 = v_{k+1}$. The sequence of vectors v_i can be extended to a sequence $\{v_i\}_{i \in \mathbb{Z}}$ of $\mathbb{Z}^2$ by $v_{\ell k+i} := A^\ell v_i$ ($\ell \in \mathbb{Z}$), which evidently correspond to loop generators of the infinite string $\dots, b_1, \dots, b_k, b_1, \dots, b_k, \dots$. Then, M is a torus bundle over S^1, (where the fiber is identified with T), and one has the extension

$$1 \to \mathbb{Z}^2 = \pi_1(T) \to \pi_1(M) \xrightarrow{q} \mathbb{Z} = \pi_1(E) \to 1, \tag{3.5.6}$$

where $\pi_1(E)$ acts on $\pi_1(T)$ via the monodromy A. Hence,

$$\pi_1(M) \simeq \langle v, t \mid v \in \mathbb{Z}^2,\ tvt^{-1} = Av \rangle$$

such that $q(v) = 0$ and $q(t) = 1_{\mathbb{Z}}$.

Remark 3.5.6 ([743]) In a sequence like (3.5.6), the projection q splits (since $\mathbb{Z}$ is free). Any splitting s is given by $s(1_{\mathbb{Z}})$, which has the form tv, for some $v \in \mathbb{Z}^2$. In particular, the splitting is not unique. Take any two splittings s and s' and write $s'(1) := s(1)u$ for some $u \in \mathbb{Z}^2$. Then one can define an automorphism ϕ_u of $\pi_1(M)$ by $\phi_u(v) = v$ for $v \in \mathbb{Z}^2$ while $\phi_u(t) = tu$. This shows that any two splittings of (3.5.6) can be identified by an automorphism of $\pi_1(M)$.

At homological level one has the exact sequence $0 \to \mathrm{coker}(A - I) \to H_1(M, \mathbb{Z}) \to H_1(E, \mathbb{Z}) = \mathbb{Z} \to 0$. The above fact reads as follows: any two splittings of this last sequence can be identified by an automorphism of $\pi_1(M)$. (This is a special property of cyclic graphs; for further consequences regarding coverings of $M(\Gamma)$ see 4.2.8.)

3.5.7 The Monodromy Operator A The monodromy operator $A \in \mathrm{SL}_2(\mathbb{Z})$ satisfies $\tau := \mathrm{trace}(A) \ge 3$. Therefore the characteristic polynomial $\lambda^2 - \tau\lambda + 1$ has two positive (non-rational) roots. We denote them $\lambda_1 > 1 > \lambda_2 > 0$. Let V_1 and V_2 in $\mathbb{R}^2$ be the corresponding eigenvectors chosen such that V_1 is in the fourth quadrant and V_2 in the second one. Then $\lim_{i \to \infty} \mathbb{R}_{>0} v_i = \mathbb{R}_{>0} V_1$, and $\lim_{i \to -\infty} \mathbb{R}_{>0} v_i = \mathbb{R}_{>0} V_2$. Let $\mathrm{Cone}(\Gamma)$ be the real (open) positive cone $\mathbb{R}_{>0}\langle V_1, V_2 \rangle$ generated by V_1 and V_2. Note that $v_i \in \mathrm{Cone}(\Gamma) \cap \mathbb{Z}^2$ for any i.

A depends on the above choice of the edge, or of $\pi_1 \to \mathbb{Z}^2$, $\gamma_0 \mapsto v_0$, $\gamma_1 \mapsto v_1$. For the edge $(1,2)$ the new monodromy operator is $A_{(1,2)} = A(b_2, \dots, b_k, b_1) = A(b_1)^{-1} \cdot A \cdot A(b_1)$, hence $A_{(1,2)}$ and A are conjugate. The same is true for any other choice of the edge. In fact, the correspondence $(b_1, \dots, b_k) \mapsto A$ from *oriented cycles* to the set of conjugacy classes of $C \in \mathrm{SL}_2(\mathbb{Z})$ with $\mathrm{trace}(C) \geq 3$ is a bijection (cf. [273, 298], [505, 6.3]). One recovers $(b_1, \dots, b_k)$ from $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ as follows. Write $\omega := (a - d + \sqrt{\tau^2 - 4})/2b$ as an infinite (negative) continued fraction. It is periodic, let $(c_1, \dots, c_t)$ be its shortest period. Set $C = A(c_1, \dots, c_t)$. Then there exists $k \in \mathbb{Z}_{>0}$ such that $\mathrm{trace}(C^k) = \tau$ (or equivalently, $\lambda_1(C)^k = \lambda_1(A)$). Then the wished b-sequence is the k-repetition of the c-sequence (hence $C^k = A$ too). (The c-sequence repeated infinitely many times can be recovered from the cone $\mathrm{Cone}(\Gamma) \cap \mathbb{Z}^2$ as well, noting that $\{v_i\}_i$ are the lattice points of the boundary of the convex closure of $\mathrm{Cone}(\Gamma) \cap \mathbb{Z}^2$, which satisfy the identities from 3.5.5).

If we change the orientation in the cycle of the graph (or, the generator of $\pi_1(E)$), then the new monodromy operator will be $\overline{A} = A(b_k, \dots, b_1)$, and

$$\overline{A} = SA^{-1}S^{-1}, \quad \text{where} \quad S = S^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

If $T(C)$ denotes the torus bundle over S^1 with monodromy C (where $\mathrm{trace}(C) \geq 3$ always), then $T(C_1)$ is orientation preserving diffeomorphic with $T(C_2)$ if and only if C_1 is conjugate in $\mathrm{SL}_2(\mathbb{Z})$ to either C_2 or $SC_2^{-1}S^{-1}$ [505, 6.2] (according to the cases when the generator of $\pi_1(E)$ is changed or not). Hence $T(A) \simeq T(\overline{A})$, and their fundamental groups are isomorphic via $t \leftrightarrow t^{-1}$, $\binom{r}{s} \leftrightarrow S\binom{r}{s}$ and $A \leftrightarrow A^{-1}$.

3.5.8 The Dual Cyclic Graph The dual graph is associated with $-\Gamma$. By plumbing calculus (in such a way that we never blow up/down the cutting edge) $-\Gamma$ can be transformed into a cyclic graph Γ^* with negative definite intersection form, with all Euler decorations $-b_1^*, \dots, -b_{k^*}^*$ satisfying $b_j^* \geq 2$, and with all edge decorations positive. One checks (see e.g. [505]) that if the b-sequence is

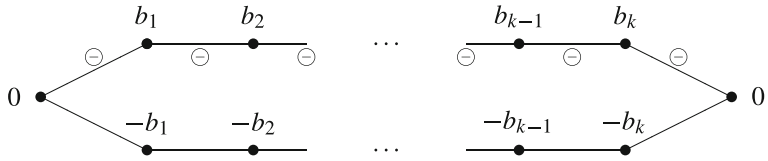
$$2^{k_1^*-1}, k_1 + 2, 2^{k_2^*-1}, k_2 + 2, \dots, k_g + 2,$$

(where 2^k represents k copies of 2), then the b^* sequence in Γ^* is

$$k_1^* + 2, 2^{k_1-1}, k_2^* + 2, 2^{k_2-1}, \dots, 2^{k_g-1}.$$

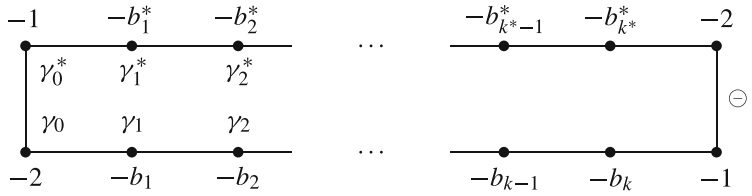
We wish to identify $\pi_1(T)$ in the two manifolds $M(\Gamma)$ and $M(-\Gamma) = -M(\Gamma)$ in a canonical way, such that both monodromy actions will act on the same $\mathbb{Z}^2$.

Since $\sum_i (b_i - 2) > 0$ we may assume $b_k \geq 3$. Consider the next plumbing graph:



By plumbing calculus (0-chain and oriented handle absorptions), this graph is equivalent to a single vertex with genus 1 and Euler number 0, hence representing the trivial torus bundle over S^1. Since the monodromy acting on the torus is trivial, the tori associated with the edges can be identified by canonical isomorphisms.

In this graph, on the 'top subchain', we can proceed the calculus, which provides the b^* sequence of Γ^* (just repeating the steps, which provide the normal form of $-\Gamma$). Then we get the graph (here we use the fact that $b_k \geq 3$):



We fix $\pi_1(T) = \mathbb{Z}^2$ under the canonical identification of Γ as in 3.5.5 with base elements $v_0 = \binom{0}{1}$ and $v_1 = \binom{1}{0}$, and we represent $\gamma_i = \gamma^{v_i}$ for $i = 1, 2$. Then, we express in the above graph the elements γ_i^* $(i = 1, 2)$ in terms of γ_0 and γ_1 as well. Namely, using the relations $\gamma_0^2 = \gamma_1 \gamma_0^*$ and $\gamma_0^* = \gamma_0 \gamma_1^*$, we get $\gamma_0^* = \gamma_0^2 \gamma_1^{-1}$ and $\gamma_0 \gamma_1^{-1}$. Hence, though in the canonical representation of Γ^* (considered independently from Γ, and repeating 3.5.5 for it), we would send γ_0^* to $\binom{0}{1}$ and γ_1^* to $\binom{1}{0}$, in this representation (imposed by the canonical representation of Γ when we send γ_0 to $\binom{0}{1}$ and γ_1 to $\binom{1}{0}$) we have to send γ_0^* to $u_0 := \binom{-1}{2}$ and γ_1^* to $u_1 := \binom{-1}{1}$. Let $\mathrm{Cone}^c(\Gamma)$ be the complementary cone of $\mathrm{Cone}(\Gamma)$, namely $\mathbb{R}_{>0}\langle -V_1, V_2 \rangle$. Then $u_0, u_1 \in \mathrm{Cone}^c(\Gamma) \cap \mathbb{Z}^2$, hence $\mathrm{Cone}^c(\Gamma)$ is 'the cone of Γ^*' in the canonical representation of Γ.

Moreover, one has the following identity, cf. [458, 505, 578].

Proposition 3.5.9 *Set* $T := \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix}$ *and consider the monodromy operators* $A = A(b_1, \ldots, b_k)$ *and* $A^* = A(b_1^*, \ldots, b_{k^*}^*)$. *Then* $AT = TA^*$.

In particular, $\pi_1(M(\Gamma))$ *and* $\pi_1(M(\Gamma^*))$ *can be canonically identified.*

Proof In the above graph the monodromy operator of the trivial torus-fibration is the identity. This reads as (starting from the left -1-vertex)

$$A(1,2)\cdot A\cdot A(1,2)\cdot A(b_{k^*}^*,\dots,b_1^*)=-I,$$

where the sign is given by the unique negative edge. Since $A(1,2)=ST^{-1}=-TS$ and by 3.5.7 one also has $A(b_{k^*}^*,\dots,b_1^*)=S(A^*)^{-1}S$, we get $AT=TA^*$. □

Hence $\pi_1(M)$ does not determine the torus bundle M, however $\pi_1(M)$ does determine the 3-manifold up to the ambiguity $M(\Gamma)\leftrightarrow M(\Gamma^*)$. Note also that $\det(T)=-1$, hence the proposition does not say that A and A^* are necessarily conjugate in $\mathrm{SL}_2(\mathbb{Z})$ (although, in special cases they can be conjugate or even equal, see e.g. the case $k=3$, $b_1=b_2=b_3=3$, which is auto-dual).

Remark 3.5.10 By Grauert's Theorem 2.5.2 any cyclic graph as in 3.5.5 is a link of a normal surface singularity; these singularities are called *cusps*. They are minimally elliptic singularities, cf. 7.2.19, and they are taut (the topological type supports a unique analytic structure). Those which can be realized as hypersurface singularities are exactly the $T_{p,q,r}$ germs with (unimodal) equations $x^p+y^q+z^r+axyz=0$ ($1/p+1/q+1/r<1$, $a\in\mathbb{C}^*$). (Their graphs can be computed by the algorithm from 5.5.23.) For some other cases see 5.5.30 and 5.5.31 below, or [174].

3.5.C Surgery 3-Manifolds

3.5.11 [213, 5.3] Assume that K is a knot in the oriented S^3. Let $T(K)$ be a tubular neighbourhood of K. Fix an orientation of K, an oriented meridian μ and longitude λ in $\partial T(K)$. Then glue the solid torus $S^1\times D^2$ to $S^3\setminus T(K)$ along their boundaries such that the circle $\alpha=\{point\}\times\partial D^2$ from $S^1\times D^2$ in $H_1(\partial T,\mathbb{Z})$ is represented by $p\mu+q\lambda$ for certain relative prime integers p and q. Changing the orientation of K or of α reverses the signs of both p and q, but the rational number p/q and the diffeomorphisms type of the glued manifold is unaffected. The resulted oriented 3-manifold, denoted by $S^3_{p/q}(K)$, depends only on the *Dehn surgery coefficient* $p/q\in\mathbb{Q}\cup\{\infty\}$. Surgery with coefficient ∞ is trivial, it recovers S^3. If $p/q\in\mathbb{Z}$ then we speak about *integral surgery*. One has $H_1(S^3_{p/q}(K),\mathbb{Z})=\mathbb{Z}/p\mathbb{Z}$.

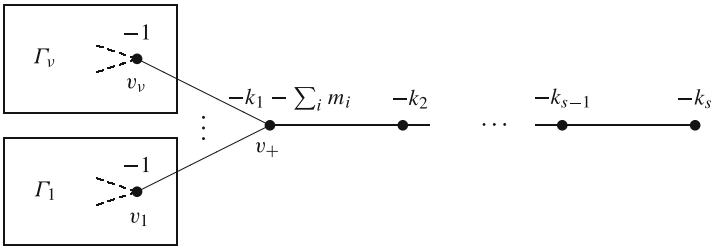
Example 3.5.12 (The Graph of $S^3_{-d}(\#_i K_i)$, K_i Algebraic) We fix an integer d and a collection of algebraic knots $\{K_i\}_{i=1}^{\nu}$ in S^3 (determined by irreducible plane curve singularities $(C_i,0)\subset(\mathbb{C}^2,0)$). Set the connected sum $K=K_1\#\cdots\#K_\nu\subset S^3$ of the knots K_i. Then $S^3_{-d}(K)$ is a plumbed 3-manifold whose plumbing graph is constructed as follows. First, let Γ_i be the minimal good embedded resolution graph of $(C_i,0)\subset(\mathbb{C}^2,0)$ (see 3.3.10) with a unique -1 vertex v_i which supports the strict transform. One also considers the cycle $Z_i=\mathrm{div}_{E(\Gamma_i)}(f_i)\in L(\Gamma_i)$ given by

the local reduced equation f_i of $(C_i, 0)$; let m_i be the multiplicity in Z_i of the -1 curve of Γ_i. (One can show that m_i, in terms of spice diagram 5.3.20, is $a_s p_s$, and in terms of the multiplicity sequence is the sum of the squares of the multiplicities, including the 1's, cf. 5.2.2.)

Then, in order to get the graph of $S^3_{-d}(K)$ from the disjoint union $\sqcup_i \Gamma_i$, one introduces a new vertex v_+, which is glued to each graph Γ_i via a new edge connecting v_+ and v_i, and one inserts the Euler decoration $-d - \sum_i m_i$ on v_+. The Euler decorations of $\{\Gamma_i\}_i$ stay unmodified. For a proof see e.g. 9.1.17.

The resulting graph is negative definite if and only if $d > 0$. (If $d < 0$ then I is semidefinite, and in general $|\det(I)| = |d|$).

Example 3.5.13 (The Graph of $S^3_{-p/q}(\#_i K_i)$, K_i Algebraic) Choose $p > 0, q > 0$ and $\gcd(p, q) = 1$ and let $K_i \subset S^3$ be an algebraic knot as in 3.5.12. Then the plumbing graph of the surgery manifold $S^3_{-p/q}(\#_i K_i)$ is the following.



where in the boxes we have Γ_i (without arrowhead) and $k_1 \geq 1$ and $k_j \geq 2$ $(2 \leq j \leq s)$ are integers determined by the continued fraction $p/q = [[k_1, k_2, \ldots, k_s]]$.

3.6 Basic Classification Theorems on 3-Manifolds and Links

In this section we collect the basic terminology and certain fundamental facts on the classification of *oriented* closed 3-manifolds. For more details see e.g. [262, 289, 736]. Then we recall relevant consequences (given by Neumann) regarding links of normal surface singularities. Some properties of the fundamental groups of links are also listed. ♣

3.6.1 We start with the original version of the Sphere Theorem.

Theorem 3.6.2 (Sphere Theorem, Papakyriakopoulos [557]) *Let M be an orientable 3-manifold such that $\pi_2(M)$ is not the trivial group. Then there exists a non-zero element of $\pi_2(M)$ having a representative that is an embedding $S^2 \hookrightarrow M$.*

An embedded 2-sphere $S^2 \hookrightarrow M$ is *incompressible* if it does not bound an embedded ball in M. M is called *irreducible* if it contains no incompressible 2-sphere.

An incompressible 2-sphere separates M if $M \setminus S^2$ is not connected. If M has an incompressible separating 2-sphere then M is a *connected sum* $M_1 \# M_2$ of the two components of $M \setminus S^2$ (M_1 and M_2 are obtained by completion of the components of $M \setminus S^2$ by balls glued on their boundaries). Recall that the connected sum $M_1 \# M_2$ of two 3-manifolds M_1 and M_2 is obtained by removing from each manifold the interior of a ball and then the resulting boundary components are glued.

M is called *prime* if it contains no incompressible separating 2-sphere. One proves that M is prime if and only if either M is irreducible or $M \simeq S^2 \times S^1$.

Theorem 3.6.3 (Prime Decomposition Theorem, Kneser [320], Milnor [428]) *Any 3-manifold has a unique connected sum decomposition into prime 3-manifolds; the list of summands is unique up to an order and homeomorphism.*

The next step is to describe the irreducible/prime 3-manifolds. They can also be decomposed/cut along incompressible surfaces.

A connected surface F, $F \neq S^2$, embedded in M is *incompressible* if $\pi_1(F) \to \pi_1(M)$ is injective. Two disjoint embedded surfaces F_1 and F_2 are *parallel* if they bound a subset of M isomorphic to $F_1 \times [0, 1]$.

Theorem 3.6.4 (Kneser–Haken Finiteness Theorem) *The number of disjoint pairwise non-parallel incompressible surfaces that can be embedded in M is bounded.*

Next we concentrate on incompressible tori. Such a torus T is called *canonical* if any other embedded incompressible torus T' can be isotoped to be disjoint from T.

We will consider in M a maximal disjoint collection $\mathscr{T} = \{T_1, \dots, T_k\}$ of canonical incompressible tori such that no two of them are parallel. The existence of a maximal collection is guaranteed by the Kneser-Haken finiteness theorem. Such a collection will be called JSJ-system (Jaco–Shalen–Johannson-system), and the splitting $M \setminus \cup_i T_i$ will be called the JSJ-decomposition of M.

Theorem 3.6.5 (Jaco–Shalen–Johannson [290, 292]) *Assume that M is compact, oriented and irreducible. Then a JSJ-system is unique up to an isotopy.*

3.6.6 JSJ-Decomposition of $M(\Gamma)$ In the case of an oriented plumbed 3-manifold associated with connected negative definite graph Γ all the tori in $\mathscr{T}$ are provided by certain edges via the plumbing construction and all the pieces of the JSJ-decomposition are Seifert, that is, plumbed 3-manifolds associated with star-shaped graphs (including strings). For more on Seifert structures see 5.1.11 and 5.1.12.

If Γ is a string or star-shaped then $\mathscr{T} = \emptyset$. If Γ is cyclic then $\mathscr{T}$ consists of one torus, which is associated with one of the edges as in 3.5.B.

Next, we assume that Γ is minimal good, and it has at least one node. Consider a maximal string of Γ whose end vertices are nodes of Γ (in the case of a loop, the

two ends meet in a unique node), and they contain no other 'interior' nodes. The tori determined by the edges of the string are parallel in $M(\Gamma)$. From each such string fix one of them. Then their collection $\mathscr{T}$ constitute a JSJ-system.

3.6.7 We include here some statements regarding singularity links.

Theorem 3.6.8 ([505])

(a) Singularity links are irreducible, hence prime (corresponding to the fact that resolution graphs are connected, cf. with 2.1.7 and with the end of 3.3.3).
(b) The dual graph of the minimal good resolution of a singularity (X, o) is determined by the oriented homeomorphism (or diffeomorphism) type of the link L_X. In particular, the link and the resolution graph contain the same information.
(c) If L is a singularity link, then $-L$ (L with reversed orientation) is also a singularity link if and only if L is either a lens space $L(n, q)$ with $n > 1$, or a torus bundle over the circle whose monodromy has trace ≥ 3. This holds if and only if the resolution graph is respectively a string or a cyclic graph; see 3.1.6 and 3.5.8.
(d) Any singularity link which is not in the list of exceptions of part (c), is determined by its fundamental group. For the exceptions, see again 3.1.6 and 3.5.8.
(e) If a Seifert manifold is a singularity link then the corresponding resolution graph is star-shaped (see also Sect. 5.1).

E.g., if M is realized as the link of (X, o), and $\pi_1(M) = 1$, then by part *(d)* $M \simeq S^3$, and by the sufficiency of the calculus the minimal resolution graph of (X, o) should be empty, hence (X, o) itself should be smooth (hence this reproves Mumford's theorem [448], see also 3.7.10 here).

The Sphere theorem and JSJ decomposition have the following consequences.

Corollary 3.6.9 *Assume that $L = M(\Gamma)$ is a singularity link with graph Γ.*

(a) $\pi_2(L) = 0$ and $H_q(\pi_1, \mathbb{Z}) = H_q(L, \mathbb{Z})$ for $q = 1, 2$.
(b) If $\pi_1(L)$ is finite then Γ is star-shaped (including the case of strings).
(c) If $\pi_1 = \pi_1(L)$ is infinite then L is the Eilenberg–Mac Lane space of π_1, that is $L = K(\pi_1, 1)$. Moreover, π_1 is torsion free.
(d) If $\pi_1(L_X)$ is abelian then it is finite cyclic.

For the description of all finite and nilpotent fundamental groups of links see 3.7.

Proof

(a) We follow [31, p.32]. By the Sphere Theorem, if N is an orientable 3-manifold with $\pi_2(N) \neq 0$, then N contains an embedded 2-sphere which is homotopically non-trivial. Hence, since L is irreducible, we must have $\pi_2(L) = 0$. Hence $K(\pi_1, 1)$ is obtained from L by attaching q–cells with $q \geq 4$.
(b) If L contains a JSJ torus T, then $\pi_1(T)$ is a subgroup of $\pi_1(L)$. Hence $\mathscr{T} = \emptyset$.

(c) When π_1 is infinite, the 3-manifold $\widetilde{L}$, the universal covering of L, has $H_3(\widetilde{L}) = 0$ (use non-compactness). Hence, by the Hurewicz theorem $\pi_i(L) = \pi_i(\widetilde{L}) = H_i(\widetilde{L}) = 0$ for any $i > 1$. Next, any finite dimensional Eilenberg–Mac Lane space $L = K(\pi_1, 1)$, has a torsion free fundamental group. Indeed, if $\mathbb{Z}_k \subset \pi_1$ is a non-trivial finite subgroup, then let $\hat{L}$ be the regular covering of L with $\pi_1(\hat{L}) = \mathbb{Z}_k$. Then $\hat{L}$ is at the same time $K(\mathbb{Z}_k, 1)$ and finite dimensional. By the first statement $H_q(\hat{L}, \mathbb{Z})$ is non-zero for infinitely many q, contradicting the second statement.

(d) If $\pi_1(L_X)$ is infinite, then (since it is torsion free) it is $\mathbb{Z}^s$. But $K(\mathbb{Z}^s, 1) = (S^1)^s$, which even for $s = 3$ cannot be represented by a negative definite plumbing graph. In the finite abelian case apply 2.3.3(ii).

□

For several other properties of the fundamental groups of 3-manifolds see [31].

3.7 The Fundamental Group of the Link

We give a 'canonical' finite presentation for $\pi_1(L_X)$ (including versions with optimal deficiency zero). The Seifert case is compared with the crystallographic and Dyck–Schwarz polygonal groups. We provide a classification of Γ whenever $\pi_1(L_X)$ is finite or infinite nilpotent. We also present Mumford's theorem: if $\pi_1(L_X) = 1$ then (X, o) is (analytically) smooth. ♣

3.7.1 Let (X, o) be a normal surface singularity. In this section we provide a finite presentation for the fundamental group π_1 of the singularity link $L = L_X$. If the graph has cycles then this presentation is more involved. Therefore, first we study the case when the link is a rational homology sphere.

Let us fix a plumbing graph Γ (e.g. the dual graph of a good resolution), a base point in $M(\Gamma)$, and also an ordering of the vertices. Furthermore, for any $v \in \mathcal{V}$ we consider the generic S^1-fibers around E_v (as in 3.3.6), and we complete this free loop with connecting path to the fixed base point. We will use the same symbol γ_v for them.

Theorem 3.7.2 (Mumford [448]) *Assume that L is a rational homology sphere. Then there exists a convenient choice of the connecting paths such that $\pi_1(L)$ has the next finite presentation: the generators are $\{\gamma_v\}_{v\in\mathcal{V}}$, while the relations are $\gamma_v\gamma_u = \gamma_u\gamma_v$ whenever $(v, u) \in \mathcal{E}$, and $\prod_u \gamma_u^{I_{v,u}} = 1$ for every $v \in \mathcal{V}$. Here in the product we list the terms in the increasing order of u according to the ordering of $\mathcal{V}$.*

Proof Consider the S^1-bundles $B_v \to S_v$ from the plumbing construction 3.3.3. Then for any v one has $\pi_1(B_v) = \langle \gamma_v \mid \gamma_v^{I_{vv}} = 1 \rangle$. On the other hand, if from B_v we remove the spaces $D_p^\circ \times S^1$ corresponding to all edges adjacent to v, and we denote the remaining space by $L_X \cap B_v$, then $\pi_1(L_X \cap B_v)$ is generated by

$\{\gamma_u \mid I_{vu} \neq 0\}$, with the relation $\prod_u \gamma_u^{I_{v,u}} = 1$. These relations are glued together via van Kampen's theorem. The relation $\gamma_v\gamma_u = \gamma_u\gamma_v$ for $(v,u) \in \mathscr{E}$ corresponds to the fact that $L_X \cap B_v \cap B_u$ is a torus with abelian fundamental group. The existence of the convenient connecting paths is guaranteed by the fact that Γ is a tree. For more details see [270, 448]. □

The above statement has the following immediate generalization.

Proposition 3.7.3 *Let Γ be a tree with ordered set of vertices, such that the genus decorations might be non-zero. Then $\pi_1(L)$ has the following presentation. The generators are $\{\gamma_v\}_{v\in\mathscr{V}}$, and for each v with $g_v > 0$ one needs $2g_v$ additional generators $\{a_{v,m}, b_{v,m}\}_{1\leq m\leq g_v}$. The relations are: $[\gamma_v, \gamma_u] = 1$ whenever $(v,u) \in \mathscr{E}$, $[\gamma_v, a_{v,m}] = [\gamma_v, b_{v,m}] = 1$ for every v and $1 \leq m \leq g_v$; and finally $[a_{v,1}, b_{v,1}]\cdots[a_{v,g_v}, b_{v,g_v}] \cdot \prod_u \gamma_u^{I_{v,u}} = 1$ for every $v \in \mathscr{V}$, where, as usual, in the (last) product we list the terms in the increasing order of u (and $[a,b] = aba^{-1}b^{-1}$).*

Remark 3.7.4 Assume that Γ is not a string and it has a leg with Seifert invariant $(\alpha,\omega) = [[b_1,\dots,b_s]]$. If $\gamma_1,\dots,\gamma_s$ are the corresponding elements in π_1 (with γ_s corresponding to the end-vertex) then $\gamma_s^{b_s} = \gamma_{s-1}, \dots, \gamma_i^{b_i} = \gamma_{i-1}\gamma_{i+1}$, and each γ_i $(1 \leq i < s)$ can be eliminated. Indeed, if $n_{is}/q_{is} = [[b_i,\dots,b_s]]$ (as in 2.2.A) then $\gamma_{i-1} = \gamma_s^{n_{is}}$.

Example 3.7.5 (The Case of Star-Shaped Graphs) If Γ is star-shaped then the above presentation via 3.7.4 has the following form. Below the loops γ_0, $\{\gamma_j\}_{j=1}^{\nu}$ are associated with (the central and end) vertices v_0 and v_{js_j} respectively.

$$\langle \gamma_0, \{\gamma_j\}_{j=1}^{\nu}, \{a_m, b_m\}_{m=1}^{g} \mid \gamma_0 \text{ central},\ \gamma_j^{\alpha_j} = \gamma_0, \forall j,\ \gamma_0^{b} = \textstyle\prod_m [a_m, b_m]\cdot\prod_j \gamma_j^{\omega_j}\rangle.$$

γ_0 is the loop corresponding to the regular/generic S^1 Seifert fiber. If $\alpha_j > 1$ then γ_j is the corresponding singular Seifert fiber. Clearly, $\overline{\pi_1} := \pi_1/\langle\gamma_0\rangle$ is

$$\overline{\pi_1} = \langle \{\bar\gamma_j\}_{j=1}^{\nu}, \{a_m, b_m\}_{m=1}^{g} \mid \bar\gamma_j^{\alpha_j} = 1, \forall j,\ \textstyle\prod_m [a_m, b_m] \cdot \prod_j \bar\gamma_j^{\omega_j} = 1\rangle.$$

By $x_j := \bar\gamma_j^{\omega_j}$, $\bar\gamma_j := x_j^{\omega_j'}$, where $\omega_j\omega_j' \equiv 1 \pmod{\alpha_j}$, $\overline{\pi_1}$ transforms into the *crystallographic group* (if $g = 0$, it is also named *Dyck–Schwarz polygonal group*)

$$\langle \{x_j\}_{j=1}^{\nu}, \{a_m, b_m\}_{m=1}^{g} \mid x_1^{\alpha_1} = \cdots = x_\nu^{\alpha_\nu} = \textstyle\prod_m [a_m, b_m] \cdot \prod_j x_j = 1\rangle.$$

One shows that π_1 is infinite if and only if $\overline{\pi_1}$ is infinite (see e.g. [289, VI.11]), if and only if γ_0 has infinite order in π_1 (use 3.6.9(c)).

For a topological interpretation in terms of a resolution space see the paragraph after 5.1.25. (Groups with presentation of $\overline{\pi_1}$ in classical literature of 'surface topology groups' are also called *F-groups*.)

Example 3.7.6 (Schwarz Triangle Group [620]) Let P be either the surface of unit 2-sphere, or the Euclidean plane, or the Bolyai-Lobachevsky plane. For any fixed integers $p,\ q,\ r \geq 2$ we consider a triangle T in P with angles $\pi/p,\ \pi/q, /\pi/r$ and geodesic edges. The *full Schwarz triangle group* $\Delta^*(p,q,r)$ is the group of isometries of P, which is generated by the reflections $\sigma_1,\ \sigma_2,\ \sigma_3$ in the three edges of T. Its presentation is

$$\Delta^*(p,q,r) = \langle \sigma_1,\ \sigma_2,\ \sigma_3 \,|\, \sigma_1^2 = \sigma_2^2 = \sigma_3^2 = (\sigma_1\sigma_2)^p = (\sigma_2\sigma_3)^q = (\sigma_3\sigma_1)^r = 1 \rangle.$$

The *Schwarz triangle group* $\Delta(p,q,r)$ is the subgroup of $\Delta^*(p,q,r)$ of index 2, consisting of all orientation preserving elements of $\Delta^*(p,q,r)$. Its presentation is

$$\Delta(p,q,r) = \langle \tau_1,\ \tau_2,\ \tau_3 \,|\, \tau_1^p = \tau_2^q = \tau_3^r = \tau_1\tau_2\tau_3 = 1 \rangle.$$

$\tau_1,\ \tau_2,\ \tau_3$ represent rotations of angle $2\pi/p,\ 2\pi/q, /, 2\pi/r$, hence their order in Δ are $p,\ q,\ r$. Δ is a finite group of order $2/(p^{-1} + q^{-1} + r^{-1} - 1)$ if $P = S^2$, or equivalently, $p^{-1} + q^{-1} + r^{-1} > 1$ (that is, $(p,q,r) \in \{(2,3,3),\ (2,3,4),\ (2,3,5),\ (2,2,r)\}$). Otherwise the group Δ is infinite. For more details and properties see [431, 620].

The hyperbolic case extends to $n_i = \infty$ as well by taking the modular triangle.

For more topological connections and for the *centrally extended triangle group* see 5.1.35. For generalization to *polygonal groups* see [168].

Remark 3.7.7 In fact, some of the relations in the presentation of π_1 from 3.7.2 and 3.7.3 are superfluous. Before we eliminate them we note the following.

If $\langle \gamma \,|\, r \rangle$ is a finite presentation of a group π, then the *deficiency* $\mathrm{def}\langle \gamma \,|\, r \rangle$ of the presentation is defined as the difference of cardinalities $|\gamma| - |r|$. By Epstein [187], for any presentation of a finitely presented group π one has $\mathrm{def}\langle \gamma \,|\, r \rangle \leq \mathrm{rank}\, H_1(\pi, \mathbb{Z}) - \rho H_2(\pi, \mathbb{Z})$, where ρH_2 is the minimal number of generators of H_2.

Since by 3.6.9 one has $H_q(\pi_1, \mathbb{Z}) = H_q(L, \mathbb{Z})$ for $q = 1, 2$, $\mathrm{def}\langle \gamma \,|\, r \rangle \leq 0$.

Define also $\mathrm{def}(\pi)$ as the supremum of $\mathrm{def}\langle \gamma \,|\, r \rangle$, where $\langle \gamma \,|\, r \rangle$ runs over all finite presentations of π. Then (see e.g. [289, V.3]), for any compact 3-manifold M without boundary, one has $\mathrm{def}(\pi_1(M)) \geq 0$. Hence, for any link L, $\mathrm{def}(\pi_1(L)) = 0$.

This motivates us to find 'optimal' presentations with $\mathrm{def}\langle \gamma | r \rangle = 0$.

Theorem 3.7.8 (Optimal Deficiency Version) *Assume that Γ is a tree. Then in the presentations of $\pi_1(L)$ from 3.7.2 and 3.7.3 one can eliminate all the relations of type $[\gamma_v, \gamma_u] = 1$ for $(v,u) \in \mathscr{E}$ (since they automatically follow from the remaining relations). The deficiency of the presentations after this elimination is zero.*

Proof If the vertex v has exactly one neighbour, say u, then $[\gamma_v, \gamma_u] = 1$ follows from the remaining relations (since $\gamma_v^{E_v^2}\gamma_u = 1$). The other commuting relations are proved by induction as follows. Fix a vertex $* \in \mathscr{V}$ and define $d(v)$ for every $v \in \mathscr{V}$ as the graph-distance in Γ of v and $*$. Fix $v \in \mathscr{V}$ and $u \in \mathscr{V}_v$. Then u is an *input*,

respectively *output* neighbour of v whenever $d(u) - d(v)$ is $+1$ or -1. $*$ has no output neighbours, any other v has exactly one. The remaining relations show that if γ_v commutes with all the input neighbours, then it necessarily commutes with the output neighbour as well. Hence the relations $[\gamma_v, \gamma_u] = 1$ for $(v, u) \in \mathscr{E}$ follow by decreasing induction on $d(v)$. □

3.7.9 $\pi_1(M(\Gamma))$ **for General** Γ Assume that Γ is a negative definite plumbing graph with $c(\Gamma) > 0$. Let Γ' be a maximal tree in Γ, that is $\Gamma' = \Gamma \setminus \mathscr{E}^\circ$, where $\mathscr{E}^\circ = \{e_1, \dots, e_{c(\Gamma)}\}$ are certain edges of Γ. Recall that each edge $e \in \mathscr{E}^\circ$ represents a embedded torus T_e in L. Let $N(T_e)$ be an open tubular neighbourhood of it in L. We endow Γ' with a set of arrowheads, for each edge $e = (u, v) \in \mathscr{E}^\circ$ we put an arrowhead $a(u)$ on u and one arrowhead $a(v)$ on v. Then each arrowhead represents a torus boundary component of the boundary of $M(\Gamma) \setminus \cup_{e\in\mathscr{E}^\circ} N(T_e)$.

Let $\mathscr{W}(\Gamma')$ be the union of the old vertices $\mathscr{V}(\Gamma)$ with the $2c(\Gamma)$ arrowheads. We order $\mathscr{W}(\Gamma')$, and we define the incidence matrix $I_{v,u}$, $v \in \mathscr{V}(\Gamma)$, $u \in \mathscr{W}(\Gamma')$ as follows. If $u, v \in \mathscr{V}(\Gamma)$, $(u, v) \notin \mathscr{E}^\circ$, then it is the entry of the usual intersection matrix of Γ. If v supports an arrowhead $a(v)$ then $I_{v,a(v)} = 1$, otherwise it is zero.

Furthermore, for each $v \in \mathscr{W}(\Gamma')$ we construct a loop γ_v as in the previous cases. For an arrowhead the starting loop is an oriented meridian of the missing solid torus (see 3.3.6). Then $\pi_1(M(\Gamma) \setminus \cup_{e\in\mathscr{E}^\circ} N(T_e)$ has the following presentation (the proof is similar as in the case of 3.7.2 and 3.7.3 combined with 3.7.8).

The generators are $\{\gamma_v\}_{v\in\mathscr{W}(\Gamma')}$, and for each $v \in \mathscr{V}(\Gamma)$ with $g_v > 0$ we take $2g_v$ additional generators $\{a_{v,m}, b_{v,m}\}_{1\le m\le g_v}$. The relations are: $[\gamma_v, \gamma_{a(v)}] = 1$ whenever a supports the arrowhead $a(v)$, $[\gamma_v, a_{v,m}] = [\gamma_v, b_{v,m}] = 1$ for every v and $1 \le m \le g_v$; and finally $[a_{v,1}, b_{v,1}] \cdots [a_{v,g_v}, b_{v,g_v}] \cdot \prod_{u\in\mathscr{W}(\Gamma')} \gamma_u^{I_{v,u}} = 1$ for every $v \in \mathscr{V}(\Gamma)$. Here, in the last product, as usual, we list the terms in the increasing order of u.

Since Γ' is a tree, the construction of the connecting paths for each γ_v is canonical, and they produce along each string of Γ' the typical relations $\gamma_v^{b_v} = \gamma_u\gamma_w$ (†) (u, w adjacent vertices of v). However, this type of relation in Γ would fail if we wish to propagate along an edge $e \in \mathscr{E}^\circ$. Indeed, a loop associated with $a(v)$ (or v) would need a different connecting path (one, which would be compatible with the paths of u and $a(u)$) if we wish to have propagation relations (†) along e. This connecting path (compatible with the one of u) and the existing path (associated with Γ') form a loop in $M(\Gamma)$. Hence, for any such $e \in \mathscr{E}^\circ$ one has to introduce a new generator of $\pi_1(M(\Gamma))$ (the above loop), which corrects by conjugation the above ambiguity.

Therefore, a possible presentation of $\pi_1(M(\Gamma))$ is the following. Let $\langle \mathrm{Gen} \mid \mathrm{Rel}\rangle$ be the finite presentation of $\pi_1(M(\Gamma) \setminus \cup_{e\in\mathscr{E}^\circ} N(T_e)$ given above. Then a presentation of $\pi_1(M(\Gamma))$ is $\langle \mathrm{Gen} \cup \{t_e\}_{e\in\mathscr{E}^\circ} \mid \mathrm{Rel} \cup \{r_e\}_{e\in\mathscr{E}^\circ}\rangle$, where for $e = (u, v)$ one has

$$r_e : \quad t_e\gamma_u t_e^{-1} = \gamma_{a(v)} \quad \text{and} \quad t_e\gamma_{a(u)}t_e^{-1} = \gamma_v.$$

Note that by the relation r_e the previously defined relations $[\gamma_v, \gamma_{a(v)}] = 1$ and $[\gamma_u, \gamma_{a(u)}] = 1$ are equivalent. So, one of them can be eliminated for each e. In this way we get a presentation of $\pi_1(M(\Gamma))$ with deficiency zero.

One can write $\pi_1(M(\Gamma))$ also in the language of *graph of groups*, see e.g. [333] based on [580, 631]. See also [35] for a presentation. Compare also with 3.5.B.

The first application of the presentation 3.7.2 is the following topological characterization of the smoothness given by Mumford.

Theorem 3.7.10 (Mumford [448]) *If the minimal good resolution graph of a normal surface singularity is non-empty then $\pi_1(L_X) \neq 1$. In other words, if $\pi_1(L_X)$ is trivial, then necessarily L_X is S^3 and (X, o) is smooth.*

Proof Since I have difficulties to understand the original proof from [448], here I modify it slightly. We start with the following group theoretical statement. Below $G * H$ is the free product of G and H. □

Proposition 3.7.11 *Suppose that G_i $(i = 1, 2, 3)$ are groups, $g_i \in G_i$, and set $G := G_1 * G_2 * G_3/\langle g_1 g_2 g_3 = 1\rangle$. Then G_i injects into G for at least one index i.*

Proof If $g_1 = 1$ in G_1 then $G = G_1 * (G_2 * G_3/\langle g_2 g_3 = 1\rangle)$ contains G_1 as a subgroup. If $g_i \neq 1$ for every i then we prove that each G_i is a subgroup of G.

Let n_i, $1 < n_i \leq \infty$, be the order of g_i in G_i. Let $\mathbb{Z}_{n_i}$ be the subgroup of G_i generated by g_i. First we note that each $\mathbb{Z}_{n_i}$ embeds into $\mathbb{Z}_{n_1} * \mathbb{Z}_{n_2} * \mathbb{Z}_{n_3}/\langle g_1 g_2 g_3 = 1\rangle$. This fact follows from the structure of the triangle group, see 3.7.6. (Here one has to analyse the $n_i = \infty$ cases as well.)

Next note that the amalgamated product of G_1 and $\mathbb{Z}_{n_1} * \mathbb{Z}_{n_2} * \mathbb{Z}_{n_3}/\langle g_1 g_2 g_3 = 1\rangle$ over the common subgroup $\mathbb{Z}_{n_1}$ is $G_1 * \mathbb{Z}_{n_2} * \mathbb{Z}_{n_3}/\langle g_1 g_2 g_3 = 1\rangle$. By the general theory, see e.g. [400], G_1 and $\mathbb{Z}_{n_1} * \mathbb{Z}_{n_2} * \mathbb{Z}_{n_3}/\langle g_1 g_2 g_3 = 1\rangle$ inject into $G_1 * \mathbb{Z}_{n_2} * \mathbb{Z}_{n_3}/\langle g_1 g_2 g_3 = 1\rangle$, hence all the factors of $G_1 * \mathbb{Z}_{n_2} * \mathbb{Z}_{n_3}/\langle g_1 g_2 g_3 = 1\rangle$ inject into this group as well. Then by a similar way we replace each $\mathbb{Z}_{n_i}$ by G_i $(i = 2, 3)$. □

Now we return back to the proof of 3.7.10. Since $H_1(L_X) = 0$ we can assume that $g(\Gamma) = c(\Gamma) = 0$ (cf. Proposition 3.4.2) and we will apply 3.7.2. We proceed by induction on the number of nodes. We verify that if Γ is a non-empty negative definite connected graph, and $\pi_1(M(\Gamma)) = 1$ then Γ cannot be minimal good (hence it can be blown down inductively to the empty graph).

If Γ has no nodes then the statement follows from 2.3.1 and 3.1.6. Assume that Γ is a minimal good graph such that $\pi_1(M(\Gamma)) = 1$, and it has at least one node.

Next, we fix a node v. If we delete v and the adjacent edges we get $\nu \geq 3$ connected graphs $\Gamma_1, \ldots, \Gamma_\nu$. Assume that $v_i \in \mathcal{V}(\Gamma_i)$ is adjacent to v in Γ. Since $\pi_1(M(\Gamma)) = 1$ one gets from 3.7.2 that $\pi_1(M(\Gamma))/\langle \gamma_v = 1\rangle = G_1 * \cdots * G_\nu/\langle \gamma_{v_1} \cdots \gamma_{v_\nu} = 1\rangle$ is trivial, where $G_i = \pi_1(M(\Gamma_i))$. Then $G_1 * G_2 * G_3/\langle \gamma_{v_1}\gamma_{v_2}\gamma_{v_3} = 1\rangle$ is trivial too. Hence, by 3.7.11 there exists i such that G_i is trivial. By induction Γ_i cannot be minimal good. Since Γ is minimal good, this can happen only if v_i is a node in Γ and it is a -1 vertex. This shows that Γ must have a node with Euler decoration -1. Let us restart the previous discussion where v is

such a node. Then by the above discussion we get another node v_i, adjacent to v, and with Euler decoration -1. But this contradicts the negative definiteness of Γ.

Corollary 3.7.12 *If a normal surface singularity (X, o) has the property that $\pi_1(L_X)$ is finite, then the universal covering of L_X is S^3, and (X, o) is a quotient singularity $(X, o) = (\mathbb{C}^2, 0)/\pi_1(L_X)$ for some linear representation $\pi_1(L_X) \subset \mathrm{GL}_2(\mathbb{C})$. In particular, the possible finite fundamental groups $\pi_1(L_X)$ are exactly the finite small subgroups of $\mathrm{GL}_2(\mathbb{C})$. Two quotient singularities are analytically isomorphic if and only if the corresponding fundamental groups are conjugate in $\mathrm{GL}_2(\mathbb{C})$.*

Proof Consider the universal covering (X_{uc}, o) of (X, o) associated with the trivial subgroup of $\pi_1(L_X)$, cf. 12.3.3. Then $L(X_{uc}, o)$ is simply connected, hence by 3.7.10 (X_{uc}, o) is smooth, $(\mathbb{C}^2, 0)$. Furthermore, the action of π_1 on $(\mathbb{C}^2, 0)$ can be linearized in some local coordinates. Indeed, the action of π_1 induces an action on the cotangent space $\mathfrak{m}/\mathfrak{m}^2$ of $(\mathbb{C}^2, 0)$ and on the exact sequence $0 \to \mathfrak{m}^2 \to \mathfrak{m} \to \mathfrak{m}/\mathfrak{m}^2 \to 0$. Since π_1 is finite, this sequence equivariantly splits, which provides the wished local coordinates in $\mathfrak{m}$. See also 12.3.6 and 12.3.7. □

Remark 3.7.13 The normality in 3.7.12 is essential. Indeed, the link of an irreducible isolated non-normal singularity is the same as the link of its normalization. A possible construction for such germs is the following. Take the subring R of $\mathbb{C}[t, s]$ generated by all the monomials of type $t^a s^b$, where $(a, b) \in \mathbb{Z}^2_{\geq 0} \setminus \{(0, 1)\}$. Then R is generated by $x = s^2$, $y = s^3$, $z = t$ and $w = st$ with relations $x^3 - y^2 = w^2 - z^2 x = wy - zx^2 = xw - yz = 0$. It defines a non-normal irreducible isolated surface singularity, whose normalization is smooth.

Remark 3.7.14 Theorem 3.7.10 can be formulated in this way too: if the link is homeomorphic/diffeomorhic to S^3 then (X, o) is smooth. This implication is lost in higher dimensions: the link of the (non smooth, but normal) isolated hypersurface singularity $\{z_0^{6k-1} + z_1^3 + z_2^2 + z_3^2 + \cdots + z_{2m}^2 = 0, 0\}$ for $m = 2$ is a topological sphere for any k, and it is even diffeomorphic to S^7 if $28|k$ [82, 271, 275]. For $k = 1, \ldots, 28$ it provides the 28 classes of exotic smooth structures on S^7. For $m = 3$ and $k = 1, \ldots, 992$ we obtain the 992 exotic structures on S^{11}.

Regarding this *higher* dimensional phenomenon it is worth to add the following: if we supplement the smooth manifold structure of the link of the Brieskorn singularity (X, o) with its natural contact structure $\xi_{X,o}$ induced by the complex analytic structure of (X, o), then (X, o) is smooth if and only if $\xi_{X,o}$ agrees with the standard contact structure of the sphere (induced by the smooth germ) [707, 708]. In a more general context, the following fact was conjectured by Seidel and proved by McLean in [414]: Assume that (X, o) is an isolated normal 3–dimensional singular germ. Then, if $(L_X, \xi_{X,o})$ is contactomorphic with $(L_{\mathbb{C}^3}, \xi_{\mathbb{C}^3,0})$ then (X, o) is smooth. Hence the analogue of Mumford's theorem works if we involve contact structures.

For links of normal surface singularities the diffeomorphism type of the link already determines uniquely the contact structure induced by any analytic type (X, o) up to a contactomorphism (in the rational homology sphere case even up to isotopy) [107].

3.7.15 Classification of Γ with $\pi_1(M(\Gamma))$ Finite The classification of conjugacy classes of finite small subgroups of $\mathrm{GL}_2(\mathbb{C})$ has been carried out by Brieskorn [83], see also [54]. Our argument below is different: it is based on two facts, namely that the quotient singularities are rational, and that the universal abelian covering of a weighted homogeneous singularity is a special Brieskorn complete intersection. Although both statements will be proved later, we prefer to place the next discussion here. (In 6.3.33 we will also show that the canonical covers of those singularities which have finite fundamental groups are ADE, hence this fact also characterizes them.)

Any finite π_1 of $\mathrm{GL}_2(\mathbb{C})$ commutes with the diagonal subgroup $\mathbb{C}^*$, hence $(X, o) = (\mathbb{C}^2, 0)/\pi_1$ is weighted homogeneous. In particular, its graph is a string or star-shaped with $g = 0$ and $\nu \geq 3$ (this follows from 3.6.9 too). The case of string happens if and only if π_1 is cyclic and the action is as in 2.3.1 producing cyclic quotient singularities. Assume next that the graph is star-shaped with $\nu \geq 3$ and with α–Seifert invariants $\{\alpha_j\}_j$. Then, by 5.1.33, the universal abelian covering (X_a, o) is a Brieskorn complete intersection in $\mathbb{C}^\nu$ with Brieskorn exponents $\{\alpha_j\}_j$. Since the (X_a, o) has a link with finite fundamental group—$[\pi_1, \pi_1]$—it is a quotient singularity by 3.7.12. On the other hand, a quotient singularity is rational, see 7.1.5. Therefore, (X_a, o) being both rational and complete intersection, it should be an ADE singularity of Brieskorn type, cf. 7.1.5(d). (Hence one of A_n, D_4, E_6, E_8.)

In particular, $\nu = 3$ and $\sum_j 1/\alpha_j > 1$. Returning back to (X, o) with this information, one verifies that any arbitrary completion of such triple $\{\alpha_j\}_j$ with $\{\omega_j\}_j$ and b to get the Seifert invariants of Γ_X (satisfying $e < 0$) is allowed. This provides all the possible graphs, and the corresponding finite fundamental groups, which agree with all the possible small finite subgroups of $\mathrm{GL}_2(\mathbb{C})$.

We mention that for fixed $\{\alpha_j\}_j$, e (mod 1) determines completely the entries $\{\omega_j\}_j$, hence e determines all the Seifert invariants (see e.g. [83, 291, 592]).

For completeness, we say a few words about the finite subgroups G of $\mathrm{SL}_2(\mathbb{C})$ as well. The existence of a G-invariant hermitian inner product on $\mathbb{C}^2$ shows that any such G is conjugate to a finite subgroup of SU_2. On the other hand, using the quaternion algebra one constructs a double covering homomorphism $\mathrm{SU}_2 \to \mathrm{SO}_3$ with kernel ± 1. Hence, any finite subgroup G of SU_2 maps onto a finite subgroup $\bar{G}$ of SO_3, and any subgroup $\bar{G}$ of SO_3 is covered by a group $G \subset \mathrm{SU}_2$ with kernel of order ≤ 2. Since -1 is the unique element of order 2 in SU_2, it is contained in any subgroup G of even order. Moreover, if the order of $G \subset \mathrm{SU}_2$ is odd, then G must by cyclic. Hence, the kernel of $G \to \bar{G}$ is trivial only if G is cyclic of odd order. On the other hand, finite subgroups of SO_3 are exactly the group of rotation symmetries of one of the regular polyhedrons, or of the dihedron and pyramid. The corresponding groups $\bar{G}$ are the cyclic (pyramid case), dihedral (dihedron), $\mathfrak{A}_4$ (tetrahedron), $\mathfrak{S}_4$

(octahedron, cube), $\mathfrak{A}_5$ (icosahedron, dodecahedron), where $\mathfrak{S}_k$ (resp. $\mathfrak{A}_k$) denotes the symmetric (resp. alternating) group of k letters. Their order is n, $2n$, 12, 24, 60. The corresponding finite subgroups G in SU_2 are the cyclic, binary dihedral, binary tetrahedral, binary octahedral and binary icosahedral groups of order n, $4n$, 24, 48, 120. The corresponding quotients are A_{n-1} ($n \geq 2$), D_{n+2} ($n \geq 2$), E_6, E_7, E_8 (for their equations see 7.1.5).

The finite subgroups of $\mathrm{SL}_2(\mathbb{C})$ have the major advantage that they preserve the volume form $dz_1 \wedge dz_2$, hence the corresponding quotients $(\mathbb{C}^2/G, o)$ are Gorenstein, cf. 7.1.5.

From a different point of view, the ADE singularities are exactly the *(numerically) canonical singularities*, see 6.3.33.

Remark 3.7.16 ([505, 6.1]) Assume that Γ is a connected negative definite minimal graph. If $M(\Gamma)$ is a torus bundle over S^1 then Γ is either a cyclic graph or it has only one vertex with $g_v = 1$ and $e_v = -b$.

Hence, in this second case ($g_v = 1$, $e_v = -b$), π_1 has two descriptions. First, since $M(\Gamma)$ is an S^1 bundle over the torus with Euler number $-b$, π_1 is the central extension of $\mathbb{Z}^2$ by $\mathbb{Z}$ with Euler class $-b \in \mathbb{Z} = H^2(\mathbb{Z}^2; \mathbb{Z})$. Or, $M(\Gamma)$ is a torus bundle over S^1 with monodromy matrix $A = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$.

In this second description, the exact sequence $0 \to \mathbb{Z}^2 \to \pi_1 \xrightarrow{q} \mathbb{Z} \to 0$ suggests that π_1 can also be represented as the Heisenberg group H_b of the unipotent matrices

$$\begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} \\ 0 & 1 & b\mathbb{Z} \\ 0 & 0 & 1 \end{pmatrix} \quad \text{with} \quad q \begin{pmatrix} 1 & t & x \\ 0 & 1 & by \\ 0 & 0 & 1 \end{pmatrix} = t.$$

This is the only case when the fundamental group of a surface singularity link is an infinite nilpotent group (this last statement follows from 3.6.9 and [676]).

Remark 3.7.17 Consider a cyclic graph Γ with decorations $-b_1, \ldots, -b_k$ as in 3.5.B. Let $\Gamma^{\ominus}$ be the graph obtained from Γ by changing one of its edges into a $\ominus$-edge. Them $M(\Gamma^{\ominus})$ is again a torus bundle over S^1, whose monodromy is $-A(b_1, \ldots, b_k)$, cf. [505, 6.1]. $M(\Gamma^{\ominus})$ cannot be a singularity link, nevertheless some of them appear as the boundary of the Milnor fiber ∂F of certain non-isolated hypersurface singularities. E.g., in the case of $T_{p,q,\infty}$ with equation $x^p + y^q + xyz = 0$, ∂F is $-M(\Gamma^{\ominus})$, where the cyclic graph has two vertices, their Euler numbers are p and q, and one of the edges is a $\ominus$-edge [502, 22.2]. (Compare with 3.5.10.)

Remark 3.7.18 Assume that an oriented 3-manifold M is the boundary of a smooth Stein 4-manifold Y.

For example, we can consider the boundary $\partial \widetilde{X} = M(\Gamma)$ of $\widetilde{X}$, a resolution of (X, o). This complex manifold $\widetilde{X}$ is not Stein, however by [66] the holomorphic

structure of $\widetilde{X}$ can be deformed into a Stein structure preserving (even the contact structure of) the 3-manifold boundary.

Also, as a different second case, one might consider an analytic smoothing of (X, o) (if there are any), where Y is the nearby smooth fiber with $\partial Y \simeq \partial X = M$.

We claim that for such Stein boundary M, the inclusion $M = \partial Y \hookrightarrow Y$ induces a surjective homomorphism $\pi_1(M) \to \pi_1(Y)$.

Indeed, there exists a (strongly pseudoconvex) Morse function $g : Y \to \mathbb{R}$ and $c_1 < c_2$ such that $g^{-1}((-\infty, c_1]) = B_4$ is the closed 4-ball, $g^{-1}((-\infty, c_2]) = Y$ with boundary $\partial Y = g^{-1}(c_2)$, and Y is obtained from B_4 via the Morse construction by attaching 1 and 2 handles. Turning the Morse function upside down we obtain that Y is obtained from ∂Y by attaching 2 and 3 handles and a 4-ball.

Chapter 4
Coverings

4.1 Cyclic Coverings

In 4.1.A we define the purely graph-theoretical notion of cyclic covering of graphs, and in some key cases we classify all the cyclic coverings of a fixed graph. This will be applied to the embedded resolution graph $\Gamma(X, f)$. First, in 4.1.B, we associate with any analytic germ $f : (X, o) \to (\mathbb{C}, 0)$ the 'universal' cyclic covering graph $G(X, f) \to \Gamma(X, f)$. This usually contains more information than $\Gamma(X, f)$, however, if L_X is a rational homology sphere than $G(X, f) \to \Gamma(X, f)$ can be recovered merely from $\Gamma(X, f)$. Then, for any such f and integer $N > 0$ we define the cyclic covering associated with f and N, and we determine its resolution graph from $G(X, f) \to \Gamma(X, f)$ and N, cf. 4.1.C. ♣

4.1.A Cyclic Coverings of Graphs

4.1.1 In this subsection we present a graph-theoretical construction. It is motivated by the structure of the resolution graphs of cyclic coverings of a normal surface singularity (ramified along a curve or ramified only along o). For more, see [475].

For any graph Γ, $\mathscr{V} = \mathscr{V}(\Gamma)$ and $\mathscr{E} = \mathscr{E}(\Gamma)$ denote the set of vertices and edges. For $v \in \mathscr{V}$ let $\mathscr{E}_v$ be the set of edges adjacent to v. Let $c(\Gamma)$ be the number of independent cycles in Γ. We assume that Γ has no loops. If e is an edge with end-vertices v and u, we write $\partial e = \{v, u\}$. Abbreviate $[n, m] = \mathrm{lcm}\{n, m\}$ and $(n, m) = \gcd\{n, m\}$.

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Definition 4.1.2

(a) A morphism of graphs $p : \Gamma_1 \to \Gamma_2$ consists of two maps $p_{\mathscr{V}} : \mathscr{V}(\Gamma_1) \to \mathscr{V}(\Gamma_2)$ and $p_{\mathscr{E}} : \mathscr{E}(\Gamma_1) \to \mathscr{E}(\Gamma_2)$, such that if $\partial e = \{v_1, v_2\}$ for $e \in \mathscr{E}(\Gamma_1)$ then $\partial p_{\mathscr{E}}(e) = \{p_{\mathscr{V}}(v_1), p_{\mathscr{V}}(v_2)\}$. If p is a morphism of graphs and $p_{\mathscr{V}}$ and $p_{\mathscr{E}}$ are isomorphisms of sets, then we say that p is an isomorphism of graphs.

(b) If Γ is a graph, we say that $\mathbb{Z}$ acts on Γ, if there are group actions $a_{\mathscr{V}} : \mathbb{Z} \times \mathscr{V} \to \mathscr{V}$ and $a_{\mathscr{E}} : \mathbb{Z} \times \mathscr{E} \to \mathscr{E}$ of $\mathbb{Z}$ such that if $\partial e = \{v_1, v_2\}$ then $\partial a_{\mathscr{E}}(h, e) = \{a_{\mathscr{V}}(h, v_1), a_{\mathscr{V}}(h, v_2)\}$. The action is trivial if both $a_{\mathscr{V}}$ and $a_{\mathscr{E}}$ are trivial actions.

One also has the natural notion of equivariant morphisms and isomorphisms.

(c) Fix a finite graph Γ with a trivial $\mathbb{Z}$ action. A $\mathbb{Z}$ (or cyclic) covering of Γ consists of a finite graph G, that carries a $\mathbb{Z}$ action, together with an equivariant morphism $p : G \to \Gamma$ such that the restriction of the $\mathbb{Z}$ action on any set of type $p^{-1}(v)$ $(v \in \mathscr{V}(\Gamma))$, respectively $p^{-1}(e)$ $(e \in \mathscr{E}(\Gamma))$, is transitive.

Fix a cyclic covering $p : G \to \Gamma$. For any $v \in \mathscr{V}(\Gamma)$ (respectively edge e with $\partial e = \{v_1, v_2\} \in \mathscr{E}(\Gamma)$) let $\mathfrak{n}_v\mathbb{Z}$ (respectively $\mathfrak{d}_e[\mathfrak{n}_{v_1}, \mathfrak{n}_{v_2}]\mathbb{Z}$) be the maximal subgroup of $\mathbb{Z}$ which acts trivially on $p^{-1}(v_i)$ $(i = 1, 2)$ (respectively on $p^{-1}(e)$). This defines a system of strictly positive integers $(\mathbf{n}, \mathbf{d}) = \left\{\{\mathfrak{n}_v\}_{v \in \mathscr{V}(\Gamma)}; \{\mathfrak{d}_e\}_{e \in \mathscr{E}(\Gamma)}\right\}$.

It is called *covering data* of p.

(d) Two cyclic coverings $p_i : G_i \to \Gamma$ $(i = 1, 2)$ are equivalent, $(G_1 \sim G_2)$, if there is an equivariant isomorphism $q : G_1 \to G_2$ such that $p_2 \circ q = p_1$.

The set of equivalence classes of cyclic coverings of Γ, associated with a fixed system of covering data $(\mathbf{n}, \mathbf{d})$, is denoted by $\mathscr{G}(\Gamma, (\mathbf{n}, \mathbf{d}))$.

Remark 4.1.3 For any fixed $(\mathbf{n}, \mathbf{d})$ there exists a unique cyclic covering $p : T \to \Gamma$ (up to an equivariant isomorphism) which admits a (non-equivariant!) morphism of graphs $s : \Gamma \to T$ with $p \circ s = \mathrm{id}_\Gamma$. It is called the *trivial cyclic covering*.

In [475] it is proved that $\mathscr{G}(\Gamma, (\mathbf{n}, \mathbf{d}))$ has the structure of an abelian group, where the trivial cyclic covering is the unit element.

Example 4.1.4 ([475]) Fix a graph Γ and $(\mathbf{n}, \mathbf{d})$ as above.

(a) $\mathscr{G}(\Gamma, (\mathbf{n}, \mathbf{d}))$ is independent of $\mathbf{d}$.

(b) If Γ has k connected components $\{\Gamma_i\}_{i=1}^k$, then the covering data $(\mathbf{n}, \mathbf{d})$ induces a covering data on each component, and $\mathscr{G}(\Gamma, (\mathbf{n}, \mathbf{d})) = \oplus_{i=1}^k \mathscr{G}(\Gamma_i, (\mathbf{n}, \mathbf{d})|_{\Gamma_i})$.

(c) If $c(\Gamma) = 0$ then $\mathscr{G}(\Gamma, (\mathbf{n}, \mathbf{d})) = 0$ for any $(\mathbf{n}, \mathbf{d})$.

(d) Denote by $\mathscr{V}^1 := \{v \in \mathscr{V}(\Gamma) : \mathfrak{n}_v = 1\}$. Let $\Gamma_{\neq 1}$ be the subgraph of Γ obtained from Γ by deleting the vertices $\mathscr{V}^1$ and all edges which have at least one endpoint in $\mathscr{V}^1$. If each connected component of $\Gamma_{\neq 1}$ is a tree, then $\mathscr{G}(\Gamma, (\mathbf{n}, \mathbf{d})) = 0$.

(e) Let Γ be a *cyclic graph*. Set $\mathfrak{n} = \gcd\{\mathfrak{n}_v : v \in \mathscr{V}(\Gamma)\}$. Then $\mathscr{G}(\Gamma, (\mathbf{n}, \mathbf{d})) = \mathbb{Z}_{\mathfrak{n}}$.

(f) More generally, assume that Γ is a graph with $c_\Gamma = 1$. Let Γ' be the *minimal cyclic subgraph* of Γ, and set $\mathfrak{n} := \gcd\{n_v : v \in \mathscr{V}(\Gamma')\}$. Then $\mathscr{G}(\Gamma, \mathbf{n}) = \mathbb{Z}_{\mathfrak{n}}$.

Remark 4.1.5 If $p : G \to \Gamma$ is a cyclic covering then $c(G) \geq c(\Gamma)$. Indeed, if $|G|$ and $|\Gamma|$ denote the topological realizations of the corresponding graphs, then the invariant subspace $H_1(|G|, \mathbb{Q})^{\mathbb{Z}}$ is isomorphic to $H_1(|\Gamma|, \mathbb{Q})$.

4.1.6 The mod_N-Construction Fix an arbitrary graph Γ and system of integers $\mathbf{n} = \{\mathfrak{n}_v\}_{v\in\mathscr{V}}$ and $\mathbf{d} = \{\mathfrak{d}_e\}_{e\in\mathscr{E}}$ as above. For any strictly positive integer N, we introduce a new set of integers $(N, \mathbf{n}) := \{(N, \mathfrak{n}_v)\}_{v\in\mathscr{V}}$ and $\mathbf{d}'$, where $\mathfrak{d}'_e := (\mathfrak{d}_e, N/(N, [\mathfrak{n}_{v_1}, \mathfrak{n}_{v_2}]))$. In other words: if $\mathfrak{n}_e = \mathfrak{d}_e \cdot [\mathfrak{n}_{v_1}, \mathfrak{n}_{v_2}]$, then $(\mathfrak{n}_e, N) = \mathfrak{d}'_e \cdot [(\mathfrak{n}_{v_1}, N), (\mathfrak{n}_{v_2}, N)]$. Using these data we define a natural map

$$\mathrm{mod}_N : \mathscr{G}(\Gamma, (\mathbf{n}, \mathbf{d})) \to \mathscr{G}(\Gamma, ((N, \mathbf{n}), \mathbf{d}')).$$

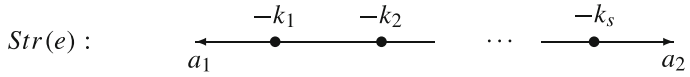
Let G be a representative of an element of $\mathscr{G}(\Gamma, (\mathbf{n}, \mathbf{d}))$. Then $\mathrm{mod}_N(G)$ is the *orbit graph* of the induced action of $N\mathbb{Z}$. More precisely, we introduce the equivalence relation $\overset{N}{\sim}$ on $\mathscr{V}(G)$, respectively on $\mathscr{E}(G)$: $v_1 \overset{N}{\sim} v_2$ (resp. $e_1 \overset{N}{\sim} e_2$) if there is an integer k such that $a_{\mathscr{V}}(kN, v_1) = v_2$ (resp. $a_{\mathscr{E}}(kN, e_1) = e_2$). Then $\mathrm{mod}_N(G) = G/\overset{N}{\sim}$.

4.1.7 Enriched Coverings We will apply the following extensions as well.

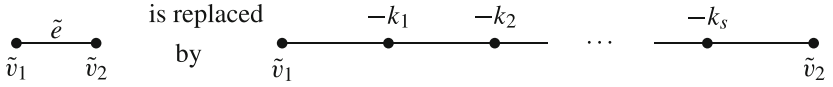
(1) Assume that the vertices of all our graphs have two types: arrowheads $\mathscr{A}$ and non-arrowheads $\mathscr{V}$, i.e. $\mathscr{W} = \mathscr{A} \sqcup \mathscr{V}$. Then in the definition of the coverings $p : G \to \Gamma$ we require: $\mathscr{A}(G) = p^{-1}(\mathscr{A}(\Gamma))$.

(2) Assume that our graphs have some decorations. Then for coverings $p : G \to \Gamma$ we require that the decoration of G must be equivariant.

(3) We might change (in an equivariant way) any edge of G into a string. If $\mathbf{Str} = \{Str(e)\}_{e\in\mathscr{E}(\Gamma)}$ is a collection of (oriented) strings (each with two end-arrows, see below), and $G \to \Gamma$ is an element of $\mathscr{G}(\Gamma, (\mathbf{n}, \mathbf{d}))$, then the new graph $G(\mathbf{Str})$ denotes that graph in which we replace equivariantly every edge of G above e by the corresponding string $Str(e)$. That is, if



then any



4.1.B *The Universal Covering of the Embedded Resolution Graph*

Fix a normal surface singularity (X, o) and a germ $f : (X, o) \to (\mathbb{C}, 0)$ of an analytic function. Let ϕ be a good embedded resolution with dual graph $\Gamma(X, f)$. For simplicity we assume that any two irreducible components of the exceptional divisor have at most one intersection point. The final goal of this subsection is to construct a cyclic covering of graphs $p : G(X, f) \to \Gamma(X, f)$, abbreviated as $p : G_f \to \Gamma_f$. ♣

4.1.8 The Construction of $p : G_f \to \Gamma_f$ via the Milnor Fibration of f [158, 475] We start with the following decomposition of the Milnor fiber.

Let T_v ($v \in \mathcal{V}$) be a small closed tubular neighborhood of E_v. Then for any edge $e = (v, u)$ connecting two non-arrowheads, the intersection $T_v \cap T_u$ is a bidisc T_e. If T_a, for $a \in \mathcal{A}$, is a small tubular neighborhood of the irreducible component S_a of the strict transform S, and a is adjacent to $v_a \in \mathcal{V}$, then corresponding to the edge $e = (a, v_a)$ we introduce the bidisc $T_e = T_a \cap T_{v_a}$. Set $T = \cup_{w \in \mathcal{W}} T_w$ ($\mathcal{W} = \mathcal{V} \cup \mathcal{A}$).

Next, consider the lifted fiber $F := (f \circ \phi)^{-1}(t)$. For sufficiently small $t > 0$ it sits in T. Set $F_w = F \cap T_w$ for any $w \in \mathcal{W}$, and $F_e = F \cap T_e$ for any $e \in \mathcal{E}$.

It is possible to choose the geometric monodromy acting on F in such a way that it preserves the subspaces $\{F_w\}_{w \in \mathcal{W}}$ and $\{F_e\}_{e \in \mathcal{E}}$. In fact, the connected components of F_w, respectively of F_e, are cyclically permuted by this action.

Let $\mathfrak{n}_w$ and $\mathfrak{n}_e$ be the number of connected components of F_w and F_e respectively. Then for any edge e with $\partial e = \{w_1, w_2\}$ we have $\mathfrak{n}_e = \mathfrak{d}_e \cdot [\mathfrak{n}_{w_1}, \mathfrak{n}_{w_2}]$ for some $\mathfrak{d}_e \geq 1$.

The covering $p : G_f \to \Gamma_f$, with these covering data, is constructed as follows. Above a vertex $w \in \mathcal{W}(\Gamma_f)$ there are $\mathfrak{n}_w$ vertices of G_f which correspond to the connected components of F_w. If w is an arrowhead in Γ then all the vertices above w are arrowheads. Above an edge e of Γ_f, there are $\mathfrak{n}_e$ edges of G_f: they correspond to the connected components of F_e. The $\mathbb{Z}$ action is generated by the monodromy.

Fix an edge $\tilde{e}$ and a vertex $\tilde{w}$ of G_f, which correspond to the connected component $F_{\tilde{e}}$ and $F_{\tilde{w}}$ respectively. Then, by definition, $\tilde{w}$ is an end of $\tilde{e}$ if and only if $F_{\tilde{e}} \subset F_{\tilde{w}}$. In particular, if $\tilde{w}_1, \tilde{w}_2$ lift the ends of e, then $F_{\tilde{w}_1} \cap F_{\tilde{w}_2}$ has $\mathfrak{d}_e$ connected components, and $\tilde{w}_1$ and $\tilde{w}_2$ are connected in G_f exactly by $\mathfrak{d}_e$ edges.

$G_f = G(X, f)$ is called the *universal cyclic covering of* $\Gamma(X, f)$.

Lemma 4.1.9 *The number of connected components of the graph G_f and of the Milnor fiber F agree. This number is $|coker(arg_*)|$. In particular, G_f is connected whenever* $\gcd\{m_w : w \in \mathcal{W}\} = 1$.

This fact has the following generalization. Fix a *connected* subgraph Γ' of Γ_f with non-arrowhead vertices $\mathcal{V}'$. Since the intersection form associated with the exceptional divisors $E' := \cup_{v \in \mathcal{V}'} E_v$ is negative definite, by Grauert theorem [223]

$E' \subset \widetilde{X}$ can be contracted to a singular point (X', o'). Moreover, since $f \circ \phi$ is zero along E', it gives rise to a germ f' defined on (X', o'). Let the corresponding representation be denoted by arg'_*. It is obvious, that Γ' (complemented with the corresponding induced arrowheads) is a possible embedded resolution graph of f'.

Proposition 4.1.10 *Fix a germ f, its embedded resolution graph Γ_f with multiplicities $\{m_w\}_w$ and genera $\{g_v\}_v$, and the universal cyclic covering $p : G_f \to \Gamma_f$. For $v \in \mathcal{V}(\Gamma)$, let $\mathcal{W}_v$ denote the set of adjacent vertices $w \in \mathcal{W}(\Gamma)$ of v.*

(a) For any connected subgraph Γ' of Γ_f, $p^{-1}(\Gamma')$ has $|coker(arg'_*)|$ connected components.

(b) In particular, for any $v \in \mathcal{V}(\Gamma)$ with $g_v = 0$, the cardinality $\mathfrak{n}_v$ of the set $p^{-1}(v)$ is $\gcd\{m_w : w \in \mathcal{W}_v \cup \{v\}\}$. Similarly, for any $a \in \mathcal{A}(\Gamma)$, the cardinality $\mathfrak{n}_a$ of $p^{-1}(a)$ is $\gcd(m_a, m_{v_a})$, where v_a is the adjacent non-arrowhead to a. Finally, the number of edges $\mathfrak{n}_e$ above e with end-vertices w_1, w_2 is $\gcd(m_{w_1}, m_{w_2})$.

In contrast, for $v \in \mathcal{V}(\Gamma)$ with $g_v > 0$ for $|p^{-1}(v)| := \mathfrak{n}_v$ only the following divisibility is guaranteed: $\mathfrak{n}_v \mid \gcd\{m_w : w \in \mathcal{W}_v \cup \{v\}\}$.

Example 4.1.11 If L_X is a rational homology sphere, then Γ_f is a tree with all $g_w = 0$. Therefore, by Proposition 4.1.10, the covering data $(\mathbf{n}, \mathbf{d})$ of $G_f \to \Gamma_f$ is determined by Γ_f. Furthermore, by 4.1.4(c), $\mathcal{G}(\Gamma_f, (\mathbf{n}, \mathbf{d})) = 0$. Hence Γ_f determines completely (and explicitly) the covering $G_f \to \Gamma_f$.

Example 4.1.12 Assume that Γ_f is a tree. Then, again, $\mathcal{G}(\Gamma_f, (\mathbf{n}, \mathbf{d})) = 0$, hence the covering is determined from Γ_f once we know $\mathfrak{n}_v$ for all $v \in \mathcal{V}(\Gamma)$ with $g_v > 0$. Nevertheless, $\mathfrak{n}_v$ for $g_v > 0$, in general, cannot be determined from Γ_f.

Indeed, set $(X, o) = (\{x^2 + y^7 - z^{14} = 0\}, 0)$ and take $f_1(x, y, z) = z^2 + y^2$ and $f_2(x, y, z) = z^2 - y + y^2$ as in 3.4.8. Then $\Gamma(X, f_1) = \Gamma(X, f_2)$, this graph is a tree, and it has a vertex with $g_v = 3$. The two graphs $G(X, f_i)$ are different, in the first case v is covered by two vertices, while in the second case by only one [475]. Hence $G(X, f_1)$ is a tree, while $c(G(X, f_2)) = 1$.

Example 4.1.13 Assume that $c(\Gamma_f) > 0$, but $g_v = 0$ for all v. Then the covering data $(\mathbf{n}, \mathbf{d})$ is completely determined by Proposition 4.1.10, hence each covering is an element of $\mathcal{G}(\Gamma_f, (\mathbf{n}, \mathbf{d}))$. Nevertheless, this set can have more than one element.

E.g., set $(X, o) = (\{z^2 + (x^2 - y^3)(x^3 - y^2) = 0\}, 0)$ and $f_1 = x^2 + y^4$ and $f_2 = x^2 - y^3 + y^4$ as in 3.4.8. Then again $\Gamma(X, f_1) = \Gamma(X, f_2)$, $g_v = 0$ for each v, nevertheless $c(\Gamma) = 1$. The two graphs $G(X, f_i)$ are different: in the first case the cycle of Γ is trivially double covered, while in the second case non-trivially double covered.

Remark 4.1.14 The author does not know which elements of $\mathcal{G}(\Gamma, (\mathbf{n}, \mathbf{d}))$ can be realized as $G_f \to \Gamma_f = \Gamma$ provided by some holomorphic germ f.

4.1.C Ramified Cyclic Coverings of Germs

4.1.15 Let (X, o) be a normal surface singularity and let $f : (X, o) \to (\mathbb{C}, 0)$ be the germ of an analytic function. For any integer $N \geq 1$, take $b : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$, $z \mapsto z^N$, and let $X_{f,N}$ be the normalization of the fiber product $\{(x, z) \in (X \times \mathbb{C}, o \times 0) : f(x) = z^N\}$. The second projection $(x, z) \mapsto z$ induces an analytic map $X_{f,N} \to \mathbb{C}$, still denoted by z. The first projection $(x, z) \mapsto x$ gives rise to a ramified cyclic $\mathbb{Z}_N$-covering $pr : X_{f,N} \to X$, ramified along $V(f)$. If f defines an isolated singularity then $X_{f,N}$ is irreducible; but, in general, it is a multigerm.

The $\mathbb{Z}_N = \{\xi \in \mathbb{C} : \xi^N = 1\}$ action of $X_{f,N}$ is induced by $\xi * (x, z) \mapsto (x, \xi z)$. The covering is guided by the monodromy representation $\rho_N : \pi_1(X \setminus V(f)) \to \mathbb{Z}_N$ of the regular covering $X_{f,N} \setminus V(z) \to X \setminus V(f)$. Since $X \setminus V(f)$ is connected and $\mathbb{Z}_N$ is abelian, ρ_N factorizes through $r_N : H_1(L_X \setminus L_{V(f)}, \mathbb{Z}) \to \mathbb{Z}_N$.

Let $arg_* : H_1(L_X \setminus L_{V(f)}, \mathbb{Z}) \to \mathbb{Z}$ be the morphism induced by f and $\mathrm{mod}_N : \mathbb{Z} \to \mathbb{Z}_N$ the natural projection. Then $r_N = \mathrm{mod}_N \circ arg_*$. Hence, if the Milnor fiber F of f has k connected components, then $X_{f,N}$ is a multigerm $(X_{f,N}, \{o_1, \ldots, o_{(k,N)}\})$ with $\gcd(k, N)$ connected components. They are isomorphic and are permuted by the action. By definition, the resolution graph $\Gamma(X_{f,N})$ is the union of the resolution graphs of $(X_{f,N}, o_i)_{i=1}^{(k,N)}$ (similarly for the embedded resolution graph $\Gamma(X_{f,N}, z)$).

In the sequel we provide an algorithm how G_f and N determine $\Gamma(X_{f,N}, z)$.

4.1.16 The Embedded Resolution Graph of $\Gamma(X_{f,N}, z)$ Recall that the universal graph covering $p : G_f \to \Gamma_f$ is an element of $\mathcal{G}(\Gamma_f, (\mathbf{n}, \mathbf{d}))$, where $\mathfrak{n}_w = |p^{-1}(w)|$ and $\mathfrak{n}_e = \mathfrak{d}_e[\mathfrak{n}_{w_1}, \mathfrak{n}_{w_2}] = |p^{-1}(e)|$ for any $e \in \mathcal{E}(\Gamma)$ with $\partial e = \{w_1, w_2\}$. Moreover, the graph Γ_f also has some decorations: multiplicities $\{m_w\}_{w \in \mathcal{W}}$, genera $[g_v]_{v \in \mathcal{V}(\Gamma)}$ (and Euler numbers, which can be recovered from the multiplicities, and will be neglected in this algorithm).

For any fixed integer $N \geq 1$, we construct a new graph in four steps.

Step 1. We take the 'orbit covering' $\mathrm{mod}_N(G_f) \to \Gamma_f$ of the subgroup $N\mathbb{Z} \subset \mathbb{Z}$ of G_f, as an element of $\mathcal{G}(\Gamma_f, ((N, \mathbf{n}), \mathbf{d}'))$. Here, $(N, \mathbf{n}) = \{\gcd(N, \mathfrak{n}_w)\}_{w \in \mathcal{W}(\Gamma_f)}$, and $\mathfrak{d}'_e = (\mathfrak{d}_e, N/(N, [\mathfrak{n}_{w_1}, \mathfrak{n}_{w_2}]))$ for any edge e with $\partial e = (w_1, w_2)$, cf. 4.1.6.

Step 2. We put some decorations on $\mathrm{mod}_N(G_f)$ as follows.

(a) The multiplicity $(m_{\tilde{w}})$ of any vertex $\tilde{w} \in \mathcal{W}(\mathrm{mod}_N(G_f))$, which lies above $w \in \mathcal{W}(\Gamma_f)$, is $m_w/(m_w, N)$.

(b) The genus $[\tilde{g}_{\tilde{v}}]$ of any vertex $\tilde{v} \in \mathcal{V}(\mathrm{mod}_N(G_f))$, which lies above $v \in \mathcal{V}(\Gamma_f)$ and has genus g_v and multiplicity m_v, is given by

$$2 - 2\tilde{g}_{\tilde{v}} = \frac{(2 - 2g_v - \kappa_v) \cdot \gcd(m_v, N) + \sum_{w \in \mathcal{W}_v(\Gamma_f)} \gcd(m_w, m_v, N)}{\gcd(N, \mathfrak{n}_v)},$$

where $\mathcal{W}_v(\Gamma_f)$ is the set of adjacent vertices of v and $\kappa_v = |\mathcal{W}_v(\Gamma_f)|$ is its cardinality.

Step 3. Any edge $\tilde{e}$ of $\mathrm{mod}_N(G_f)$, with end-vertices $\tilde{w}_1$ and $\tilde{w}_2$, lying above e with end-vertices w_1 and w_2, will be replaced by a string $\overline{Str}(e)$ as follows (cf. 4.1.7(3)).

If $t_e = \gcd(m_{w_1}, m_{w_2}, N)$, then set $N' = N/t_e$ and $m'_{w_i} = m_{w_i}/t_e$ for $i = 1, 2$. Consider the unique $0 \le \lambda < N'/(m'_{w_1}, N')$ and $m_1 \in \mathbb{N}$ with:

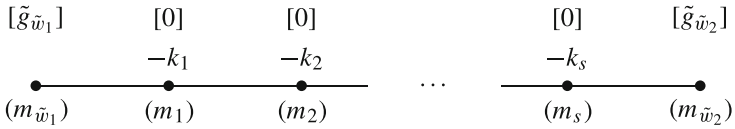
$$m'_{w_2} + \lambda \cdot \frac{m'_{w_1}}{(m'_{w_1}, N')} = m_1 \cdot \frac{N'}{(m'_{w_1}, N')}. \tag{4.1.1}$$

If $\lambda = 0$, then the edge $\tilde{e}$ remains unchanged.

If $\lambda \neq 0$, then take the continuous fraction:

$$\frac{N'/(m'_{w_1}, N')}{\lambda} = [[k_1, k_2, \ldots, k_s]], \quad k_1, \ldots, k_s \ge 2.$$

If both w_1 and w_2 are non-arrowhead vertices, then $\overline{Str}(e)$ is the string



with genera [0], self intersection numbers $-k_1, \ldots, -k_s$, and multiplicities $m_1, \ldots, m_s$. (In the diagram we inserted also the genera $[\tilde{g}_{\tilde{w}_1}]$, $[\tilde{g}_{\tilde{w}_2}]$ and multiplicities $(m_{\tilde{w}_1})$, $(m_{\tilde{w}_2})$ of the vertices $\tilde{w}_1$ and $\tilde{w}_2$ already determined in step 2.) The integer m_1 is the number given by the the congruence (4.1.1). The multiplicities $m_2, \ldots, m_s$ are determined using (2.2.1), namely $m_2 = k_1 m_1 - m_{\tilde{w}_1}$, and $m_{i+1} = k_i m_i - m_{i-1}$ for $i \ge 2$.

Then any edge $\tilde{e}$ of $\mathrm{mod}_N(G_f)$, lying above e, is replaced by the string $\overline{Str}(e)$. Furthermore, if $w_2 \in \mathscr{A}(\Gamma_f)$, then we do a similar replacement, with the only modification that in the string $\overline{Str}(e)$ given above the right vertex corresponding to $\tilde{w}_2$ is replaced by an arrowhead with only one decoration, its multiplicity $(m_{\tilde{w}_2})$.

This new graph is denoted by $\mathrm{mod}_N(G_f)(\mathbf{Str})$.

Step 4. The decoration of $\mathrm{mod}_N(G_f)(\mathbf{Str})$ is not complete: the Euler numbers corresponding to the vertices $\tilde{v}$ of $\mathrm{mod}_N(G_f))$ are missing. We determine them by (2.2.1).

Theorem 4.1.17 ([475]) *The decorated graph* $\mathrm{mod}_N(G_f)(\mathbf{Str})$ *is (a possible) embedded resolution graph* $\Gamma(X_{f,N}, z)$. *In particular, if we delete all the arrows and multiplicities of* $\mathrm{mod}_N(G_f)(\mathbf{Str})$, *then we obtain a resolution graph* $\Gamma(X_{f,N})$ *of* $X_{f,N}$.

The proof is based on the identification of the quotient singularities appearing in the 'Jungian diagram' (cf. 2.4.2), for details see [475].

The construction, in general, gives a non-minimal resolution graph.

For the appreciation of the role of G_f (versus Γ_f), the reader is invited to construct (say) the double covers of the examples from 4.1.12 and 4.1.13.

4.1.18 The Case When L_X Is a Rational Homology Sphere If L_X is a rational homology sphere, then the covering $G_f \to \Gamma_f$ can be reconstructed from Γ_f, cf. Theorem 4.1.10. In particular, $\Gamma(X_{f,N}, z)$ is completely determined by the embedded resolution graph Γ_f of f and the integer N.

In particular, if $(X, o) = (\mathbb{C}^2, 0)$, and f is a plane curve singularity, then $X_{f,N}$ is the normalization of $(\{f(x, y) = z^N\}, 0) \subset (\mathbb{C}^3, 0)$. If f defines an isolated singularity, then $(\{f(x, y) = z^N\}, 0)$ is automatically normal.

For different particular cases of the algorithm see [22, 358, 472, 536, 541].

Remark 4.1.19 Using the formula of (Step 2), one shows that $\tilde{g}_{\tilde{v}} \geq g_v$. Moreover, $c(\mathrm{mod}_N(G_f)) \geq c(\Gamma_f)$, see 4.1.5. Hence rank $H_1(L_{X_{f,N}}) \geq$ rank $H_1(L_X)$ too.

Remark 4.1.20 Assume that L_X is a rational homology sphere link, and let $f : (X, o) \to (\mathbb{C}, 0)$ be as above. Let $g : (X, o) \to (\mathbb{C}, 0)$ be another holomorphic germ, and we assume that Γ_f might serve as the embedded resolution graph of g as well (with different set of arrowheads and multiplicities). Then the above algorithm, which produces $\Gamma(X_{f,N})$ form Γ_f and N, can be improved in order to obtain on the graph $\Gamma(X_{f,N})$ the divisor of germ $g \circ pr : X_{f,N} \to \mathbb{C}$ as well.

4.1.D Algebraic Monodromy and Cyclic Coverings

We relate the algebraic monodromy of $f : (X, o) \to (\mathbb{C}, 0)$ with the properties of the universal covering $G(X, f) \to \Gamma(X, f)$ and also with induced properties of the resolution graphs of cyclic coverings $\{X_{f,N}\}_N$. ♣

4.1.21 As the construction of G_f shows, the fibration of f and the structure of the cyclic covering is strongly related. This has an effect on the algebraic monodromy of f as well. In this section we support this fact. We adopt the notations of 3.2.A.

If the Milnor fiber F has k connected components, then $H_0(F) \simeq \mathbb{Z}^k$, $h_0(x_1, \ldots, x_k) = (x_2, \ldots, x_k, x_1)$ and $\Delta_0(t) = 1 - t^k$.

Proposition 4.1.22

(a) The characteristic polynomial $\Delta_1(t)$ is determined by the graph Γ_f of a good resolution via A'Campo's formula [6, 7]:

$$\zeta(t) = \Delta_0(t)/\Delta_1(t) = \prod_{v\in\mathscr{V}} (1-t^{m_v})^{2-2g_v-\kappa_v(\Gamma_f)},$$

where $\kappa_v(\Gamma_f)$ is the number of edges adjacent to v in Γ_f.

In particular, if F is connected then $1-\mu(f) = \sum_{v\in\mathscr{V}} m_v(2-2g_v-\kappa_v(\Gamma_f))$.

(b) The maximal size of the Jordan block is 2.

Proof We use the decomposition 4.1.8. Note that F has the homotopy type of $\cup_{v\in\mathscr{V}} F_v$. We also write $F'_v := F_v \setminus \cup_{w\in\mathscr{W},\, w\neq v} F_w$ for $v \in \mathscr{V}$.

(a) follows from the multiplicativity property of the zeta-function with respect to this decomposition.

(b) The m_v power of the restriction of h_{geo} to F'_v is homotopic to the identity. Therefore, if $M = \mathrm{lcm}_v\{m_v\}$, then $\mathrm{im}(h_1^M - I)$ is in the subspace $\mathrm{im}\,(\cup_{e\in\mathscr{E}} H_1(F_e) \to H_1(F))$ of $H_1(F)$. But each F_e is homotopic to a union of circles, which are cyclically permuted by the h_{geo}. Hence $h_1^M|\mathrm{im}(h_1^M - I)$ is the identity.

□

4.1.23 Derivatives of Characteristic Polynomials/Rational Functions In several (e.g. surgery) formulae we will use certain identities satisfied by the derivatives of Δ_1 at $t = 1$. Here are the needed elementary algebraic statements.

Assume that Δ is a rational function of type $\prod_i (1-t^{m_i})^{k_i}$ ($m_i \in \mathbb{Z}_{>0}$, $k_i \in \mathbb{Z}$) such that $\sum_i k_i = 0$. Then the derivative $\Delta'(t)$ of $\Delta(t)$ satisfies $\Delta'(1)/\Delta(1) = \deg(\Delta)/2$.

Furthermore, define the symmetrized rational function $\Delta_s(t) := t^{-\mu/2}\cdot\Delta(t)$, where $\mu := \deg(\Delta)$. Then $\Delta_s(1) = \Delta(1)$, $\Delta'_s(1) = 0$, and

$$\Delta''(1)/\Delta(1) - \mu(\mu-2)/4 = \Delta''_s(1)/\Delta_s(1) = \tfrac{1}{12}\cdot\textstyle\sum_i k_i m_i^2.$$

4.1.24 For an interpretation of Δ_1 **as the Alexander polynomial of the link** $L_{V(f)} \subset L_X$ see [178], or 9.1.2 here. For some additional properties and special formulae in the case of plane curve singularities see 5.2.2 and 5.3.20.

4.1.25 We write $H_{1,\lambda}$ for the generalized eigenspace corresponding to the eigenvalue λ of h_1, and let $\#^l_\lambda$ be the number of Jordan blocks of size l of the restriction $h_1|_{H_{1,\lambda}}$. Obviously, $\#^1_\lambda + 2\#^2_\lambda = \dim(H_{1,\lambda})$ and $\dim(\ker(h_1|_{H_{1,\lambda}} - \lambda I)) = \#^1_\lambda + \#^2_\lambda$.

For $\lambda = 1$ A'Campo's formula and the Wang exact sequence (cf. 3.2.14) give the following.

Corollary 4.1.26 ([500]) *For a good embedded resolution one has:*

(a) $\dim(H_{1,1}) = 2g(\Gamma_f) + 2c(\Gamma_f) + |\mathscr{A}| - 1$;
(b) $\dim(\ker(h_1 - 1)) = 2g(\Gamma_f) + c(\Gamma_f) + |\mathscr{A}| - 1$.
Therefore, $\#_1^2 = c(\Gamma_f) = c(\Gamma_X)$. *In particular,* $\#_1^2$ *is independent of the germ* f, *and depends only on the topology of the link* L_X.
(c) Assume that f *defines an isolated singularity with* $|\mathscr{A}| = 1$, *and* L_X *is a rational homology sphere. Then* $\Delta_1(1) = |\mathrm{coker}(h_1 - 1)| = |\mathrm{coker}(I)|$. *Hence* $\Delta_1(1)$ *is independent of* f, *it depends only on* L_X.

The number of Jordan blocks associated with eigenvalues $\lambda \neq 1$, in general, cannot be determined from Γ_f. Assume e.g. that F (or, the graph G_f) is connected. Then $X_{f,N}$ is also connected for any N. Moreover,

$$\sum_{\lambda^N=1} \#_\lambda^2(f) = \#_1^2(z) = c(\Gamma(X_{f,N})) = c(\mathrm{mod}_N(G_f)). \tag{4.1.2}$$

This says that h_1 has finite order if and only if there is no cycle in any of the cyclic coverings of $(X, V(f))$ (fact noticed in [161]). Hence $\#_\lambda^2(f)$ follows by Möbius inversion. (4.1.2) combined with 4.1.C gives $\sum_\lambda \#_\lambda^2(f) = c(G_f)$ as well.

The combinatorial $\mathbb{Z}$ action on G_f can be interpreted topologically as well. Let $|G_f|$ be the topological realization of G_f. The generator of $\mathbb{Z}$ acts on $|G_f|$ too. At homological level, this induces a finite morphism on $H_1(|G_f|)$. Let $H_1(|G_f|)_\lambda$ be the corresponding λ-eigenspace. Then, $\#_\lambda^2(f) = \dim H_1(|G|)_\lambda$, see [475].

Example 4.1.27 In the situation of 4.1.12 $h_1(f_1)$ is finite, but $h_1(f_2)$ has a Jordan block of size 2, with eigenvalue $\lambda = -1$. In the case of 4.1.13 both operators for $i = 1, 2$ have the same structure. That is, $h_1(f_i)$ has a Jordan block of size 2 with eigenvalue $\lambda = 1$, because Γ_f has a cycle. Since G_f has two independent cycles, there is one more Jordan block of size 2 and this one has eigenvalue $\lambda = -1$.

Example 4.1.28 (The Case When L_X *Is a Rational Homology Sphere. [178, 508])*
In this case $c(\Gamma(X_{f,N})$ can be computed merely from Γ_f:

$$\sum_{\lambda^N=1} \#_\lambda^2(f) = \sum_{e \in \mathscr{E}_n(\Gamma_f)} ((\mathfrak{n}_e, N) - 1) - \sum_{v \in \mathscr{V}(\Gamma_f)} ((\mathfrak{n}_v, N) - 1) = c(\Gamma_{f,N}), \tag{4.1.3}$$

where $\mathscr{E}_n$ is the set of edges connecting two non-arrowheads.

Remark 4.1.29 The integers $\{\#_\lambda^2(f)\}_\lambda$ are not totally independent of each other. E.g., if f defines an isolated singularity and $c(\Gamma_X) = 0$ then one has the next fact [500]. Assume that $\#_\lambda^2 \neq 0$ for λ with $\lambda^k = 1$. Write $k = pq$ with $\gcd(p, q) = 1$. Then there exists certain η with $\#_\eta^2 \neq 0$ such that either $\eta^p = 1$ or $\eta^q = 1$.

4.1.30 Examples When h_1 Is Finite In [369] Lê Dũng Tráng proved that the monodromy of an *isolated irreducible* germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ is always finite.

In Lê's result the irreducibility assumption is really essential. Indeed, if we take the non-irreducible germ $f(x, y) = (x^2 + y^3)(x^3 + y^2)$, then the resolution of the double covering has a cycle, hence $h_1(f)$ has Jordan block of size 2 with $\lambda = -1$ [6].

Similarly, if $f : (X, o) \to (\mathbb{C}, 0)$ has a finite monodromy $h_1(f)$, then $c(\Gamma_X)$ necessarily must be zero as well (see 4.1.26).

Proposition 4.1.31 ([500]) *Assume that L_X is a rational homology sphere and $f : (X, o) \to (\mathbb{C}, 0)$ defines an isolated singularity with $|\mathscr{A}(\Gamma)| = 1$. Then, if $\#^2_\lambda \neq 0$ for some λ of order k, then there exists a prime number $p \geq 2$ such that $p|k$ and $p^2| \, |H_1(L_X, \mathbb{Z})|$.*

In particular, if $|H_1(L_X, \mathbb{Z})|$ is square free, then $h_1(f)$ is finite for any f.

The above result is sharp. Indeed, take $(X, o) = (\{x^{10} + x^3y^2 + y^3 + z^2 = 0\}, o)$ in $(\mathbb{C}^3, 0)$ and $f(x, y, z) = x$. Then $|H_1(L_X, \mathbb{Z})| = 4$ and h_1 contains one Jordan block of size 2 with eigenvalue -1. (See [500].)

4.1.E o–Ramified Cyclic Coverings

We say that a finite map $(Y, o) \to (X, o)$ of normal surface singularities is a (Galois) o–covering if the restriction $Y \setminus \{o\} \to X \setminus \{0\}$ is a regular (i.e. unramified) (Galois) covering. Any surjective character $\alpha : H_1(L_X, \mathbb{Z}) \to \mathbb{Z}_N$ provides a Galois o–covering $(X_\alpha, o) \to (X, o)$ with Galois group $\mathbb{Z}_N$. In this subsection we extend the previous algorithms to obtain the plumbing graph of L_{X_α}. ♣

4.1.32 The Setup of an o-Ramified Cyclic Covering In the next discussion it is instructive to distinguish the next two related levels. In the topological setup we start with a negative definite plumbing graph Γ of a plumbed 3–manifold and we consider $H_1 := H_1(M(\Gamma), \mathbb{Z})$. In the analytic situation we start with a normal surface singularity (X, o) with link L_X and we take $H_1 := H_1(L_X, \mathbb{Z})$.

In both cases, we fix a character $\alpha \in \widehat{H_1}$ of finite order $N > 1$, or equivalently, an epimorphism $\alpha : H_1 \to \mathbb{Z}_N$. In the analytic context, by covering theory (cf. 12.3), α determines a unique normal surface singularity (X_α, o), and a Galois covering $(X_\alpha, o) \to (X, o)$, which is regular above $X \setminus \{o\}$. At the level of links this induces a regular Galois $\mathbb{Z}_N$–covering $L_{X_\alpha} \to L_X$ of smooth manifolds. In the topological context we can consider ('merely') the regular Galois $\mathbb{Z}_N$-covering M_α of $M(\Gamma)$ associated with the representation $\pi_1(M(\Gamma)) \to H_1 \xrightarrow{\alpha} \mathbb{Z}_N$.

The topological type of (X_α, o) depends only on L_X and on the character α. The analytic structure of (X_α, o) is the 'pullback' of the analytic structure of (X, o); in particular, it depends only on the topological covering and the analytical type of (X, o). Hence, *for fixed (X, o) and N*, the following data are equivalent: (1) a $\mathbb{Z}_N$

analytic o-ramified covering $(X_\alpha, o) \to (X, o)$, or (2) an epimorphism $\alpha : H_1 \to \mathbb{Z}_N$.

The main aim is to determine the graph Γ_{X_α} (or, in the topological context, the plumbing graph of M_α). Or, even more, if we know the analytic type/equations of (X, o) then we wish to determine the equations of X_α as well.

Case 1 Let us start with the following observation. Assume first that we have an analytic germ $f : (X, o) \to (\mathbb{C}, 0)$ such that $\mathrm{mod}_N \circ arg_* : H_1(L_X \setminus L_{V(f)}) \to \mathbb{Z}_N$ is onto. Then, from 3.4.6, if $\mathrm{mod}_N \circ arg_*|_{\mathbb{Z}^{|\mathscr{A}|}} = 0$ (that is, $N|m_a$ for every $a \in \mathscr{A}$), then $\mathrm{mod}_N \circ arg_*$ induces a well-defined character $\alpha \in \widehat{H_1}$ of order N.

Next, for our fixed character $\alpha \in \widehat{H_1}$ of order N, assume that we can find a function f such that $\mathrm{mod}_N \circ arg_*|_{\mathbb{Z}^{|\mathscr{A}|}} = 0$ and the induced character on H_1 is exactly α. Then we claim that (X_α, o) is exactly $(X_{f,N}, o)$. Hence in this way we obtain explicitly even the analytic singularity (X_α, o) (even at equation level).

In order to prove the claim we have to show that $(X_{f,N}, o) \to (X, o)$ is unramified along the generic points of $V(f)$. In a neighbourhood U of a generic point of $V(f)_a$ the covering can be modelled as follows. Let us choose in $(\mathbb{C}^2, o)$ some local coordinates (x, y), where $\{x = 0\} = V(f)_a \cap U$. Then its $\mathbb{Z}_N$–covering is the normalization of $\{(x, y, z) : x^{m_a} = z^N\}$, which is unramified over U (since $N|m_a$).

Since X_α depends only on α, the isomorphism class of the singularity $(X_{f,N}, o)$ is independent of the choice of f (under the properties considered above). The plumbing graph of L_{X_α} is determined via the algorithm from 4.1.C. In particular, L_{X_α} is determined from the multiplicity system of Γ_f and from the covering data of $G_f \to \Gamma_f$. Again, the output of the algorithm is independent of the choice of f.

Case 2 If we are interested merely in the topology of X_α (or, if it is hard to find some germ f, or, if its existence is even obstructed by the analytic type of (X, o)) then in order to find Γ_{X_α} we proceed by the topological parallel of **Case 1**.

Namely, we find an open book decomposition (L_X, L, p) with binding L (the analogue of $V(f)$), with fiber F, geometric monodromy h_{geo} and plumbing graph Γ, such that $\mathrm{mod}_N \circ p_*|_{\mathbb{Z}^{|\mathscr{A}|}} = 0$ and the induced character of H_1 is α. Then L_{X_α} is the $\mathbb{Z}_N$-covering of (L_X, L, p): the total space of the open book with the same fiber F and geometric monodromy h_{geo}^N. The plumbing grap Γ_{X_α} can be computed via the algorithm 4.1.C from the multiplicity system of (L_X, L, p) and from the covering data of $G \to \Gamma$ (constructed similarly as $G_f \to \Gamma_f$ from the open book decomposition).

We make this construction explicit in 4.1.33–4.1.35 whenever L_X is a rational homology sphere (hence $G \to \Gamma$ is not needed, cf. 4.1.11). In this case Γ_{X_α} can be determined from the multiplicity system of Γ and from N.

4.1.33 Assume that $H_1(L_X, \mathbb{Q}) = 0$, hence $H := L'/L = \mathrm{Tors}(H_1)$ equals H_1.

Then α becomes a character of H. Since $\theta : L'/L = H \to \widehat{H}$ is an isomorphism, there exists $l' \in L'$ such that $\theta([l']) = \alpha$. A possible l' can be found as follows.

If D_v is a transversal disc to E_v, then the homology classes $\{[\partial D_v]\}_v$ generate H (see 3.4.3 and note that $[D_v] = -[E_v^*]$ in L'). Let us write $\alpha([\partial D_v]) = e^{2\pi i m_v/N}$ for some integers m_v. We set $l' := \sum_v m_v E_v/N \in L \otimes \mathbb{Q}$. Taking the composition $\alpha' : L' \to H \xrightarrow{\alpha} \mathbb{Z}_N$, for any v one has

$$\alpha'(E_v^*) = \alpha(-[\partial D_v]) = e^{-2\pi i m_v/N} = e^{2\pi i (l', E_v^*)_{\mathbb{Q}}},$$

i.e., $\alpha'(\,\cdot\,) = e^{2\pi i (l', \,\cdot\,)_{\mathbb{Q}}}$. Since $\alpha'|_L = 1$ we obtain that $l' \in L'$ and $\theta([l']) = \alpha$.

Note that the choice of the integers $\{m_v\}_v$ above is not unique.

Lemma 4.1.34 *Assume that $H_1(L_X, \mathbb{Q}) = 0$, fix a character $\alpha \in \widehat{H}$, and construct a possible $l' \in L'$ as in 4.1.33. Set also $m_v^* := (l', E_v) \in \mathbb{Z}$ for every v. Then there is a choice of the integers $\{m_v\}_v$ such that $m_v > 0$ and $m_v^* \le 0$ for all $v \in \mathscr{V}$.*

Proof Let $o_u \in \mathbb{Z}_{>0}$ be the order of $[E_u^*]$ in H, hence $o_u E_u^* = \sum_v c_{uv} E_v$ for some positive integers c_{uv}, cf. 2.1.19. Then in $H_2(\widetilde{X}, \partial \widetilde{X})$ one has $-o_u D_u = \sum_v c_{uv} E_v$, hence adding $\sum_v c_{uv} E_v$ to l' the coefficient m_v is modified into $m_v + N c_{uv}$ for every v, while m_u^* into $m_u^* - o_u$. Repeated application of this step proves the statement. □

Assume that the choice of l' in 4.1.33 satisfies the properties of the above lemma. For any fixed vertex v, if $m_v^* < 0$ then we put a unique arrowhead $a = a(v)$ on the graph, supported by v, with multiplicity $m_a := -N m_v^*$.

In particular, we constructed a plumbing graph Γ with non-arrowhead vertices $\mathscr{V}(\Gamma_X)$ and arrowheads $\mathscr{A}$, and a multiplicity system $\{m_v\}_{v \in \mathscr{V}(\Gamma_X)}$, $\{m_a\}_{a \in \mathscr{A}}$.

Theorem 4.1.35 *Assume that $H_1(L_X, \mathbb{Q}) = 0$. Then a possible resolution graph of (X_α, o) is given by the algorithm 4.1.16 (the version applied in the situation when L_X is rational homology sphere, cf. 4.1.18) where the multiplicity system of the divisor of f should be replaced by the system $\{m_v\}_{v \in \mathscr{V}(\Gamma_X)}$, $\{m_a\}_{a \in \mathscr{A}}$ defined above, and the covering degree is N. The multiplicity system on the new graph (the analogs of the multiplicities of z) are used to determine some of the Euler decorations of the covering graph, but at the end must be deleted together with all the arrowheads of the covering graph.*

Proof The proof is similar to the proof of Theorem 4.1.17, if we notice that analytic realization of the divisor $Nl' - \sum_v N m_v^* D_v$ (as the divisor of an analytic function f) is not essential in the construction of the topological covering $L_{X_\alpha} \to L_X$. [Hence, naturally, the algorithm does not guarantee the analytic realization of the output divisor, the analogue of the divisor of z in Theorem 4.1.17.] □

The above discussion shows that if $H_1(L_X, \mathbb{Q}) = 0$ then any $l' \in L'$ provides a well-defined topological o–ramified covering $\Gamma_\alpha \to \Gamma$ associated with $\theta([l']) \in \widehat{H}$. This fact will provide later the *topological canonical covering*, cf. 6.1.10, 6.3.22 and 6.1.19.

Remark 4.1.36 The coverings associated with l' and $-l'$ agree. Indeed, the corresponding characters $\theta([l'])$ and $\theta([-l']) = \overline{\theta([l'])}$ have the same kernel.

Example 4.1.37 (Quotient-Cusps) Consider the following graph Γ, with all $b_i \geq 2$, and at least one $b_i \geq 3$.

Define $l' \in L'$ as follows: its E_v-coefficients are $1/2$ for end-vertices, and 1 for others. Then $\alpha := \theta([l'])$ determines a $\mathbb{Z}_2$ o-ramified covering (X_α, o). Since l' satisfies the output condition of Lemma 4.1.34, using the algorithm from 4.1.35 applied for the multiplicities of $2l'$, we get that (X_α, o) is the cyclic graph with decorations $2 - 2b_1, -b_2, \ldots, -b_{k-1}, 2 - 2b_k, -b_{k-1}, \ldots, -b_2$. Since this is the graph of a cusp singularity, singularities with graph Γ are called *quotient-cusps*. They are taut rational singularities [361]. For several of their properties see e.g. [514].

Remark 4.1.38 As we mentioned above, if $H_1(L_X, \mathbb{Q}) = 0$ then the correspondence $\theta : H \to \widehat{H}$, $[l'] \mapsto \alpha$ from 4.1.33 provides a character of $H = L'/L$ for any $l' \in L'$, hence a well-defined covering. However, if $H_1(L_X, \mathbb{Q}) \neq 0$, $\theta([l'])$ still provides a character $\alpha \in \widehat{H}$, but a morphism $H \to \mathbb{Z}_N$ fails to provide a unique/canonical morphism $H_1(L_X, \mathbb{Z}) \to \mathbb{Z}_N$, since there is no canonical projection $H_1 \to H = \mathrm{Tors}(H_1)$.

This fact, in some cases, can be remedied in the presence of some additional properties of (X, o). E.g., in the case of cusps (cyclic graphs), the ambiguity of the splitting of $0 \to H \to H_1 \to \mathbb{Z} \to 0$ is unraveled by the existence of an automorphism of $\pi_1(L_X)$, cf. 3.5.6 and 4.2.8.

4.1.39 Group Extensions Associated with the Covering Consider the covering $c : L_{X_\alpha} \to L_X$ from 4.1.32. The exact sequence of the fundamental groups $1 \to \pi_1(L_{X_\alpha}) \to \pi_1(L_X) \to \mathbb{Z}_N \to 0$ induces the homological exact sequence

$$H_1(L_{X_\alpha}, \mathbb{Z}) \xrightarrow{c_*} H_1(L_X, \mathbb{Z}) \xrightarrow{\alpha} \mathbb{Z}_N \to 0. \tag{4.1.4}$$

In fact, since the Galois group $\mathbb{Z}_N$ acts freely on L_{X_α}, we can apply the homological Cartan–Leray spectral sequence $E^2_{p,q} = H_p(\mathbb{Z}_N, H_q(L_{X_\alpha}, \mathbb{Z})) \Rightarrow H_*(L_X, \mathbb{Z})$ (cf. [104, Ch XVI, §9]), which identifies the kernel of c_* with the module generated by elements of $H_1(L_{X_\alpha}, \mathbb{Z})$ of type $t \cdot x - x$, where $t \in \mathbb{Z}_N$ and $x \in H_1(L_{X_\alpha}, \mathbb{Z})$ and $\cdot$ denotes the action. In other words, $\ker(\alpha)$ is the module of coinvariants of the action.

Later in the study of the divisor class group of (X, o) and $\widetilde{X}$ we will need certain cohomological properties of the covering as well. The exact sequence (4.1.4) can

be dualized in two different ways. Taking $\mathrm{Hom}(\cdot, \mathbb{Z})$, or $\mathrm{Hom}(\cdot, \mathbb{Z}_N)$ and using $\mathrm{Ext}(\mathbb{Z}, \mathbb{Z}_N) = 0$, we obtain the inclusion $H^1(L_X, \mathbb{Z}) \hookrightarrow H^1(L_{X_\alpha}, \mathbb{Z})$, respectively the exact sequence

$$0 \to \mathbb{Z}_N \to H^1(L_X, \mathbb{Z}_N) \to H^1(L_{X_\alpha}, \mathbb{Z}_N). \tag{4.1.5}$$

However, in Chern class computations (see e.g. 6.1.A) it is more useful to consider the identifications $H^2(L_X, \mathbb{Z}) = H_1(L_X, \mathbb{Z})$ and $H^2(L_{X_\alpha}, \mathbb{Z}) = H_1(L_{X_\alpha}, \mathbb{Z})$ given by Poincaré duality [645, page 297]. E.g., $H^2(L_X, \mathbb{Z})$ fits in the exact sequence $0 \to H^2(\widetilde{X}, \partial\widetilde{X}\mathbb{Z}) \to H^2(\widetilde{X}, \mathbb{Z}) \to H^2(L_X, \mathbb{Z})$, which is Poincaré dual to $0 \to H_2(\widetilde{X}, \mathbb{Z}) \to H_2(\widetilde{X}, \partial\widetilde{X}, \mathbb{Z}) \to H_1(L_X, \mathbb{Z})$, i.e. it is another interpretation of $0 \to L \to L' \to L'/L \to 0$ with $L'/L = \mathrm{Tors}(H_1(L_X, \mathbb{Z}))$.

Now, one can read from the cohomological Cartan–Leray spectral sequence $E_2^{p,q} = H^p(\mathbb{Z}_N, H^q(L_{X_\alpha}, \mathbb{Z})) \Rightarrow H^*(L_X, \mathbb{Z})$ the following facts (below $A^{\mathbb{Z}_N}$ denotes the module of $\mathbb{Z}_N$–invariants):

$$H^1(L_X, \mathbb{Z}) = H^1(L_{X_\alpha}, \mathbb{Z})^{\mathbb{Z}_N}, \tag{4.1.6}$$

and

$$\begin{gathered} 0 \to \ker(c^*) \to H^2(L_X, \mathbb{Z}) \xrightarrow{c^*} H^2(L_{X_\alpha}, \mathbb{Z})^{\mathbb{Z}_N} \xrightarrow{d_2} H^2(\mathbb{Z}_N, H^1(L_{X_\alpha}, \mathbb{Z})), \\ 0 \to \mathbb{Z}_N \to \ker(c^*) \xrightarrow{q} H^1(\mathbb{Z}_N, H^1(L_{X_\alpha}, \mathbb{Z})) \to 0. \end{gathered} \tag{4.1.7}$$

Here, if $\alpha = \theta([l'])$ then $\mathbb{Z}_N$ is generated by $[l'] \in L'/L = H^2(L_X, \mathbb{Z})$.

In particular, the kernel of $c^* : H^2(L_X, \mathbb{Z}) \to H^2(L_{X_\alpha}, \mathbb{Z})^{\mathbb{Z}_N}$ is the cyclic group $\mathbb{Z}_N$ if and only if $H^1(\mathbb{Z}_N, H^1(L_{X_\alpha}, \mathbb{Z})) = 0$, otherwise it is larger.

Recall also that if t denotes the generator of $\mathbb{Z}_N$ then $H^1(\mathbb{Z}_N, H^1(L_{X_\alpha}, \mathbb{Z}))$ equals

$$\frac{\ker\big(T : H^1(L_{X_\alpha}, \mathbb{Z}) \to H^1(L_{X_\alpha}, \mathbb{Z})\big)}{(1-t) \cdot H^1(L_{X_\alpha}, \mathbb{Z})}, \tag{4.1.8}$$

where $T := 1 + t + \cdots + t^{N-1}$. In particular, if $\mathbb{Z}_N$ acts trivially on $H^1(L_{X_\alpha}, \mathbb{Z})$ then $H^1(\mathbb{Z}_N, H^1(L_{X_\alpha}, \mathbb{Z})) = 0$, and $\ker(c^*) = \mathbb{Z}_N$. (Use the fact that $H^1(L_{X_\alpha}, \mathbb{Z})$ is torsion free.) But in general $\ker(c^*) \neq \mathbb{Z}_N$, cf. 4.1.40.

Recall also that $H^2(\mathbb{Z}_N, H^1(L_{X_\alpha}, \mathbb{Z})) = H^1(L_{X_\alpha}, \mathbb{Z})^{\mathbb{Z}_N}/T \cdot H^1(L_{X_\alpha}, \mathbb{Z})$. Then, via (4.1.6) and (4.1.7), if L_X is a rational homology sphere then c^* is onto.

Finally, let us identify the restriction $c^* : \mathrm{Tors}(H^2(L_X, \mathbb{Z})) \to \mathrm{Tors}(H^2(L_{X_\alpha}, \mathbb{Z}))$, or the morphism $c^* : (L'/L)(\Gamma(L_X)) \to (L'/L)(\Gamma(L_{X_\alpha}))$, where the graphs are connected by the algorithm from 4.1.16. If v is a vertex of $\Gamma(L_X)$ and $\widetilde{v}$ is one of the vertices of $\Gamma(L_{X_\alpha})$ above v given by the algorithm 4.1.16, then the $E_{\widetilde{v}}$–coefficient in the pullback $c^*(E_v)$ is $N/(N, m_v)$. [Indeed, if in local coordinates (x, y) of $\widetilde{X}$ the curve E_v has local equation $x = 0$ and $f = x^{m_v}$, then the covering is the

normalization of $x^{m_v} = z^N$. This has $d = (N, m_v)$ components, one of them is normalized by (w, y) via $x = w^{N/d}$, $z = w^{m_v/d}$. Hence $c^*(\mathrm{div}(x)) = \mathrm{div}(w^{N/d})$.]

Example 4.1.40 Consider a **quotient–cusp** singularity with $k = 2$, $b_1 = 2$, $b_2 = 3$, cf. 4.1.37. Consider the $\mathbb{Z}_2$–covering described in 4.1.37. In this case (X_α, o) is a cusp singularity whose graph has two vertices decorated by -2 and -4. Moreover, $H^2(L_X, \mathbb{Z}) = \mathbb{Z}_4 \oplus \mathbb{Z}_4$ and $\mathrm{Tors}\, H^2(L_{X_\alpha}, \mathbb{Z}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. The kernel of c^* is generated by two elements, say l'_1 and l'_2, where l'_1 has coefficients $1/2$ on the two left end–vertices and all the other coefficients are integers, while l'_2 is defined by symmetric properties. (Compare with the last paragraph of 4.1.39.) Both generate elements of order 2, and $\ker(c^*) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, where $([l'_1], [l'_2])$ corresponds to α.

In particular, in this case $\ker(c^*)$ is not even cyclic.

One also sees directly that $H^1(\mathbb{Z}_2, H^1(L_{X_\alpha}, \mathbb{Z})) = \mathbb{Z}_2$. Indeed, $H^1(L_{X_\alpha}, \mathbb{Z}) = \mathbb{Z}$, the $\mathbb{Z}_2$–action is nontrivial, hence (4.1.8) gives $\mathbb{Z}_2$.

4.1.41 $H^1\widetilde{X}, \mathscr{O}_{\widetilde{X}})$ **and Cyclic Coverings** Assume that $c : (X_\alpha, o) \to (X, o)$ is an o–covering with Galois group $\mathbb{Z}_N$ associated with $\alpha \in \widehat{H_1}$ as in 4.1.32. Fix a resolution $\phi : \widetilde{X} \to X$, and let $\widetilde{c} : \widetilde{X}_\alpha \to \widetilde{X}$ be the normalized pullback of c via ϕ. Then $\widetilde{c}^*$ injects $H^1(\widetilde{X}, \mathscr{O}_{\widetilde{X}})$ into the $\mathbb{Z}_N$–invariants of $H^1(\widetilde{X}_\alpha, \mathscr{O}_{\widetilde{X}_\alpha})$. We claim that in fact

$$H^1(\widetilde{X}_\alpha, \mathscr{O}_{\widetilde{X}_\alpha})^{\mathbb{Z}_N} = H^1(\widetilde{X}, \mathscr{O}_{\widetilde{X}}). \tag{4.1.9}$$

Since $\widetilde{c}$ is finite, by the Leray spectral sequence $H^1(\widetilde{X}_\alpha, \mathscr{O}_{\widetilde{X}_\alpha}) = H^1(\widetilde{X}, \widetilde{c}_*\mathscr{O}_{\widetilde{X}_\alpha})$. If $\oplus_{\rho\in\widehat{\mathbb{Z}_N}}(\widetilde{c}_*\mathscr{O}_{\widetilde{X}_\alpha})_\rho$ is the eigenspace decomposition of $\widetilde{c}_*\mathscr{O}_{\widetilde{X}_\alpha}$, then $H^1(\widetilde{X}, \widetilde{c}_*\mathscr{O}_{\widetilde{X}_\alpha}) = \oplus_\rho H^1(\widetilde{X}, (\widetilde{c}_*\mathscr{O}_{\widetilde{X}_\alpha})_\rho)$. On the other hand, each $(\widetilde{c}_*\mathscr{O}_{\widetilde{X}_\alpha})_\rho$ is a (nonempty) line bundle (see the first paragraph of the proof of 6.2.9, or [328]). Furthermore, $(\widetilde{c}_*\mathscr{O}_{\widetilde{X}_\alpha})^{\mathbb{Z}_N} = \mathscr{O}_{\widetilde{X}}$. This last fact can also be verified locally over $\widetilde{X}$, e.g. over a small neighborhood of an intersection of two exceptional divisors in $\widetilde{X}$ we have the local model of the natural projection of a Hirzebruch–Jung singularity, cf. 2.3.1.

4.2 Abelian Coverings

The results regarding cyclic coverings are generalized for (ramified and o-ramified) abelian coverings. The o-ramified coverings of cyclic graphs (including the discriminant covering) are exemplified. ♣

4.2.1 Abelian Coverings of Graphs We fix a finite abelian group H. All the definitions and results of Sect. 4.1.A regarding cyclic coverings of graphs can be extended naturally to H-coverings. We provide briefly the needed material, the details can be completed by the reader.

Definition 4.2.2 If Γ is a graph, we say that H acts on Γ, if there are group actions $a_{\mathscr{V}} : H \times \mathscr{V} \to \mathscr{V}$ and $a_{\mathscr{E}} : H \times \mathscr{E} \to \mathscr{E}$ of H such that if $\partial e = \{v_1, v_2\}$ then $\partial a_{\mathscr{E}}(h, e) = \{a_{\mathscr{V}}(h, v_1), a_{\mathscr{V}}(h, v_2)\}$. The action is trivial if $a_{\mathscr{V}}$ and $a_{\mathscr{E}}$ are trivial.

Fix a finite graph Γ with a trivial H-action. A H-covering of Γ consists of a finite graph G, that carries a H-action, together with an equivariant morphism $p : G \to \Gamma$ such that the restriction of the H-action on any set of type $p^{-1}(v)$ ($v \in \mathscr{V}(\Gamma)$), respectively $p^{-1}(e)$ ($e \in \mathscr{E}(\Gamma)$), is transitive.

Two H-coverings $p_i : G_i \to \Gamma$ $(i = 1, 2)$ are equivalent if there is an H-equivariant isomorphism $q : G_1 \to G_2$ such that $p_2 \circ q = p_1$.

For any $v \in \mathscr{V}(\Gamma)$ and for any edge e let Stab_v and Stab_e be the stabilizer subgroup of v and e respectively. By the definition of the action it follows that Stab_e is a subgroup of Stab_v whenever $v \in \partial e$. The system of subgroups of H, $\mathbf{Stab} := \left\{\{\mathrm{Stab}_v\}_{v \in \mathscr{V}(\Gamma)}; \{\mathrm{Stab}_e\}_{e \in \mathscr{E}(\Gamma)}\right\}$, is called the *covering data* of p.

Similarly as in the cyclic case, one has the following facts.

Theorem 4.2.3 *Let $\mathscr{G}(\Gamma, \mathbf{Stab})$ be the set of equivalence classes of H-coverings of Γ associated with a system of covering data* **Stab**. *Then $\mathscr{G}(\Gamma, \mathbf{Stab})$ is a finite abelian group. It is trivial whenever $c(\Gamma) = 0$.*

4.2.4 Abelian Ramified Coverings of Normal Surface Singularities Let (X, o) be a normal surface singularity and $f_k : (X, o) \to (\mathbb{C}, 0)$ $(1 \le k \le \kappa)$ germs of analytic functions. Furthermore, we fix κ positive integers $N_1, \ldots, N_\kappa$. Let $X_{f,N}$ ($f = \{f_k\}_k$ and $N = \{N_k\}_k$) be the multigerm defined as the normalization of

$$\{(x, z_1, \ldots, z_\kappa) \in (X \times \mathbb{C}^\kappa, o \times 0) \,:\, f_k(x) = z_k^{N_k} \text{ for all } k\}.$$

The group $\Pi := \prod_{k=1}^{\kappa} \mathbb{Z}_{N_k}$ acts on $X_{f,N}$ by $(\xi_1, \ldots, \xi_\kappa) * (x, z) = (x, \xi_1 z_1, \ldots, \xi_\kappa z_\kappa)$.

We wish to determine the resolution graph of $X_{f,N}$ from the common embedded resolution graph Γ_f of the germs $\{f_k\}_k$ and the collection $\{N_k\}_k$. However, as we already know from the cyclic case ($\kappa = 1$), this is not possible in general: if the link of (X, o) is not a rational homology sphere then one needs some extra information about the open book decomposition provided by the Milnor fibration of f. Since the analogue of this extra structure for $\kappa > 1$ is rather technical (and we will not need later), in the sequel we will assume that L_X is a rational homology sphere.

We emphasize the following fact too. Though $X_{f,N}$ can be constructed inductively by κ consecutive cyclic coverings starting from X (at step k introducing $f_k = z_k^{N_k}$), the resolution graph of $X_{f,N}$ cannot be determined inductively using blindly the algorithm valid for cyclic coverings with base space/graph a rational homology sphere link. Indeed, even if we start with a rational homology sphere link, it can happen that after the first cyclic cover the output graph is not a rational homology sphere, hence the continuation of the cyclic algorithm at the next steps is obstructed.

Let Γ_f be the embedded resolution graph of the collection of functions $\{f_k\}_k$; this means that the resolution ϕ is a good resolution of the product $\prod_k f_k$, and each vertex $w \in \mathscr{W} = \mathscr{V} \cup \mathscr{A}$ is decorated by $(m_{w1}, \dots, m_{w\kappa})$ representing the vanishing orders of the functions along the corresponding divisor. If $a \in \mathscr{A}$ then some of the multiplicities m_{ak} can be zero. Set $d_{wk} := \gcd\{m_{wk}, N_k\}$ for $1 \le k \le \kappa$ and $w \in \mathscr{W}$.

We consider the projection q to $\widetilde{X}$ of the normalized fiber product of the projection $p : X_{f,N} \to X$ and the resolution $\phi : \widetilde{X} \to X$.

Let C_w ($w \in \mathscr{W}$) be either an exceptional curve E_v or a strict transform S_a. The local model of q over a small neighbourhood of a generic point of C_w is the normalization of $\{(x, y, z_1, \dots, z_\kappa) : x^{m_{wk}} = z_k^{N_k} \text{ for all } k\}$. Then the monodromy around $\{x = 0\}$ is $\mathscr{M}_w := (e^{2\pi i m_{w1}/N_1}, \dots, e^{2\pi i m_{w\kappa}/N_\kappa}) \in \Pi$. We denote its order $\mathrm{lcm}_k\{N_k/d_{wk}\}$ by M_w. Hence the local neighbourhood is covered by $|\Pi|/M_w$ connected components in the normalized fiber product. In some local coordinates (t, y) of one of them, the map q has the form $x = t^{M_w}$, $y = y$, $z_k = t^{M_w m_{wk}/N_k}$. In particular, the pullbacks of the functions have multiplicities $\{M_w m_{wk}/N_k\}_k$ along $q^{-1}(C_w)$.

In the previous paragraph, if $w = a$ is an arrowhead, then above $C_w = S_a$ there are exactly $|\Pi|/M_a$ irreducible components (discs) (which will be codified by arrowheads). Hence $Stab_a$ is the subgroup in Π generated by $\mathscr{M}_a$.

Next consider an intersection point of two divisors corresponding to $e = (v, w) \in \mathscr{E}$. The local model above a small neighbourhood of this point is the normalization of $\{(x, y, z_1, \dots, z_\kappa) : x^{m_{vk}} y^{m_{wk}} = z_k^{N_k} \text{ for all } k\}$. All the irreducible components are isomorphic to each other, and they are of Hirzebruch–Jung type. They are described in Example 2.3.6. The image $\mathrm{im}\rho$ of the monodromy representation is generated in Π by $\mathscr{M}_v$ and $\mathscr{M}_w$. Above the point there are $|\Pi|/|\mathrm{im}\rho|$ points, the stabilizer of any of them is exactly $Stab_e = \mathrm{im}\rho$. Furthermore, $\ker\rho$ can be generated as follows. Let s_v be the smallest positive integer for which the system of congruences $\Lambda_v m_{vk} + s_v m_{wk} \equiv 0 \pmod{N_k}$ (for all k) has a solution $\Lambda_v \in \mathbb{Z}$. Chose the unique solution $0 \le \Lambda_v < M_v$. Then $\ker\rho$ is generated by $(M_v, 0)$ and (Λ_v, s_v). In particular, $|Stab_e| = M_v s_v$, and the negative continued fraction of M_v/Λ_v provides the singularity type.

Finally, consider a non-arrowhead vertex v of Γ_f codifying a component E_v. Then above E_v there are $|\Pi/\sum_{e:\partial e \ni v} Stab_e|$ irreducible components. (This number equals $|\Pi/\langle \mathscr{M}_v, \{\mathscr{M}_w\}_{w \in \mathscr{W}_v}\rangle|$ too.) The stabilizer of any of these components is $Stab_v := \sum_{e:\partial e \ni v} Stab_e$. (Here the fact that $E_v \simeq \mathbb{P}^1$ is crucial.) The genus $\tilde{g}_v$ of such a component is given by

$$2 - 2\tilde{g}_v = \frac{(2 - \kappa_v) \cdot |\Pi|/M_v + \sum_{w \in \mathscr{W}_v} |\Pi/Stab_{(w,v)}|}{|\Pi/Stab_v|}. \tag{4.2.1}$$

The gluing of the local data in a unique global graph is guaranteed by the vanishing of the group $\mathscr{G}(\Gamma, \mathbf{Stab})$, since $c(\Gamma_f) = 0$, cf. Theorem 4.2.3.

Hence we proved the following theorem.

Theorem 4.2.5 *The resolution graph of $X_{f,N}$ can be determined as follows. We consider the unique covering graph $G \to \Gamma_f$ associated with the covering data* **Stab** *described above. According to this, above $v \in \mathcal{V}(\Gamma_f)$ we put $|\Pi/Stab_v|$ vertices, each with genus decoration $\tilde{g}_v$ provided by (4.2.1), and (multi-)multiplicities $\{M_v m_{vk}/N_k\}_{k=1}^{\kappa}$. Similarly, above any arrowhead $a \in \mathcal{A}(\Gamma_f)$ we put $|\Pi/Stab_a|$ arrowheads with (multi-)multiplicities $\{M_a m_{ak}/N_k\}_{k=1}^{\kappa}$.*

Above each edge $e = (v, w)$ we insert $|\Pi/Stab_e|$ strings where the Euler decorations are the negatives of the entries of the continued fraction of M_v/Λ_v (where the first entry is glued to the vertices above v). If $\Lambda = 0$ then the strings are 'empty', they are edges. The gluing of the strings with the vertices above $\mathcal{W}(\Gamma_f)$ are determined by the corresponding surjection of the orbits $H/Stab_e \to H/Stab_w$ $(w \in \partial e)$.

There is a unique extension of the multiplicities along the strings such that the system satisfies (2.2.1). Finally, the multiplicity system just constructed determines the missing Euler decorations of the vertices above $\mathcal{V}(\Gamma_f)$ (using again (2.2.1)).

Finally we delete all the multiplicities, arrowheads, and their supporting edges.

The monodromy group $\mathrm{im}\rho$ *of the Π-covering $X_{f,N} \to X$ is the subgroup of Π generated by all $\{\mathcal{M}_w\}_{w\in\mathcal{W}(\Gamma_f)}$. Then the above algorithm provides a graph with $|\Pi/\mathrm{im}\rho|$ connected (identical) components, corresponding to the irreducible components of $X_{f,N}$.*

4.2.6 Abelian o-Ramified Coverings In this subsection we generalize Theorem 4.1.35, which describes the graphs of o-ramified cyclic coverings, for the general case of arbitrary o-ramified abelian coverings. We use (the proof of) Theorem 4.1.35 and Theorem 4.2.5 proved in the previous subsection.

Again, we assume that L_X is a rational homology sphere.

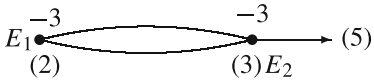
Fix characters $\alpha_1, \ldots, \alpha_\kappa$ of order $r_1, \ldots, r_\kappa$ of $H = H_1(L_X, \mathbb{Z})$. This provides a map $A = (\alpha_1, \ldots, \alpha_\kappa) : H \to \Pi := \prod_k \mathbb{Z}_{r_k}$. The composition $\pi_1(L_X) \to H \to \Pi$ determines a o-ramified covering with Galois group Π, which will have $|\Pi/\mathrm{im}(A)|$ identical connected components (each associated with $\ker A \subset H$).

For example, if A is an isomorphism, then we get a connected space, the *universal abelian covering* of L_X. (This, at the level of analytic germs, provides the *universal abelian covering* $(X_a, o) \to (X, o)$ of (X, o).)

Similarly as in 4.1.33, for any k we choose multiplicities $\{m_{kv}\}_{v\in\mathcal{V}(\Gamma(L_X))}$ such that $\alpha_k([\partial D_v]) = e^{2\pi i m_{vk}/r_k}$. By Lemma 4.1.34 we can assume that for each k and v one has $m_{vk} > 0$ and $m^*_{vk} := (\sum_u m_{uk}E_u, E_v)/r_k \in \mathbb{Z}_{\le 0}$. We complete the graph $\Gamma(L_X)$ with arrowheads $\mathcal{A}$, one arrowhead $a = a(v)$ for each $v \in \mathcal{V}(\Gamma(L_X))$, and we define their multiplicities by $\{m_{a(v)k}\}_k := \{-rm^*_{vk}\}_k$.

Theorem 4.2.7 *The graph of the regular Π-covering associated with $A : H \to \Pi$ is given by the algorithm of Theorem 4.2.5, applied for the multiplicity system $\{m_{vk}\}_{v\in\mathcal{V}(\Gamma(L_X))}$ and $\{m_{ak}\}_{a\in\mathcal{A}}$, and covering degrees $N_k = r_k$.*

Example 4.2.8 (Certain Coverings of Cyclic Graphs (Cusp Singularities)) Consider the following cyclic graph with an arrow and multiplicity system.



Then the multiplicities determine a character $\alpha \in \widehat{H}$, namely $\alpha := \theta([l'])$, where $l' := (2E_1 + 3E_2)/5 \in L'$. However, in general (when $H_1(L_X, \mathbb{Q}) \neq 0$), there is no canonical epimorphism $H_1 \simeq H \oplus \mathbb{Z} \to H$. Therefore, in the general context, such an l' (or α) does not determine canonically a character $\alpha_1 \in \widehat{H_1}$ ($H_1 = H_1(L_X, \mathbb{Z})$) (that is, an epimorphism $\pi_1 \to H_1 \to \mathbb{Z}_5$, see the discussion from 3.4 and 3.5.6).

On the other hand, *in the case of cusps*, by 3.5.6, different epimorphisms $H_1 \to H$ are related by an automorphism of $\pi_1(L_X)$, hence by an automorphism of the link, or of the singularity germ. In particular, l', or α, determine a character $\alpha_1 \in \widehat{H_1}$, hence, in our case, a $\mathbb{Z}_5$ o-ramified covering.

Also, with the above multiplicity system, Γ determines uniquely a (Milnor) fibration as in 3.4.10, hence the universal covering graph G as well, see 4.1.13. In fact, the graph covering $G \to \Gamma$ is a bijection of graphs. Therefore, the graph of the $\mathbb{Z}_5$ covering of Γ can be computed via 4.1.10 using the above multiplicity system. It turns out that the $\mathbb{Z}_5$ covering graph of Γ is the very same graph as Γ itself!

More generally, let Γ be any negative definite cyclic graph. Then by the above argument, the projections $H_1 \to H$ are equivalent up to an automorphism of π_1, hence there is a natural o-ramified covering associated with $\pi_1 \to H_1 \to H$ with Galois transformation group H. It is called the *discriminant covering* (the analogue of the universal abelian covering valid for QHS^3's). We claim that the graph of the discriminant covering of the cyclic graph Γ is exactly its dual graph Γ^* (cf. 3.5.8). (In the previous example, with b-sequence $-3, -3$, one has $\Gamma = \Gamma^*$ and $H = \mathbb{Z}_5$.)

Indeed, consider the torus bundle $M(\Gamma)$, and fix a generator of $\pi_1(E)$, hence the orientation of the torus (or of $\pi_1(T) = \mathbb{Z}^2$) as well. Let A be the monodromy, and consider the group extension 3.5.6. Then $H = \operatorname{coker}(A - I)$, hence the fundamental group π_1^c of the discriminant covering has a similar group extension, where $\pi_1(T) = \mathbb{Z}^2$ is replaced by another $\mathbb{Z}^2$, namely by the image of $A - I$, which is preserved by A (since A commutes with $A - I$), and whose restriction (the very same A) becomes the new monodromy operator. Hence $\pi_1^c = \pi_1$. Recall that in general the fundamental group and the monodromy determine the graph (torus bundle) only up to the ambiguity of duality (orientation) (in our case up to the ambiguity $\Gamma \leftrightarrow \Gamma^*$), cf. 3.5.9 and the paragraph after it. However, here we have some extra information. Indeed, since the generator of $\mathbb{Z} = \pi_1(S^1)$ is fixed, the ambiguity can be dissolved if we clarify whether $\pi_1(T)$ has the same orientation as $\operatorname{im}(A - I)$ or not. Since $\det(A - I) = 2 - \operatorname{trace}(A) < 0$, the graph of the covering is Γ^*.

For more on covering properties of cusps and quotient cusps see [514].

Chapter 5
Examples

5.1 Weighted Homogeneous Singularities

In this part we define weighted homogeneous normal surface singularities. We introduce the Seifert invariants and we show that the minimal good resolution graph is star-shaped and it can be determined from the Seifert invariants. We discuss the Seifert 3-manifold structure of the link and the topological classification of such singularities. We introduce the analytic Seifert line bundles (they appear from a good resolution of a weighted homogeneous singularity by contracting the 'legs' into quotient singularities). They lead to the analytic classification of weighted homogeneous singularities. A convenient covering of the total space $T \to C$ of the associated Seifert line bundle is a genuine line bundle; in this way $T \to C$ appears as an automorphic factor. This fact provides the Dolgachev–Pinkham–Demazure description of the graded algebra of the singularity, and concrete formulae for the Poincaré series and geometric genus.

Then we provide two constructions whenever the link is a rational homology sphere. The first one (a result of Neumann) realises the universal abelian covering as a Brieskorn–Hamm complete intersection. The second one is an analytic plumbing construction, which will have several computational consequences later. ♣

5.1.1 [541, 542] Fix some positive integers $(w_1, \dots, w_n)$. One defines the action of $\mathbb{C}^*$ on $\mathbb{C}^n$ with weights $(w_1, \dots, w_n)$ by $t \cdot (x_1, \dots, x_n) = (t^{w_1}x_1, \dots, t^{w_n}x_n)$. A polynomial $f \in \mathbb{C}[x]$ is called weighted homogeneous of degree ℓ with respect to the weights $(w_1, \dots, w_n)$ if $f(t \cdot x) = t^{\ell} f(x)$, where $\ell \in \mathbb{Z}_{\geq 0}$.

Let us fix an affine algebraic variety $X \subset \mathbb{C}^n$. X is called weighted homogeneous with weights $\{w_i\}_i$ if it is stable with respect to the above action of $\mathbb{C}^*$. Since the weight are all positive the action on X is *good*, that is, the origin is contained in

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the closure of any orbit. If additionally we assume that $\gcd_i\{w_i\} = 1$ and $X \not\subseteq \cup_i\{x_i = 0\}$ then the action is *effective* too, that is, if $t \cdot x = x$ for all $x \in X$ then $t = 1$. If X is weighted homogeneous then its defining ideal is generated by weighted homogeneous polynomials. In particular, its affine coordinate ring is $\mathbb{Z}_{\geq 0}$-graded: $R = \oplus_{\ell \geq 0} R_\ell$. In fact, all finitely generated $\mathbb{Z}_{\geq 0}$-graded $\mathbb{C}$-algebras correspond to affine varieties with good $\mathbb{C}^*$-action.

In this section we will assume that X is an affine surface, which admits a good and effective $\mathbb{C}^*$ action (with $w_i > 0$ and $\gcd_i\{w_i\} = 1$) and the origin is its isolated singularity. The normality of $R = \oplus_{\ell \geq 0} R_\ell$ is not automatically guaranteed.

Example 5.1.2 (Homogeneous Affine Cones [258, Ex. II.5.14]) Let $C \subset \mathbb{P}^{n-1}$ be an irreducible smooth projective curve as in 2.1.5(b). Let X be the (homogeneous) affine cone over it with graded affine coordinate ring R. In general, R is not normal: its normalization is $R' := \oplus_{\ell \geq 0} H^0(C, \mathscr{O}_C(\ell))$, where $\mathscr{O}_C(\ell) = \mathscr{O}_{\mathbb{P}^{n-1}}(\ell)|_C$. In fact, $R_\ell = R'_\ell$ for $\ell \gg 0$, which also shows that $R^{(d)} := \oplus_{\ell \geq 0} R_{\ell d}$ ($d \gg 0$) is normal. (It is of type R' associated with the d-uple Veronese embedding of C.)

Example 5.1.3 (Algebraic Cones) Let C be a smooth compact curve, and $\mathscr{L}$ is an ample line bundle on C. Then the graded algebra $R = \oplus_{\ell \geq 0} H^0(C, \mathscr{L}^\ell)$ is a normal finitely generated graded $\mathbb{C}$-algebra. (In fact, there exists an isomorphism $f : C \to P$, where $P := \mathrm{Proj}(R)$, and $f^*(\mathscr{O}_P(\ell)) = \mathscr{L}^\ell$ for all $\ell \geq 0$.) By contracting the zero section in the total space $\widetilde{X}$ of $\mathscr{L}^{-1}$ we get a normal surface X with graded coordinate ring R. Multiplication in the fibers of $\mathscr{L}^{-1}$ induces a good $\mathbb{C}^*$-action on X.

If $s \in H^0(C, \mathscr{L}^\ell)$ then the pullback of s to the resolution $\widetilde{X}$ in convenient local (C, fiber)-coordinates (x, u) has the form $a(x)u^\ell$, hence $\mathrm{div}_C(s) = \ell C$. (Indeed, using the above property of f^*, it is enough to verify the case of Example 5.1.2 and $\mathscr{L} = \mathscr{O}_{\mathbb{P}^{n-1}}(1)|_C$. In this case the total space of $\mathscr{L}^{-1}$ is the blow up $\widetilde{X}$ of $0 \in X \subset \mathbb{C}^n$.)

Lemma 5.1.4 *Let C, $\mathscr{L}$ be as above and let $\pi : \widetilde{X} \to C$ be the projection. Then*

(a) $\pi^*(\mathscr{L}^{-1}) = \mathscr{O}_{\widetilde{X}}(C)$, $\mathscr{L}^{-1} = \pi^*(\mathscr{L}^{-1})|_C = \mathscr{O}_{\widetilde{X}}(C)|_C$, $\pi_*(\mathscr{O}_{\widetilde{X}}) = \oplus_{\ell \geq 0} \mathscr{L}^\ell$,
(b) $K_{\widetilde{X}}(C) = \pi^*(K_C)$ *(compare it with the adjunction $K_{\widetilde{X}}(C)|_C = K_C$).*

Proof For the first statement of *(a)* we construct a section of $\pi^*(\mathscr{L}^{-1})$. This in local (C, fiber)-coordinates (x, u) of $\widetilde{X}$ is the tautological $(x, u) \mapsto u$. Its divisor is C.

Next we prove *(b)*. We cover C with charts $\{U_\alpha\}_\alpha$ with local coordinates $\{z_\alpha\}_\alpha$, such that the transitions are $z_\beta = \varphi_{\alpha\beta}(z_\alpha)$. Therefore, sections of K_C in local charts have the form $\{\varphi_\alpha\}_\alpha$ (or, $\{\varphi_\alpha(z_\alpha)dz_\alpha\}_\alpha$), where $\varphi_\alpha = \varphi_\beta(\varphi_{\alpha\beta}) \cdot \varphi'_{\alpha\beta}$. Let the transition functions of $\mathscr{L}^{-1}$ be $t_\beta = h_{\alpha\beta}(z_\alpha)t_\alpha$. Then as sections of $\Omega^2_{\widetilde{X}}$, $\psi_\alpha dz_\alpha \wedge dt_\alpha = \psi_\beta dz_\beta \wedge dt_\beta$ if and only if $\psi_\beta = \psi_\alpha/(\varphi'_{\alpha\beta}h_{\alpha\beta})$, hence $K_{\widetilde{X}} = \pi^*(K_C \otimes \mathscr{L})$. □

Definition 5.1.5 In the sequel we assume that the origin is a normal point of X.

A normal analytic surface singularity (X_{an}, o) is called weighted homogeneous if there exists a normal affine surface X, which admits a good $\mathbb{C}^*$ action and a singular point $o \in X$ such that (X_{an}, o) is analytically isomorphic with the (induced analytic germ) (X, o).

In order to avoid the redundance of notations, we will use X or (X, o) for both objects, when we need the global action we think about X as affine variety, when we discuss local properties then we handle it as a germ.

5.1.6 Seifert (Or Singular) Line Bundles Consider the cyclic quotient singularity $X_{\alpha,\omega} = \mathbb{C}^2/\mathbb{Z}_\alpha$ as in 2.3.1(d) (with $(\alpha, \omega) = (n, q)$). We recall that $\xi \in \mathbb{Z}_\alpha$ acts as $\xi \cdot (z_1, z_2) = (\xi z_1, \xi^\omega z_2)$. We replace $\mathbb{C}^2$ by $D \times \mathbb{C}$, where D is the unit disc. Let $[z_1, z_2] \in (D \times \mathbb{C})/\mathbb{Z}_\alpha$ be the class of $(z_1, z_2) \in D \times \mathbb{C}$, and consider the projection $\pi : X^D_{\alpha,\omega} := (D \times \mathbb{C})/\mathbb{Z}_\alpha \to D$, $[z_1, z_2] \mapsto z_1^\alpha$. All the fibers are isomorphic to $\mathbb{C}$. $\mathbb{C}^*$ acts on $X^D_{\alpha,\omega}$ by $t \cdot [z_1, z_2] = [z_1, tz_2]$.

Definition 5.1.7 Let C be a smooth compact curve. An analytic space of dimension two T and an analytic map $\pi : T \to C$ with all the fibers isomorphic to $\mathbb{C}$ is called *Seifert line bundle*, if any point $c \in C$ has an open neighbourhood U such that $\pi^{-1}(U) \to U$ is isomorphic (by an isomorphism linear on fibres) either with $D \times \mathbb{C} \to D$ or with $X^D_{\alpha,\omega} \to D$ for some pair (α, ω) (and c corresponds to $0 \in D$). In this second case, $p^{-1}(c)$ is called a singular fiber with Seifert invariants (α, ω).

Theorem 5.1.8 (See e.g. [447]) *Let X be a weighted homogeneous normal surface, cf. 5.1.1. Then $C := (X \setminus 0)/\mathbb{C}^*$ is a smooth compact curve. Let T be the closure of the graph of $X \setminus 0 \to C$ in $X \times C$. Then the first projection ϕ is a modification $T \to X$, which is an isomorphism above $X \setminus 0$ and the exceptional curve is isomorphic to C. The second projection $\pi : T \to C$ is a Seifert line bundle (with zero section being the exceptional curve). T itself is not smooth, it has a cyclic quotient singularity at the intersection of exceptional curve with each singular fiber.*

The Seifert invariants can be interpreted as *orbit invariants* of the singular fibers. Orbit invariants are defined as follows. We can assume that $X \subset \mathbb{C}^n$ is not contained in any coordinate hyperplane, and let $\{c_k\}_k$ be the images in C of the orbit closures sitting in $X \cap \{\prod_i x_i = 0\}$. Since $\gcd_i\{w_i\} = 1$, π is a trivial line bundle over $C \setminus \cup_k c_k$. Let $x = (x_1, \ldots, x_n)$ be a point in $X \cap \{\prod_i x_i = 0\}$, $x \neq 0$. Set $\alpha := \gcd\{w_i : x_i \neq 0\}$. Then, the isotropy group $G_x \subset \mathbb{C}^*$ of x is $\mathbb{Z}_\alpha$; let $\lambda = e^{2\pi i/\alpha}$ denote its generator. Let N_x be a G_x-invariant normal section at x to the orbit, hence multiplication by $\lambda \in G_x \subset \mathbb{C}^*$ preserves N_x. Define $\gamma \in \mathbb{Z}$ such that λ acts on N_x by multiplication by λ^γ. The integers (α, γ) are the *orbit invariants* of the singular fiber given by x.

On the other hand, by Holmann's Slice Theorem [279], for a small neighbourhood U of $\pi(x) = c$ one can identify $\pi^{-1}(U) \to U$ with a certain $X^D_{\alpha,\omega} \to D$.

The numerical invariants (α, γ) and (α, ω) are related as follows.

In the model $X^D_{\alpha,\omega} \to D$ the isotropy subgroup of the point $[0, 1]$ is $\{t : t^\alpha = 1\}$, and on $N_{[0,1]} = \{[z_1, 1]\}_{z_1}$ the generator λ acts by $[z_1, 1] \mapsto [z_1, \lambda]$ (as $\mathbb{C}^*$ acts), which equals $[z_1\lambda^\gamma, 1]$ (by the definition of γ). Hence $\lambda^\gamma = \xi$ and $1 = \lambda\xi^\omega$ for a certain ξ. This shows that the orbit invariants (α, γ) and Seifert invariants (α, ω) are related by $\gamma\omega \equiv -1 \pmod{\alpha}$.

5.1.9 The Resolution [541] T is not smooth, but resolving its cyclic quotient singularities we get a minimal good resolution $\widetilde{X}$. Its graph is *star-shaped* with (the strict transform of) C central. Set g for the genus of C and $-b_0 = (C, C)_{\widetilde{X}}$. (The proof of) 2.3.1 shows that if we resolve a singular point of type (α, ω) with continued fraction $\alpha/\omega = [[b_1, \ldots]]$, then C intersects the $(-b_1)$-curve. For the resolution graph Γ we use the notations from 3.5.A. Recall also the definitions of the orbifold/virtual Euler number $e := -b_0 + \sum_j \omega_j/\alpha_j$.

5.1.10 Covering Property of e [291] In fact, since T is a $\mathbb{Q}$-homology manifold, by $\mathbb{Q}$-Poincaré duality one defines the (rational) self-intersection of C in T, which equals $e \in \mathbb{Q}$ [291, Th. 3.3]. Moreover, by the projection formula, if $T' \to C'$ and $T \to C$ are Seifert line bundles, and the following diagram commutes (with ρ onto)

$$\begin{array}{ccc} T' & \xrightarrow{\tau} & T \\ \downarrow & & \downarrow \\ C' & \xrightarrow{\rho} & C \end{array}$$

then $\deg(\tau|_{\mathrm{gf}})e' = (\deg(\tau|_{\mathrm{gf}})C', C')_{\mathbb{Q}} = (\tau^*C, C')_{\mathbb{Q}} = (C, \deg(\rho)C)_{\mathbb{Q}} = e \cdot \deg(\rho)$, where $\tau|_{\mathrm{gf}}$ denoted the restriction to the generic fiber. Note that $\deg(\tau) = \deg(\tau|_{\mathrm{gf}}) \cdot \deg(\rho)$ too. This can be compared with the interpretation from 5.1.24 and covering properties from 9.7.38 and the comparison of weights in 6.2.12; for more see [509].

5.1.11 The Link. Seifert S^1-Bundles To any Seifert line bundle one can associate a natural Seifert disc bundle: in the construction 5.1.6 the fiber $\mathbb{C}$ is replaced by the unit disc D. The boundary of this Seifert disc bundle is a Seifert S^1-bundle. For the convenience of the reader we (re)write the definition. $\mathbb{Z}_\alpha$ acts on $D \times S^1$ by $\xi \cdot (z_1, z_2) = (\xi z_1, \xi^\omega z_2)$. Let $[z_1, z_2]$ be the class of $(z_1, z_2) \in D \times S^1$, and consider the projection $\pi : L^D_{\alpha,\omega} := D \times S^1/\mathbb{Z}_\alpha \to D$, $[z_1, z_2] \mapsto z_1^\alpha$. All the fibers are isomorphic to S^1, and S^1 acts on $L^D_{\alpha,\omega}$ by $t \cdot [z_1, z_2] = [z_1, tz_2]$.

Let C be a C^∞ oriented compact Riemann surface (without boundary). A C^∞ map $\pi : L^{S^1} \to C$ with all the fibers isomorphic to S^1 is called *Seifert S^1-bundle*, if any $c \in C$ has an open neighbourhood U such that $\pi^{-1}(U) \to U$ is isomorphic (S^1-equivariantly on fibres) either with $D \times S^1 \to D$ or with $L^D_{\alpha,\omega} \to D$ for some (α, ω). In the second case, $p^{-1}(c)$ is called a singular fiber with Seifert invariants (α, ω).

The S^1 action globalizes to an action on L^{S^1}, the orbits being the fibers of π.

Theorem 5.1.12 (Topological Classification Theorem [624]) *The graph, hence the link L_X of (X, o) too, is completely determined by the Seifert invariants*

$$\mathrm{Sf} := (b_0, g; (\alpha_j, \omega_j),\ 1 \le j \le \nu).$$

The link L_X is the underlying total space of the S^1-bundle with Seifert invariants Sf. *In particular, it admits an induced Seifert S^1 action and Seifert S^1-bundle structure $L_X^{S^1}$. The invariant* Sf *determines the Seifert structure $L_X^{S^1}$ up to an orientation preserving S^1-equivariant homeomorphism.*

Example 5.1.13 (Algebraic Cones) If X is a cone singularity associated with $(C, \mathscr{L})$ as in 5.1.3, then $\nu = 0$, $g = g(C)$ and $-b_0 = \deg_C \mathscr{L}^{-1}$.

Example 5.1.14 ((Non)uniqueness of the Seifert Structures) On the same 3-manifold L one might produce several non-equivalent Seifert S^1 actions. E.g., on $S^3 = \{(x, y) : |(x, y)| = 1\}$ the Seifert invariant of the action $t * (x, y) = (tx, ty)$ is $(1, 0;\ )$, of the action $t * (x, y) = (t^3 x, ty)$ is $(1, 0; (3, 2))$, while of the action $t * (x, y) = (t^2 x, t^3 y)$ is $(1, 0; (2, 1), (3, 1))$. Similar phenomenon appears for lens spaces as well (even on the minimal graph with different choices of the central vertex). In fact, the only 3-manifolds, which are diffeomorphic with links of normal surface singularities, and which admit non-equivalent (even non-isotopic) Seifert structures are exactly the lens spaces [67, 505].

Example 5.1.15 (Hypersurface Singularities) Topological types (that is, resolution graphs) of isolated weighted homogeneous hypersurface singularities in $\mathbb{C}^3$ are completely classified. In this case the three weights and the degree d provides a plane $\mathfrak{P} = \{\sum_i x_i w_i = d\}$, which supports all the monomials of the equation. $\mathfrak{P}^+$, the part of $\mathfrak{P}$ in the closed first quadrant, should satisfy certain restrictions in order to get an isolated singularity. These conditions are: $\mathfrak{P}^+$ must have a lattice point on every coordinate plane, and for every coordinate axis $\mathfrak{P}^+$ must have a lattice point at most 1 far from the axis, cf. [345, 1.13]. The monomials on such $\mathfrak{P}^+$ (with generic coefficients) provide an equisingular family of singularities with constant resolution graph. The possible types are represented by the following normal forms [541, 543].

(1) $x_1^{a_1} + x_2^{a_2} + x_3^{a_3}$,
(2) $x_1^{a_1} + x_2^{a_2} + x_2 x_3^{a_3}$, $a_2 > 1$,
(3) $x_1^{a_1} + x_2^{a_2} x_3 + x_2 x_3^{a_3}$, $a_2 > 1$, $a_3 > 1$,
(4) $x_1^{a_1} + x_1 x_2^{a_2} + x_2 x_3^{a_3}$, $a_1 > 1$,
(5) $x_1^{a_1} x_2 + x_2^{a_2} x_3 + x_3^{a_3} x_1$,
(6) $x_1^{a_1} + x_2 x_3$,
$(a_1 - 1)(a_2 b_3 + a_3 b_2) = a_1 a_2 a_3$,
(7) $x_1^{a_1} + x_1 x_2^{a_2} + x_1 x_3^{a_3} + x_2^{b_2} x_3^{b_3}$,
$(a_1 - 1)(a_2 b_3 + a_3 b_2) = (a_1 a_2 - 1) a_3$.
(8) $x_1^{a_1} x_2 + x_1 x_2^{a_2} + x_1 x_3^{a_3} + x_2^{b_2} x_3^{b_3}$,

For each case, the corresponding Seifert invariants can be found in [541, 543].

Elements of (1) are called *Brieskorn polynomials/singularities*.

In the next list of invariants the dimension can be arbitrary, that is, $\{f = 0\}$ is a hypersurface weighted homogeneous singularity in $\mathbb{C}^{n+1}$ ($n \ge 1$) of weights

$\{w_i\}_i$ and degree d. Then $R = \mathbb{C}[x]/(f)$ is graded according to these weights. The Poincaré series of R (associated with this grading), $P_X(t) = \sum_\ell \dim R_\ell \cdot t^\ell$, is $(1-t^d)/\prod_i(1-t^{w_i})$ (see e.g. 8.1.2).

Some of the invariants from the Milnor homological package are the following (cf. 3.2.13).

(a) [432] $\mu(f) = \sum_i(\frac{d}{w_i} - 1)$.

(b) [432] If $\Delta_n(t)$ has roots $\det(1 - th_n) = \prod_{k=1}^{\mu}(\xi_k - t)$, then we can write $\mathrm{Div}(\Delta_n) := \sum_k(\xi_k) \in \mathbb{Z}[S^1]$. For each $a \in \mathbb{Z}_{>0}$ we also set $\Lambda_a := \sum_{j=1}^{a}(e^{2\pi ij/a}) \in \mathbb{Z}[S^1]$. Then, if we write d/w_i as u_i/v_i with $u_i, v_i > 0$ and $\gcd(u_i, v_i) = 1$ then $\mathrm{Div}(\Delta_n) = \prod_i(v_i^{-1}\Lambda_{u_i} - (1))$.

(c) [654] Let $M(f)$ be the Milnor algebra $\mathbb{C}\{x\}/(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_{n+1}})$. (As a vector space, it can be identified with $H_n(F, \mathbb{C})$.) Let us redefine the (rational) degree of a monomial x^α by $\ell(\alpha) := \sum_i(\alpha_i + 1)w_i/d$, and consider the graded vector space $M(f) = \oplus_\ell M_\ell$ graded by ℓ. Then the Poincaré polynomial $P_{M(f)}(t) = \sum_\ell t^\ell \dim M_\ell$ of $M(f)$ is

$$P_{M(f)}(t) = \prod_i \frac{t^{w_i/d} - t}{1 - t^{w_i/d}}.$$

[Though in this book we do not plan to discuss the Hodge theoretical aspects of the vanishing (co)homology of hypersurface singularities, here it is worthy to mention the following connection. If $\{s_1, \dots, s_\mu\}$ is the collection of spectral numbers in the sense of the mixed Hodge theory developed by Steenbrink and Varchenko [654, 657, 712, 714], then $P_{M(f)}(t) = \sum_i t^{s_i}$.]

(d) If $P_{M(f)}(t) = \sum_k t^{\ell_k}$, then $\mathrm{Div}(\Delta_n) = \sum_k(e^{2\pi i\ell_k})$.

(e) Moreover, $\mu_0 = \sum_{\ell\in\mathbb{Z}} \dim M_\ell$ and $\mu_\pm = \sum_{\ell\notin\mathbb{Z}} \dim M_\ell$, sum over $(-1)^{[\ell]} = \pm 1$.

(f) Each $\ell_k \in (0, n+1)$ and $P_{M(f)}(t)$ is symmetric with respect to $(n+1)/2$.

(g) [602, 603] Consider the graded subvector space $M(f)_{(0,1]}$ of $M(f)$ generated by monomial x^α with $\ell(\alpha) \in (0, 1]$. Then its dimension is p_g. (Hence, the number of spectral number in the interval $(0, 1]$ is p_g, a fact valid even for non-weighted homogeneous hypersurface isolated singularities as well [602].)

If $n = 2$ (case of surface singularities) we have additionally the following facts. By symmetry $\dim M_1 = \dim M_2$, hence the link is a rational homology sphere if and only if $M_1 = 0$ (via (d)). M_1 is freely generated by monomials x^α with $\ell(\alpha) = 1$, which stay in bijective correspondence (via $\alpha \mapsto \alpha + (1, 1, 1)$) with the lattice points in $\mathfrak{P}^+$ with all entries positive. Similarly, p_g appears as the number of lattice points under and on the plane $\mathfrak{P}$ with all entries positive.

For invariants of weighted homogeneous isolated complete intersections see [234, 254] and 9.7.24 below.

Example 5.1.16 A germ $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ is called quasihomogeneous if f belongs to the Jacobian ideal $J(f) = (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_{n+1}})$. Weighted homogeneous singularities, by the Euler relation $\sum_i w_i x_i (\partial f/\partial x_i) = d \cdot f$, are quasihomogeneous. By a theorem of K. Saito [601] if f is quasihomogeneous and defines an isolated singularity then after a biholomorphic change of coordinates it becomes a weighted homogeneous polynomial.

Recall that the Milnor number μ of a hypersurface germ is the dimension of the Milnor algebra $M(f) = \mathbb{C}\{x\}/J(f)$, while the Tjurina number τ is the dimension of the Tjurina algebra $T(f) = \mathbb{C}\{x\}/(f, J(f))$. In particular, for isolated hypersurface germs, $\mu = \tau$ if and only if f in some coordinates is weighted homogeneous.

For some other characterizations, properties (and conjectures) of weighted homogeneous hypersurfaces see [769].

Example 5.1.17 (Brieskorn–Hamm Complete Intersections) Fix $n \geq 3$ positive integers $a_i \geq 2$ $(i = 1, \ldots, n)$. For a generic set of complex numbers $C = \{c_{ji}\}$, $i = 1, \ldots, n$, $j = 1, \ldots, n-2$, (in fact it is enough that all the $(n-2)$-minors are nonzero)

$$X_C(a_1, \ldots, a_n) := \left\{x \in (\mathbb{C}^n, 0) \,:\, c_{j1}x_1^{a_1} + \cdots + c_{jn}x_n^{a_n} = 0 \text{ for all } \; 1 \leq j \leq n-2\right\}$$

is a complete intersection, whose link $L_X = \Sigma(a_1, \ldots, a_n)$ is independent of the choice of coefficients C, cf. [253]. The Seifert invariants are (cf. [291, 511]):

$$\Big(b_0, g; \underbrace{(\alpha_1, \omega_1), \ldots, (\alpha_1, \omega_1)}_{q_1}; \ldots; \underbrace{(\alpha_n, \omega_n), \ldots, (\alpha_n, \omega_n)}_{q_n}\Big)$$

where

$$\begin{aligned} A &:= \prod_{i=1}^{n} a_i, \quad a := \operatorname{lcm}\{a_i; \quad 1 \leq i \leq n\}, \\ \alpha_i &:= \frac{a}{\operatorname{lcm}\{a_j; \quad j \neq i\}}, \quad q_i := \frac{A\alpha_i}{aa_i} \quad (1 \leq i \leq n), \\ \gamma_i &:= \frac{a}{a_i}, \quad \omega_i\gamma_i \equiv -1 \pmod{\alpha_i} \quad (1 \leq i \leq n), \\ 2g - 2 &:= (n-2)A/a - \textstyle\sum_i q_i, \quad -e = A/a^2. \end{aligned}$$

Additionally, the following facts hold:

- $H_1(L_X, \mathbb{Q}) = 0$ iff $g = 0$; this in terms of $\{a_i\}_i$ reads as follows (cf. [82, 253, 489]): $(a_1, \ldots, a_n)$ (after a possible permutation) has one of the following forms.

(i) $(a_1, \dots, a_n) = (kp_1, kp_2, p_3, \dots, p_n)$, where $k \geq 1$ and the integers $\{p_j\}_{j=1}^n$ are pairwise coprime, and $\gcd(k, p_j) = 1$ for any $j \geq 3$;
(ii) $(a_1, \dots, a_n) = (2^c p_1, 2p_2, 2p_3, p_4 \dots, p_n)$, where the integers $\{p_j\}_{j=1}^n$ are odd and pairwise coprime, and $c \geq 1$.

- L_X is an integral homology sphere if and only if $\{a_j\}_j$ are pairwise coprime.

5.1.18 Analytic Classification of Seifert Line Bundles We fix a smooth curve C, distinct points $P_1, \dots, P_\nu$ on C, and positive integers $\alpha_1, \dots, \alpha_\nu$.

We wish to provide the analytic classification of the Seifert line bundles $\pi : T \to C$ over C, with the property that the set of singular fibers are included in $\cup_i \pi^{-1}(P_i)$, and the isotropy group of $\pi^{-1}(P_i)$ is a factor of $\mathbb{Z}_{\alpha_i}$. We denote this set by $\mathrm{Pic}(C, P, \alpha) = \mathrm{Pic}(C, \{P_i, \alpha_i\}_i)$.

Let us define local models as in 5.1.6: $\pi : X^D_{\alpha,\omega} := D \times \mathbb{C}/\mathbb{Z}_\alpha \to D$ is defined similarly, with the only exception that now we allow any ω with $0 \leq \omega < \alpha$, not necessarily relative prime with α. Let $d := \gcd(\alpha, \omega)$. Then the Seifert invariants of the central fiber are $(\alpha/d, \omega/d)$. Then, by definition, $\mathrm{Pic}(C, P, \alpha)$ consists of Seifert line bundles over C such that over points from $C \setminus \{P_1, \dots, P_\nu\}$ one has usual local trivialization, while at any P_i one takes a local model of type $\pi : X^D_{\alpha_i,\omega_i} \to D$, for certain $0 \leq \omega_i < \alpha_i$, and where the origin of D is identified with P_i.

First we note that there is a natural tensor product of elements of $\mathrm{Pic}(C, P, \alpha)$. The tensor product of two models X_{α_i,ω_i} and $X_{\alpha_i,\omega_i'}$ over D is $X_{\alpha_i,\widetilde{\omega}_i}$, where $\widetilde{\omega}_i \equiv \omega_i + \omega_i' \pmod{\alpha_i}$, $0 \leq \widetilde{\omega}_i < \alpha_i$. This makes $\mathrm{Pic}(C, P, \alpha)$ a semigroup. Each ω_i (associated with the fiber $\pi^{-1}(P_i)$) defines a map $\omega_i : \mathrm{Pic}(C, P, \alpha) \to \mathbb{Z}_{\alpha_i}$, which is (semigroup) multiplicative. Furthermore, $\prod_i \omega_i : \mathrm{Pic}(C, P, \alpha) \to \prod_i \mathbb{Z}_{\alpha_i}$ is onto, and its kernel $(\prod_i \omega_i)^{-1}(0)$ is $\mathrm{Pic}(C)$, the group of (genuine) line bundles on C. Since $\mathrm{Pic}(C)$ is a group, $\mathrm{Pic}(C, P, \alpha)$ becomes a group as well. The identity component in $\mathrm{Pic}(C, P, \alpha)$ is the identity component $\mathrm{Pic}^0(C)$ of $\mathrm{Pic}(C)$. Set $P := \mathrm{Pic}(C, P, \alpha)/\mathrm{Pic}^0(C)$.

Then one has the following commutative diagram (where i and j are injective and the last two vertical arrows are surjective):

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Pic}^0(C) & \longrightarrow & \mathrm{Pic}(C) & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow{=} & & \downarrow{j} & & \downarrow{i} & & \\
0 & \longrightarrow & \mathrm{Pic}^0(C) & \longrightarrow & \mathrm{Pic}(C, P, \alpha) & \xrightarrow{c} & P & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \prod_i \mathbb{Z}_{\alpha_i} & = & \prod_i \mathbb{Z}_{\alpha_i} & &
\end{array}$$

The group P has the following abelian presentation $\langle h_0, h_1, \dots, h_\nu \mid \alpha_j h_j = h_0, \quad j = 1, \dots, \nu \rangle$. Moreover, $i(n) = nh_0 \in P$. If $c(T \to C) = -bh_0 + \sum_i \omega_i h_i$ (written in a unique way with $0 \leq \omega_i < \alpha_i$), then the Seifert invariants of $T \to C$ are $(b, g; (\alpha_i', \omega_i')_i)$, where $\alpha_i' = \alpha_i/d$, $\omega_i' = \omega_i/d$, $d := \gcd(\alpha_i, \omega_i)$. (Hence P provides the 'topological' classification, and c takes $T \to C$ to the topological

type of the corresponding Seifert bundle.) If $g = 0$ then $\mathrm{Pic}^0(C) = 0$ too, and $\mathrm{Pic}(C, P, \alpha) \simeq P$.

5.1.19 Let us consider $C = (X \setminus 0)/\mathbb{C}^*$ again, as in 5.1.8, and let $\{P_j\}_j$ be the collection of points of C parametrizing the singular fibers (with $\alpha_j > 1$).

Theorem 5.1.20 (Analytic Classification Theorem [125, 507, 543, 571])
The set of isomorphism types of normal surface weighted homogeneous singularities X with fixed topology (Seifert invariants) and fixed analytic type of $(C, \{P_j\}_j)$ is isomorphic to $\mathrm{Pic}^0(C)$ modulo an action of $\mathrm{Aut}(C, \{P_j\}_j)$. (The correspondence is given by Theorem 5.1.8 and the description of $\mathrm{Pic}(C, P, \alpha)$ in 5.1.18 above.)

5.1.21 Following [742] we define the **virtual Euler characteristic** of a Seifert S^1 bundle by

$$\chi = 2 - 2g - \sum_j \frac{\alpha_j - 1}{\alpha_j}.$$

The words 'virtual' is motivated by the fact that χ behaves like the Euler characteristic with respect to coverings (as it will be explained after Theorem 5.1.23).

The sign $+, 0, -$ of χ subdivides Seifert manifolds into three, geometrically rather different families. This is reflected (almost faithfully) by the fundamental group π_1 of the link as well. For the following facts see [125, 540, 676, 685], and 3.7.15, 6.3.30, 6.3.33 here.

(i) $\chi > 0$ if and only if π_1 is finite. This happens exactly if $g = 0$ and $\nu \le 2$, or $g = 0$, $\nu = 3$, and $\sum_j 1/\alpha_j > 1$, cf. 3.7.15. These are exactly the quotient singularities, compare with (6.3.6) and 6.3.33. (Note that π_1 is finite if and only if the order of the generic fiber γ_0 in π_1 is finite, cf. 3.6.9).

(ii) $\chi = 0$ if and only if either (a) $g = 1$ and $\nu = 0$, or (b) $g = 0$ and $\sum_j 1/\alpha_j = \nu - 2$ (the possible solutions are $(3, 3, 3)$, $(2, 3, 6)$, $(2, 4, 4)$, $(2, 2, 2, 2)$). π_1 is infinite nilpotent if and only if $\chi = 0$ and (a) holds. In such a case π_1 is determined by $-b_0$, it is a Heisenberg group, and L_X is a torus bundle over S^1, as described in 3.7.16.

5.1.22 $T \to C$ **as Automorphic Factor** Consider the notations of Theorem 5.1.20, let (α_j, ω_j) be the Seifert invariant at P_i $(\alpha_j > 1)$.

Theorem 5.1.23 ([102, 125, 152, 153, 155, 431, 503, 571]) *There exists a smooth compact curve C' and a Galois (ramified) covering $\rho : C' \to C$ with Galois group G such that the ramification locus is exactly $\cup_j P_j$, and the ramification index at any point Q_{ji} above P_j is α_j (that is, the stabilizer subgroup of each Q_{ji} has order α_j).*

5.1.24 The set $\rho^{-1}(P_j)$ consists of $\deg\rho/\alpha_j$ points Q_{ji}. Set g' for the genus of C'. Then the canonical divisors of C' and C satisfy $K' = \rho^*(K) + \sum_{ji}(\alpha_j - 1)Q_{ji}$, hence

$$2 - 2g' = \deg\rho \cdot (2 - 2g - \textstyle\sum_j(\alpha_j - 1)/\alpha_j). \tag{5.1.1}$$

This identity $\chi_{top}(C') = \chi \cdot \deg\rho$ motivates the 'virtual Euler characteristic' name of χ (here χ_{top} denotes the topological Euler characteristic).

Let D denote the *rational Dolgachev–Pinkham–Demazure divisor*

$$D := -C|_C - \sum_j \frac{\omega_j}{\alpha_j} P_j,$$

where $-C|_C$ denotes the divisor of the conormal bundle of C in the resolution $\widetilde{X}$. Then $\deg D = -e$. Furthermore, set $D' := \rho^*(D) = \rho^*(-C|_C) - \sum_{i,j}\omega_j Q_{ji}$. It is an integral divisor invariant under the action of the Galois group G of the covering. Define the ample line bundle $\mathscr{L}' := \mathscr{O}_{C'}(D')$. The degree of $\mathscr{L}'^{-1}$ is $\deg\rho \cdot e$, which will be denoted by $-b'$. Let $T' \to C'$ be the total space of $\mathscr{L}'^{-1}$ with its projection, and X' the cone singularity T'/C', cf. 5.1.3. The identity $\deg\mathscr{L}'^{-1} = \deg\rho \cdot e$ explains the terminology 'orbifold/virtual Euler number (or degree)' of e (see also 5.1.25).

We recall that if B is a curve and G is a group of automorphisms of B, then a G-invariant line bundle $L \to B$ such that G acts freely outside the zero section is called an *automorphic factor* with respect to G.

Theorem 5.1.25 ([102, 125, 152, 153, 155, 431, 503, 571]) *The line bundle $T' \to C'$ is an automorphic factor with respect to G and the quotient $T'/G \to C$ is isomorphic to the Seifert line bundle $T \to C$. The actions of G and $\mathbb{C}^*$ on X' commute inducing a good $\mathbb{C}^*$ action on X'/G. The surfaces X and X'/G are isomorphic $\mathbb{C}^*$ equivariantly.*

In particular, the restriction of $T' \to T$ to the generic fiber has degree one (compare with 5.1.10).

By Van Kampen theorem $\pi_1(T \setminus \cup_j P_j) = \overline{\pi_1}$. (Indeed, write $T \setminus \cup_j P_j$ as the union of $T \setminus C$ and $\pi^{-1}(C \setminus \cup_j P_j)$. For the definition of $\overline{\pi_1}$ see 3.7.5.) Note also that $\pi_1(T' \setminus \cup_{ij} Q_{ij}) = \pi_1(T')$. The regular covering $T' \setminus \cup_{ji} Q_{ji} \to T \setminus \cup_j P_j$ gives the extension of groups $1 \to \pi_1(T') \to \overline{\pi_1} \to G \to 1$. Hence, $\overline{\pi_1}$ admits a normal subgroup of finite index isomorphic with the surface fundamental group $\pi_1(C') = \pi_1(T')$. (Since the fundamental group $\pi_1(T')$ is torsion free, we have the following: $\overline{\pi_1}$ admits a normal torsion free subgroup with finite index. This is the statement of the classical so-called *Fenchel's conjecture*. Hence, the affirmative answer to Fenchel's conjecture (see [92, 204]) is equivalent to the existence of the covering from 5.1.23.)

Definitely, one can pull back the line bundle $T' \to C'$ further to the universal covering $\tilde{C}'$. Since $\chi_{top}(C') = \chi \cdot \deg\rho$, if the sign of χ is $+, 0$ or $-$, then

$\tilde{C}' = \mathbb{P}^1$, $\mathbb{C}$ or the upper half plane respectively. In the last two cases the pullback line bundle is trivial, hence $T' \to C'$ (or $T \to C$) appears as the automorphic factor of $\mathbb{C} \times \tilde{C}' \to \tilde{C}'$ with respect to the discrete properly discontinuous group of automorphisms $\pi_1(C')$ (or $\overline{\pi_1}$) of $\tilde{C}'$. Hence the graded ring of X appears as the graded ring of automorphic forms.

5.1.26 The Dolgachev–Pinkham–Demazure Formulae [571] Fix X normal, and let $R = \oplus_{\ell \geq 0} R_\ell$ be the graded algebra of X, and $P_X(t) = \sum_{\ell \geq 0} \dim R_\ell \cdot t^\ell$ the corresponding Poincaré series. Then (using the cone singularity T'/C' and 5.1.4)

$$R_\ell = H^0(C', \mathscr{O}_{C'}(\ell D'))^G. \tag{5.1.2}$$

Set $D^{(\ell)} = \ell(-C|_C) - \sum_j \lceil \ell \omega_j/\alpha_j \rceil P_j$. Then $D^{(\ell)}$ is the greatest divisor on C with $\rho^*(D^{(\ell)}) \leq \ell D'$, hence $H^0(C', \mathscr{O}_{C'}(\ell D'))^G = H^0(C, \mathscr{O}_C(D^{(\ell)}))$. Therefore

$$R_\ell = H^0(C, \mathscr{O}_C(D^{(\ell)})) \text{ and } R = \oplus_{\ell \geq 0} H^0(C, \mathscr{O}_C(D^{(\ell)})). \tag{5.1.3}$$

Similarly, using again 5.1.4(a), the geometric genus $p_g := h^1(\mathscr{O}_T)$ of X is the dimension of

$$H^1(\mathscr{O}_T) = H^1(\mathscr{O}_{T'})^G = \oplus_{\ell \geq 0} H^1(C', \mathscr{O}_{C'}(\ell D'))^G.$$

By duality $H^1(C', \mathscr{O}_{C'}(\ell D'))^* = H^0(C', \mathscr{O}_{C'}(-\ell D' + K'))$. Since $-D^{(\ell)} + K$ is the greatest divisor on C with $\rho^*(-D^{(\ell)} + K) \leq -\ell D' + K'$, one also has

$$H^0(C', \mathscr{O}_{C'}(-\ell D' + K'))^G = H^0(C, \mathscr{O}_C(-D^{(\ell)} + K)) = H^1(C, \mathscr{O}_C(D^{(\ell)}))^*. \tag{5.1.4}$$

Therefore,

$$p_g = \oplus_{\ell \geq 0} h^1(C, \mathscr{O}_C(D^{(\ell)})). \tag{5.1.5}$$

Assume next that L_X is a rational homology sphere, that is $C = \mathbb{P}^1$, and set

$$N(\ell) := \deg D^{(\ell)} = \ell b_0 - \sum_j \lceil \ell \omega_j/\alpha_j \rceil. \tag{5.1.6}$$

Since $e < 0$ one has $\lim_{\ell \to \infty} N(\ell) = \infty$. Moreover,

$$P_X(t) = \sum_{\ell \geq 0} \max\{0, N(\ell) + 1\} t^\ell, \tag{5.1.7}$$

and $p_g = P_{H^1}(t)|_{t=1}$, where $P_{H^1}(t)$ is the polynomial

$$P_{H^1}(t) := \sum_{\ell \geq 0} h^1(C, \mathscr{O}_C(D^{(\ell)}))t^\ell = \sum_{\ell \geq 0} \max\{0, -N(\ell) - 1)\}\, t^\ell. \tag{5.1.8}$$

In particular, P_X and p_g are topological (whenever $C = \mathbb{P}^1$).

Note that $1 + \ell b_0 - \sum_j \lceil \ell\omega_j/\alpha_j \rceil \leq 1 - e\ell$, hence $1 + N(\ell) \leq 0$ for $\ell < 0$.

Example 5.1.27 (Continuation of 5.1.15) Assume that (X, o) is a weighted homogeneous hypersurface normal surface singularity with rational homology sphere link. Let $\{w_i\}_i$ be the weights and d the degree. Then (cf. 8.1.2) the Poincaré series is $P_X(t) = (1 - t^d)/\prod_i (1 - t^{w_i})$. This fact together with (5.1.7) show that the topological type of (X, o) determines (via $P_X(t)$) the data $(w_1, w_2, w_3; d)$ too.

Remark 5.1.28 (P_{H^1} as 'Polynomial Part' of P_X, cf. 8.7)

From $P_X(t)$ one can recover the polynomial $P_{H^1}(t)$ completely. Indeed, take

$$P(t) := \sum_{\ell \geq 0} \chi(\mathscr{O}_C(D^{(\ell)}))t^\ell = \sum_{\ell \geq 0} (N(\ell) + 1)t^\ell.$$

Then (see 8.1.9) $P(t)$ can be written as a rational function $p(t)/q(t)$ with $p, q \in \mathbb{C}[t]$ and $\deg p < \deg q$. Hence, if one writes (in a unique way) $P_X(t)$ as $p(t)/q(t) + r(t)$ with $p,\ q,\ r \in \mathbb{C}[t]$ and $\deg p < \deg q$, then $p/q = P$ and $r = P_{H^1}$.

Remark 5.1.29

(a) Recall the following result, see e.g. [449, Theorem 6] (compare also with 6.5.8): If $\mathscr{L}$ and $\mathscr{M}$ are invertible sheafs on the smooth curve C of genus g, and $\deg \mathscr{L} \geq 2g + 1$, $\deg \mathscr{M} \geq 2g$, then $H^0(C, \mathscr{L}) \otimes H^0(C, \mathscr{M}) \to H^0(C, \mathscr{L} \otimes \mathscr{M})$ is onto. Since $\deg D^{(\ell)}$ tends to infinity (cf. (5.1.6)), one can identify easily an ℓ_0 such that $\oplus_{\ell > \ell_0} R_\ell \subset \mathfrak{m}_{X,0}^2$, and $\oplus_{\ell \leq \ell_0} R_\ell$ contains a set of generators for R. (In this way we get a concrete ℓ_0; the existence of a non-concrete ℓ_0 follows more rapidly from the fact that R is finitely generated R_0-algebra.) See part (b) as well.

Take, e.g. $g = 0$, and define $\mathfrak{r} = \chi/e$. Then by the formulae from 5.1.24 $\mathfrak{r} = \deg K_{C'}/\deg D'$. Hence, using (5.1.4), for $\ell > \mathfrak{r}$ one has $h^1(\mathscr{O}_C(D^{(\ell)})) = 0$, or $N(\ell) \geq -1$. This also shows that $P_{H^1} \equiv 0$ if $\mathfrak{r} < 0$, and $\deg P_{H^1} \leq \mathfrak{r}$ otherwise.

Note also that $N(\alpha) = \mathfrak{o} \geq 1$ (where $\mathfrak{o}$ is the order of h_0 in H, cf. (3.5.5)), and for any $\ell \geq 0$ one has $N(\ell + \alpha) = N(\ell) + N(\alpha)$. Hence for $\ell > \alpha + \mathfrak{r}$ one has $N(\ell) \geq 0$, and by the above theorem $R_\ell \subset \mathfrak{m}_{X,0}^2$ for all $\ell > 2\alpha + \mathfrak{r}$. (This bound $2\alpha + \mathfrak{r}$ in general is not sharp.)

For other interpretations of $\mathfrak{r}$ in terms of the canonical cycle see 6.3.30. For formulae of e and $\mathfrak{r}$ in the case of complete intersections see 9.7.24.

(b) The optimal ℓ_0 from part (a) is subject of the representation theory as well. E.g., if the finite group H acts linearly on $\mathbb{C}[x_1, \cdots, x_d]$, then $\mathbb{C}[x_1, \cdots, x_d]^H$

is generated in degree $\leq |H|$ [521]. This is called the *Noether bound* (and it is sharp for the cyclic group) and it has several generalizations in the literature.

Example 5.1.30 ([571])

(a) Assume that $g = 0$, $b = 1$, $\nu = 3$ such that each pair (α, ω) is $(4, 1)$. Let P_1, P_2, P_3 be the points of $C = \mathbb{P}^1$ corresponding to the singular fibers, we identify them with $0, 1, \infty$ on $\mathbb{P}^1$. We also write $\mathscr{O}_C(-C) = \mathscr{O}_{\mathbb{P}^1}(P_3)$. Then $D^{(\ell)} = \ell P_3 - \sum_j \lceil \ell/4 \rceil P_j$. The graded ring R is generated by an element x in degree 3 and by y and z in degree 4. Indeed, $D^{(3)} = 2P_3 - P_1 - P_2$, and we take $x = st(t - s) \in R_3$, while $D^{(4)} = 3P_3 - P_1 - P_2$, and we take $y = s^2t(t - s)$, $z = st^2(t - s)$ from R_4. The relation between them (that is, the equation of X is) $yz(z - y) = x^4$, a hypersurface singularity with $p_g = 1$. The above formulae for x, y, z are considered via the monomorphism $\oplus_\ell R_\ell \subset \oplus_\ell H^0(C, \mathscr{O}_C(\ell P_3)) \simeq \oplus_\ell \mathbb{C}[s, t]_\ell$.

(b) Assume that $g = 1$, $b = 1$, $\nu = 1$ and $(\alpha, \omega) = (3, 2)$. Let P be the points of C corresponding to the singular fibers, and we set $\mathscr{O}_C(-C) = \mathscr{O}_C(R)$ for some $R \in C$. Then $D^{(\ell)} = \ell R - \lceil 2\ell/3 \rceil P$. One can distinguish three cases.

If $R \sim P$ then R is generated by three elements in degrees 1, 6 and 9, $p_g = 3$. A possible equation is e.g. $z^2 = y^3 + x^{18}$. (Here and below $\sim$ means 'linear equivalent'.)

If $R \not\sim P$ but $2R \sim 2P$, then R is generated by three elements in degrees 2, 3 and 7, $p_g = 2$, a possible equation is $z^2 = x(y^4 + x^6)$.

If $2R \not\sim 2P$, then X is not a hypersurface, and $p_g = 1$.

Example 5.1.31 For any negative definite star-shaped graph Γ there exists an analytic weighted homogeneous structure such that the corresponding resolution graph is Γ. This fact follows from 5.1.20. More directly, one can also argue as follows. Having the Seifert invariants $\mathrm{Sf}(b_0, g : (\alpha_j, \omega_j)_j)$, one can take a curve C of genus g and a rational divisor D on it of degree $-e$, which plays the role of the rational Dolgachev–Pinkham–Demazure divisor $-C|_C - \sum_j (\omega_j/\alpha_j) P_j$. Then (X, o) is $\mathrm{Spec}(R)$, where R is the graded algebra defined as in (5.1.3), for details see [144, 748]. (See also 5.1.36–5.1.37 for $g = 0$ case as well.)

On the other hand, fixing an analytic singularity (X, o), whose dual resolution graph Γ is star-shaped, usually it is not automatical that (X, o) is weighted homogeneous, or that it is an equisingular (that is, p_g-constant) deformation of a weighted homogeneous singularity. We consider the following two examples.

The first example is the hypersurface isolated singularity $(Y, o) = (x^2 + y^2 + z^2)^2 + x^6 + y^6 + z^6 = 0$. Its minimal resolution graph contains only one vertex with Euler number -4 and genus 5, though the equation of Y is not (semi-)quasi-homogeneous. However, there exists a family of isolated complete intersection singularities $(X_\lambda, o)_{\lambda \in \mathbb{C}} : \{x^2 + y^2 + z^2 + \lambda w = 0,\ x^6 + y^6 + z^6 + w^2 = 0\}$, with (X_0, o) weighted homogeneous (with $\mu = 63$ and $p_g = 9$), such that the

deformation is equisingular (p_g and also the link stays constant), and $(X_{-1}, o) = (Y, o)$.

The second example is the singularity (X_a, o) from 6.8.13, which has the same rational homology sphere link as a Brieskorn hypersurface singularity, nevertheless the two germs are not equisingular (for details see 6.8.13). Similar phenomenon can appear even with integral homology sphere links, see [399].

5.1.32 The Universal Abelian Covering If the resolution graph of X is a string of rational curves then its universal abelian covering is smooth, cf. 2.3.1.

Theorem 5.1.33 ([506]) *Assume that X is a weighted homogeneous surface singularity with $g = 0$ and $\nu \geq 3$. Then its universal abelian covering X_a is a Brieskorn complete intersection singularity*

$$\left\{z \in \mathbb{C}^\nu \;\middle|\; f_i := c_{1i} z_1^{\alpha_1} + \cdots + c_{\nu i} z_\nu^{\alpha_\nu} = 0, \quad i = 1, \ldots, \nu - 2\right\},$$

where every maximal minor of (c_{ji}) is non zero (that is, (c_{ji}) has 'full rank'). The coefficients (c_{ji}) and the action of H are described in the next proof.

Proof We show that there is a choice of the coefficients (c_{ji}) such that the quotient singularity with respect to H is exactly (X, o). Let us fix first any set of coefficients (c_{ji}) of full rank. By row operations, the matrix (c_{ji}) can be transformed into

$$\begin{pmatrix} c_1 & c_1' & 1 & 0 & \cdots & 0 \\ c_2 & c_2' & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \\ c_{\nu-2} & c_{\nu-2}' & 0 & 0 & \cdots & 1 \end{pmatrix}$$

such that all the entries c_i and c_i' are nonzero and $c_i c_k' - c_k c_i' \neq 0$ if $i \neq k$.

In this way the complete intersection remains unmodified. Let us denote it by Z.

The integral weights of the equations are $w_j = \alpha/\alpha_j$, hence the $\mathbb{C}^*$ action on $\mathbb{C}^\nu$ and Z is $t \cdot (z_1, \ldots, z_\nu) = (t^{\alpha/\alpha_1} z_1, \ldots, t^{\alpha/\alpha_\nu} z_\nu)$.

We invite the reader to recall the structure of the group H from 3.5.4 and also the lattice identities from 3.5.2. Next we define an H-action on Z.

The group H acts on $\mathbb{C}^\nu$ by $h * z_j = \theta([E^*_{js_j}])(h) z_j$. (For the definition of θ see 2.1.16.) Since the equations f_i are eigenvectors we obtain an induced action on Z too. The action on Z is free off the origin. Indeed, assume that $z \in Z \setminus 0$ is fixed by $h \in H$. Since at most one coordinate of z can be zero, say z_j, and $\{[E^*_{is_i}]\}_{i:i\neq j}$ generate H (use 3.5.4), we get $\theta(h) = 1$, hence $h = 1$ too.

The $\mathbb{C}^*$-action and the H-action on both $\mathbb{C}^\nu$ and Z commute (since both are diagonal), hence we obtain an induced $\mathbb{C}^*$-action on Z/H. However, note that the $\mathbb{C}^*$-action restricted to $\mathbb{Z}_{\mathfrak{o}} = \langle t = e^{2\pi i/e\alpha} \rangle \subset \mathbb{C}^*$ coincides with the restriction of the H-action to the subgroup $\langle h_0 \rangle$. Hence the induced $\mathbb{C}^*$ action on Z/H has an isotropy group $\mathbb{Z}_{\mathfrak{o}}$; factorization by it we get the effective good $\mathbb{C}^*$-action on Z/H.

Our goal is to verify that Z/H is isomorphic to X (if we make the right choice of the coefficients (c_{ji})). First we verify that for any choice of (c_{ji}) the quotient Z/H has the same Seifert invariants as X.

The Seifert invariants of Z are given in 5.1.17 (by changing $(a_1, \dots, a_\nu)$ into $(\alpha_1, \dots, \alpha_\nu)$). *We use for them the standard notations with tilde.* E.g., $\tilde{A} = \prod_k \alpha_k$, $\tilde{a} = \alpha$, $\tilde{\alpha}_j = \alpha/\alpha_j^\vee$, where $\alpha_j^\vee = \operatorname{lcm}\{\alpha_k,\ k \neq j\}$; $\tilde{q}_j = \prod_k \alpha_k/(\alpha_j\alpha_j^\vee)$; $|\tilde{e}| = (\prod_k \alpha_k)/\alpha^2$.

All the singular $\mathbb{C}^*$ orbits of Z are in the coordinate hyperplanes. Let us analyze those in $z_1 = 0$. This is the complete intersection curve $-z_2^{\alpha_2} = z_3^{\alpha_3}/c_1' = \cdots = z_\nu^{\alpha_\nu}/c_{\nu-2}'$. This has exactly $\tilde{q}_1$ components. Using the identities 3.5.2, one sees that h_1 preserves each component, while H acts transitively. Furthermore, using the group relation of H and 3.5.4 one has that $|H/\langle h_1\rangle| = \prod_{j\neq 1} \alpha_j/\alpha_1^\vee = \tilde{q}_1$. Hence the set of components is an $H/\langle h_1\rangle$ torsor. In particular, in Z/H these components are glued together into a unique orbit.

Each nonsingular orbit of Z covers its image in Z/H by degree $\mathfrak{o}$, hence $\tilde{C} \setminus \{\text{singular orbits}\} \to C_{Z/H} \setminus \{\nu \text{ points}\}$ is an $|H|/\mathfrak{o}$ regular covering. Since $2 - 2\tilde{g} - \sum_j \tilde{q}_j = (2-\nu)\tilde{A}/\tilde{a} = (2-\nu)|H|/\mathfrak{o}$ (cf. 5.1.17), we get that $g(C_{Z/H}) = 0$. Similarly, since $\tilde{C} \to C_{Z/H}$ is a (branched) $|H|/\mathfrak{o}$-covering, by 5.1.10 $e_{Z/H} = \tilde{e}\mathfrak{o}^2/|H| = e$.

The isotropy subgroup of a point $z \in Z$ with $z_1 = 0$ has order $\tilde{\alpha}_1 = \alpha/\alpha_1^\vee$, cf. 5.1.17. Since this point has an $\langle h_1\rangle$-orbit of length $\operatorname{ord}_H(h_1)$, the isotropy group in Z/H has order $\tilde{\alpha}_1 \cdot \operatorname{ord}_H(h_1)$. In fact, this should be divided by $\mathfrak{o}$, the order of covering of the generic orbits. Hence, $(\alpha_1)_{Z/H} = \tilde{\alpha}_1 \cdot \operatorname{ord}_H(h_1)/\mathfrak{o} = \alpha_1$.

The invariants $(\omega_{Z/H})_i$ will be determined via the orbit invariants.

Take $z = (0, 1, r_3, r_4, \dots)$ on $z_1 = 0$, where $r_i^{\alpha_i} = -c_{i-2}'$ for $i \geq 3$. Next, take the normal section N_z of type $z(u) := (u, 1, \phi_3(u), \dots, \phi_\nu(u))$ in Z, where $u \in (\mathbb{C}, 0)$. Here $\phi_i(u)$ is the local branch of $(-c_{i-2}' - c_{i-2}u^{\alpha_1})^{1/\alpha_i}$ with $\phi_i(0) = r_i$, $i \geq 3$.

The action of h_1 identifies z with $h_1 * z$ in Z/H. The generator of the isotropy group in Z/H of the $\mathbb{C}^*$-action is $\lambda = e^{2\pi i/\alpha_1}$. This lifted to Z acts as

$$\lambda \cdot (z_1, \dots, z_\nu) = (e^{2\pi i \frac{\alpha}{\alpha_1\alpha_1}} z_1, \dots, e^{2\pi i \frac{\alpha}{\alpha_1\alpha_\nu}} z_\nu).$$

This should be modified by the argument $\mathfrak{o} = \alpha|e|$ of the isotropy group $\mathbb{Z}_\mathfrak{o}$ into

$$(e^{2\pi i \frac{1}{\alpha_1\alpha_1|e|}} z_1, \dots, e^{2\pi i \frac{1}{\alpha_1\alpha_\nu|e|}} z_\nu).$$

This, by the action of h_1 can be mapped back to $N_z \subset Z/H$, this happens by multiplication by the diagonal action (cf. 3.5.2)

$$(e^{2\pi i(\frac{1}{\alpha_1\alpha_1 e} - \frac{\omega_1'}{\alpha_1})}, e^{2\pi i \frac{1}{\alpha_1\alpha_2 e}}, \dots, e^{2\pi i \frac{1}{\alpha_1\alpha_\nu e}}).$$

Hence, the action on N_z transforms $(u, 1, \phi_3(u), \dots, \phi_\nu(u))$ into

$$(e^{2\pi i(\gamma_{H/Z})_1/\alpha_1}u, 1, \phi_3(u), \dots, \phi_\nu(u)) = (e^{-2\pi i\frac{\omega_1'}{\alpha_1}}u, 1, \phi_3(u), \dots, \phi_\nu(u)).$$

(Note that $\phi_i(\xi u) = \phi_i(u)$ for any ξ with $\xi^{\alpha_1} = 1$.) Thus the orbit invariant $(\gamma_{H/Z})_1$ is $-\omega_1'$ (mod α_1), hence $(\omega_{H/Z})_1 = \omega_1$, cf. the paragraphs after Theorem 5.1.8.

Hence, the Seifert invariants of Z/H and X agree. Next, we wish to apply the Analytic Classification Theorem 5.1.20: we show that with the right choice of (c_{ji}) the analytic types of Z/H and X agree.

Note that $\tilde{C}$ (the base of the Seifert fibration of Z) has equations $\{f_i = 0\}_i$ in the weighted projective space $\mathbb{P}^{\nu-1}_w$. Consider $\mathbb{C}^\nu$ with coordinates $(w_1, \dots, w_\nu)$ and the plane $S := \{\sum_j c_{ij}w_j = 0,\ i = 1, \dots, \nu-2\}$. The fibers of the map $\mathbb{C}^\nu \to \mathbb{C}^\nu$, $w_i = z_i^{\alpha_i}$ are the orbits of the action $(z_1, \dots, z_\nu) \mapsto (\xi_1 z_1, \dots, \xi_\nu z_\nu)$ of $\prod_i \mathbb{Z}_{\alpha_i}$. This induces a map $Z \to S$ and a covering $\tilde{C} \to \mathbb{P}S = \mathbb{P}^1$ at the level of projective spaces. In this way $\mathbb{P}S$ appears as the quotient of $\tilde{C}$ under the induced action of $\prod_i \mathbb{Z}_{\alpha_i}/\mathbb{Z}_{\mathfrak{o}}$. Since $\prod_i \mathbb{Z}_{\alpha_i}/\mathbb{Z}_{\mathfrak{o}} = H/\langle h_0\rangle$, the covering $\tilde{C} \to \mathbb{P}S$ can be identified with the covering $\tilde{C} \to C_{Z/H}$ together with the base points of the Seifert fibers. In the case of $\mathbb{P}S = \mathbb{P}^1$ they are the intersection points with the coordinate hyperplanes $w_i = 0$, which can be determined from the coefficients (c_{ji}). In particular, with the right choice of these coefficients we can match the base points of the Seifert fibers of Z/H and X. □

Remark 5.1.34 Motivated by the above proof (having in mind the weights of $X = Z/H$) and also by certain combinatorial/topological reasons (see e.g. 6.2.12, or 8.1.B and 8.6.13) we can renormalize the weights $\{\alpha/\alpha_i\}_i$ of $\mathbb{C}^\nu$ (and of the universal abelian covering and Z) by dividing them by $\mathfrak{o} = \alpha|e|$. Accordingly, we define the new weights $\mathrm{wt}(z_j) = (|e|\alpha_j)^{-1} \in \frac{1}{\mathfrak{o}}\mathbb{Z}$. In fact, $\mathrm{wt}(z_i) = -(E_0^* \cdot E_{ir_i}^*)$ too, This induces a grading on the polynomial ring $\mathbb{C}[z]$, and all f_i are weighted homogeneous of degree $|e|^{-1}$.

Example 5.1.35 ([152, 153, 431]) Consider the integers $p, q, r \geq 2$ with $1/p + 1/q + 1/r < 1$, and the Schwarz triangle group $\Delta = \Delta(p, q, r)$ acting on the Bolyai–Lobachevsky plane, as in 3.7.6. Then $\Delta \subset \mathrm{PSL}_2(\mathbb{R})$. It is known that $\pi_1(\mathrm{PSL}_2(\mathbb{R})) = \mathbb{Z}$. Let $\phi : \mathrm{SL}_2(\mathbb{R})\tilde{} \to \mathrm{PSL}_2(\mathbb{R})$ be the universal covering (the universal coverings of $\mathrm{PSL}_2(\mathbb{R})$ and $\mathrm{SL}_2(\mathbb{R})$ agree). Set $\Delta\tilde{} = \Delta\tilde{}(p, q, r) := \phi^{-1}(\Delta(p, q, r))$. It is called the *centrally extended triangle group*, and it satisfies $0 \to \mathbb{Z} \to \Delta\tilde{} \to \Delta \to 1$. Its presentation is $\{\tau_1^p = \tau_2^q = \tau_3^r = \tau_1\tau_2\tau_3\}$ (where $\tau_1\tau_2\tau_3$ generates the central subgroup $\mathbb{Z}$), cf. [431]. This can be compared with 3.7.5: $\Delta\tilde{}$ agrees with the fundamental group of the plumbed 3-manifold associated with the start-shaped graph $\Gamma(p, q, r)$ with $g = 0$ and central vertex -1, and the three legs of length one decorated by $-p$, $-q$, $-r$ respectively. (This is the link of the *triangle singularity of Dolgachev* $D_{p,q,r}$, cf. 7.2.20.) In particular, the compact 3-manifold $\mathrm{PSL}_2(\mathbb{R})/\Delta = \mathrm{SL}_2(\mathbb{R})\tilde{}/\Delta\tilde{}$ is the link of $D_{p,q,r}$. (Indeed, $\mathrm{SL}_2(\mathbb{R})\tilde{}$ is diffeomorphic to $\mathbb{R}^3$, and the link is a $K(\pi_1, 1)$ space, cf. 3.6.9.)

On the other hand, $\mathrm{SL}_2(\mathbb{R})\tilde{}/[\Delta\tilde{}, \Delta\tilde{}\,]$ is the link $\Sigma(p,q,r)$ of the Brieskorn singularity $x^p + y^q + z^r = 0$ [431] and $\pi_1(\Sigma(p,q,r)) = [\Delta\tilde{}, \Delta\tilde{}\,]$. In particular, $\{x^p + y^q + z^r = 0\} \to D_{p,q,r}$ is a universal abelian covering with group $H = \Delta\tilde{}/[\Delta\tilde{}, \Delta\tilde{}\,]$; a fact compatible with the construction of 5.1.33.

The case $1/p + 1/q + 1/r > 1$ is analogous. In that case the graph $\Gamma = \Gamma(p,q,r)$ described above is not negative definite, hence we have to consider the negative definite plumbing graph associated with $-\Gamma$. It has three legs and all decorations are -2's. This is the graph of some DE singularity (X, o). The central extension is the $\mathbb{Z}_2$ extension $\mathrm{SU}_2 \to \mathrm{SO}_3$, hence $0 \to \mathbb{Z}_2 \to \Delta\tilde{} \to \Delta \to 1$. Finally, again, $\{x^p + y^q + z^r = 0\} \to X$ is a universal abelian covering [431]. The order of Δ^* and $\Delta\tilde{}$ is $4/(p^{-1} + q^{-1} + r^{-1} - 1)$, while the order of $[\Delta\tilde{}, \Delta\tilde{}\,]$ is $4/(pqr(p^{-1} + q^{-1} + r^{-1} - 1)^2)$. (For the definition of Δ^* see 3.7.6.)

5.1.36 Special Analytic Plumbing Let us fix a star-shaped plumbing graph with given Seifert invariants Sf with $g = 0$. We determine a normal surface singularity (X, o) with this fixed resolution graph by the construction of its resolution $\widetilde{X}$ by an 'analytic plumbing'. Corresponding to the legs we fix the distinct complex numbers $\{p_j\}_{1 \le j \le \nu}$ in $\mathbb{C}$ (the affine coordinate of the points P_j). Each leg (with divisors $\{E_{ji}\}_{i=1}^{s_j}$, $1 \le j \le \nu$) will be covered by open sets $\{U_{j,i}\}_{i=0}^{s_j}$, copies of $\mathbb{C}^2$ with coordinates $(u_{j,i}, v_{j,i})$, exactly as in the proof $(a \Rightarrow c)$ of 2.3.1 (with a p_j shift in v_0). The gluing maps are $v_{j,i} = u_{j,i-1}^{-1}$ $(1 \le i \le s_j)$ and $u_{j,i}$ equals $u_{j,i-1}^{b_{ji}} v_{j,i-1}$ for $2 \le i \le s_j$ and $u_{j,0}^{b_{j1}}(v_{j,0} - p_j)$ for $i = 1$. Furthermore, all $U_{j,0}$ charts will be identified with each other: $u_{j,0} = u_{k,0}$, $v_{j,0} = v_{k,0}$; denoted simply by U_0, with coordinates (u_0, v_0). Until now, the curve E_0 appears only in U_0, it has equation $u_0 = 0$. To cover E_0 completely we need another copy U_{-1} of $\mathbb{C}^2$ with coordinates (u_{-1}, v_{-1}) as well; the gluing of $U_0 \setminus \{v_0 = 0\}$ with $U_{-1} \setminus \{u_{-1} = 0\}$ is $v_0 = u_{-1}^{-1}$, $u_0 = u_{-1}^{b_0} v_{-1}$.

Proposition 5.1.37 *The above analytic structure admits a $\mathbb{C}^*$ action. Moreover, the minimal good resolution of any weighted homogeneous singularity with $g = 0$ admits such an analytic plumbing representation for certain constants $\{p_j\}_j$. (The equation of the lifted $\mathbb{C}^*$ orbits to the resolution constructed by such an analytic plumbing are given by $v_0 = \text{constant} \neq p_j$ in the case of generic orbits and $v_{j,s_j} = 0$ for $j = 1, \ldots, \nu$ in the case of singular ones.)*

Proof We blow down the legs in $\widetilde{X}$. We obtain a space T, which contains E_0, and it has a cyclic quotient singularity X_{α_j,ω_j} at the point $P_j \in E_0$. We show that there is a natural Seifert line bundle structure $T \to C = E_0$. For this we analyze different local charts in a neighbourhood of P_j organized in the next diagram. For simplicity

we drop the index j; by D_u (resp $\mathbb{C}_u$) we mean a disc (affine line) with coordinate u.

$$\begin{array}{ccccc}
(z_1,z_2)\in & D_{z_1}\times\mathbb{C}_{z_2} & \xrightarrow{\psi_0} & D_{v_0}\times\mathbb{C}_{u_0}\subset U_0 \\
\downarrow & \downarrow\mathbb{Z}_\alpha & & \downarrow\phi_0 \\
[z_1,z_2]\in D_{z_1}\times\mathbb{C}_{z_2}/\mathbb{Z}_\alpha & \xrightarrow{\simeq} & X_{\alpha,\omega} & =\{(x,y,z):\ xy^{\alpha-\omega}=z^\alpha\}\ \text{normalized}
\end{array}$$

Above the left hand column is the factorization from 5.1.6; ϕ_0 (the restricted resolution) is $x = u_0^\alpha v_0^\omega$, $y = v_0$, $z = u_0 v_0$; the $\mathbb{Z}_\alpha$-equivariant ψ_0 is $v_0 = z_1^\alpha$, $u_0 = z_2 z_1^{-\omega}$, which factorizes through $(z_1, z_2) \mapsto [z_1, z_2]$. Note also that $\phi_0 \circ \psi_0$ is the map $x = z_2^\alpha$, $y = z_1^\alpha$, $z = z_2 z_1^{\alpha-\omega}$ from the proof *(a)*⇔ *(d)* of Theorem 2.3.1. These spaces project to C, in compatible local charts the maps are $(z_1, z_2) \mapsto z_1^\alpha$, $(v_0, u_0) \mapsto v_0$, $(x, y, z) \mapsto y$. Furthermore, they admit the following compatible $\mathbb{C}^*$ actions: $t \cdot (z_1, z_2) = (z_1, tz_2)$, $t \cdot (v_0, u_0) = (v_0, tu_0)$, $t \cdot (x, y, z) = (t^\alpha x, y, tz)$. This identifies the projection $X_{\alpha,\omega} \to D_{v_0}$ and the $\mathbb{C}^*$ action in $X_{\alpha,\omega}$ with the wished local model from 5.1.6 and 5.1.18. This extends naturally to $T \to C$ (as the v_0-projection). The generic $\mathbb{C}^*$ orbits are given by $v_0 = const. \neq p_j$.

The special $\mathbb{C}^*$ orbit can be seen in the chart U_s. By the transformation $u_s = u_0^\alpha v_0^\omega$ and $v_s = u_0^{-\omega'} v_0^{-\tau}$ (or $u_0 = u_s^{-\tau} v_s^{-\omega}$, $v_0 = u_s^{\omega'} v_s^\alpha$) in U_s the projection to C is $(v_s, u_s) \mapsto u_s^{\omega'} v_s^\alpha$ and the $\mathbb{C}^*$ action is $t \cdot (u_s, v_s) = (t^\alpha u_s, t^{-\omega'} v_s)$. Hence the special orbit is given by $\{v_s = 0\}$. (For the notation $\omega\omega' - 1 = \alpha\tau$ see (2.2.5).)

The $\mathbb{C}^*$ action on the other charts can also be detected.

This shows the existence of the $\mathbb{C}^*$ action on the constructed analytic type.

Conversely, if we start with a weighted homogeneous singularity X, the existence of the plumbing is guaranteed by the Analytic Classification Theorem. Indeed, for any such X we construct an analytic plumbing X_{ap} with the same graph and the same meeting points P_j (via the appropriate choice of numbers p_j). Then, by 5.1.20, the two analytic structures X and X_{ap} agree. □

5.1.38 In this model, several statements and formulae of the Dolgachev–Pinkham theory can be reproved. Let us explain first in this language the contribution of the singular points in the Dolgachev–Pinkham–Demazure divisor, that is, the term $-\lceil \ell\omega_j/\alpha_j \rceil$ in each $D^{(\ell)}$. The ℓ-homogeneous element $f = u_0^\ell (v_0 - p_j)^m \in \mathbb{C}[u_0, v_0]$ of the chart U_0 has the form $u_{j,s_j}^{-\tau_j \ell + \omega_j' m} v_{j,s_j}^{-\omega_j \ell + \alpha_j m}$ in U_{j,s_j}. Since $\alpha_j \tau_j = \omega_j \omega_j' - 1$, f has a regular (holomorphic) extension in $\cup_i U_{j,i}$ if and only if $m \geq \lceil \ell\omega_j/\alpha_j \rceil$, $1 \leq j \leq \nu$.

The element $u_0^\ell v_0^n$ has a regular extension to $U_0 \cup U_{-1}$ if and only if $n \leq \ell b_0$. In particular, a basis of R_ℓ, the vector space of ℓ homogeneous elements, consists of

$$u_0^\ell \cdot \prod_j (v_0 - p_j)^{\lceil \ell\omega_j/\alpha_j \rceil} \cdot v_0^k,$$

where $0 \leq k \leq \ell b_0 - \sum_j \lceil \ell\omega_j/\alpha_j \rceil = \deg D^{(\ell)}$ (compatibly with (5.1.7)).

Remark 5.1.39 (End Curve Functions) Theorem 5.1.33 and its proof shows that any weighted homogeneous singularity with $g = 0$ satisfies the 'end curve condition'. This condition requires for each fixed 'end exceptional component' of the exceptional locus the existence of an 'end curve function', whose reduced strict transform is irreducible and it intersects only that exceptional component, and the intersection is transversal. (This property will be discussed in detail in 8.5.)

Indeed, take the coordinate function z_j from 5.1.33. Then $g_j := z_j^{\mathrm{ord}(h_j)}$ is H-invariant, hence $g_j \in R$. The reduced strict transform of its zero set is a singular $\mathbb{C}^*$ fiber in $T \to C$, hence in the good minimal resolution it satisfies the above requirements.

This can be seen directly in the model 5.1.36 of the analytic plumbing as well. First note that the multiplicity of $\mathrm{ord}(h_j) \cdot E^*_{js_j}$ along E_0 is $\mathrm{ord}(h_j)/(|e|\alpha_j) = \alpha_j^\vee$. Hence g_j should be in $R_{\alpha_j^\vee}$. Indeed, in U_0 it is

$$u_0^{\alpha_j^\vee} \cdot \prod_{k \neq j} (v_0 - p_k)^{\alpha_j^\vee \omega_k/\alpha_k} \cdot (v_0 - p_j)^m,$$

where $m := \alpha_j^\vee(b_0 - \sum_{k\neq j} \omega_k/\alpha_k)$. The vanishing order of g_j along $\{v_{j,s_j} = 0\}$, the strict transform of the singular orbit, is $-\omega_j\alpha_j^\vee + \alpha_j m = \alpha_j\alpha_j^\vee|e| = \mathrm{ord}(h_j)$.

5.1.40 Is the Hilbert–Samuel Function Topological if $g = 0$? Assume that $g = 0$. Since p_g and the Poincaré series P_X are topological, cf. (5.1.7) and (5.1.8), it is natural to ask the analogue question for other analytic invariants, e.g. for the local Samuel–Hilbert function (for its definition see 6.7.16). The Hilbert–Samuel function determines several other invariants, as the multiplicity or the embedding dimension. Theorem 8.5.31 shows that the multiplicity is indeed topological. However, in general, the whole Samuel–Hilbert function is not, e.g. already the embedding dimension $\dim(\mathfrak{m}_{X,o}/\mathfrak{m}_{X,o}^2)$ cannot be recovered from the Seifert invariants.

Indeed, consider the Seifert invariants $b_0 = 2$, $\nu = 6$, $\alpha_1 = 2, 2, 3, 3, 7, 7$, and $\omega_j = 1$ for all j. Then $\mathfrak{o} = 2$. We apply Theorem 5.1.33 with $c_i' = 1$ and parameters $\{c_i\}_{i=1}^4$ (which by the full rank condition are all non-zero and distinct); all of them determine certain analytic structures. Following the algorithm of Theorem 5.1.33 one can write the equations of the universal abelian covering and its quotient. (For the computation see [494].) One finds the following two defining equations for X:

$$y_3^4 - y_2^3 y_4 + (c_1 + c_2) y_3^2 y_4 + c_1 c_2 y_4^2,$$

$$y_1^7 - y_2^3 + (c_1 + c_2 - c_3 - c_4) y_3^2 + (c_1 c_2 - c_3 c_4) y_4.$$

Hence, if $c_1c_2 - c_3c_4 \neq 0$ then the embedding dimension is 3, otherwise it is 4. (Note that the restriction $c_1c_2 - c_3c_4 = 0$ is non-trivial in the space of analytic structures.)

On the other hand we have the following positive results; note that via the above example the assumptions are rather sharp.

Proposition 5.1.41 ([494]) *Let $P_{\mathfrak{m}_{X,o}/\mathfrak{m}_{X,o}^2}(t)$ be the Poincaré series of the induced grading on $\mathfrak{m}_{X,o}/\mathfrak{m}_{X,o}^2$. Then $P_{\mathfrak{m}_{X,o}/\mathfrak{m}_{X,o}^2}(t)$ has a (precise) combinatorial characterization in terms of Seifert invariants if any of the following assumptions holds: (a) $\mathfrak{o} = 1$; (b) $\mathfrak{o} \leq \alpha/\alpha_j^\vee$ (equivalently, $\mathrm{ord}(h_j) \leq \alpha_j$) for all j; (c) $\nu \leq 5$; (d) $b_0 \geq \nu - 2 \geq 1$, and $\omega_j = 1$ for all j.*

Example 5.1.42 (Some Algebraically Defined Coverings) Consider the graded ring $R = \oplus_{\ell \geq 0} R_\ell$ as in 5.1.26. For any positive integer k take the graded ring $R^{(k)} := \oplus_{\ell \geq 0} R_{k\ell}$ as well. Then $R^{(k)}$ is normal and the inclusion $R^{(k)} \hookrightarrow R$ gives a regular map between the corresponding affine varieties $X \to X^{(k)}$. In fact, we can regard $X^{(k)}$ as the quotient of X with respect to a cyclic action of $\mathbb{Z}_k$. Indeed, regard $\mathbb{Z}_k$ as a subgroup of $\mathbb{C}^*$, hence $\mathbb{Z}_k$ acts naturally on X, and $R^{\mathbb{Z}_k} = R^{(k)}$, hence $X^{(k)} = X/\mathbb{Z}_k$.

E.g., if $k = l\alpha + 1$ for certain integer l, then $X \to X^{(k)}$ is an o-ramified cyclic covering (we will give some more details regarding this case in 5.1.43). However, in general, the covering $X \to X^{(k)}$ is not *o-ramified*. Take e.g. for X the germ from 5.1.30(a) and $k = 4$. Then $X^{(k)}$ is smooth with local coordinates (y, z) and the covering is the cyclic ramified covering $x^4 = yz(z - y)$. Note that if $\alpha | k$ then $X^{(k)}$ is an algebraic cone (use (5.1.3) and 5.1.3).

Example 5.1.43 (Some Topologically Defined Coverings) First we construct a topological regular covering of a Seifert 3-manifold. For simplicity we will assume that $g = 0$ (the general case runs similarly).

Start with Seifert invariants $(b; (\alpha_i, \omega_i)_{i=1}^\nu)$ and with the corresponding Seifert manifold. For any positive integer l set $k = l\alpha + 1$ and $b' := b + |e|l\alpha$. Then the fundamental groups associated with $(b; (\alpha_i, \omega_i)_{i=1}^\nu)$ and $(b'; (\alpha_i, \omega_i)_{i=1}^\nu)$ are $\pi_1 = \langle g_0,\ g_j\,;\, g_j^{\alpha_j} = g_0,\ \prod_j g_j^{\omega_j} = g_0^b \rangle$ and $\pi_1' = \langle p_0,\ p_j\,;\, p_j^{\alpha_j} = p_0,\ \prod_j p_j^{\omega_j} = p_0^{b'} \rangle$. Then π_1 embeds into π_1' by $g_0 = p_0^k$ and $g_j = p_j p_0^{a_j}$, where $a_j = l\alpha/\alpha_j$. There is a similar embedding $H \hookrightarrow H'$ of first homology groups as well. In fact, there is an epimorphism $H' \to \mathbb{Z}_k$ given by $[p_0] \mapsto \widehat{1}$, $[p_j] \mapsto -\widehat{a_j}$ with kernel H; and the kernel of $\pi_1' \to H' \to \mathbb{Z}_k$ is π_1. (The corresponding topological $\mathbb{Z}_k$ covering at level of plumbing graphs can also be constructed as in 4.1.35 by putting an arrowhead on the central vertex of the graph of X' with multiplicity $\mathfrak{o}' = k\mathfrak{o}$).

In this way we get a topological regular covering of *links* of singularities, however, the construction does not guarantees that if (X, o) is a singularity, whose link is the Seifert manifold associated with $(b; (\alpha_i, \omega_i)_{i=1}^\nu)$ then the Galois action of the link can be extended in an analytic way to (X, o) (hence, $X/\mathbb{Z}_k$, as an analytic space would make sense). In general this is not happening. However, if X is weighted homogeneous, then the Galois action of the covering will be compatible with the $\mathbb{C}^*$ action, hence we get a o-ramified covering $X \to X'$, whose topology is given above. In fact, this covering can be obtained algebraically by the choice of

$k = l\alpha + 1$ in 5.1.42. In this algebraic situation the induced covering $C \to C'$ at the level of central curves is trivial.

In conclusion: for any weighted homogeneous singularity whose link has Seifert invariants $(b, g; (\alpha_i, \omega_i)_{i=1}^{\nu})$, there exists another weighted homogeneous singularity X' such that X is an o-ramified cyclic covering of X', the universal abelian coverings of X and X' coincide, and the link of X' has Seifert invariants $(b', g; (\alpha_i, \omega_i)_{i=1}^{\nu})$, where b' can be arbitrarily large. (Hence, to any X one can associate an infinite tower of singularities $\{X'\}$ with b' larger and larger, such that all the universal abelian coverings are the same.)

In particular, any weighted homogeneous singularity with $g = 0$ is an o-ramified cyclic covering of a weighted homogeneous singularity with $p_g = 0$ (this vanishing is guaranteed by $b' \gg 0$, cf. (5.1.8)).

This fact can be used e.g. as follows. Using a diagram of type (6.2.1) associated with the covering $X \to X'$ (instead of the universal abelian covering) and a statement as in (6.2.4), the cohomology of certain invariant line bundles on the resolution of X can be computed via the cohomology of line bundles on the resolution of a rational singularity; and this last part is solved completely, see Corollary 7.1.12.

Problem 5.1.44 *Characterise those normal surface singularities (X, o) which admit a finite abelian/cyclic o-ramified covering $(X, o) \to (X', o)$ with (X', o) rational.*

5.2 Superisolated Singularities

This part contains an introduction to the theory of superisolated singularities with a special emphasise to the subtle relationship with the theory of projective rational cuspidal curves. We present the 'Semigroup Distribution Property' (SDP) of such curves. Several properties of the superisolated singularities (as testing properties for general conjectures/theorems, like the Seiberg–Witten Invariant Conjecture) are deeply related with the SDP.
We list several rational cuspidal curves, many of them will produce key examples later in the next chapters. ♣

5.2.1 Hypersurface superisolated singularities achieved historically the reputation of being an interesting class of singularities providing a series of nice examples and counterexamples. The class 'contains' in a canonical way the theory of complex projective plane curves, a connection which will be emphasized in this book as well. Superisolated singularities were introduced by I. Luengo [398] in order to show that the μ-constant stratum in the semiuniversal deformation space of an isolated hypersurface singularity, in general, is not smooth. Later Artal Bartolo in [23] used them to provide a counterexample for S. S.-T. Yau's conjecture (showing

that, in general, the link of an isolated hypersurface surface singularity and its characteristic polynomial do not determine the embedded topological type of the singular germ). Based on superisolated singularities and their universal abelian coverings, [399] presents counterexamples of the 'Universal abelian covering conjecture' of Neumann and Wahl [513] and of the 'Seiberg–Witten invariant conjecture' of Nicolaescu and the author [489], see 9.7.B. On the other hand, Durfee's Conjecture and the Monodromy Conjecture of Denef and Loeser has been proved for them, see [24, 416].

Before we start the definition of superisolated germs we review some basic facts and notations about plane curve singularities.

5.2.2 Invariants of Plane Curve Singularities Let us fix first an irreducible plane curve singularity $(C, o) \subset (\mathbb{C}^2, 0)$. For its minimal good embedded resolution graph in terms of Newton pairs $\{(p_i, q_i)\}_i$ see 3.3.10. For the characteristic polynomial $\Delta_1(t)$ (or zeta-function $\zeta(t)$) of the monodromy in terms of the resolution graph see 4.1.22. For the splice diagram, splice decorations, semigroup $\mathscr{S}_{C,o}$, and formulae for $\Delta_1(t)$ and $\mathscr{S}_{C,o}$ in terms of splice decorations see 5.3.20.

Sometimes it is more convenient to code the local embedded topological type of an *irreducible* plane curve singularity (C, o) by its multiplicity sequence. It is the sequence of multiplicities of the consecutive strict transforms in the blowing up procedure of the minimal good embedded resolution. Since the number of occurrences of the multiplicity 1 equals the last multiplicity greater than 1, we omit the multiplicities 1: we denote such a sequence by $[m_1, \ldots, m_l]$, where $m_1 \geq m_2 \geq \cdots \geq m_l > m_{l+1} = 1$ for a suitable $l \geq 1$. In fact, we will write $[n_{1_{r_1}}, \ldots, n_{k_{r_k}}]$ for a multiplicity sequence, which means that the multiplicity n_i occurs r_i times for $i = 1, \ldots, k$. E.g., a germ with Newton pair $(4, 11)$ has sequence $[4_2, 3]$. [Similarly, if (C, o) has more components, then m_i's are the multiplicities of the corresponding strict transforms at all the infinitely closed points in the minimal good resolution procedure.]

We also will need the delta-invariant of (C, o). For an irreducible (C, o), its delta-invariant $\delta(C)$ is the codimension of $n^*\mathscr{O}_{C,o} \subset \mathscr{O}_{\mathbb{C},o} = \mathbb{C}\{t\}$, where n is the normalization of (C, o). In general, $\delta(C)$ of (C, o), with irreducible components $\{(C_k, o)\}_{k=1}^r$, is the codimension of $\mathscr{O}_{C,o} \subset \prod_k \mathbb{C}\{t_k\}$. It satisfies $\delta(C) = \sum_k \delta(C_k) + \sum_{k<l}(C_k, C_l)_o$ and the Jung/Milnor's formula $\mu = 2\delta(C) - r + 1$ [294, 430]. In terms of a multiplicity sequence one has (for a proof see below)

$$2\delta(C) = \sum_i m_i(m_i - 1). \tag{5.2.1}$$

If (C, o) is irreducible, its semigroup $\mathscr{S}_{C,o} \subset \mathbb{N}$ is the set of all the possible intersection multiplicities $(h, C)_o$, where $h \in \mathscr{O}_{\mathbb{C}^2,0}$. The delta-invariant $\delta(C)$ appears also as the cardinality of the finite set $\mathbb{N} \setminus \mathscr{S}_{C,o}$. The largest element of $\mathbb{N} \setminus \mathscr{S}_{C,o}$ is $\mu - 1$, and for $0 \leq k \leq \mu - 1$ one has the following 'gap-symmetry': $k \in \mathscr{S}_{C,o}$ if and only if $\mu - 1 - k \notin \mathscr{S}_{C,o}$. For the generators of $\mathscr{S}_{C,o}$ see 5.3.20.

For an irreducible plane curve singularity the embedded resolution graph, the splice diagram, Δ_1, $\mathscr{S} := \mathscr{S}_{C,o}$ are all (equivalent) complete invariants of the embedded topological type. For some additional formulae see 5.3.20 as well.

Some of the interplays are the following. By [94]

$$\Delta_1(t)/(1-t) = \sum_{k \in \mathscr{S}} t^k. \tag{5.2.2}$$

Since $\Delta(1) = 1$ and $\Delta'(1) = \delta$ (use e.g. (5.2.2) or the formulae from 5.3.20), one gets $\Delta(t) = 1 + \delta(t-1) + (t-1)^2 \cdot Q(t)$ for some polynomial $Q(t) = \sum_{i=0}^{\mu-2} \alpha_i t^i$ with integral coefficients. In fact, all the coefficients $\{\alpha_i\}_{i=0}^{\mu-2}$ are strictly positive, and $\delta = \alpha_0 \geq \alpha_1 \geq \cdots \geq \alpha_{\mu-2} = 1$. Indeed, by the above identity (5.2.2), one has $\delta + (t-1)Q(t) = \sum_{k \notin \mathscr{S}} t^k$, or $Q(t) = \sum_{k \notin \mathscr{S}} (t^{k-1} + \cdots + t + 1)$. This shows that

$$\alpha_i = \#\{k \notin \mathscr{S} \ : \ k > i\}. \tag{5.2.3}$$

[The identity (5.2.2) has another interesting consequence too: the polynomial Δ_1 is 'alternating', that is all the nonzero coefficients are ± 1 and they alternate. This fact cannot be generalized to germs $f : (X, o) \to (\mathbb{C}, 0)$, for more see [491].]

Assume that C is a projective curve on a smooth surface S, $p \in C$ such that (C, p) is locally irreducible with multiplicity m_p and delta-invariant $\delta(C, p)$. Let $\pi : S' \to S$ be the blow up of S at p, C' the strict transform of C, E_{new} the created exceptional (-1)-curve, $p' := E_{new} \cap C'$. Then using the relations $\pi^*(K_S) = K_{S'} - E_{new}$, $\pi^*(C) = C' + m_p E_{new}$ (and (12.4.4) and 12.4.8) we get $(C', E_{new}) = m_p$, $C^2 = (C')^2 + m_p^2$, $\delta(C', p') = \delta(C, p) - (m_p^2 - m_p)/2$ (which proves (5.2.1) as well).

5.2.3 Definition of Superisolated Singularities [398] A hypersurface singularity $(X, o) \subset (\mathbb{C}^3, 0)$ is called superisolated if the modification $\widetilde{X}$ of (X, o), induced by the blow up $0 \in \mathbb{C}^3$, is smooth. The definition guarantees that (X, o) is isolated. In fact, if X is not smooth, this $\widetilde{X}$ is exactly the *minimal* resolution of X, cf. 5.2.4(a).

Assume that (X, o) is the zero set of $f : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$, $f = f_d + f_{d+1} + \cdots$, where f_j is homogeneous of degree j, $f_d \not\equiv 0$. Then (X, o) is superisolated if and only if the projective plane curve $C := \{f_d = 0\} \subset \mathbb{P}^2$ is reduced with (isolated) singularities $\{p_i\}_i$, and these points are not situated on the projective curve $\{f_{d+1} = 0\}$ (cf. 5.2.4(a)). In this case the embedded topological type (and the equisingularity type) of f does not depend on the choice of f_j's for $j > d$, as long as f_{d+1} satisfies the above requirement. Hence, if one replaces f_j by 0 for all $j > d+1$ and f_{d+1} by l^{d+1}, where l is a linear form not vanishing at the points $\{p_i\}_i$, then the embedded topological type stays unmodified. In particular, any reduced projective plane curve C with equation $f_d = 0$ provides (the equisingularity type) of a superisolated singularity given by equation $f_d + l^{d+1}$ (l generic). Therefore, those

invariants of (X, o), which are stable with respect to equisingular deformations, depend only on C.

Strictly speaking, $\mathrm{Sing}(C) = \emptyset$ is also allowed, hence all isolated homogeneous (or semi-homogeneous) hypersurface singularities are superisolated (and in this case the equisingularity type remains unchanged even if we delete l^{d+1}). But the main point in the study of superisolated germs is exactly the way how the local singularity types (C, p_i) perturb the invariants of an isolated homogeneous singularity.

We denote by μ_i and Δ_i (with the choice $\Delta_i(1) = 1$) the Milnor number and the characteristic polynomial of the local plane curve singularities $(C, p_i) \subset (\mathbb{P}^2, p_i)$.

We denote the irreducible components of C by $\{C_\alpha\}_\alpha$, set $\deg(C_\alpha) = d_\alpha$.

5.2.4 Invariants (For some proofs and other invariants see also 9.7.B)

(a) [398] The exceptional curve E in $\widetilde{X}$ can be identified with C. In fact, if $\mathbb{P}^2$ is the exceptional surface of the blow up $B \to \mathbb{C}^3$ of $0 \in \mathbb{C}^3$, $p \in E \subset \mathbb{P}^2$, and (u, v, w) are local coordinates in B at p such that $\mathbb{P}^2$ locally is $u = 0$, then $E \subset \mathbb{P}^2$ is given by the local equation $g(v, w)$ of the curve C, and $\widetilde{X}$ by $u = g(v, w)$. This also shows that the intersection number (C_α, C_β) of two different components of $E = C$ considered in $\mathbb{P}^2$ or in $\widetilde{X}$ agree (hence by Bézout theorem it is $d_\alpha d_\beta$). In particular, the intersection matrix of the curves $\{C_\alpha\}_\alpha$ in $\widetilde{X}$ is

$$(C_\alpha, C_\beta) = d_\alpha d_\beta \text{ for } \alpha \neq \beta, \text{ and } C_\alpha^2 = -d_\alpha(d - d_\alpha + 1). \tag{5.2.4}$$

[The strict transform of the generic linear function intersects each C_α transversely in d_α points and it has multiplicity 1 along C, hence the second part of (5.2.4) follows from (2.2.1).] Note also that $C_\alpha^2 \leq -2$ unless $d = 1$, hence $\widetilde{X}$ is minimal.

Therefore, if C is irreducible, then $\widetilde{X}$ has one irreducible exceptional divisor, which is isomorphic to C, and C^2 in $\widetilde{X}$ is $-d$. Hence, when C is irreducible, the link of (X, o) is a rational homology sphere if and only if C is rational and all the plane curve singularities $(C, p_i) \subset (\mathbb{P}^2, p_i)$ are irreducible. We use the terminology *cusp* for them. Such an irreducible curve C is called *rational cuspidal plane curve*. In this case the minimal good resolution is obtained from $\widetilde{X}$ by resolving the plane curve singularities $(C, p_i) \subset (\widetilde{X}, p_i)$. Note that by the above local equations (first paragraph of (a)) the embedded topological types $(C, p_i) \subset (\widetilde{X}, p_i)$ and $(C, p_i) \subset (\mathbb{P}^2, p_i)$ agree. Furthermore, $\sum_i \mu_i = (d - 1)(d - 2)$ from the rationality of C.

Hence, under the condition that C is irreducible and the link L_X is a rational homology sphere, the minimal good resolution graph Γ of (X, o) is the surgery graph described in 3.5.12. That is, the link of (X, o) is the oriented surgery 3-manifold $S^3_{-d}(\#_i K_i)$, where $(K_i \subset S^3)$ are the local knots of $(C, p_i) \subset (\mathbb{P}^2, p_i)$. In the minimal good resolution E_+ stays for the exceptional curve corresponding to C (realized by the very first blow up), and v_+ is the corresponding vertex in the graph.

(b) Fix any resolution graph Γ of $\{f = 0\}$. Let $K \in L'$ be the *canonical cycle* associated with Γ, and $|\mathscr{V}|$ the number of vertices. Then $K^2 + |\mathscr{V}| = -d(d-2)^2 + 1$. (In the resolution $\widetilde{X}$ by the adjunction formula $-K_{\widetilde{X}} = (d-2)C$, cf. 6.3.)

(c) [23, (3.6.4)] The *Milnor number* μ and *characteristic polynomial* Δ_f of f are

$$\mu = (d-1)^3 + (d-1)(d-2), \quad \Delta_f(t) = \frac{t^d - 1}{t-1} \cdot \prod_i \Delta_i(t^{d+1}).$$

$\Delta_i(1) = 1$ implies $|H| = \Delta_f(1) = d$. In fact, $H = \mathbb{Z}_d$, and it is generated by $[E_+^*]$.

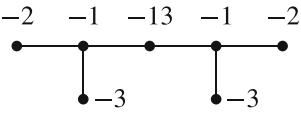
For the Jordan block structure of the monodromy of f, or for the equivariant mixed Hodge numbers of the vanishing cohomology of f see [23, 25].

Note that both the link L_f and characteristic polynomial Δ_f depend only on the integer d and the local topological types of (C, p_i) (that is, on the *combinatorial type* of the projective curve C). On the other hand, the Jordan block structure of the monodromy of f usually cannot be determined from the combinatorial type, it might depend essentially on the homeomorphism type of the pair $(\mathbb{P}^2, C)$. In particular, there exist (so called) Zariski pairs of curves C_1, C_2 with common combinatorial types, which determine two superisolated singularities with identical links and characteristic polynomials, but with different monodromy operators (hence different embedded topological types) [23].

(d) Since $12p_g = \mu - (K^2 + |\mathscr{V}|)$, see [362] or 6.9.6 here, one has $p_g = d(d-1)(d-2)/6$ (which equals the geometric genus of the isolated homogeneous singularity of degree d.)

Remark 5.2.5 (The Link as a Surgery 3-Manifold) By 3.5.12 the link of the superisolated singularity associated with a rational cuspidal curve is the surgery 3-manifold $S^3_{-d}(\#_i K_i)$. In the general surgery construction we require no additional assumptions which connect the knots K_i and the integer d. However, in the present superisolated singularity case, $L_X = S^3_{-d}(\#_i K_i)$ satisfies $(d-1)(d-2) = \sum_i \mu_i$.

In fact, by Grauert Theorem (see Sect. 2.5) for any $d > 0$ and any $\{K_i\}_i$, the surgery 3-manifold is the link of a certain surface singularity. Nevertheless, the concrete construction of a certain 'distinguished' analytic type (with topology identified by $S^3_{-d}(\#_i K_i)$) is not known except for the superisolated case or some sporadic cases. E.g., such a sporadic case is the complete intersection $\{x^2 = u^3 + v^2 y,\ y^2 = v^3 + u^2 x\} \subset (\mathbb{C}^4, 0)$ with $L_X = S^3_{-1}(T_{2,3}\#T_{2,3})$ with graph



$T_{2,3} =$ (2,3) torus knot

Here $H = 0$ and the equations are of splice type, cf. 5.4. [Usually, a negative definite graph given by the above surgery procedure does not satisfy the semigroup or monomial condition 5.4.10, hence $S^3_{-d}(\#_i K_i)$ cannot be the link of a splice quotient singularity. This happens e.g. if the number of cusps is ≥ 3.]

5.2.6 The Universal Abelian Covering Let $S^3_{-d}(K_1\# \cdots \# K_\nu)$ be a surgery manifold as above, where $d > 0$ and K_i are algebraic knots. Assume that the local equations of K_i are $\{f_i = 0\} \subset (\mathbb{C}^2, 0)$. Then $H = H_1(S^3_{-d}(K), \mathbb{Z}) = \mathbb{Z}_d$, and H is generated by $[E^*_+]$. In particular, the universal abelian covering is a cyclic covering and its plumbing graph can be determined via the algorithm of Theorem 4.1.35 (taking the multiplicities of dE^*_+ and the unique arrowhead with multiplicity d supported by v_+). This algorithm can be identified on the subgraphs Γ_i with the algorithm which provides the graphs $\Gamma_i(d)$ of the link of the suspension singularities $f_i(x, y) + z^d = 0$, see 4.1.C. Let v_i be the vertex of $\Gamma_i(d)$ which supports the strict transform of z and $m_i(d)$ the multiplicity of z along v_i (both given by the algorithms in 4.1.C). Then the graph Γ_a of the universal abelian covering $S^3_{-d}(\#_i K_i)_a$ of $S^3_{-d}(\#_i K_i)$ can be constructed as follows. Consider a 'central vertex' v with genus decoration 0 and all the graphs $\Gamma_i(d)$; for each i connect v with v_i by an edge, keep all the decorations of $\Gamma_i(d)$, and add the new Euler decoration $e_v = -1 - \sum_i m_i(d)$ to v. Furthermore, one has $\det(\Gamma_a) = \prod_i \det(\Gamma_i(d))$ (use 2.2.5(6), or see [64]). Therefore, $S^3_{-d}(\#_i K_i)_a$ is rational (integral) homology sphere if and only if all suspension graphs $\Gamma_i(d)$ provide rational (integral) homology spheres. [E.g., $S^3_{-5}(T_{2,3}\#T_{2,3}\#T_{2,3}\#T_{2,7})_a$, the UAC of the superisolated singularity link considered in 5.2.8(VI) is an integral homology sphere.]

By the above discussion $S^3_{-d}(\#_i K_i)_a$ is the link of the universal abelian covering (X_a, o) of a superisolated germ. On the other hand, the 'identification' of the analytic type (e.g. to find equations, or analytic invariants) of (X_a, o) is very hard (this is an open problem). For some concrete examples see e.g. [665].

5.2.7 On the Classification Problem of Rational Cuspidal Plane Curves The classification of projective plane curves (up to the action of the automorphism group $\mathrm{PGL}_3(\mathbb{C})$) is a very difficult open problem. Even the classification of the *combinatorial types* is still open; this aims to identify those positive integers d and collection of embedded topological types of local plane curve singularities, which admit analytic realization, that is, for which there exists an irreducible projective curve of degree d having local singularities of prescribed embedded topological types. If d is very large compared with the sum of local Milnor numbers then the realization is guaranteed (see e.g. [237–239]). The smallest possible d allowed by the genus formula is when the curve is rational. In this case the 'asymptotic methods' do not work.

In the sequel we focus on the classification of *irreducible rational cuspidal curves*. For examples and partial results see e.g. [202, 203, 302, 412, 538, 539, 771–773], and the citations from the next discussions (and the references therein).

The problem remains very difficult even if we aim much less, e.g. the determination of the maximal number of cusps among all the rational cuspidal plane curves

(a problem proposed by F. Sakai in [250]). In [698] K. Tono proved that it is less than 9; the expected number was 4, a fact recently proved by Koras and Palka [341]. (This bound is realized, see below in **VI**.)

This classification problem is not only important for its own sake, but it is also connected with crucial properties, problems and conjectures in the theory of open surfaces, and in the classical algebraic geometry.

For instance, the open surface $\mathbb{P}^2 \setminus C$ is $\mathbb{Q}$-acyclic if and only if C is a rational cuspidal curve. For the famous *Rigidity Conjecture* valid for such surfaces, formulated by Flenner and Zaidenberg, see [201].

Another related, very old problem has its roots in early algebraic geometry, and wears the name of Coolidge and Nagata, see [126, 450]. It predicts that every rational cuspidal curve can be transformed by a Cremona transformation into a line. It was proved recently in [340, 553].

Palka proposed the 'Negativity Conjecture' for $\mathbb{Q}$-acyclic surfaces [554], which implies the Coolidge-Nagata conjecture and the Rigidity Conjecture, see 5.2.8.**IX**.

5.2.8 Some (Partial) Classifications

I. One of the guiding objects of the classification is the logarithmic Kodaira dimension $\bar{\kappa} := \bar{\kappa}(\mathbb{P}^2 \setminus C)$. (Its definition from [284] is the following. Let $V \to \mathbb{P}^2$ be an embedded resolution of $C \subset \mathbb{P}^2$ and let D be the reduced total transform of C. Then $\bar{\kappa} := \kappa(V, K_V + D)$ is an isomorphism invariant of $\mathbb{P}^2 \setminus C$.) Its values are $\{-\infty, 0, 1, 2\}$. It has a subtle relationship with the combinatorial type of C: by [733] if $\nu \geq 3$ then $\bar{\kappa} = 2$, while if $\nu = 2$ then $\bar{\kappa} \geq 0$. One has the following facts:

(a) If $\bar{\kappa} = -\infty$ then $\nu = 1$ by [733], and all the curves are classified by M. Miyanishi and T. Sugie [435] (see also H. Kashiwara [302]).
(b) The case $\bar{\kappa} = 0$ cannot occur by a result of Sh. Tsunoda [701] (see also [537]).
(c) The curves with $\bar{\kappa} = 1$ were classified by K. Tono [695, 697, 698].

The case $\bar{\kappa} = 2$ is still open. Recently K. Palka and T. Pełka achieved a complete classification under the assumption that the Palka's Negativity Conjecture holds for complements of rational cuspidal curves, see **IX**.

II. One can classify triples (d, a, b) such that there exists a unicuspidal rational plane curve C of degree d whose singularity has only one Puiseux pair of type (a, b), where $1 < a < b$. Let $\{\varphi_j\}_{j\geq 0}$ denote the Fibonacci numbers $\varphi_0 = 0$, $\varphi_1 = 1$, $\varphi_{j+2} = \varphi_{j+1} + \varphi_j$. Then the following triples are realized [197]:

(a) $(a, b) = (d - 1, d)$;
(b) $(a, b) = (d/2, 2d - 1)$, where d is even;
(c) $(a, b) = (\varphi_{j-2}^2, \varphi_j^2)$ and $d = \varphi_{j-1}^2 + 1 = \varphi_{j-2}\varphi_j$, where j is odd and ≥ 5;
(d) $(a, b) = (\varphi_{j-2}, \varphi_{j+2})$ and $d = \varphi_j$, where j is odd and ≥ 5;
(e) $(a, b) = (\varphi_4, \varphi_8 + 1) = (3, 22)$ and $d = \varphi_6 = 8$;
(f) $(a, b) = (2\varphi_4, 2\varphi_8 + 1) = (6, 43)$ and $d = 2\varphi_6 = 16$.

In the first four cases $\bar{\kappa} = -\infty$. The last two sporadic cases have $\bar{\kappa} = 2$ and were found by Orevkov and Artal Bartolo, cf. [537]. These two types are denoted by C_4 and C_4^* respectively.

III. Based on the Thesis of T. Liu [392], J. Bodnár in his Thesis [62] provided the following list of combinatorial data for the rational unicuspidal curves of degree d, whose singularity has two Newton pairs $(p_1, q_1)(p_2, q_2)$ [61, 62]. (In (i) $k = 2, l = 0$ degenerates to one Newton pair type.) In the first four cases $\bar{\kappa} = -\infty$, in the cases (v)-(vi) $\bar{\kappa} = 1$, while in the last two cases $\bar{\kappa} = 2$. These last two families are constructed by Orevkov [537], we will refer to them later as C_{4k} (case (vii)) and C_{4k}^* (case (viii)).

(i) $(l\varphi_{2k-1}^2 + \varphi_{2k-3}^2, l\varphi_{2k+1}^2 + \varphi_{2k-1}^2 + 2)(\varphi_{2k-1}^2, l\varphi_{2k-1}^2 + \varphi_{2k-3}^2)$, $d = \varphi_{2k-1}\varphi_{2k+1}(l\varphi_{2k-1}^2 + \varphi_{2k-3}^2)$, $k \geq 2, l \geq 0$.
(ii) $(l\varphi_{2k-1}^2 + \varphi_{2k-3}^2, l\varphi_{2k+1}^2 + \varphi_{2k-1}^2 + 2)(\varphi_{2k-1}, l\varphi_{2k-1} + \varphi_{2k-5})$, $d = \varphi_{2k+1}(l\varphi_{2k-1}^2 + \varphi_{2k-3}^2)$, $k \geq 3, l \geq 0$.
(iii) $(n-1, n)(m, nm-1)$, $d = nm$, $n \geq 3, m \geq 2$.
(iv) $(n, 4n-1)(m, nm-1)$, $d = 2nm$, $n, m \geq 2$.
(v) $(n-1, n)(n, (n+1)^2)$, $d = n^2 + 1$, $n \geq 3$.
(vi) $(n, 4n+1)(4n+1, (2n+1)^2)$, $d = 8n^2 + 4n + 1$, $n \geq 2$.
(vii) $(\varphi_{4k}/3, \varphi_{4k+4}/3)(3, 1)$, $d = \varphi_{4k+2}$, $k \geq 2$.
(viii) $(\varphi_{4k}/3, \varphi_{4k+4}/3)(6, 1)$, $d = 2\varphi_{4k+2}$, $k \geq 2$.

The case of unicuspidal curves with more Newton pairs and $\bar{\kappa} = 2$ is still open.

IV. For a list of two-cuspidal rational curves see e.g. [62, 69]. Though the completeness of this list is not proved, in [573] it is conjectured that there is no (new) rational curve with two cusps, which is not listed in the present literature.

V. The next list presents three infinite series of combinatorial types of rational three-cuspidal curves given by Flenner, Zaidenberg and Fenske, cf. [194, 202, 203], see also [436, 573]. There is another 'sporadic' curve of degree 5 not contained in any of the three series with cusp types $[2_2]$, $[2_2]$, $[2_2]$. It is conjectured that these are the only types of rational three-cuspidal curves [573, Conj. 4].

(i) $C_{d,u}$ (with $d \geq 4$ and $1 \leq u \leq d-3$) of degree d and cusp types: $[d-2], [2_u], [2_{d-2-u}]$. (Here it is enough to take $u \leq \left\lfloor \frac{d-2}{2} \right\rfloor$, as $C_{d,u} = C_{d,d-2-u}$.)
(ii) D_l (with $l \geq 1$) of degree $d = 2l + 3$ and has cusp types: $[2l, 2_l]$, $[3_l]$, $[2]$.
(iii) E_l (with $l \geq 1$) of degree $d = 3l+4$ and has cusp types: $[3l, 3_l]$, $[4_l, 2_2]$, $[2]$.

VI. In the literature there is only one rational cuspidal curve with more than three cusps, it has $d = 5$ and types $[2_3]$, $[2]$, $[2]$, $[2]$. It was conjectured that no other can occur. This was proved recently in [341].

VII. For the classification of cuspidal rational curves with degree $d \leq 5$ see Namba's book [459], for $d = 6$ see Fenske [193]. The corresponding lists are given below in 8.6.22. For $d = 7$ there is no complete classification (known by the author).

VIII. Another approach is to classify rational cuspidal curves C such that the highest multiplicity of the singular points m is close to the degree d. If $m = d - 1$ then C is of type **II**(a). Flenner and Zaidenberg classified the curves with $m = d - 2$ in [202] and $m = d - 3$ in [203]. The case $m = d - 4$ is partially solved by Fenske [194].

IX. If $\bar{\kappa} \neq 2$ then there is a classification, cf. **I**, so assume $\bar{\kappa} = 2$. With the notation as in **I** let $\kappa_{1/2} = \kappa(V, 2K_V + D) \in \{-\infty, 0, 1, 2\}$. We have $-\infty = \kappa(V) \leq \kappa_{1/2} \leq \bar{\kappa} = 2$.

The Palka Negativity Conjecture [554], stated for arbitrary $\mathbb{Q}$-acyclic surfaces, asserts that $\kappa_{1/2} = -\infty$. Recently, using the theory of almost minimal models of quasi-projective surfaces K. Palka and T. Pełka achieved a complete classification of rational cuspidal curves for which the Negativity Conjecture holds [555, 556]. In particular, they confirm the Rigidity Conjecture and list all possible multiplicity sequences of rational cuspidal curves. For $\bar{\kappa} = 2$ each sequence is realized by a curve which is unique up to a projective equivalence. There are no 'new curves' with three cusps. With these results the last piece of the classification problem for rational cuspidal curves is the Negativity Conjecture itself. It remains open, but its formulation makes it well suited for the methods of the Minimal Model Program. The key role played in the classification by the divisor $2K_V + D$ is related to the fact that for $\bar{\kappa} = 2$ the Kawamata-Viehweg vanishing gives $h^1(2K_V + D) = 0$; its topological meaning remains unclear. The above conditional classification of rational cuspidal curves has been extended to smooth $\mathbb{Q}$-acyclic surfaces in the Thesis of T. Pełka [560].

5.2.9 Restrictions Satisfied by the Combinatorial Type Next we list some properties which are necessarily satisfied whenever the combinatorial type is realized by a curve. Usually these are inequalities connecting invariants of the local topological type with some universal object depending on d. (If in the analytic proof $\bar{\kappa}$ appears then the inequalities are sharper whenever $\bar{\kappa} = 2$.)

I. Bogomolov–Miyaoka–Yau Type Inequality Let m be the highest multiplicity of the singular points. Then $3m > d$ by Matsuoka–Sakai [412]. This has the following improvement given by Orevkov [537]. If $\alpha = \frac{(3+\sqrt{5})}{2}$ then $\alpha(m + 1) + 1/\sqrt{5} > d$. Furthermore, if $\bar{\kappa} = -\infty$ then $\alpha m > d$, and if $\bar{\kappa} = 2$ then $\alpha(m + 1) - 1/\sqrt{5} > d$.

Regarding the inequality $\alpha m > d$ in general, Orevkov verified that the curves C_{4k} and C^*_{4k} $(k \geq 1)$ satisfy $d > \alpha m$, and he conjectured that these are the only rational cuspidal curves with $d > \alpha m$.

II. Spectrum Semicontinuity Assume that $C \subset \mathbb{P}^2$ is given by $f_d(x, y, z) = 0$, where $z = 0$ is a generic line identified with the line at infinity. Then the deformation $\{f_d(x, y, t)\}_t = 0$ provides a specialization of the multisingularity $\sum_i (C, p_i)$ to a universal plane germ (equisingular to) $(U, 0) := (x^d + y^d, 0)$. In particular, the collection of all spectral numbers of the local plane curve singularities $(C, p_i)_i$ satisfies the semicontinuity property compared with the spectral numbers of $(U, 0)$ for any interval $(\alpha, \alpha + 1)$ [713, 714]. (For more on spectrum see [654, 656, 657].)

III. Semigroup Distribution Inequality [198] Let $\mathscr{S}_{C,p_i}$ be a semigroup of the local singularities (C, p_i). Fix an integer $0 \le l < d$. Then

$$\min_{j_1+\cdots+j_\nu=ld+1} \sum_{i=1}^{\nu} \#\{\, \mathscr{S}_{C,p_i} \cap [0, j_i)\,\} \ge (l+1)(l+2)/2.$$

Indeed, $(l+1)(l+2)/2$ is the dimension of the space of homogeneous polynomials P in three variables of degree l. Let $\gamma_i(t)$ be a local parametrisation of C at p_i, and write $P(\gamma_i(t))$ as $\sum_{h=1}^{\infty} L_{i,h}(P)t^h$. The condition, imposed to P, that the local intersection multiplicity $(C, \{P=0\})_{p_i}$ is $\ge j_i$, is given by the equations $L_{i,h}(P) = 0$ for $0 \le h < j_i$, and the number of independent conditions is at most $\#\{\mathscr{S}_{C,p_i} \cap [0, j_i)\}$. Therefore, if the above inequality failed then there would exist a non-zero polynomial P of degree l with $(C, \{P = 0\})_{p_i} \ge j_i$, or $(C, \{P = 0\}) \ge \sum_i j_i = ld + 1$, contradicting Bézout's theorem.

In [198] the authors conjectured under the name *Semigroup Distribution Property*, that in the above inequality one has equality in any unicuspidal case.

In [198] the conjecture was verified for several unicuspidal cases (for $\bar{\kappa} < 2$ and for 'all known cases in the literature' with $\bar{\kappa} = 2$). [The reader might verify the property for some examples listed above (or check the proofs from [198]) to see its richness (and mystery).] The general proof for any number of cusps was obtained by Borodzik and Livingston based on the d-invariant of Heegaard Floer theory (and it will not be reproduced in this book), for the statement see below. (Note that whilst the proof of the Semigroup Distribution Inequality is elementary, this definitely cannot be said for the Semigroup Distribution Property.)

IV. Semigroup Distribution Property [68] With the previous notations,

$$\min_{j_1+\cdots+j_\nu=ld+1} \sum_{i=1}^{\nu} \#\{\, \mathscr{S}_{C,p_i} \cap [0, j_i)\,\} = (l+1)(l+2)/2$$

for any rational cuspidal curve. In the unicuspidal case this reads as

$$\#\{\, \mathscr{S}_{C,p} \cap (\,(l-1)d, ld\,]\,\} = \min\{l+1, d\} \quad (l \ge 0).$$

The distribution property provides a series of inequalities connecting the local semigroup and the global invariant d, in this sense is has formal similarities with

the inequalities provided by the semicontinuity of the spectrum. Nevertheless, these two restrictions are independent. E.g., $d = 17$ and Newton pairs (2,7), (4,17) satisfy the 'Semigroup Distribution Property' but not the spectrum semicontinuity [199]; $d = 30$ and the pairs (3,7), (6,23) satisfy the spectrum semicontinuity but already fail the Semigroup Distribution Inequality in the first interval [68].

It is surprising that the restriction on the very first intervals is already rather strong: e.g. it implies the Matsuoka–Sakai inequality $3m > d$, cf. [68, 198].

For more connection with surface singularity theory, and for the motivation of Semigroup Distribution Property, see here 8.6.B, 9.7.B and 11.3.C.

For generalizations to higher genus $g(C) > 0$ case see [65, 70].

Remark 5.2.10 The above proof from part III can be improved to obtain the following reinterpretation suggested by Campillo (for details see [199, Th. 6.8]).

Let C be an unicuspidal rational plane curve of degree d. The curve C satisfies the 'Semigroup Distribution Property' if and only if the elements of the semigroup $\mathscr{S}_{C,p}$ in $[0, ld]$ are realized via intersection multiplicities at p by projective (possible non-reduced) curves of degree l (for $l \leq d - 3$).

Remark 5.2.11 The *weighted Lê–Iomdin singularities* generalize the superisolated germs. They are isolated hypersurface singularities of the form $f_d + f_{d+k}$, where f_i are weighted homogeneous of degree i (associated with the same weights), and $f_d = 0$ has a one-dimensional singular locus. It would be rather interesting to extend all the results and computations of this book from the superisolated case to this more general context. For certain invariants (e.g. for the Milnor package) see [26].

5.3 Splice Diagrams

In this part we introduce the splice diagrams. They codify (negative definite) integral homology sphere graph 3-manifolds in a more condensed/conceptual way as plumbing graphs do: the diagrams reflect the JSJ-decomposition into Seifert integral homology spheres. The 'diagram calculus' is also discussed.
To any negative definite plumbing tree (with zero genus decorations) we associate a splice diagram. Furthermore, we establish a bijection between such plumbing graphs with determinant one and splice diagrams satisfying the 'relative prime condition' and 'edge-determinant positivity'. We follow [178]. ♣

5.3.1 Integral homology sphere oriented plumbed 3-manifolds can equivalently be codified by their *splice diagrams*, see the book of Eisenbud and Neumann [178], or the article of Siebenmann [635]. Later Neumann and Wahl associated splice diagrams with rational homology sphere plumbed 3-manifolds as well [515, 516].

In this section we assume that all 3-manifolds are (irreducible) rational homology spheres, hence all the plumbing graphs are trees with zero genus decorations.

Definition 5.3.2 A splice diagram $\mathfrak{G}$ is a tree with vertices $\mathscr{V}^{\mathfrak{G}}$ and edges $\mathscr{E}^{\mathfrak{G}}$. We assume that the valency $\kappa_v^{\mathfrak{G}}$ of any vertex v is different than 2. If $\kappa_v^{\mathfrak{G}} = 1$ then v is called *end-vertex*, otherwise it is a *node*. The tree $\mathfrak{G}$ has certain decorations as well: for each node v and adjacent edge e we insert a weight $d_{ve} \in \mathbb{Z}_{>0}$.

If one studies the splice diagrams associated with arbitrary graph manifolds then one has to consider a more general set of decorations, e.g. the weights d_{ve} might be negative as well. However, for rational homology sphere singularity links $d_{ve} \in \mathbb{Z}_{>0}$ always, hence in this book we assume this setup from the beginning.

5.3.3 From Plumbing Graphs to Splice Diagrams [178, §20], [513, 515, 516] Let Γ be a *connected, negative definite* plumbing tree. One constructs from Γ a splice diagram $\mathfrak{G} = \mathfrak{G}(\Gamma)$ as follows. We keep all end-vertices and nodes of Γ (they will serve as end-vertices and nodes in $\mathfrak{G}$ as well). However, a string of Γ connecting two vertices with $\kappa_v \neq 2$ is replaced in $\mathfrak{G}$ by a single edge. In particular, if Γ is a string, then $\mathfrak{G}$ is the diagram •——•

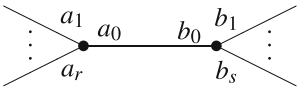
Next we assume that Γ has at least one node. For each node $v \in \mathscr{V}^{\mathfrak{G}}$ and adjacent edge $e \in \mathscr{E}^{\mathfrak{G}}$, let Γ_{ve} be the connected component of the graph $\Gamma \setminus v$ adjacent to e. Then set $d_{ve} := \det(\Gamma_{ve})$ for the edge-vertex decoration required by 5.3.2.

To the 'empty plumbing graph' we associate the 'empty splice diagram'.

Lemma 5.3.4 *Let $\mathfrak{G}$ be obtained from Γ as in 5.3.3.*

(a) If $\det(\Gamma) = 1$, then for any fixed node $v \in \mathscr{V}^{\mathfrak{G}}$ the integers $\{d_{ve}\}_e$ are pairwise relative prime. (This last property will be called 'Relative prime condition'.)

(b) ('Edge-determinant positivity') Consider in $\mathfrak{G}$ the weights in the 'neighborhood of an edge' $e \in \mathscr{E}^{\mathfrak{G}}$:



Then the expression $a_0b_0 - \prod_{i>0} a_i \prod_{j>0} b_j$, called the edge determinant of e, is positive. (In particular, $\prod_{I>0} a_i \prod_{j>0} b_j/(a_0b_0) \in (0, 1)$.)

(c) ('Ideal condition') Fix a node v in $\mathfrak{G}$, an adjacent edge e, and the connected component $\mathfrak{G}_{ve}$ of $\mathfrak{G}$ which is adjacent to e. For any end-vertex w of $\mathfrak{G}$, which is in $\mathfrak{G}_{ve}$, let m_{vw} be the product of weights of $\mathfrak{G}$ along but not on the path p, where p is the minimal path connecting v and w (and the weights around v are not counted). Let I_{ve} be the ideal of $\mathbb{Z}$ generated by $\{m_{vw}\}_w$. (If Γ_{ve} is a string then $I_{ve} = \mathbb{Z}$.) Then $d_{ve} \in I_{ve}$.

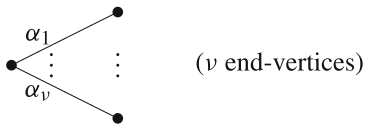
Proof Part *(a)–(b)* follows from 2.2.5(6) applied for Γ. For *(a)* we apply it to the node v and an adjacent vertex u. Then we get an expression of type $1 = A \cdot d_{ve} - B \cdot \prod_{e' \neq e} d_{ve'}$ (with $A, B \in \mathbb{Z}$), where the product is over all the edges adjacent to

v different from e. In the case of *(b)* let u and v be the corresponding nodes of Γ and let G denote the string connecting them. Then by 2.2.5(6) the edge determinant is $\det(\Gamma)\det(G)$, which is positive since Γ is negative definite.

(c) Consider the graph Γ_{ve} as in 5.3.3, and complete it with arrowhead a on the place of the vertex v. Put multiplicity d_{ve} on the arrow, and consider the corresponding open book decomposition (topological Milnor fibration) as in Sect. 3.4. Since d_{ve} is the determinant of the graph, if u is the supporting vertex of the arrow, then $d_{ve}E_u^*$ is an integral cycle, its positive entries are the multiplicities of the open book. The open book exists since it can be realized even analytically by 2.5.8. The multiplicity system of $d_{ve}E_u^*$ on end-vertices w (and even on nodes) coincides with the multiplicities $\{m_{vw}\}_w$ by 2.2.5(5), see also 5.3.17 and 5.3.19. We wish to determine $\mathrm{im}(\arg_*)$. It is generated by elements of type $\arg(\gamma_v)$, where γ_v are loops as in 3.3.6 (see also 3.4). However, they satisfy certain homological relations similar to the multiplicity system (2.2.1). We claim that $\mathrm{im}(\arg_*)$ is generated by elements of type $\arg(\gamma_w)$, with w ends of Γ_{ve} (as in the statement). (Use an inductive argument as in the proof of 3.7.8 with $* = a$, see also 3.4.3.) In particular, $I_{ve} \subset \mathbb{Z}$ is exactly the image of $\arg_*$. On the other hand, $d_{ve} = \arg_*(\gamma_a) \in \mathrm{im}(\arg_*) = I_{ve}$. □

Note that for a diagram the 'relative prime condition' implies the 'ideal condition'.

Example 5.3.5 Assume that Γ is star-shaped with ν legs and leg-determinants $\{\alpha_j\}_j$, cf. 3.5.1. Then $\mathfrak{G}$ has the shape



Remark 5.3.6

(a) Usually, from $\mathfrak{G}(\Gamma)$ one cannot recover Γ.

E.g. if Γ is star-shaped, then one cannot recover the Seifert invariants $\{(\alpha_j, \omega_j)\}_j$ and b_0 only from the leg-determinants $\{\alpha_j\}_j$.

(b) The converse of Lemma 5.3.4(a) does not hold in general: even if for all v all the weights $\{d_{ve}\}_e$ are pairwise relative prime, $\det(\Gamma) = 1$ is not guaranteed.

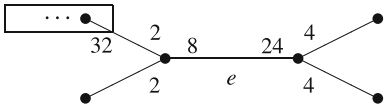
Indeed, take again the star-shaped case, with pairwise relative prime leg-determinants $\{\alpha_j\}_j$, but with $b_0 \gg 0$. Then $\det(\Gamma) \neq 1$, that is, $M(\Gamma)$ is not an integral homology sphere. Nevertheless (and this is an important point we wish to emphasize), by 5.1.33 and 5.1.17 the *universal abelian covering* of $M(\Gamma)$ is an integral homology sphere. This fact will be generalized in 5.3.24 and in the paragraph after it.

(c) In the star-shaped case, if $\det(\Gamma) = 1$ (hence if the leg-determinants are pairwise relative prime too, cf. 5.3.4(a)) then Γ can be recovered from $\mathfrak{G}(\Gamma)$. Indeed, by 5.1.33 if $\det(\Gamma) = 1$ then Γ is the link of a Brieskorn–Hamm

complete intersection. Then, with the notation of 5.1.17, $\det(\Gamma) = 1$ implies $\gcd(a_i, a_j) = 1$ too. From the formulae of 5.1.17, $a_j = \alpha_j$ and the Seifert invariants are computable from $\{\alpha_j\}_j$. [By an independent argument, the identity $1 = |e| \prod_i \alpha_i = \prod_i \alpha_i (b_0 - \sum_i \omega_i/\alpha_i)$, cf. (3.5.5), determines b_0 and $\{\omega_i\}_i$ in terms of $\{\alpha_i\}_i$.]

Hence, in this case, *the splice diagram (as in 5.3.5) represents the integral homology sphere link $\Sigma(\alpha_1, \dots, \alpha_\nu)$ of the Brieskorn–Hamm complete intersection.*

(d) There exist splice diagrams (satisfying even the 'edge-determinant positivity' and 'ideal condition') which are not associated with any plumbing graph by the procedure of 5.3.3. The main reason is that for given integers d and $\{\alpha_i\}_i$, a star-shaped graph Γ with $\det(\Gamma) = d$ and leg-determinants $\{\alpha_i\}_i$ usually does not exist. E.g. $d = 8$ and $\alpha_1 = \alpha_2 = \alpha_3 = 4$ cannot be realized (use (3.5.5), namely $\det(\Gamma) = |H| = |e| \cdot \prod_j \alpha_j$, and the expression (3.5.1) for e). In particular, if $\mathfrak{G}$ is associated with some Γ then it cannot have the following sub-diagram:



Indeed, the subgraph corresponding to the above diagram (without the upper-left vertex) should have determinant 32, and by the edge-determinant formula from the proof of 5.3.4*(b)*, the determinant of the string corresponding to e should be 4. Hence the right node represents a non-existing Seifert piece with determinant $d = 8$ and $\alpha_1 = \alpha_2 = \alpha_3 = 4$. (For determinant computations use again 2.2.5(6).)

5.3.7 Splice Diagrams Codifying Splice Operation on Integral Homology Spheres Consider a splice diagram, which has at least one node, and such that *for any node v the edge weights $\{d_{ve}\}_e$ are pairwise relative prime*. Let us fix one of its nodes v and the adjacent edges together with their weights $\{d_{ve}\}_e$. These data determine a star-shaped splice diagram $\mathfrak{G}(\{d_{ve}\}_e)$. By convention, this represents the integral homology sphere $\Sigma_v := \Sigma(\{d_{ve}\}_e)$ (compare with 5.3.6(c)). Then the original diagram $\mathfrak{G}$ represents an oriented integral homology sphere, which is obtained from the pieces $\{\Sigma_v\}_v$ by *splicing*, where v runs over all the nodes of $\mathfrak{G}$, cf. [178].

The meaning of this is the following. Let e be an edge of $\mathfrak{G}$ connecting two nodes v and u. Consider the two Seifert manifolds Σ_v and Σ_u. Recall that Σ_v is given by the complete intersection equations, where the number of variables $\{z_{ve}\}_e$ is exactly the number of edges, cf. 5.1.33. Each zero set $K_{ve} := \{z_{ve} = 0\} \cap \Sigma_v$ is an oriented Seifert fiber in Σ_v (special if $d_{ve} > 1$). Let $T_{ve} \subset \Sigma_v$ be a small tubular neighbourhood of K_{ve}, whose oriented boundary ∂T_{ve} is a 2-torus. Since Σ_v is an integral homology sphere, it has a well defined oriented meridian $M(K_{ve})$ in ∂T_{ve} (defined in such a way that its linking number $\ell n(M(K_{ve}), K_{ve}) = 1$, and $M(K_{ve})$ is

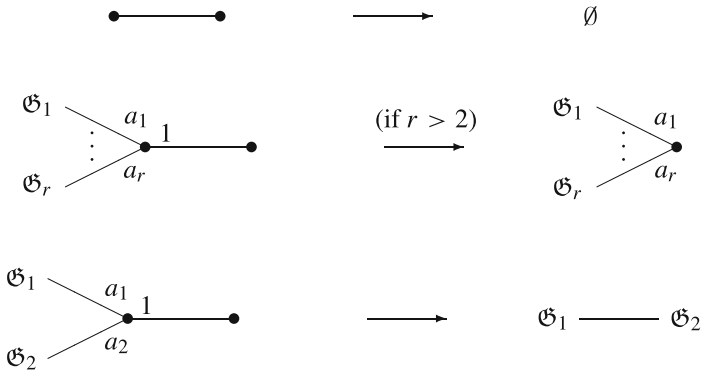
homologically trivial in T_{ve}), and a well-defined oriented longitude $L(K_{ve}) \subset \partial T_{ve}$ (defined by $\ell n(L(K_{ve}), K_{ve}) = 0$ and $L(K_{ve})$ is homologous with K_{ve} in T_{ve}). Then the *splicing* of Σ_v and Σ_u along e (or, along the knots K_{ve} and K_{ue}) means to glue $\Sigma_v \setminus T^\circ_{ve}$ with $\Sigma_u \setminus T^\circ_{ue}$ along their boundaries ∂T_{ve} and ∂T_{ue} by matching $M(K_{ve})$ to $L(K_{ue})$ and $M(K_{ue})$ to $L(K_{ve})$.

If we perform all these splicings corresponding to all node-connecting edges, we obtain an oriented integral homology sphere $M(\mathfrak{G})$, whose splice components are the Seifert manifolds $\{\Sigma_v\}_v$, indexed by the nodes of $\mathfrak{G}$.

On the other hand, by convention, similarly as for plumbing graphs, the empty splice diagram, or the diagram •——•, represent S^3.

In 5.3.10 and 5.3.15 we will show that if Γ is a plumbing graph as in 5.3.3 with $\det(\Gamma) = 1$ and $\mathfrak{G}$ is the splice diagram associated with it, then $M(\Gamma) = M(\mathfrak{G})$.

5.3.8 Splice Diagram Calculus [178] Consider the set of splice diagrams as in 5.3.7. By the construction of 5.3.7 different splice diagrams might represent the same (integral homology sphere) oriented 3-manifold. The corresponding calculus of splice diagrams lists several operations connecting splice diagrams, such that along these steps the oriented diffeomorphism type $M(\mathfrak{G})$ is preserved. Similarly to the case of plumbing calculus, we will need a shorter list of operations, which already satisfies the *stability* and the *sufficiency* of the calculus (cf. 3.3.11) for those splice diagrams which are associated with connected negative definite plumbing graphs. These operations are:



On the left hand side, the end-vertex supported by an edge with weight 1 represents a generic S^1-fiber in the corresponding Seifert component. The corresponding 'leg' can be deleted, unless we wish to highlight the corresponding (generic) knot K_{ve} (e.g., if we wish to use it in the future for a splicing with another component).

If we do not have such edges, and the graph is not of type •——•, then the diagram is called *minimal*. The Seifert pieces indexed by the nodes of a minimal splice

diagram provide a minimal Jaco–Shalen–Johannson decomposition of $M(\mathfrak{G})$. (As mentioned above, the empty splice diagram represents S^3.)

5.3.9 Now we return back to the correspondence between plumbing graphs and splice diagrams started in 5.3.3. Several graphs Γ might produce the same $\mathfrak{G}$ via the algorithm 5.3.3. Nevertheless, one has the following generalization of 5.3.6(c).

Theorem 5.3.10 ([178, §22]) *The algorithm 5.3.3 realizes a bijection between minimal (connected, negative definite) plumbing graphs with determinant 1 and minimal splice diagrams satisfying the 'relative prime condition' and the 'edge-determinant positivity'. The correspondence preserves the number of nodes.*

The statement can be compared with the remark from 5.3.6(d): the 'edge positivity' and 'ideal condition' do not guarantee that $\mathfrak{G}$ equals $\mathfrak{G}(\Gamma)$ for some Γ, however, if $\mathfrak{G}$ satisfies additionally the 'relative prime condition' then $\mathfrak{G} = \mathfrak{G}(\Gamma)$ for some Γ.

We start the proof (which is done in several steps) by the construction of such Γ.

5.3.11 From Splice Diagram to Plumbing Graph We describe the inversion algorithm of 5.3.3: we associate to any diagram $\mathfrak{G}$ (as in the statement of Theorem 5.3.10) a graph Γ such that $\mathfrak{G}(\Gamma) = \mathfrak{G}$. (For more details see [178].)

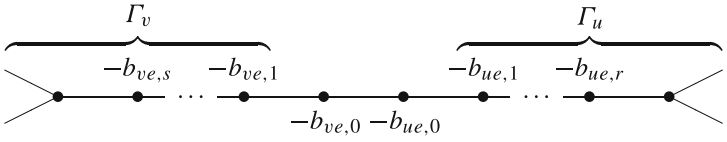
Let $\{\Sigma_v\}_v$ be the Seifert splice pieces of the splice diagram $\mathfrak{G}$. Each Σ_v is the link of a Brieskorn–Hamm complete intersection. Its minimal good plumbing graph Γ_v (together with the corresponding Seifert invariants) is given in 5.1.17. Next, we will glue these plumbing graphs $\{\Gamma_v\}_v$ in a special way, as it will be described below.

Let $e \in \mathscr{E}^{\mathfrak{G}}$ be an edge connecting the nodes v and u in $\mathfrak{G}$. Let $\{d_{ve'}\}_{e'}$ be the weights in $\mathfrak{G}$ around v, and take the (negative) continued fraction

$$\frac{\prod_{e' \neq e} d_{ve'}}{d_{ve}} = [[b_{ve,0}, b_{ve,1}, \ldots, b_{ve,s}]] \tag{5.3.1}$$

with $b_{ve,0} \geq 1$ and $b_{ve,i} \geq 2$ for $i \geq 2$. In particular, $b_{ve,0} = \lceil \prod_{e' \neq e} d_{ve'}/d_{ve} \rceil$.

Furthermore, $[[b_{ve,s}, \ldots, b_{ve,2}, b_{ve,1}]]$ equals α_{ve}/ω_{ve}, where $(\alpha_{ve}, \omega_{ve})$ is the Seifert invariant of the e-leg in Σ_v (use 5.1.17). Moreover, $d_{ve} = \alpha_{ve}$ and the determinant of the string with decorations $-b_{ve,0}, \ldots, -b_{ve,s}$ is $\prod_{e' \neq e} d_{ve'}$. The entry $b_{ve,0}$ will serve as the Euler number of an additional vertex needed in the gluing. Consider analogously the corresponding data for Γ_u too, and glue Γ_v and Γ_u as follows:



(∗)

This plumbing graph is not minimal (and it is not even negative definite). 5.3.4*(b)* implies that $1 \in \{b_{ve,0}, b_{ue,0}\}$. Hence one starts to blow down consecutively intermediate (-1)-vertices on this string. It turns out that after several blow downs one necessarily creates a 0-vertex, and after the 0-chain absorption the graph becomes minimal, good, and negative definite. One shows that the nodes survive as nodes. This procedure repeated for all the edges e (connecting nodes) provides a graph Γ.

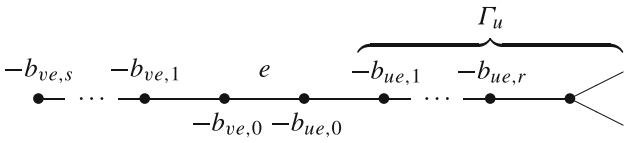
Let us verify that for this Γ one has $\det(\Gamma) = 1$, and the determinant Γ_{ve} (cf. 5.3.3) agrees with the original integer d_{ve}. These follow from the next Lemmas.

Lemma 5.3.12 $b_{ue,0} = \left\lceil \frac{\prod_{e' \neq e} d_{ue'}}{d_{ue}} \right\rceil$ *equals any of the following integers: (i) the unique integer such that the determinant of the extended graph* Γ_u^{ext} *of* Γ_u *with an additional vertex decorated by* $-b_{ue,0}$ *(and glued by edge* e_u *to the vertex* $v_{ue,1}$*) is zero; (ii)* $-(E^*_{ue,1}, E^*_{ue,1})$ *(computed in* Γ_u*); (iii)* $\det(\Gamma_u \setminus v_{ue,1})$.

Proof Use 2.2.5(6) for Γ_u^{ext}: in the case (i) let G be the string with vertices $v_{ue,1}, \ldots, u_{ue,r}$, while in case (ii) and (iii) let G be the edge e_u. □

Lemma 5.3.13

(a) The determinant of the next graph is $-d_{ve}$.



Hence, after blowing downs and 0-chain absorption (as in 5.3.11) the corresponding graph-determinant becomes $d_{ve} = \alpha_{ve}$.

(a') The statement of Lemma 5.3.12 and part (a) of the present Lemma remain valid if Γ_u *is replaced by a graph obtained from* Γ_u *by gluing to it several other Seifert components (by the above procedure).*

(b) In part (a) one can replace the string situated on the left of $v_{ve,1}$ *by any subgraph* Γ'_v *of* Γ_v*, which contains this string. Indeed, let* $\widetilde{\Gamma}$ *be the graph obtained by gluing* Γ'_v *(via* $v_{ve,1}$ *and* e*) to* Γ_u*, and* $\widetilde{\Gamma}_0$ *the graph obtained from* $\widetilde{\Gamma}$ *by blowing downs and* 0*-chain absorptions ('near* e*'). Then* $\det(\widetilde{\Gamma}_0) = -\det(\widetilde{\Gamma}) = \det(\Gamma'_v)$.

(c) If Γ *is obtained from* $\mathfrak{G}$ *as above then* $\det(\Gamma) = 1$.

Proof *(a)* Use 2.2.5(6) (where G will stay for e) combined with 5.3.12. For the sign change see 3.3.14. For *(a')* and *(b)* proceed similarly (and by induction). For *(c)* assume first that Γ is obtained by gluing Γ_v and Γ_u. Then use 2.2.5(6) in $(*)$ of 5.3.11 for the edge $(v_{ve,0}, v_{ue,0})$ and the vanishing from 5.3.12*(i)*. For the general case use induction. □

5.3.14 The above discussion shows that for any $\mathfrak{G}$ there exists $\Gamma(\mathfrak{G})$ (with $\det(\Gamma(\mathfrak{G})) = 1$) such that $\mathfrak{G}(\Gamma(\mathfrak{G})) = \mathfrak{G}$. This proves the surjectivity of the operation $\Gamma \rightsquigarrow \mathfrak{G}$ (from 5.3.3). The injectivity of $\Gamma \rightsquigarrow \mathfrak{G}$ follows from the next statement. Moreover, its proof explains the topological reason of taking the graph $(*)$ as well.

Proposition 5.3.15 *The oriented 3-manifolds $M(\Gamma)$ (constructed via plumbing from Γ as in 3.3.3) and $M(\mathfrak{G}(\Gamma))$ (constructed by splicing from $\mathfrak{G}(\Gamma)$ as in 5.3.7) agree.*

Proof We will compare the plumbing construction associated with the diagram $(*)$ and the splicing of Γ_v and Γ_u. Let Γ_v^{ext} be the extended graph of Γ_v as in 5.3.12. Let $\pi_{ve,i} : B_{ve,i} \to S_{ve,i}$ be the S^1-bundle used in the plumbing of $M(\Gamma_v^{ext})$ associated with the vertex $v_{ve,i}$. In $(*)$ we have to plumb $M(\Gamma_v^{ext})$ and (the similarly defined) $M(\Gamma_u^{ext})$. For this we take a disc $D_v \subset B_{ve,0}$ and a trivialization of $\pi_{ve,0}^{-1}(D_v) = D_v \times S^1$. (The center of D_v will stay for $E_{ve,0} \cap E_{ue,0}$. Later we will need the disc $D_v' \subset B_{ve,0}$ used in the plumbing of $B_{ve.0}$ with $B_{ve,1}$ as well.) Then, in the plumbing construction of $(*)$ we glue $M(\Gamma_v^{ext}) \setminus \pi_{ve,0}^{-1}(D_v^\circ)$ with its u-analogue.

First we compare $M(\Gamma_v^{ext}) \setminus \pi_{ve,0}^{-1}(D_v^\circ)$ with $\Sigma_v \setminus T_{ve}^\circ$. Note that $\Sigma_v \setminus T_{ve}^\circ$ can be seen via plumbing in $(*)$ as well (it is associated with the subgraph Γ_v of $(*)$). In particular, $M(\Gamma_v^{ext}) \setminus \pi_{ve,0}^{-1}(D_v^\circ)$ equals the union of $\Sigma_v \setminus T_{ve}^\circ$ with $M(K_{ve} \times (B_{ve,0} \setminus (D_v \cup D_v'))$. This latter space is $S^1 \times S^1 \times [0, 1]$, and it is glued to the boundary of $\Sigma_v \setminus T_{ve}^\circ$ via one of its boundaries. Hence the boundaries of $M(\Gamma_v^{ext}) \setminus \pi_{ve,0}^{-1}(D_v^\circ)$ and $\Sigma_v \setminus T_{ve}^\circ$ are also canonically identified.

Next notice that in the boundary of $M(\Gamma_v^{ext}) \setminus \pi_{ve,0}^{-1}(D_v^\circ)$ the 'vertical S^1', $p_v \times S^1$ ($p_v \in \partial D_v$), corresponds exactly to the meridian $M(K_{ve})$. Furthermore, we claim that the 'horizontal S^1', $\partial D_v \times \{pt\}$, corresponds to the longitude $L(K_{ve})$. To prove it, it is convenient to consider the corresponding disc bundle $\mathscr{D}_{ve,0}$ as well (cf. 3.3.5). Then take a C^∞ section in $\mathscr{D}_{ve,0}$, which above D_v is $D_v \times \{pt\}$, intersects the tube $\mathscr{D}_{ve,1}$ in a disc parallel to $\mathscr{D}_{ve,1} \cap E_{ve,0}$ (hence along a generic fiber of $\mathscr{D}_{ve,1}$), and (by Euler obstruction) it has intersection number $-b_{ve,0}$ with $E_{ve,0}$. But the linking number (in Σ_v) of K_{ve} with the boundary $\partial_{ve,1}^{gen}$ of the generic fiber of $\mathscr{D}_{ve,1}$) is $b_{ve,0}$. [Indeed, consider the Milnor fibration of $\Sigma_v \setminus K_{ve}$, which is a Seifert surface of K_{ve}. It intersects $\partial_{ve,1}^{gen}$ in $-(E_{ve,1}^*, E_{ve,1}^*)$ points, then use 5.3.12.] Therefore, $\partial D_v \times \{pt\}$ transforms via the contraction along $S^1 \times S^1 \times [0, 1]$ into the longitude of K_{ve} sitting in $\partial T_{ve} \subset \Sigma_v$. Hence the claim follows.

Hence plumbing along $(*)$ or splicing Σ_v and Σ_u along K_{ve} and K_{ue} is the same. Then use induction on the number of $\{\Sigma_v\}_v$ components. □

5.3.16 Now we are ready to **finish the proof of Theorem 5.3.10**, namely the injectivity of the operation $\Gamma \rightsquigarrow \mathfrak{G}$. Let us fix a minimal plumbing graph Γ with $\det(\Gamma) = 1$. Then $M(\Gamma)$ is an integral homology sphere, hence the components of its (minimal) JSJ-decomposition are Seifert integral homology spheres. In particular, there exists a splice diagram $\mathfrak{G}$ such that $M(\Gamma) = M(\mathfrak{G})$. Let $\Gamma(\mathfrak{G})$ be

the plumbing graph obtained from $\mathfrak{G}$ as in 5.3.11. Then by 5.3.15 one has $M(\mathfrak{G}) = M(\Gamma(\mathfrak{G}))$, hence $M(\Gamma) = M(\Gamma(\mathfrak{G}))$. Since both Γ and $\Gamma(\mathfrak{G})$ are minimal, by plumbing calculus (cf. 3.3.12) $\Gamma = \Gamma(\mathfrak{G})$. In particular, $\mathfrak{G}(\Gamma) = \mathfrak{G}(\Gamma(\mathfrak{G})) = \mathfrak{G}$, hence $\Gamma = \Gamma(\mathfrak{G}(\Gamma))$ too.

5.3.17 I^{-1} and Splice Decorations Let us consider the situation of Lemma 5.3.13*(b)*. Take e.g. any two vertices w, w' of Γ_v, which are not sitting on the string (situated left to $v_{ve,1}$ in the graph from part *(a)*). Let Γ'_v be the subgraph obtained from Γ_v by deleting the minimal path connecting them. Since $\det(\Gamma'_v)$ is $-I^{-1}_{ww'}(\Gamma_v)$ (cf. 2.2.5(5)), we get that $I^{-1}_{ww'}$ computed in Γ_v, or in the graph obtained by splicing Γ_v and Γ_u, is the same. The same property remains true even if we splice together arbitrary many components, and consider any two vertices w, w'.

Therefore, if w and w' are nodes or ends in a graph, then $-I^{-1}_{ww'}$ can be computed very conveniently in terms of the splice decorations: it is the product of all d_{ve}'s along the path but not on the path connecting w and w' (provided that $\det(\Gamma) = 1$), compare with 2.2.5(5).

5.3.18 Arrows on Splice Diagrams Assume that $\det(\Gamma) = 1$, hence Γ and $\mathfrak{G}$ determine each other. This allows us to carry any arrow (arrowhead and its supporting edge) supported by a node or end-vertex in Γ to an arrow in $\mathfrak{G}$. An arrow in Γ supported by a node v will become an arrow supported by the corresponding node in $\mathfrak{G}$. Its weight (of the supporting edge) near v will be 1, and it represents a generic S^1-orbit in Σ_v. Next, assume that we have a leg in Γ supported by a node v and it has w as its end-vertex. It is represented by an edge e and an end-vertex w in $\mathfrak{G}$ with some weight d_{ve}. An arrowhead supported by w in Γ is coded in $\mathfrak{G}$ by the following modification: the end-vertex w is replaced by an arrowhead (representing the corresponding S^1-special orbit in Σ_v with Seifert α-invariant d_{ve}).

In particular, a set of arrows on $\mathfrak{G}$ represents a link in $M(\mathfrak{G})$.

The splice calculus from 5.3.8 extends naturally to diagrams with arrows.

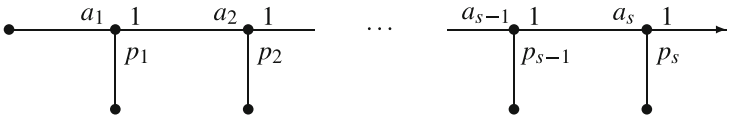
5.3.19 Multiplicities Fix Γ as above. By 3.3.6 the multiplicities along the nodes and end-vertices of a function (or of an open book decomposition) whose arrowhead is supported by a node n can be determined from the data $\{-I^{-1}_{n,v}\}$, where v runs over nodes and end-vertices. In particular, by property 2.2.5(5) or by 5.3.17, they are determined by the splice diagram weights. (The same is true if the arrowhead is supported by an end-vertex, but in that case the multiplicity of that end-vertex is not determined by the above data.) In the case of a plumbing graph this set of multiplicities (m_v with $\kappa_v \neq 2$) determines several numerical invariants (see e.g. A'Campo's formula, 4.1.22). This shows that such invariants can be recomputed in the splice diagrams as well. This fact also motivates the definition of the splice diagram (and makes it very convenient in such multiplicity computations).

E.g., assume that $\mathfrak{G}$ has a unique arrow with multiplicity $m_a = 1$ supported by a node v. Then the multiplicity of any vertex w of $\mathfrak{G}$ is the product of all weights *along but not on* the minimal path connecting w and v (or w and the arrow).

Example 5.3.20 (Plane Curve Singularities [178])

Assume that (C, o) is an irreducible plane curve singularity with Newton pairs $\{(p_i, q_i)\}_{i=1}^{s}$. For its embedded (minimal good) resolution graph see 3.3.10.

Define the integers $\{a_i\}_{i=1}^{s}$ by $a_1 := q_1$ and $a_{i+1} := q_{i+1} + p_{i+1}p_i a_i$ for $i \geq 1$. Then the splice diagram is



The diagram represents S^3 (since the minimal splice diagram obtained from the above diagram is empty), while the arrow represents the link of C in S^3.

The weights of the above splice diagram can be deduced from the resolution graph described in 3.3.10 via the procedure 5.3.3. The verification of a_1 and $p_1, \ldots, p_s$ is immediate. The weights 1 are determinants of subgraphs, which can be blown down. The determinants of the strings connecting two nodes in Γ are $q_2, \ldots, q_s$, hence the weights a_i follow via 2.2.5(6) (where G is one of these strings).

Set $\beta_i := a_i p_i p_{i+1} \cdots p_s$ for $1 \leq i \leq s$, $\bar{\beta}_0 := p_1 p_2 \cdots p_s$, and $\bar{\beta}_i := a_i p_{i+1} \cdots p_s$ for $1 \leq i \leq s$. Since they are the multiplicities of f along the corresponding vertices of $\mathfrak{G}$ (cf. 5.3.19), A'Campo's formulae 4.1.22 reads as

$$\zeta^{-1}(t) = \frac{\Delta_1(t)}{t-1} = \frac{\prod_{1 \leq i \leq s}(t^{\beta_i} - 1)}{\prod_{0 \leq i \leq s}(t^{\bar{\beta}_i} - 1)}. \tag{5.3.2}$$

Hence, if $\Delta_1(a, b)(t)$ is the characteristic polynomial of $x^a + y^b$, then

$$\Delta_1(C, o)(t) = \Delta_1(a_s, p_s)(t) \cdot \prod_{i=1}^{s-1} \Delta_1(a_{s-i}, p_{s-i})(t^{p_s p_{s-1} \cdots p_{s-i+1}}). \tag{5.3.3}$$

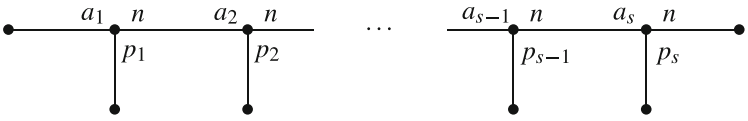
Furthermore, the semigroup $\mathscr{S}_{C,o} \subset \mathbb{N}$ has the elements $\bar{\beta}_i$ $(0 \leq i \leq s)$ as a minimal set of generators. [These generators $\bar{\beta}_i$ $(0 \leq i \leq s)$ are the intersection multiplicities of f with the germs $x, y, g_2, \ldots, g_s$, which will be constructed in Example 5.4.7—the 'end curve functions' of the diagram.] Each element $\gamma \in \mathscr{S}_{C,0}$ can be represented in a unique way in the form $\gamma = k_0\bar{\beta}_0 + \sum_{1 \leq i \leq s} k_i \bar{\beta}_i$ with $k_0 \geq 0$ and $0 \leq k_i \leq p_i - 1$ for $1 \leq i \leq s$, see [684]. Recall (cf. 5.2.2) that $\{k \in \mathbb{Z},\ k \geq \mu\} \subset \mathscr{S}_{C,o}$.

For any $j \in \{1, \ldots, s\}$ let (C_j, o) be the plane curve singularity with Newton pairs $\{(p_i, q_i)\}_{i=1}^{j}$. Then, using the above identities, one verifies that for $j < s$ one has $a_{j+1} > \mu(C_j)$, hence $a_{j+1} \in \mathscr{S}_{C_j,o}$. (This can be, and will be compared with the 'semigroup condition' from 5.4.2.)

Example 5.3.21 (Suspensions of Plane Curve Singularities [512])

Fix an isolated plane curve singularity $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$, and let M be the link of the suspension singularity $f(x, y) + z^n = 0$. If M is an integral homology sphere then f is irreducible. Indeed, one has the covering $M \to S^3$ branched along the link L_f of f, and the covering group $\mathbb{Z}_n$ acts on M with fixed point set L_f. But by Smith theory, such an action must have a connected fixed point set. Hence $L_f = S^1$.

Next, assume that f is irreducible. Then M is an integral homology sphere if and only if n is relative prime with all entries of the Puiseux pairs (equivalently, of the Newton pairs). This also means that the integers (n, a_i, p_i) are pairwise relative prime for any i. Furthermore, the splice diagram of M in this case is



This follows either by reinterpretation of the algorithm from 4.1.C for splice diagrams, or by an inductive argument (as in [512]).

Example 5.3.22 (General Cyclic Coverings) The previous example has the following generalization [512]. Assume that the integral homology sphere M is represented by the splice diagram $\mathfrak{G}$, and it has a knot $L \subset M$ represented by an arrow on $\mathfrak{G}$. A weight of $\mathfrak{G}$ on an edge e is called 'near' or 'far' according to whether it is on the end of the edge nearest to or furthest from the arrow of $\mathfrak{G}$. Then the $\mathbb{Z}_n$-cyclic covering of M branched along L is an integral homology sphere if and only if n is relatively prime to all near weights of $\mathfrak{G}$, and its splice diagram can be obtained from $\mathfrak{G}$ by multiplying each far weight by n.

Here are two examples:

Here $M = \{u^5 + v^{22} + w^{39} = 0\}$, $L = \{u - v^3 - w^7 = 0\}$, $n = 2$, hence the covering has equation $u^5 + v^{22} + w^{39} = u - v^3 - w^7 - z^2 = 0$, or $(v^3 + w^7 + z^2)^5 + v^{22} + w^{39} = 0$.

Here $M = \{u^5 + z^{13} + w^{44} = 0\}$, $L = \{u - z^2 - w^7 = 0\}$, $n = 3$, hence the covering has equation $u^5 + z^{13} + w^{44} = u - z^2 - w^7 - v^3 = 0$, or $(v^3 + w^7 + z^2)^5 + z^{13} + w^{44} = 0$.

The two output 3-manifolds have the same splice components, but they are glued differently.

5.3.23 Before we state the next theorem, let us mention that usually the universal abelian covering of $M(\Gamma)$ is not an integral homology sphere.

Theorem 5.3.24 ([559]) *Assume that a rational homology sphere 3-manifold $M(\Gamma)$ is given by the plumbing graph Γ. Let $\mathfrak{G} = \mathfrak{G}(\Gamma)$ be the splice diagram associated with Γ. Then the universal abelian covering of $M(\Gamma)$ depends only on $\mathfrak{G}$.*

This has the following application. Assume additionally that for any node v of $\mathfrak{G}$ the weights $\{d_{ve}\}_e$ are pairwise relative prime. Then we claim that the universal abelian covering $M(\Gamma)_a$ of $M(\Gamma)$ is the integral homology sphere $M(\mathfrak{G})$. That is, the plumbing graph of $M(\Gamma)_a$ is that unique graph, which has determinant 1 and corresponds to $\mathfrak{G}$ by the algorithms 5.3.3 and 5.3.10. Indeed, let $\Gamma(\mathfrak{G})$ be the graph given by 5.3.10. Then, since Γ and $\Gamma(\mathfrak{G})$ have the same splice diagram, $M(\Gamma)_a = M(\Gamma(\mathfrak{G}))_a$ by the above theorem. But $M(\Gamma(\mathfrak{G})) = M(\mathfrak{G})$, cf. 5.3.15, hence $M(\Gamma(\mathfrak{G}))_a = M(\mathfrak{G})_a = M(\mathfrak{G}) = M(\Gamma(\mathfrak{G}))$.

5.4 Splice Quotient Singularities

This part is an introduction to the theory of splice quotient singularities of Neumann and Wahl. There are three basic constructions of such germs: the 'original' construction of Neumann and Wahl (based on 'semigroup and congruence conditions'), its modification by Okuma (based on the 'monomial condition'), and the presentation based on the 'end curve condition'. Analytic germs satisfying this condition are splice quotients by the 'End Curve Theorem'. ♣

5.4.1 Splice quotient singularities were introduced by Neumann and Wahl in [515]. From any fixed graph Γ (which has some special arithmetical properties, cf. below) one constructs a family of singularities with common equisingularity type, such that any member admits a distinguished resolution, whose dual graph is exactly Γ. The construction suggests that the analytic properties of the singularities constructed in this way are strongly linked with the fixed resolution and with its graph Γ. (Hence, the expectation is that certain analytic invariants might be computable from Γ.)

A starting assumption is that Γ is a tree with rational vertices, that is, the link of the constructed singularity is a rational homology sphere. In this section this assumption always holds.

Presently, there are three different approaches how one can introduce and study splice quotient singularities; each of them is based on a different geometric property. They are: (a) the 'original' construction of Neumann–Wahl, (b) the 'modified' version by Okuma, and (c) considering singularities satisfying the 'end-curve condition'. It turns out that all these approaches provide the same family of singularities.

In the first two cases we start with a topological type (that is, with Γ), which satisfies certain restrictions, and we endow it with an analytic structure, the 'splice quotient' analytic type. In the third case we start with an analytic structure, which satisfies a certain analytic property, and it turns out that in fact it can be represented by splice quotient equations as in the cases (a)–(b).

5.4.2 The Construction of Neumann–Wahl [515] The construction starts with a given negative definite graph Γ, and one imposes two combinatorial restriction on it, the *semigroup and congruence conditions*. The congruence condition is empty if $\det(\Gamma) = 1$. Using the first condition one writes the equations of a complete intersection. The equations depend only on the splice diagram associated with the graph, in particular they are called 'splice diagram equations'. Then one defines an action of H on this complete intersection: here the congruence condition is needed.

Though in the next parts we prefer to use the second approach (of Okuma), we will present in short the Neumann–Wahl construction as well.

Fix a plumbing graph Γ and the associated splice diagram $\mathfrak{G} = \mathfrak{G}(\Gamma)$, cf. 5.3.3. For a node v and adjacent edge e of $\mathfrak{G}$ let $\mathfrak{G}_{ve}$ be the maximal connected subdiagram of $\mathfrak{G} \setminus v$ adjacent to e. For any end-vertex w of $\mathfrak{G}$, which is in $\mathfrak{G}_{ve}$, let m_{vw} be the product of weights of $\mathfrak{G}_{ve}$ along but not on the minimal path connecting v and w. Let S_{ve} be the *semigroup* of $\mathbb{Z}_{\geq 0}$ generated by $\{m_{vw}\}_w$. Then the 'semigroup condition' requires that $d_{ve} \in S_{ve}$. Note that this is a stronger condition than the corresponding 'ideal condition' from 5.3.4*(c)*: the 'ideal condition' is always satisfied, while the semigroup condition may not be satisfied (see e.g. the diagram Γ in 5.4.5).

Assume that the 'semigroup condition' is satisfied in $\mathfrak{G}$ for any node v and adjacent e. Let $\mathcal{V}^e$ be the set of ends. We introduce a complex variable z_v for each $v \in \mathcal{V}^e$, and we write $|\mathcal{V}^e| - 2$ local equations in $(\mathbb{C}^{|\mathcal{V}^e|}, 0)$.

Let us fix a node v. Write d_{ve} as $\sum \alpha_w m_{vw}$, where the sum is over the ends of $\mathfrak{G}$ sitting in $\mathfrak{G}_{ve}$, and $\alpha_w \in \mathbb{Z}_{\geq 0}$. Consider the corresponding monomial $M_{ve} := \prod z_w^{\alpha_w}$. Furthermore, consider a $\kappa_v^{\mathfrak{G}} \times (\kappa_v^{\mathfrak{G}} - 2)$ matrix $(a_{v,e,j})_{e,j}$ of full rank (this means that all maximal minors are non-zero). Then for this node v one writes $\kappa_v^{\mathfrak{G}} - 2$ equations of the following type (the sum runs over all edges e adjacent to v)

$$\sum_e a_{v,e,j} M_{ve} = 0 \quad (1 \leq j \leq \kappa_v^{\mathfrak{G}} - 2).$$

Neumann and Wahl in [515] proved that all these equations define an isolated complete intersection singularity in $\mathbb{C}^{|\mathcal{V}^e|}$. If $\det(\Gamma) = 1$, then it admits a resolution with resolution graph identical to Γ. If $\det(\Gamma) \neq 1$ then this complete intersection is

the universal abelian covering of a normal surface singularity (X, o), which admits a resolution with graph Γ. More precisely, (X, o) is realized as the quotient of the above isolated complete intersection singularity (X_a, o) by H (that is, by L'/L associated with Γ). The action of H on $\mathbb{C}^{|\mathcal{V}_e|}$ is defined by $h * z_v = \theta([E_v^*])(h)z_v$. This induces an action (free off o) on (X_a, o) too. The existence of monomials M_{ve} with the property that the splice diagram equations are H-quasi-invariants is guaranteed by the 'congruence condition'.

Different choices of M_{ve}, or adding to the equation any power series $g_{v,e,j}$ (in variables $\{z_e\}_{v\in\mathcal{V}_e\cap\mathfrak{G}_{ve}}$) having monomials of type $\prod z_w^{\beta_w}$ with 'higher degree' $\sum \beta_w m_{vw} > d_{ve}$ (and satisfying compatibility with the H-action), produce an equisingular deformation of the singularity (with the same resolution graph Γ).

Following Neumann and Wahl, the singularity (X, o) defined in this way is called *splice quotient singularity* (and the defining equations are the *splice equations*). If $\det(\Gamma) = 1$ then the complete intersection $(X, o) = (X_a, 0)$ is called a *splice (type) singularity*.

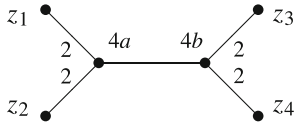
For an explicit computation of an example (equations and action) see 8.5.32.

Example 5.4.3 Both particular suspensions considered in 5.3.22 are splice singularities. E.g. in the first case the equations $u^5 + v^{22} + w^{39} = v^3 + w^7 + z^2 = 0$ are of splice type, and the deformation $tu + v^3 + w^7 + z^2 = 0$ ($t \in \mathbb{C}$) is a 'higher degree' equisingular deformation of the original equation.

Hence, we obtain an equisingular family of complete intersections (with stable integral homology sphere link given by $\mathfrak{G}$), such that the embedding dimension for $t = 0$ is 4 while for $t \neq 0$ is 3. This shows that (although we expect that several discrete analytic invariants of a splice quotient is readable from Γ), this is not true for any invariant.

Example 5.4.4 (The Universal Abelian Covering of a Quotient-Cusp)

Consider the graph Γ of a quotient-cusp as in 4.1.37. As usual, cf. 2.2.A, we write $[[b_1, \dots, b_k]] = \alpha/\omega$ and $[[b_1, \dots, b_{k-1}]] = \omega'/\tau$, hence $\omega\omega' = \alpha\tau + 1$. Set $a := \omega - \tau$, $b := \omega' - \tau$ and $d := \alpha - \omega - \omega' + \tau$. Then $|H| = 16d$ and the splice diagram has the following form (where we already indicated the variables z_v as well):



Hence a possible set of equation of the UAC is $z_1^2 + z_2^2 = z_3^n z_4^m$ and $z_3^2 + z_4^2 = z_1^k z_2^l$, with $n+m = 2a$ and $k+l = 2b$. Let ξ be a primitive $4d$-th root of unity. Then H acts on $\mathbb{C}^4$ as follows: $[E_1^*]$ acts diagonally by $(-\xi^a, \xi^a, \xi, \xi)$, $[E_2^*]$ by $(\xi^a, -\xi^a, \xi, \xi)$, $[E_3^*]$ by $(\xi, \xi, -\xi^b, \xi^b)$, and $[E_4^*]$ by $(\xi, \xi, \xi^b, -\xi^b)$. The universal abelian covering is a cusp with dual b^*-sequence $(2a, 2b, 2a, 2b)$ whenever $a \neq 1 \neq b$, and b^*-

sequence $(2a, 2, 2a, 2)$ if $b = 1$, and b^*-sequence $(2b, 2, 2b, 2)$ if $a = 1$ (note that $ab > 1$ always). For more see [514] and [299].

Example 5.4.5 (Suspension as Quotient of Splice Type) Consider the plane curve singularity with two irreducible branches ϕ_1 and ϕ_2 and equation

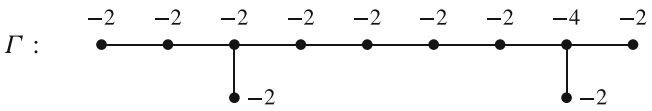
$$a^3b^3 + \Big(\frac{b+a}{2}\Big)^{11} = \phi_1(a,b)\cdot\phi_2(a,b) = (a^3 + \frac{b^8}{2^{11}} + \cdots)(b^3 + \frac{a^8}{2^{11}} + \cdots).$$

Define the isolated complete intersection in variables (r, s, a, b) by $(X_a, o) = \{r^2 = \phi_1(a,b),\ s^2 = \phi_2(a,b)\}$. $\mathbb{Z}_4$ acts on it via its generator G by $r \mapsto s$, $s \mapsto -r$, $a \mapsto b$ and $b \mapsto a$. The action of G^2 is $r \mapsto -r$, $s \mapsto -s$, $a \mapsto a$ and $b \mapsto b$, and its invariants are $a, b, R := r^2, S := s^2, T := rs$ with additional equation $T^2 = RS$. Hence

$$X' := X_a/G^2 = \{RS = T^2, R = \phi_1(a,b), S = \phi_2(a,b)\} = \{T^2 = \phi_1(a,b)\phi_2(a,b)\}.$$

By the change of variables $a = u - v$, $b = u + v$ it becomes $\{T^2 = (u^2 - v^2)^3 + u^{11}\}$. The induced action of G is $T \mapsto -T$, $u \mapsto u$, $v \mapsto -v$ with invariants u, $X := T^2$, $Y := v^2$ and $Z := Tv$ and new relation $Z^2 = XY$. Hence, $X_a/\mathbb{Z}_4$ is given by $X = (u^2 - Y)^3 + u^{11}$ and $XY = Z^2$. After substitution it becomes $Z^2 = Y((u^2 - Y)^3 + u^{11})$, and by change of variables $w := u^2 - Y$ it has the form $Z^2 = (u^2 - w)(w^3 + u^{11})$.

This is a hypersurface singularity (X, o) of suspension type $z^2 = f(u, w)$ (even with non-degenerate Newton principal part), its minimal resolution graph Γ is



Note that, because of the E_8-subgraph with determinant 1, Γ does not satisfy the semigroup condition. This can happen even with f irreducible. E. Sell classified those irreducible germs f and integers n, for which the graph of the suspension $z^n = f(u, w)$ satisfies the semigroup and congruence conditions [625–627].

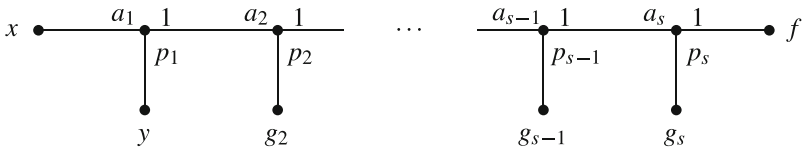
In fact, even the following situation can happen: a suspension graph satisfies the semigroup and congruence conditions—hence the splice quotient analytic structure can be considered—, nevertheless the suspension equations (analytic type) cannot be realized by the splice quotient equations, see again [625].

The reader is invited to compute the graph of the $\mathbb{Z}_2$-covering (X', o) of (X, o) (which is a suspension too) and verify that it does satisfy the semigroup and congruence conditions, furthermore, the equations of (X_a, o) given above are particular splice diagram equations. For such germs (X, o), which are not splice

type, but some of their coverings are of splice type, we can expect that they share the basic properties of splice quotients. Note also that $H(X,o) = \mathbb{Z}_4$ and $H(X',o) = \mathbb{Z}_2$, and (X_a,o) is the UAC of both of them. Hence the fact that the universal abelian covering is a complete intersection is not the property merely of splice quotients. (Here the action of $H = \mathbb{Z}_4$ on (X_a,o) is not diagonal, as it is in the case of splice quotients.)

Remark 5.4.6 The starting point of splice quotient singularities was that theorem of Neumann [506], which says that the universal abelian covering of a weighted homogeneous singularity with $g = 0$ is a Brieskorn complete intersection, cf. 5.1.33 here. This phenomenon is generalized by the splice quotient singularities. Later Neumann and Wahl verified that quotient-cusps are also splice quotients [514]. Based on this positive results they conjectured in [513] that any $\mathbb{Q}$-Gorenstein singularity with $\mathbb{Q}HS^3$-link is a splice quotient. Several counterexamples were provided in [399], one of them is the suspension from 5.4.5. For others (namely, for superisolated germs with $\nu \geq 3$) see the end of 5.2.5.

Example 5.4.7 Besides that they define the splice quotient (X,o), the splice equations can be used to produce equations of holomorphic germs on (X,o) as well. We exemplify this fact in the case $(X,o) = (\mathbb{C}^2,o)$. Consider the following splice diagram $\mathfrak{G}$ obtained from the splice diagram of a plane curve singularity by replacing its arrowhead with a vertex; for notations and numerical invariants see 5.3.20.



Via splice diagram calculus 5.3.8, the diagram is equivalent to the empty diagram, hence it represents the 3-manifold S^3, or the space-germ $(\mathbb{C}^2,0)$. Nevertheless, we will write the corresponding splice equations connecting the variables $x, y, g_2, \ldots, g_s, f$ associated with the end-vertices. First note that for any $2 \leq j \leq s$, $a_j \in \mathscr{S}_{C_{j-1},o}$ (see end of 5.3.20), hence there exist positive integers $n_{j,x}$, $n_{j,y}$, and $n_{j,2}, \ldots, n_{j,s-1}$ such that

$$a_j = n_{j,x}\bar{\beta}_0(C_{j-1}) + n_{j,y}\bar{\beta}_1(C_{j-1}) + \sum_{i=2}^{j-1} n_{j,i}\bar{\beta}_i(C_{j-1}).$$

Hence, a possible set of splice equations are

$$\begin{aligned} x^{a_1} + y^{p_1} &= g_2, \\ x^{n_{2,x}} y^{n_{2,y}} + g_2^{p_2} &= g_3, \\ x^{n_{3,x}} y^{n_{3,y}} g_2^{n_{3,2}} + g_3^{p_3} &= g_4, \\ \cdots & \\ x^{n_{s,x}} y^{n_{s,y}} \textstyle\prod_{i=2}^{s-1} g_i^{n_{s,i}} + g_s^{p_s} &= f. \end{aligned} \tag{5.4.1}$$

Clearly, $\mathbb{C}\{x, y, g_2, \ldots, g_s, f\}/(\text{equations})$ is $\mathbb{C}\{x, y\}$. But the point is that $g_2, \ldots, g_s, f$ can be expressed inductively as germs in $\mathbb{C}\{x, y\}$, hence we can regard them as elements of $\mathbb{C}\{x, y\}$. Moreover, the construction provides automatically the simultaneous embedded resolution of all $x, y, g_2, \ldots, g_s, f \in \mathbb{C}\{x, y\}$: they are all irreducible, and their strict transforms in the resolution having dual graph Γ (associated with $\mathfrak{G}$) are transversal cuts of the corresponding irreducible exceptional curves corresponding to the end-vertices. We will see, that in the language of 5.4.13 this means that these germs are 'end curve functions'. Note also that the constructed germ $f(x, y)$ is a possible equation of a plane curve singularity with the given Newton pairs. In fact, considering different (allowed) choices of the integers $n_{j,x}, n_{j,y}, n_{j,i}$, and different (allowed) coefficients in the equations, and adding higher degree monomials in the splice equations, we obtain all the possible equations of the plane curve singularities with given fixed Newton pairs, cf. [344].

5.4.8 Okuma's Version In the next review we follow Okuma's approach from [533] (which slightly differs from Neumann–Wahl construction [515, (7.1)]). According to this, Γ has to satisfy a combinatorial restriction: the *monomial condition* (see 5.4.10). This condition is equivalent to the semigroup and congruence conditions of Neumann-Wahl (see [515, §13]). The combinatorially defined equations and H-action are similar to the Neumann–Wahl case, although now they involve the monomial condition. We call the equations *Γ-combinatorial equations*.

5.4.9 Fix a resolution graph Γ (a tree with all genus decorations zero). Let $\mathcal{V}^e$ denote the set of end-vertices, and $\mathbb{C}\{z\}$ the ring of local convergent power series in variables $\{z_i\}_{i\in\mathcal{V}^e}$. As before, H acts on $\mathbb{C}\{z\}$ by $h * z_i = \theta([E_i^*])(h)z_i$. For each node v ($\kappa_v \geq 3$) the connected components Γ_{ve} of $\Gamma \setminus v$ are called *branches* of v (they are indexed by the adjacent edges e of v). Let $\mathcal{V}^e_{ve}$ be the set of end-vertices contained in a branch Γ_{ve}.

Definition 5.4.10 ([533])

(a) A *monomial cycle* has the form $D(\alpha) = \sum_{v\in\mathcal{V}^e} \alpha_v E_v^* \in L'$, where $\alpha_v \in \mathbb{Z}_{\geq 0}$.
(b) We say that Γ satisfies the *monomial condition* if for any node v and any edge e adjacent to v, there exists a monomial cycle $D(\alpha)$ such that $D(\alpha) - E_v^*$ is an effective integral cycle supported on Γ_{ve}. (In particular, $\alpha_v = 0$ for $v \notin \mathcal{V}^e_{ve}$.) In this case, $z^\alpha \in \mathbb{C}\{z\}$ is called an *admissible monomial* belonging to Γ_{ve}.

Fix a vertex $v \in \mathscr{V}$. For any monomial $z^\alpha \in \mathbb{C}\{z\}$ one defines its *v-degree* by $v\text{-deg}(z^\alpha) = -(D(\alpha), E_v^*) \in \mathbb{Q}$. The v-degree of any power series is the minimum of the v-degrees of its monomials having non-zero coefficient. From definitions, any admissible monomial belonging to any branch of v has v-degree $-(E_v^*, E_v^*)$ and it is a $\theta([E_v^*])$-eigenvector of the H-action.

Definition 5.4.11 (Γ-Combinatorial Equations [515, 533]) Assume that Γ satisfies the monomial condition. For any fixed node v we make some arbitrary choices.

We select an admissible monomial M_{ve} for each branch Γ_{ve} of v, and we select a matrix $(a_{v,e,j})$ (e adjacent edge of v, $1 \le j \le \kappa_v - 2$) of full rank. Finally, we might choose convergent power series $g_{v,j} \in \mathbb{C}\{z\}$ ($1 \le j \le \kappa_v - 2$) which are $\theta([E_v^*])$-eigenvectors of the H-action (as in 5.4.9) and have v-degree greater than $-(E_v^*, E_v^*)$. Then, associated with v, we write $\kappa_v - 2$ equations

$$\sum_e a_{v,e,j} M_{ve} + g_{v,j} = 0 \quad (1 \le j \le \kappa_v - 2).$$

The set of these equations for all nodes v form the *Γ-combinatorial equations* of Γ.

Theorem 5.4.12 ([515, (7.2)]) *A set of Γ-combinatorial equations of Γ define an isolated complete intersection singularity in $(\mathbb{C}^{|\mathscr{V}^e|}, 0)$ on which H acts freely off the origin. It is the universal abelian covering of its factor by H. The factor (X, o) has a good resolution with dual graph Γ.*

A singularity defined as in (5.4.12) is called *splice quotient*. Its equisingularity type is independent of the different choices (cf. [515, (10.1)], [532]); in particular, one can take $g_{v,j} = 0$ to get 'simpler' analytic representatives.

5.4.13 The 'End Curve Condition' The third approach defines a family of singularities with a special analytic property, with the *end curve condition*. This requires the existence of a resolution of the singular germ, which has the following property: For each exceptional irreducible component E_v, which corresponds to an end-vertex of the resolution graph Γ, there exists an analytic function whose reduced strict transform is irreducible, it intersects the exceptional curve only along E_v, and this intersection is transversal. (These are called 'end curve functions', for details see 8.5). The fact that splice quotient singularities associated with a graph Γ (in the sense of the first two definitions of Neumann–Wahl or Okuma) satisfy the 'end curve condition' follows from an addendum of Theorem 5.4.12, cf. [515], which says that the coordinate functions of $(\mathbb{C}^{|\mathscr{V}^e|}, 0)$ (lifted to the resolution with graph Γ) are 'end curve functions'.

The converse is harder; it says that a singularity, which has a resolution with graph Γ and in this resolution it satisfy the 'end curve condition' is, in fact, a splice quotient (associated with Γ). This is the subject of the 'End Curve Theorem' [75, 517, 534]. A proof will be given here in Theorem 8.5.34.

Several analytic properties of singularities satisfying the end curve condition will be given in Sect. 8.5.

5.5 Newton Non-degenerate Singularities

In the first Sect. 5.5.A we review certain facts and invariants regarding hypersurface singularities with non-degenerate Newton principal part. In 5.5.B we present the general case of a Newton non-degenerate (analytic) Weil divisor embedded into an affine toric variety. We focus on the case of curves and surfaces, on the delta invariant, geometric genus, and in the surface case on (generalized) Oka's algorithm, which provides the resolution graph from the Newton diagram. Normality and Cartier properties are also discussed. ♣

5.5.A *The Classical Case of Hypersurfaces in* $(\mathbb{C}^{n+1}, 0)$

Though starting with the next subsections we will discuss the general case of non-degenerate singularities, as (local analytic) Weil divisors in affine toric varieties, here we prefer to collect some facts relevant to the themes of the book about the classical case of *hypersurfaces*. This serves also as a model for the general case.

5.5.1 Basic Definitions (The Hypersurface Case) [345] Fix an integer $n \geq 1$. For any set $S \subset \mathbb{N}^{n+1}$ denote by $N\Gamma_+(S) \subset \mathbb{R}^{n+1}$ the convex closure of $\bigcup_{p\in S}(p + \mathbb{R}^{n+1}_{\geq 0})$. By definition, the *Newton boundary* (or *diagram*) $N\Gamma(S)$ associated with S is the union of compact boundary faces (of any dimension) of $N\Gamma_+(S)$. Let $N\Gamma_-(S)$ denote the (real) cone $\cup_{x\in N\Gamma(S)}[0, x]$ with base $N\Gamma(S)$ and vertex 0.

Let $f\colon (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be an analytic function germ defined by a convergent power series $\sum_p a_p x^p$, where $x^p = \prod_i x_i^{p_i}$. By definition, the *Newton boundary* $N\Gamma(f)$ of f is $N\Gamma(\mathrm{supp}(f))$, where $\mathrm{supp}(f)$ is the support $\{p \in \mathbb{N}^{n+1} : a_p \neq 0\}$ of f. We also write $N\Gamma_-(f)$ for $N\Gamma_-(\mathrm{supp}(f))$. The *Newton principal part* of f is $\sum_{p\in N\Gamma(f)} a_p x^p$. Similarly, for any q-face F of $N\Gamma(f)$ (of any dimension q), set $f_F(x) := \sum_{p\in F} a_p x^p$. We say that f is *non-degenerate* on F if the system of equations $\partial f_F/\partial x_1 = \cdots = \partial f_F/\partial x_{n+1} = 0$ has no solution in $(\mathbb{C}^*)^{n+1}$. When f is non-degenerate on every q-face of $N\Gamma(f)$, we say (after Kouchnirenko [345]) that f has a *non-degenerate Newton principal part*. The diagram $N\Gamma(f)$ (or f) is called *convenient* if $N\Gamma(f)$ intersects all the coordinate axes.

5.5.2 When Is the Singularity Isolated? If we fix a Newton boundary $N\Gamma$ (i.e. $N\Gamma = N\Gamma(S)$ for some S), then the set of coefficients $\{a_p : p \in N\Gamma\}$ for which $f(x) = \sum_{p\in N\Gamma} a_p x^p$ is Newton non-degenerate (as its own principal part) form a non-empty Zariski open set, cf. [345, 1.10(iii)]. Nevertheless, even for generic coefficients $\{a_p\}_{p\in N\Gamma}$, the germ $f = \sum_{p\in N\Gamma} a_p x^p$ (or any f with $N\Gamma(f) = N\Gamma$), in general, does not define an isolated singularity. If $n = 1$ or 2 the characterizations are the following (for the general case see [345, 1.13(ii)]). For convenience, let us assume that f is singular, i.e. $\partial f(0) = 0$.

If $n = 1$ then $S = \mathrm{supp}(f)$ supports the equation of an isolated singularity if and only if for any of the two coordinate axis, $N\Gamma$ has a lattice point at most 1 far from the axis (in Euclidean distance). If f defines an isolated singularity and it has a *convenient* non-degenerate Newton principal part then the number of irreducible components is the combinatorial length (the number of primitive segments) of $N\Gamma$. Furthermore, any of the local irreducible components cannot have more than one Puiseux pair.

If $n = 2$ then $N\Gamma$ supports an isolated singularity if and only if

- $\{(0,0,0),(0,0,1),(0,1,0),(1,0,0)\} \cap N\Gamma = \emptyset$ (i.e. $\partial f(0) = 0$),
- the diagram $N\Gamma$ has a lattice point on every coordinate plane, and
- for any coordinate axis, $N\Gamma$ has a lattice point at most 1 far from the axis.

(5.5.1)

E.g., a convenient f with generic coefficients defines an isolated singularity.

The geometric meaning of (5.5.1) is the following. If the second point is not satisfied then $f = 0$ contains a coordinate plane (and some other components), hence cannot have an isolated singularity. Otherwise, under the non-degenerate condition, it might have singularities only along the coordinate axes. E.g., the x_3 axis is in $V(f)$ whenever f is not convenient along this axis. Moreover, in this case, the curve transversal type of $V(f)$ along this axis has a Newton diagram, which is associated with the projection of $\mathrm{Supp}(f)$ to the x_1x_2 plane. Hence, $V(f)$ is not singular along the axis if and only if this transversal type is smooth, that is, the projected $\mathrm{Supp}(f)$ contains at least one of the points $(0,1)$ or $(1,0)$.

Example 5.5.3 Assume that $n = 2$ and (5.5.1) is satisfied. Then $N\Gamma$ cannot be merely one vertex. Moreover, if $N\Gamma$ has no 2-faces then (modulo a permutation of the coordinates) it is the segment $[(0,1,1),(k,0,0)]$ for some $k \geq 2$. Otherwise, any edge of the boundary $\partial N\Gamma$ of $N\Gamma$ should lie either on a coordinate plane or be (after permuting coordinates) of the form $AB = [(a,0,c),(0,1,b)]$ with $a > 0$ and $b > 0$.

5.5.4 Some Discrete Invariants Determined from the Newton Boundary If f defines an isolated singularity and has a non-degenerate Newton principal part, then its Newton boundary $N\Gamma(f)$ determines almost all its discrete analytic and embedded topological invariants. In such discussions one inevitably faces the arithmetical properties of integral polytopes.

(a) The *Milnor number* $\mu(f)$ of f is given by Kouchnirenko [345]. For any $N\Gamma$ let V_{n+1} be the $(n+1)$-dimensional volume of $N\Gamma_-$, and for $1 \leq q \leq n$, let V_q be the sum of the q-dimensional volumes of all the intersections of $N\Gamma_-$ with q-dimensional coordinate planes. Set $\nu(N\Gamma) := \sum_{i=0}^{n+1}(n+1-i)! \cdot V_{n+1-i}$. Then $\mu(f)$ of any *convenient* Newton non-degenerate germ f equals $\nu(N\Gamma(f))$.

In fact, the same formula is valid for non-convenient isolated singularities as well. Indeed, assume e.g. that $n = 2$, and the diagram $N\Gamma(f)$ does not intersect

the x_3 axis. Let AB be an edge as in 5.5.3. Then the deformation $f_d := f + tz_3^d$ with $d \geq \mu(f) + 2$ has a uniform stable radius for the Milnor fibration [523], hence $\mu(f) = \mu(f_d)$. Moreover, $N\Gamma_-(f_d) = N\Gamma_-(f) \cup W_d$, where W_d is the 3-simplex with vertices 0, A, B and $(0, 0, d)$. Since $(6V_3 - 2V_2 + V_1)(W_d) = 0$, one obtains that $\nu(N\Gamma(f_d)) = \nu(N\Gamma(f))$. (Since f is finitely determined, f and f_d are right-equivalent for $d \gg 0$ and their other invariants agree too. Hence, being convenient is not really essential, see also [745].)

(b) The *characteristic polynomial* of the algebraic monodromy is determined by Varchenko [711]; the set of *spectral numbers* is computed in [135, 314, 603, 654]; the *multiplicity* of f by $\min_{p=(p_i)_i \in N\Gamma(f)} \sum p_i$.

(c) The *geometric genus* of the surface singularity $(\{f = 0\}, 0)$ is computed by Merle and Teissier as $\#(N\Gamma_-(f) \cap \mathbb{Z}^3_{>0})$, cf. [418], see also [443].

(d) The *embedded topological type* and the *Milnor fibration* of f (with its homological 'package' as in 3.2.A) is determined by $N\Gamma(f)$, cf. [523, 2.1].

(e) An embedded resolution of $(V(f), 0) \subset (\mathbb{C}^{n+1}, 0)$ can be done via toric modifications associated with regular subdivisions of the dual fan of f, see e.g. [711] (or the next 5.5.B). If $n = 2$, an explicit (and canonical) construction of the *dual resolution graph* $\Gamma_{V(f)}$ of the surface singularity $(V(f), 0)$ is given by Oka [524]. We recall this below in 5.5.23.

For certain generalizations to the complete intersection case see [234, 312, 313, 443, 526, 527] and the references therein, and also 10.6.20 here.

5.5.5 Equivalence of Newton Boundaries Assume that $n = 2$. We say that two Newton boundaries $N\Gamma_1$ and $N\Gamma_2$ are *equivalent* if both define isolated singularities and $N\Gamma_1 \subset N\Gamma_2$. This definition generates an equivalence relation of boundaries (see [76]). E.g., if $N\Gamma_1$ has a boundary segment AB as in 5.5.3, then adding a monomial of type tx_3^d ($t \in \mathbb{C}^*$, $d \gg 1$) provides a new equivalent diagram (however, $N\Gamma_1$ can be completed near the x_3 axis even by more than one 2-faces). The equations of germs with equivalent boundaries can be connected by (sequence of) μ-constant linear deformations (of type $f + tg$). Furthermore, such deformations admit weak simultaneous resolutions [525], are equimultiple [700], and along them the embedded topological types are constant [558]. In particular, the whole Milnor fibrations with their homological packages, and the links associated with equivalent diagrams are the same.

5.5.6 The Case $H_1(L_{V(f)}, \mathbb{Q}) = 0$ Assume that $n = 2$. If $H_1(L_{V(f)}, \mathbb{Z}) = 0$ then the singularity is necessarily of Brieskorn type 5.1.15(1), hence typically $H_1(L_{V(f)}, \mathbb{Z})$ is non-zero. By Oka's algorithm for $\Gamma_{V(f)}$, cf. 5.5.23, (or via integral spectral numbers [603]) we obtain that

$$\text{the link is a rational homology sphere if and only if } N\Gamma(f) \cap \mathbb{Z}^3_{>0} = \emptyset. \tag{5.5.2}$$

If this happens then $N\Gamma(f)$ has a very rigid structure, which is described in [76]. E.g., $N\Gamma$ can contain at most one non-triangular 2-face, which must be a trapezoid.

In general, from the link it is impossible to recover the Newton boundary (see e.g. the pair of examples from 6.7.11). However, one has the following result.

Theorem 5.5.7 ([76]) *Assume that $f : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ is isolated and it has a non-degenerate principal part. If the link is a rational homology sphere, then from the link one can recover the Newton boundary (up to 'boundary' equivalence), hence the equations of the singularity as well (up to an equisingular deformation).*

In particular, the abstract 3-dimensional link (whenever it is realized by a Newton non-degenerate equation) determines uniquely the embedded topological type, the multiplicity, the Milnor homological package and geometric genus of the germ.

Let us comment on this statement. For Newton non-degenerate germs, usually we construct the embedded resolution of the pair $(V(f), 0) \subset (\mathbb{C}^3, 0)$ by a toric modification of $\mathbb{C}^3$ associated with a regular subdivision of the fan determined by the Newton diagram. In fact, there is a subdivision associated in a very precise and canonical way to $N\Gamma(f)$; for details see e.g. [524, 711] (or the next 5.5.B). Oka's algorithm, describes the resolution graph from the combinatorics of $N\Gamma(f)$ based on this canonical resolution (see 5.5.23). Since this subdivision is in some sense minimal, one expects that the obtained resolution graph is not very far from the minimal graph as well. Indeed, for the minimal representative of the Newton boundary (in its equivalence class) Oka's algorithm provides the minimal good resolution graph (see [76, Prop 4.2.5]). Furthermore, one might hope that the above procedure is invertible (though the algorithm is rather complex).

The above theorem 5.5.7 basically says that for Newton boundaries satisfying both (5.5.1) and (5.5.2) this is indeed the case, the association $N\Gamma(f) \mapsto \Gamma$ is injective. Braun and Némethi [76] provides even an 'inversion algorithm'. This fact also guarantees that the Milnor number or the geometric genus, in principle, are computable from the graph; the exact expressions in terms of Γ will be given later in 11.6.A.

Part of this inversion is the following fact: the minimal resolution graph is a string if and only if $N\Gamma$ is equivalent to a segment (cf. 5.5.3) if and only if $(X, 0)$ is an A-singularity. Otherwise, the (non-zero) number of nodes of a minimal resolution graph coincides with the number of 2-faces in a minimal representative $N\Gamma$ in its equivalence class.

5.5.B Weil Divisors in Affine Toric Singularities

For the presentation of the Newton non-degenerate Weil divisors we follow [499]. First we start to review some notations from affine toric geometry [208, 522].

5.5.8 Let N be a free abelian group of rank $r \in \mathbb{N}$ and set $M = \mathrm{Hom}(N, \mathbb{Z}) = N^\vee$. Abridge $M_\mathbb{R} = M \otimes \mathbb{R}$ and $N_\mathbb{R} = N \otimes \mathbb{R}$. If $\sigma \subset N_\mathbb{R}$ is a finitely generated rational

cone we define its $\sigma^\perp = \{u \in M_\mathbb{R} \mid \langle u, v\rangle = 0 \text{ for all } v \in \sigma\}$. The *dual cone* by $\sigma^\vee = \{u \in M_\mathbb{R} \mid \langle u, v\rangle \geq 0 \text{ for every } v \in \sigma\}$.

To any cone $\sigma \subset N_\mathbb{R}$ we set the semigroup $S_\sigma = \sigma^\vee \cap M$, the algebra $A_\sigma = \mathbb{C}[S_\sigma]$ and the *affine toric variety* $U_\sigma = \mathrm{Spec}(\mathbb{C}[S_\sigma])$. The r-torus $\mathbb{T}^r = (\mathbb{C}^*)^r$ acts on U_σ.

5.5.9 A *fan* $\triangle$ in N is a collection of cones in $N_\mathbb{R}$ satisfying the following two conditions: any face of a cone from $\triangle$ is in $\triangle$, and the intersection of two cones in $\triangle$ is a face of each of them. The *support* of a fan $\triangle$ is $|\triangle| = \cup_{\sigma\in\triangle}\sigma$. If $\tau, \sigma \in \triangle$ and $\tau \subset \sigma$, then one has a morphism $U_\tau \to U_\sigma$. These morphisms form a direct system whose limit is denoted by $Y_\triangle$ and it is called the *toric variety* associated with $\triangle$. Each U_σ is an open set of $Y_\triangle$. The actions of $\mathbb{T}^r$ on the affine varieties U_σ for $\sigma \in \triangle$ glue together to form an action on $Y_\triangle$. In fact, $\mathbb{T}^r$ itself embeds into $Y_\triangle$ as $U_{\{0\}}$.

Assume that $\triangle'$ is another fan in a lattice N' and $\phi : N' \to N$ is a linear map such that for any $\sigma' \in \triangle'$ there is a $\sigma \in \triangle$ so that $\phi(\sigma') \subset \sigma$. Then this induces maps $U'_{\sigma'} \to U_\sigma \to Y_\triangle$, which glue together to form a map $Y_{\triangle'} \to Y_\triangle$. This map is proper if and only if $|\phi^{-1}(\triangle)| = |\triangle'|$.

Definition 5.5.10

1. For a cone $\sigma \subset N_\mathbb{R}$, let $\triangle_\sigma$ denote the fan consisting of faces of σ. We also write Y_σ instead of $Y_{\triangle_\sigma}$.
2. If $\triangle$ is a fan and $i \in \mathbb{N}$, define $\triangle^{(i)} = \{\sigma \in \triangle \mid \dim\sigma = i\}$.
3. A *regular cone* (resp. *simplicial cone*) is a cone generated by an integral (resp. rational) basis of N.
4. A *subdivision* of a fan $\triangle$ is a fan $\tilde{\triangle}$ such that $|\tilde{\triangle}| = |\triangle|$ and each cone in $\triangle$ is a union of cones in $\tilde{\triangle}$. If $\tilde{\triangle}_1$ and $\tilde{\triangle}_2$ are two subdivisions, then we say that $\tilde{\triangle}_2$ *refines* $\tilde{\triangle}_1$ if $\tilde{\triangle}_2$ is a subdivision of $\tilde{\triangle}_1$. A subdivision is called *regular* if it consists of regular cones.
5. If $\Sigma \subset N_\mathbb{R}$ is a cone and $\tilde{\triangle}$ is a subdivision of $\triangle_\Sigma$, then denote by $\tilde{\triangle}^*$ the fan consisting of $\sigma \in \tilde{\triangle}$ for which $\sigma \subset \partial\Sigma$. Here we regard $\partial\Sigma$ as the union of proper faces of Σ. Hence, $\tilde{\triangle}^*$ is a subdivision of $\triangle_\Sigma \setminus \{\Sigma\}$.
6. For $\sigma \in \triangle$, let O_σ be the closed subset of U_σ defined by the ideal generated by monomials x^p where $p \in (\sigma^\vee \setminus \sigma^\perp) \cap M$. We identify this set with its image in $Y_\triangle$. The closure of O_σ in $Y_\triangle$ is denoted by $V(\sigma)$. In the case when σ is a ray, $V(\sigma)$ is a Weil divisor and we write $D_\sigma = V(\sigma)$. The orbits of the $\mathbb{T}^r$ action on $Y_\triangle$ are precisely the sets O_σ for $\sigma \in \triangle$.
7. For $\sigma \in \triangle$, let N_σ be the subgroup of N generated by $\sigma \cap N$ and define $M_\sigma := M/\sigma^\perp \cap M$ too. Note that $M_\sigma \cong N_\sigma^\vee$. Let $\varpi_\sigma : M \to M_\sigma$ be the canonical projection. Assuming $\sigma \in \triangle$ has dimension s, we have $(U_\sigma, O_\sigma) \cong (Y_\sigma \times (\mathbb{C}^*)^{r-s}, (\{0\} \times (\mathbb{C}^*)^{r-s})$. In particular, $O_\sigma \subset Y_\triangle$ has Y_σ as a transverse type.

Example 5.5.11 Assume that $\Sigma \subset N$ is a non-regular simplicial simplex defining Y_Σ. Let $\ell_1, \ldots, \ell_r \in N$ be primitive generators of Σ and set $N' := \mathbb{Z}\langle\ell_1, \ldots, \ell_r\rangle$.

Then the inclusion $N' \hookrightarrow N$ induces a finite abelian covering $\mathbb{C}^r = Y_{\Sigma \subset N'} \to Y_{\Sigma \subset N}$ realizing Y_Σ as an abelian quotient singularity.

5.5.12 Weil Divisors in Affine Toric Varieties [499] Let us fix a fan $\triangle$. For any $p \in M$ there is an associated rational function on U_σ. These glue together to form a rational function x^p on $Y_\triangle$. We refer to these functions as *monomials*. A monomial x^p is a regular function on $Y_\triangle$ if $p \in |\triangle|^\vee = \cap_{\sigma \in \triangle} \sigma^\vee$.

Next we fix $\triangle = \triangle_\Sigma$, where Σ is a fixed rational top-dimensional finitely generated strictly convex cone in a lattice N of rank r. The (maximal) ideal generated by all $\{x^p \mid p \in \Sigma^\vee \cap M \setminus 0\}$ determines a point, denoted by 0, and it is called the *origin* of $Y_\triangle$. Write also $Y := Y_\triangle$. It is known that Y is Cohen–Macaulay, cf. [89, 278].

Consider a function f with (local analytic) expansion

$$f(x) = \sum_{p \in M \cap \Sigma^\vee} a_p x^p \text{ and support } \operatorname{supp}(f) = \left\{ p \in M \cap \Sigma^\vee \,\middle|\, a_p \neq 0 \right\}.$$

Here f can be the germ of any holomorphic function at the origin. Thus, $(X, 0)$ (defined below via f) will be the germ of an analytic space. Correspondingly, we will consider $(Y, 0)$ as a germ of an analytic space, though Y is an algebraic variety.

Definition 5.5.13 The Newton polyhedron of f with respect to Σ is the polyhedron $N\Gamma_+(f)$, the convex closure in $M_\mathbb{R}$ of $\operatorname{supp}(f) + \Sigma^\vee$. The union of compact faces of $N\Gamma_+(f)$ is denoted by $N\Gamma(f)$ and is called the *Newton boundary (or diagram)* of f with respect to Σ.

Σ is precisely the set of linear functions on M, which have a minimal value on $N\Gamma_+(f)$. Denote by $F(\ell)$ the minimal set of $\ell \in \Sigma$ on $N\Gamma_+(f)$. For $\ell_1, \ell_2 \in \Sigma$ we write $\ell_1 \sim \ell_2$ if and only if $F(\ell_1) = F(\ell_2)$. Then $\sim$ is an equivalence relation on Σ having finitely many equivalence classes, each of whose closure is a finitely generated rational strictly convex cone. These cones form a fan, a subdivision of $\triangle_\Sigma$, which is denoted by $\triangle_f$, and is called the *dual fan associated with f and Σ*.

For any $\sigma \in \triangle_f$, the face $F(\ell)$ is independent of the choice of $\ell \in \sigma^\circ$, in the relative interior $\sigma^\circ \subset \sigma$. Write $F_\sigma = F(\ell)$ for such ℓ. Define also $f_\sigma = \sum_{p \in F_\sigma} a_p x^p$.

For any $\sigma \in \triangle_f^{(1)}$ the semigroup $\sigma \cap N$ is generated by a unique element: denote it by ℓ_σ. Set also $m_\sigma = \min_{N\Gamma_+(f)} \ell_\sigma$. In particular, $\ell_\sigma|_{F_\sigma} \equiv m_\sigma$.

Definition 5.5.14 Denote by $(X, 0) \subset (Y, 0)$ the union of those local primary components of the germ defined by f (with their non-reduced structure), which are not invariant by the torus action.

We call (X, o) the (non-invariant) Weil divisor associated with f.

Note that for any $p \in M$, the function $x^p f$ defines the same germ $(X, 0)$. Furthermore, $(X, 0)$ is the empty germ if and only if $N\Gamma_+(f) = p + \Sigma^\vee$ for some $p \in M$.

Proposition 5.5.15 *The ideal $I_{X,0} \subset \mathcal{O}_{Y,0}$, which defines $(X,0)$ in $(Y,0)$, is generated by the functions $x^p f$ for $p \in M$ satisfying $\ell_\sigma(p) + m_\sigma \geq 0$ for all $\sigma \in \Delta_\Sigma^{(1)}$.*

Note that there are some small differences in the definitions considered in 5.5.A and 5.5.B. In the classical hypersurface case we allow irreducible components of $(X,0)$ given by the coordinate hyperplanes, in the present new situation we eliminate them (as they are invariant). E.g., in $r = 2$ case, the germ $f = xy$ represents the ordinary node in the classical case, and the empty set in the new situation. On the other hand, $xy + x^n + y^n$ $(n \geq 2)$ represents similar germs in both definitions.

In the sequel we assume that $(X,0)$ is non-empty.

Example 5.5.16 Let f be as above. We say that $N\Gamma_+(f)$, or f, is ($\mathbb{Q}$-)*pointed* if there exists a $p \in M$ ($p \in M_\mathbb{Q} = M \otimes \mathbb{Q}$) such that $\ell_\sigma(p) = m_\sigma$ for every $\sigma \in \Delta_\Sigma^{(1)}$.

For example, if Σ is regular (resp. simplicial), then any Newton polyhedron (with respect to Σ) is pointed at some $p \in M$ (resp. $p \in M_\mathbb{Q}$).

In [499] it is proved that f is pointed at $p \in M$ if and only if $(X,0)$ in $(Y,0)$ is defined by a single equation $x^{-p} f$ (cf. 5.5.15). In other words, f is pointed if and only if $(X,0)$ is a Cartier divisor in $(Y,0)$. More generally, f is pointed at $p \in M_\mathbb{Q}$ if and only if $(X,0)$ is a $\mathbb{Q}$-Cartier divisor in $(Y,0)$.

For example, if Σ is regular, then $(Y,0)$ is smooth and f is pointed, hence $(X,0)$ is a hypersurface singularity. If, however, Σ is not regular, then $(X,0)$ is not necessarily a Cartier divisor. This can happen even if Σ is simplicial, see several examples below.

Definition 5.5.17 We say that f has a *non-degenerate Newton principal part* with respect to Σ if for every $\sigma \in \Delta_f$ such that F_σ is compact, the variety $\mathrm{Spec}(\mathbb{C}[M]/(f_\sigma))$ (that is, $\{x \in \mathbb{T}^r : f_\sigma(x) = 0\}$ with its non-reduced structure) is smooth.

If this happens then $(X,0)$ will be called a *Newton non-degenerate singularity*.

5.5.18 Embedded Resolutions Assume that f is Newton non-degenerate. Let $\tilde{\Delta}_f$ be a *regular subdivision* of Δ_f. Then $\widetilde{Y} = Y_{\tilde{\Delta}_f}$ is a smooth variety, and we have a modification $\pi : \widetilde{Y} \to Y$. As a result of the nondegeneracy of f, the strict transform $\widetilde{X}$ in $\widetilde{Y}$ of a small representative X of $(X,0)$ intersects all toric orbits in $\widetilde{Y}$ smoothly. In particular, $\widetilde{X}$ is smooth, and π is an embedded resolution of $(X,0) \subset (Y,0)$.

5.5.19 The Singular Locus $\mathrm{Sing}(X,0)$ **of** $(X,0)$ Assume $(X,0) \subset (Y,0)$ is a Newton non-degenerate Weil divisor. Then, the singular locus of the germ $(X,0)$ is contained in the union of orbits of codimension ≥ 2 (in $(Y,0)$). Indeed, let $Y^{(\leq r-2)}$ be the union of orbits of dimension $\leq r-2$. Let $\pi : \widetilde{Y} \to Y$ be as in 5.5.18. The restriction $\pi^{-1}(Y \setminus Y^{(\leq r-2)}) \to Y \setminus Y^{(\leq r-2)}$ is an isomorphism, and $\widetilde{X}$ is smooth. Therefore, $X \setminus Y^{(\leq r-2)}$ is smooth. For example, if $r = 3$, then the components of $\mathrm{Sing}(X,0)$ should sit in the orbits O_σ, where σ is a 2-cone in $\partial\Sigma$.

Even without non-degeneracy assumption, for a $\sigma \in \triangle_\Sigma$, we have $O_\sigma \subset X$ if and only if the normal fan $\triangle_f$ does subdivide σ into smaller cones. However, if $(X,0) \subset (Y,0)$ is Newton non-degenerate and $O_\sigma \subset X$, then the generic transverse type of X to O_σ is a Newton non-degenerate singularity too, and its Newton polyhedron is $\varpi_\sigma(\Gamma_+(f)) \subset M_\sigma$. (Hence, any criterion which in the $r = 2$ case assures smoothness can be transformed into a criterion which in $r = 3$ case assures that $(X,0)$ has an isolated singularity; see below.)

Example 5.5.20 (Newton Non-degenerate Curve Singularities [499]) Assume that $r = 2$, and $\Sigma \subset N_{\mathbb{R}}$ is a two dimensional finitely generated strictly convex rational cone. If Σ is not regular then, by taking a convenient basis for N, we can assume that Σ is generated by $\ell = (0,1)$ and $\ell' = (n,-q)$, where $0 < q < n$ and $\gcd(n,q) = 1$, cf. 2.3.5. Therefore, $Y = X_{n,q}$. In this case the *minimal regular subdivision*, called the *canonical subdivision*, is formed by the vectors $\{W_j\}_{j=0}^{s+1}$ considered in 2.3.5 (where $W_0 = \ell$, $W_{s+1} = \ell'$).

Regarding some properties/invariants we have the following facts:

1. The germ $(X,0)$ is irreducible if and only if $N\Gamma(f)$ is a single interval with no integral interior points. In general, the number of components in $(X,0)$ is precisely the combinatorial length of $N\Gamma(f)$.
2. Assume that $(X,0)$ is irreducible with $N\Gamma(f) = F_\sigma$ for the unique $\sigma \in \Delta_f^{(1)}$ for which $\sigma \not\subset \partial\Sigma$. Set $m := \min_{N\Gamma_+(f)}(\ell)$ and $m' := \min_{N\Gamma_+(f)}(\ell')$. Then the curve $(X,0)$ is smooth if and only if the following condition holds: For any $p \in M$ with $\ell(p) > m$ and $\ell'(p) > m'$ we necessarily have $\ell_\sigma(p) > m_\sigma$ too. This happens exactly when ℓ_σ is one of the vectors $\{W_j\}_{j=1}^{s}$, or it equals either $aW_0 + W_1$ or $aW_{s+1} + W_s$ for some $a \in \mathbb{Z}_{>0}$.
3. Assume again that $(X,0)$ is irreducible with $N\Gamma(f) = F_\sigma$. Then $\mathcal{O}_{X,0} = \mathbb{C}[M \cap \Sigma^\vee]/(f)$ is graded: the class of x^p has degree $\ell_\sigma(p)$. Hence, $\mathcal{O}_{X,0} = \oplus_{s\in\mathcal{S}} R_s$, where $\mathcal{S} \subset \mathbb{N}$ is the semigroup of $(X,0)$, $\mathcal{S} = \{\ell_\sigma(p) : p \in M \cap \Sigma^\vee\}$. Since F_σ is primitive one gets that $\dim(R_s) = 1$ whenever $s \in \mathcal{S}$. The delta-invariant $\#(\mathbb{N} \setminus \mathcal{S})$ of $(X,0)$ can be identified with the number of lattice points in the following open 'triangle' T;

$$\delta(X,0) = \#\{p \in M : \ell(p) > m,\ \ell'(p) > m',\ \ell_\sigma(p) < m_\sigma\}.$$

   Indeed, the lattice points of T can be identified with those in the open triangle $T' := \{p \in M : \ell'(p) < 0,\ \ell_\sigma(p) > 0,\ \ell(p) < \max_{F_\sigma} \ell - m\}$. Furthermore, $p \in T'$ if and only if $\ell_\sigma(p) \in \mathbb{N} \setminus \mathcal{S}$.
4. In general, in the not necessarily irreducible case,

$$\delta(X,0) = \#\{p \in M : \ell(p) > m,\ \ell'(p) > m',\ p \notin N\Gamma_+(f)^\circ\}.$$

Example 5.5.21 (Newton Non-degenerate Surface Singularities [499]) Assume that $r = 3$, and let $(X, 0)$ be a Newton non-degenerate Weil divisor in $Y = Y_\Sigma$.

1. $(X, 0)$ has an isolated singularity if and only if the following condition holds: If $p \in M$ and $\ell_\sigma(p) > m_\sigma$ for all $\sigma \in \triangle_\Sigma^{(1)}$, then $\ell_\sigma(p) > m_\sigma$ for all $\sigma \in \triangle_f^{*(1)} := \Delta_f^{(1)} \cap \partial\Sigma$. In particular, if $\triangle_f^{*(1)} = \triangle_\Sigma^{(1)}$, then $(X, 0)$ has an isolated singularity.
   Surprisingly, even if $\mathrm{Sing}(X, 0) = \{0\}$, in general $(X, 0)$ is not normal. In particular, though $(Y, 0)$ is always Cohen–Macaulay, $(X, 0)$ in general is not, see 5.5.32. In fact, even if $\mathrm{Sing}(X, 0) = \{0\}$, $(X, 0)$ might not be (locally) irreducible either, see 5.5.32.
2. Let $\overline{\mathscr{O}_{X,0}}$ be the normalisation of $\mathscr{O}_{X,0}$. Then $(X, 0)$ is isolated if and only if the sum
$$S := \sum_{p \in M} \operatorname{rank} \widetilde{H}_0(N\Gamma_+(x^p f) \setminus \Sigma^\vee, \mathbb{Z})$$
   is finite. In this case $\dim(\overline{\mathscr{O}_{X,0}}/\mathscr{O}_{X,0}) = S$. The vanishing of the above expression of S can also be used as a combinatorial criterion for normality.
3. If f is pointed, then $(X, 0)$, as a Cartier divisor in the Cohen–Macaulay $(Y, 0)$, is Cohen–Macaulay. Thus, if f is pointed and $\mathrm{Sing}(X, 0) = \{0\}$ then $(X, 0)$ is automatically normal. It is harder to prove that if f is $\mathbb{Q}$-pointed and $\mathrm{Sing}(X, 0) = \{0\}$ then $(X, 0)$ is still normal.
4. Define $N\Gamma_+(f, \Sigma) := \{p \in M_\mathbb{R} \mid \ell_\sigma(p) \geq m_\sigma \text{ for all } \sigma \in \tilde{\triangle}_f^{*(1)}\}$. Then we have two (rather different) formulae
$$\begin{aligned} h^1(\widetilde{X}, \mathscr{O}_{\widetilde{X}}) &= \sum_{p \in M} \operatorname{rank} H_1(N\Gamma_+(x^p f) \setminus \Sigma^\vee, \mathbb{Z}) \\ &= \#\{(N\Gamma_+(f, \Sigma)^\circ \setminus N\Gamma_+(f)^\circ) \cap M\}. \end{aligned}$$
   The second one generalizes Merle–Teissier's formula [418], cf. 5.5.4(c).

5.5.22 The Resolution of Newton Non-degenerate Normal Surface Singularities Let $(X, 0)$ be a surface singularity as in 5.5.21, and assume that $(X, 0)$ is normal. A subdivision $\tilde{\triangle}_f$ of $\triangle_f$ is 'canonical' if it is a regular subdivision, and if for any $\sigma \in \triangle_f^{(2)} \setminus \triangle_f^*$ the restriction $\tilde{\triangle}_f$ to the rank two $\triangle_\sigma$ is the canonical subdivision of $\triangle_\sigma$. (For the definition of the canonical subdivision of a cone of rank two see 5.5.20.) The existence of a canonical subdivision is proved in [524, §3].

Since the restriction of a canonical subdivision to the 2-skeleton of $\triangle$ is uniquely determined, the dual graph Γ of the induced resolution $\widetilde{X} \to X$ is canonically determined from $N\Gamma(f)$ too, cf. [524, §3]. This is the subject of the next algorithm.

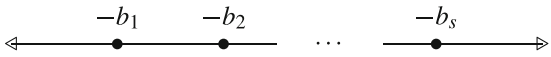
5.5.23 Oka's Algorithm for the 'Canonical' Resolution Graph Γ [499, 524] The graph Γ is a subgraph of a larger graph Γ^{ext}, whose construction is the following. We start with a set of vertices, they correspond bijectively to the set $\mathscr{F}$ of 2-faces

of $\partial N\Gamma_+(f)$: we will call them *face vertices*. Next, consider two adjacent 2-faces $\triangleleft$ and $\triangleright$ from $\mathscr{F}$, such that their intersection is a compact 1-face. Let $t_{\triangleleft,\triangleright}$ be the combinatorial length of this 1-face, and $n_{\triangleleft,\triangleright}$ be the determinant of the dual primitive vectors $\ell_\triangleleft$, $\ell_\triangleright \in \Sigma \cap N$ of the 2-faces. (This is the greatest common divisor of the 2-minors.) Then we connect the vertices corresponding to $\triangleleft$ and $\triangleright$ by $t_{\triangleleft,\triangleright}$ copies of the following 'canonical' string.

If $n_{\triangleleft,\triangleright} > 1$ then let $0 < q_{\triangleleft,\triangleright} < n_{\triangleleft,\triangleright}$ be the unique integer for which

$$\ell_{\triangleleft,\triangleright} := (\ell_\triangleleft + q_{\triangleleft,\triangleright}\ell_\triangleleft)/n_{\triangleleft,\triangleright} \in \mathbb{N}.$$

We also write $n_{\triangleleft,\triangleright}/q_{\triangleleft,\triangleright}$ as a continued fraction $[[b_1, \dots, b_s]]$ where each $b_i \geq 2$. Then the string with the corresponding Euler numbers (and 0 genus decorations) is



The left ends of all the $t_{\triangleleft,\triangleright}$ copies of the chain (marked by $\triangleleft$) are identified with the face vertex corresponding to $\triangleleft$, and similarly for the right ends marked by $\triangleright$.

If $n_{\triangleleft,\triangleright} = 1$ then any such string consists of an edge connecting the vertices $\triangleleft$ and $\triangleright$, hence we put $t_{\triangleleft,\triangleright}$ such edges. Also, in this case we set $q_{\triangleleft,\triangleright} := 0$ and $\ell_{\triangleleft,\triangleright} := \ell_\triangleright$.

Next, consider a vertex associated with a *compact* 2-face $\triangleleft$. We define its genus by the number of interior lattice points of $\triangleleft$, and its Euler number $-b_\triangleleft$ by

$$-b_\triangleleft\ell_\triangleleft + \sum_{\triangleright \in \mathscr{F}_\triangleleft} t_{\triangleleft,\triangleright}\ell_{\triangleleft,\triangleright} = 0,$$

where $\mathscr{F}_\triangleleft$ is the collection of all (compact and non-compact) 2-faces of $\partial N\Gamma_+(f)$ adjacent to $\triangleleft$. In this way we obtain the graph Γ^{ext}. Notice that the face vertices corresponding to *non-compact faces* are not decorated.

If we delete all the vertices corresponding to non-compact 2-faces (and all the edges adjacent to them) we get a dual resolution graph Γ associated with $(X, 0)$.

Note that besides the dual vectors $\ell_\triangleleft$ associated with 2-faces we have vectors associated with all the vertices of Γ^{ext}. Indeed, along the strings we can associate to each vertex a corresponding vector $\ell \in N$ (the canonical primitive sequence $\{W_j\}_j$, cf. 5.5.20). In this way, to each $v \in \mathscr{V}(\Gamma^{ext})$ we associate a vector $\ell_v \in N$ in a unique way such that the identity $-b_w\ell_w + \sum_v \ell_v = 0$ holds for any $w \in \mathscr{V}(\Gamma)$, where the sum runs over all adjacent vertices in Γ^{ext}. (Compare with (2.2.1).)

This is used as follows. For any $p \in \Sigma^\vee$, if we delete all the vertices of Γ^{ext} corresponding to non-compact faces, and we replace the edges adjacent to them by arrowheads, and if we add the multiplicity decoration $\ell_v(p)$ to each (arrowhead and non-arrowhead) vertex v, then we get the embedded resolution graph of $(X, V(x^p))$.

Remark 5.5.24 By the above construction, the link of $(X, 0)$ is a rational homology sphere if and only if all the lattice points of $N\Gamma(f)$ belong to $\partial N\Gamma(f)$.

Remark 5.5.25 If $(X, 0)$ is not normal (but it has an isolated singularity), Oka's algorithm still runs: it provides the graph of the normalization.

5.5.26 In the examples below the Newton diagrams are not always in the cone Σ. This inclusion can be realized by a convenient integral shift (cf. 5.5.15). Also, in the identification of the singularity $(X, 0)$ we first determine the resolution graph by Oka's algorithm. If this is a graph of a *taut* singularity [361] (e.g. it is a string or it is cyclic) then the analytic structure is identified uniquely as well.

Example 5.5.27 (Cyclic Quotients) Let M be $\mathbb{Z}^3$ and define the Newton diagram $N\Gamma$ as the triangle with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$. Set $V = (0, 0, q/n)$, with $q/n > 0$, and let $\Sigma^\vee$ be the cone over $V + N\Gamma$. Then, using 5.5.23, we get that $(X, 0)$ is the cyclic quotient singularity $(X_{n,q}, 0)$.

Example 5.5.28 (Triangle Singularities) Set $M = \mathbb{Z}^3$ and let $N\Gamma$ be the triangle with vertices $(2, 0, 0)$, $(0, 3, 0)$, $(0, 0, 7)$. If $\Sigma^\vee = (\mathbb{R}_{\geq 0})^3$, then $(X, 0)$ is the hypersurface singularity $x_1^2 + x_2^3 + x_3^7 = 0$. Its graph Γ is star-shaped, all $g_v = 0$, the central vertex is -1, all the tree legs contain only one vertex, whose Euler numbers are $-2, -3, -7$ respectively.

Next, fix integers $r \geq 2$, $q \geq 3$, $p \geq 7$, and with the same $N\Gamma$ redefine $\Sigma^\vee$ by: $x_1/2 + x_2/3 + x_3(p-6)/(6p-42) \geq 0$, $x_1/2 + (5q-14)x_2/(14q-42) + x_3/7 \geq 0$, $(11r - 21)x_1/(21r - 42) + x_2/3 + x_3/7 \geq 0$. (If $p = 7$ the first equation reads as $x_3 \geq 0$, the same for the others.) Then the only modification in the new graph is that the Euler numbers $-2, -3, -7$ are replaced by $-r, -q, -p$.

Remark 5.5.29 If $N\Gamma$ is contained in an affine plane, then $(X, 0)$ is weighted homogeneous. (However, not any weighted homogeneous singularity can be obtained in this way as a Weil divisor in a toric 3-dimensional singularity.)

Example 5.5.30 (Cusp Singularities ($\#\mathcal{V} = 3$)) Set $M = \mathbb{Z}^3$. Assume that $N\Gamma$ consists of the four points $o = (0, 0, 0)$ and $(1, 0, 1)$, $(0, 1, 1)$, $(0, 0, 1)$; we think about $N\Gamma$ as three triangles having o as a common vertex. $\Sigma^\vee$ is given by three planes: their oriented normal vectors are $(0, a + 1, -1)$, $(b + 1, 0, -1)$ and $(-c - 1, -c - 1, c)$. We assume that $a \geq 2$, $b \geq 2$ and $c \geq 3$. Then $(X, 0)$ is a cusp singularity with a cyclic graph decorated by $-a, -b, -c$.

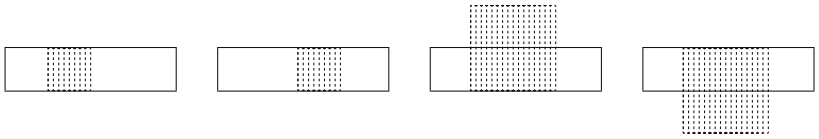
Since in all the examples above $\Sigma^\vee$ is simplicial (hence $\mathbb{Q}$-pointed) and $\triangle^*_\Sigma = \triangle^*_f$, the germs $(X, 0)$ are automatically normal, cf. 5.5.21.2. In the next example $\Sigma^\vee$ is not simplicial.

Example 5.5.31 (Cusp Singularities ($\#\mathcal{V} = 4$)) Set $M = \mathbb{Z}^3$, Assume that $M\Gamma$ consists of the five points $o = (0, 0, 0)$ and $(1, 0, 1)$, $(0, 1, 1)$, $(0, 0, 1)$ and $(1, 1, 1)$: $N\Gamma$ consists of four triangles having o as a common vertex. $\Sigma^\vee$ is given by four planes with oriented normal vectors $(0, a, -1)$, $(b, 0, -1)$, $(0, -c, c - 1)$ and $(-d, 0, d - 1)$. We assume that a, b, c, d are positive with $(a - 1)(c - 1) > 1$

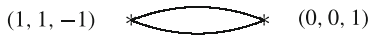
and $(b-1)(d-1) > 1$. Then (via the criterion above) one verifies that $(X, 0)$ is normal. It is a cusp singularity whose graph has four vertices decorated cyclically by $-a, -b, -c, -d$.

Example 5.5.32 (Non-normal $(X, 0)$ *[499])* Assume that Σ is generated by $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ and $(1, 1, -1)$, and $f = x_1^4 + x_1^4 x_3 + x_2^4 + \lambda x_2^4 x_3$ with λ generic. Then the ideal of $(X, 0)$ is generated by $x_3^i f$ for $0 \le i \le 3$, $(X, 0)$ has an isolated singularity, it is not normal, its delta-invariant is 4, and its normalization is $(X_{5,1}, 0)$.

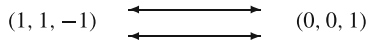
The next pictures show the two connected components of $N\Gamma_+(f) \setminus x^p \Sigma^\vee$ for the points $p = (2, 1, 0),\ (1, 2, 0),\ (1, 1, 0),\ (1, 1, -1)$:



Consider now the same Σ and $f = x_1^2 + x_2^2$. Since $\Delta_\Sigma^* = \Delta_f^*$, $(X, 0)$ has an isolated singularity, cf. 5.5.21.1. Note that the Newton diagram has no compact 2-faces, it has one 1-face with combinatorial length 2. The graph Γ^{ext} is



This means that the embedded resolution graph of $(X, 0)$ and of the germs x_1, x_2, x_3 respectively is



That is, $(X, 0)$ has two irreducible components, and both of them are smooth. In fact, we can see that Y is the cone over $\mathbb{CP}^1 \times \mathbb{CP}^1$ and $(X, 0)$ is a cone over two disjoint parallel copies of type $\{\text{point}\} \times \mathbb{CP}^1$.

In particular, $(X, 0)$ is not normal. Its delta invariant is 1.

Chapter 6
Invariants Associated with a Resolution

Let (X, o) be a complex normal surface singularity and let $\phi : \widetilde{X} \to X$ be a good resolution. By normality $\phi_*(\mathscr{O}_{\widetilde{X}}) = \mathscr{O}_X$, cf. 2.1.11. Consider the lattices $L \subset L'$ as in 2.1.15. We can identify L with the lattice of integral divisors supported on E and L' with a sublattice of rational divisors supported on E.

In cohomological considerations, e.g. in the computation of $H^*(\widetilde{X}, \mathbb{Z})$ or $H^*(\widetilde{X}, \mathscr{F})$ (where $\mathscr{F}$ is a coherent sheaf), we might take for $\widetilde{X}$ the space $\phi^{-1}(\rho^{-1}([0, \epsilon]))$ (cf. 3.4), or more generally, $\phi^{-1}(X_{St})$, where X_{St} is a small (topologically cone-like) Stein representative of (X, o). Note that if $\mathscr{F}$ is coherent analytic sheaf on $\widetilde{X}$ then $\phi_*\mathscr{F}$ is coherent on X too, cf. [222, 322, 323], or [359, p. 599]. Hence, if X is Stein, then $H^{\geq 1}(X, \phi_*\mathscr{F}) = 0$. Therefore, by Leray spectral sequence, for $q \geq 1$ and for an analytic coherent sheaf $\mathscr{F}$, $H^q(\widetilde{X}, \mathscr{F})$ agrees with $(R^q\phi\mathscr{F})_o = \lim_{\to U} H^q(\phi^{-1}(U), \mathscr{F})$, where U runs over open sets $o \in U \subset X$.

In fact, by a more general argument, if $N \subset \widetilde{X}$ is a (conveniently small) strictly Levi pseudoconvex neighborhood of E then $H^{\geq 1}(N, \mathscr{F})$ is finite dimensional by [249, Th. IX,B.6]. Furthermore, the restriction $H^{\geq 1}(\widetilde{X}, \mathscr{F}) \to H^{\geq 1}(N, \mathscr{F})$ is an isomorphism [359, Lemma 3.1]. In particular, $H^{\geq 1}(\widetilde{X}, \mathscr{F})$ is finite dimensional, and we can even assume that $\widetilde{X}$ is a strictly Levi pseudoconvex neighborhood of E.

By 'Theorem of formal functions', cf. [258] (or 6.4.4 for a line bundle $\mathscr{F}$), $(R^q\phi\mathscr{F})_o = \lim_{\leftarrow Z} H^q(Z, \mathscr{F} \otimes_{\mathscr{O}_{\widetilde{X}}} \mathscr{O}_Z)$, where Z runs over (larger and larger) effective cycles supported on E. In fact, for a line bundle $\mathscr{F}$ we directly will verify that $H^{\geq 2}(\widetilde{X}, \mathscr{F}) = 0$ and $H^1(\widetilde{X}, \mathscr{F}) = H^1(Z, \mathscr{F} \otimes \mathscr{O}_Z)$ for $Z \gg 0$, hence (again) $\dim H^1(\widetilde{X}, \mathscr{F}) < \infty$. Furthermore, by Serre duality, for a locally free sheaf $\mathscr{F}$, $H^1_c(\widetilde{X}, \mathscr{F}) = H^1(\widetilde{X}, \mathscr{F}^\vee \otimes \Omega^2_{\widetilde{X}})^*$, where H^1_c denotes cohomology with compact support.

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Let $\mathrm{Pic}(\widetilde{X}) = H^1(\widetilde{X}, \mathscr{O}^*_{\widetilde{X}})$ denote the Picard group of $\widetilde{X}$, the group of isomorphism classes of analytic line bundles on $\widetilde{X}$, cf. 12.4.1. For any effective cycle $Z \in L_{>0}$ let $\mathrm{Pic}(Z) = H^1(Z, \mathscr{O}^*_Z)$ be the group of isomorphism classes of line bundles on Z.

Recall that the geometric genus of (X, o) is defined as $p_g := h^1(\widetilde{X}, \mathscr{O}_{\widetilde{X}})$. The germ (X, o) is called rational if $p_g = 0$ (for more see 7.1).

6.1 Local Divisor Class Group

For a resolution $\widetilde{X} \to X$ we consider the Picard groups $\mathrm{Pic}(\widetilde{X})$ and $\mathrm{Pic}^0(\widetilde{X})$ and the corresponding exponential exact sequence. We also discuss their relations with the local divisor class group $\mathrm{Cl}(X, o)$. For more see [83, 389, 448]. ♣

6.1.1 By Poincaré duality, $L' = H_2(\widetilde{X}, \partial\widetilde{X}, \mathbb{Z})$ is isomorphic to $H^2(\widetilde{X}, \mathbb{Z})$, hence it is the target of the first Chern class $c_1 : \mathrm{Pic}(\widetilde{X}) \to H^2(\widetilde{X}, \mathbb{Z})$. This morphism is part of the following exact sequence (induced by the exponential exact sequence of sheaves, cf. 12.4.1):

$$0 \to H^1(\widetilde{X}, \mathbb{Z}) \xrightarrow{\iota} H^1(\widetilde{X}, \mathscr{O}_{\widetilde{X}}) \xrightarrow{\epsilon} \mathrm{Pic}(\widetilde{X}) \xrightarrow{c_1} H^2(\widetilde{X}, \mathbb{Z}) \to 0. \tag{6.1.1}$$

The morphism ι is injective since $H^0(\widetilde{X}, \mathscr{O}_{\widetilde{X}}) \to H^0(\widetilde{X}, \mathscr{O}^*_{\widetilde{X}})$ is surjective: this follows from the fact that any section of these groups is constant along E. Set

$$\mathrm{Pic}^0(\widetilde{X}) := \ker(c_1) \simeq H^1(\widetilde{X}, \mathscr{O}_{\widetilde{X}})/H^1(\widetilde{X}, \mathbb{Z}) \simeq \mathbb{C}^{p_g}/H^1(E, \mathbb{Z}).$$

Since $H^q(\widetilde{X}, \mathbb{Z}) = \lim_{\to U} H^q(U, \mathbb{Z})$ and $H^1(\widetilde{X}, \mathscr{O}_{\widetilde{X}}) = \lim_{\to U} H^1(U, \mathscr{O}_U)$, $E \subset U$, from (6.1.1) we also have $H^1(\widetilde{X}, \mathscr{O}^*_{\widetilde{X}}) = \lim_{\to U} H^1(U, \mathscr{O}^*_U)$. Furthermore, by [448], for any line bundle $\mathscr{L} \in H^1(\widetilde{X}, \mathscr{O}^*_{\widetilde{X}})$ there exists a sufficiently small neighbourhood U of E in $\widetilde{X}$ such that $\mathscr{L}|_U$ admits a meromorphic section over U. Thus, $\mathrm{Pic}(\widetilde{X})$ can be identified with the group $\mathrm{Cl}(\widetilde{X})$ of *local analytic divisors* near E modulo linear equivalence. More precisely, by a local analytic divisor we mean a sum $\sum_i n_i D_i$ of irreducible analytic divisors defined in a neighbourhood of E. Such a divisor is locally linear equivalent to zero if there exists a neighbourhood U of E, where all D_i are defined, and a meromorphic function on U such that $\mathrm{div}(f) = \sum_i n_i (D_i \cap U)$.

The lattice L embeds into both $L' = H^2(\widetilde{X}, \mathbb{Z})$ and $\mathrm{Pic}(\widetilde{X})$. For the 'homological embedding' $H_2(\widetilde{X}, \mathbb{Z}) \hookrightarrow H_2(\widetilde{X}, \partial\widetilde{X}, \mathbb{Z})$ see 2.1.15. Here we consider the Poincaré dual of this pair, namely $H^2(\widetilde{X}, \partial\widetilde{X}, \mathbb{Z}) \hookrightarrow H^2(\widetilde{X}, \mathbb{Z})$, still denoted by $L \subset L'$. The embedding of L into $\mathrm{Pic}(\widetilde{X})$ (where L is considered here as integral divisors supported on E, identified with homological integral cycles) is given by $l \mapsto \mathscr{O}(l)$. Since $c_1(\mathscr{O}(l))$ is the Poincaré dual of l, the two embeddings commute with c_1.

Similarly to the group $L'/L = \mathrm{Tors}(H^2(X\setminus\{o\},\mathbb{Z}))$ (cf. 3.4.1), $\mathrm{Pic}(\widetilde{X})/L$ is also independent of the choice of the resolution $\widetilde{X}$. Indeed, the sequence

$$0 \to L \to \mathrm{Pic}(\widetilde{X}) \xrightarrow{r} \mathrm{Cl}(X,o) \to 0$$

is exact (cf. [448]), where $\mathrm{Cl}(X,o)$ denotes the *local divisor class group* of (X,o), that is, the class group of local Weil divisors of (X,o) modulo local Cartier divisors. If D is a local irreducible analytic divisor on $\widetilde{X}$, then its restriction to $\widetilde{X}\setminus E$ can be mapped to $X\setminus\{o\}$ by ϕ, and the class of its closure is $r(\mathscr{O}_{\widetilde{X}}(D))$. [This is exactly the definition of the natural map $\phi_*:\mathrm{Cl}(\widetilde{X})\to\mathrm{Cl}(X,o)$, a reinterpretation of r.]

Hence we obtain the exact sequence

$$0 \to H^1(L_X,\mathbb{Z}) \to \mathbb{C}^{p_g} \to \mathrm{Cl}(X,o) \xrightarrow{\bar{c}_1} \mathrm{Tors}(H^2(L_X,\mathbb{Z})) \to 0. \tag{6.1.2}$$

The Chern class morphism $\bar{c}_1$—in the language of divisors and homology—has the form $\bar{c}_1' : \mathrm{Cl}(X,o) \to \mathrm{Tors}(H_1(L_X,\mathbb{Z}))$, where $\bar{c}_1'$ assigns to a Weil divisor the homological class of its intersection with the link.

$\mathrm{Cl}(X,o)$ can also be reinterpreted as the group of isomorphism classes of divisorial sheaves on (X,o), cf. 12.4.1. [$\mathscr{F}$ is a divisorial sheaf if and only if $\mathscr{L} = (\phi^*(\mathscr{F}))^{\vee\vee}$ is a locally free sheaf on $\widetilde{X}$, such that $\mathscr{L}|_{\widetilde{X}\setminus E} \simeq \mathscr{F}|_{X\setminus\{o\}}$ (by ϕ) and $\mathscr{F} = j_*(\mathscr{F}|_{X\setminus\{o\}})$, where $j : X\setminus\{o\} \hookrightarrow X$ is the inclusion. By the above discussion $\mathscr{L}$ has the form $\mathscr{O}_{\widetilde{X}}(D)$, hence $\mathscr{F} = r(\mathscr{O}_{\widetilde{X}}(D))$, that is, $\mathscr{F}$ can be associated with a Weil divisor $\phi_*(D)$.] For a ring-theoretical reinterpretation see also 6.1.4.

Example 6.1.2 The sheaf $\omega_X := j_*(\Omega^2(X\setminus\{o\}))$ is divisorial (cf. 12.4.1), it is called the sheaf of regular differentials on (X,o). One can also write it in the form $\mathscr{O}_X(K_X)$ for a certain Weil divisor K_X. More generally, for any positive integer n, the sheaf $\omega_X^{[n]} = \mathscr{O}_X(nK_X) = j_*((\Omega^2(X\setminus\{o\})^{\otimes n})$ (of regular n-differentials on X) is divisorial too.

If $K_{\widetilde{X}}$ is a canonical divisor on $\widetilde{X}$, then K_X can be taken as $\phi_*(K_{\widetilde{X}})$ (or, $r(\Omega^2_{\widetilde{X}})$).

Example 6.1.3 By (6.1.2) $\mathrm{Cl}(X,o) = 0$ if and only if $p_g = 0$ and L_X is an integral homology sphere. These imply that (X,o) is rational and numerically Gorenstein (cf. 6.3.18), hence an E_8 singularity or smooth, see 7.1.5(d).

6.1.4 Algebraic Interpretation of $\mathrm{Cl}(X,o)$. Factoriality of $\mathscr{O}_{X,o}$ Fix a normal surface singularity (X,o), and its local ring $\mathscr{O} := \mathscr{O}_{X,o}$. Ring-theoretically it is a local, Noetherian, and integrally closed domain. Let $\mathscr{K}$ be the field of fractions of $\mathscr{O}$. A fractional ideal of $\mathscr{O}$ is a sub-$\mathscr{O}$-module $\mathfrak{a}$ of $\mathscr{K}$ such that there exists $d\neq 0$ in $\mathscr{O}$ with $d\mathfrak{a}\subset\mathscr{O}$. The module $\mathfrak{a}$ is called *invertible*, or *divisorial*, if there exists another fractional ideal $\mathfrak{b}$ such that $\mathfrak{a}\mathfrak{b} = \mathscr{O}$. If this happens then the inverse is unique, namely $\mathfrak{b} = \mathscr{O} : \mathfrak{a}$, hence $\mathfrak{a}$ is invertible if and only if $\mathfrak{a}\cdot(\mathscr{O}:\mathfrak{a}) = \mathscr{O}$. For any fractional ideal $\mathfrak{a}$, $\mathfrak{a}^{**} := \mathscr{O} : (\mathscr{O} : \mathfrak{a})$ is divisorial.

Divisorial ideals can also be introduce as follows. We say that an $\mathscr{O}$-module A has rank one if A is finitely generated over $\mathscr{O}$ and $\dim_{\mathscr{K}}(A \otimes_{\mathscr{O}} \mathscr{K}) = 1$. For any $\mathscr{O}$-module A write $A^* = \mathrm{Hom}_{\mathscr{O}}(A, \mathscr{O})$, and we say that A is reflexive if the canonical map $A \to A^{**}$ is bijective. If A has rank one then so does A^*, and the $\mathscr{O}$-tensor product of two rank one modules has also rank one. Moreover, for a rank one $\mathscr{O}$-module A, the natural map $A \to A \otimes_{\mathscr{O}} \mathscr{K}$ is injective, hence A is isomorphic to an $\mathscr{O}$-submodule $\mathfrak{a}$ of $\mathscr{K}$. One identifies $\mathrm{Hom}_{\mathscr{O}}(A_1, A_2)$ with $\mathfrak{a}_2 : \mathfrak{a}_1 = \{x \in \mathscr{K} : |\, x\mathfrak{a}_1 \subset \mathfrak{a}_2\}$, hence $\mathfrak{a}$ is reflexive if and only if $\mathscr{O} : (\mathscr{O} : \mathfrak{a}) = \mathfrak{a}$, that is, if $\mathfrak{a}$ is divisorial.

Multiplication of divisorial ideals (or tensor product of rank one reflexive modules) provides a group structure. Principal ideals (of type $x\mathscr{O}$, $x \in \mathscr{K}$) form a subgroup of divisorial ideals, and the quotient group is exactly $\mathrm{Cl}(\mathscr{O})$.

An integral ideal $\mathfrak{a}$ of $\mathscr{O}$ is divisorial if and only if the prime ideals associated with $\mathscr{O}/\mathfrak{a}$ are of height 1. Furthermore, $\mathscr{O}$ is factorial if every prime ideal of height one is principal. In particular, the ring $\mathscr{O}_{X,o}$ is factorial if and only if $\mathrm{Cl}(X, o) = 0$.

Corollary 6.1.5 ([83]) *$\mathscr{O}_{X,o}$ is factorial if and only if $p_g = 0$ and $H_1(L_X, \mathbb{Z}) = 0$. This holds if and only if (X, o) is either smooth or it is an E_8 singularity (cf. 6.1.3).*

Remark 6.1.6

(1) [83] $\mathscr{O}_{X,o}$ is factorial if and only if its completion is factorial.
(2) [83, 611][71, §3, p. 99] In the category of algebraic normal surface singularities the statement of the above Corollary is not true. For example, the algebraic local ring $(\mathbb{C}[x_1, x_2, x_3]/(x_1^{a_1} + x_2^{a_2} + x_3^{a_3}))_{\mathfrak{m}}$ (where $\mathfrak{m}$ is the maximal ideal generated by x_i, and the natural numbers a_i are pairwise relative prime) is factorial. Nevertheless, the completion of this ring fails to be factorial whenever $a_i > 1$ and $(a_1, a_2, a_3) \neq (2, 3, 5)$.

Remark 6.1.7 By 6.1.2 (X, o) is rational if and only if $\mathrm{Cl}(X, o)$ is finite, that is, $\mathscr{O}_{X,o}$ has only finitely many rank one reflexive modules (cf. 6.1.4). This can be compared with the following statement [188]: (X, o) is a quotient singularity if and only if $\mathscr{O}_{X,o}$ has only finitely many indecomposable reflexive modules (of any rank).

6.1.8 $\mathrm{Pic}(Z)$ **for an Effective Cycle** Z We mention that all the statements from above, valid for $\widetilde{X}$ and the sheaf $\mathscr{O}_{\widetilde{X}}$, have their analogues for a non-zero effective cycle supported on E. Indeed, fix such an effective cycle $Z \in L_{>0}$. Denote by $\mathscr{O}_Z^*$ the sheaf of units of $\mathscr{O}_Z$. Then $\mathrm{Pic}(Z) = H^1(Z, \mathscr{O}_Z^*)$ is the group of isomorphism classes of invertible sheaves on Z. It appears in the exact sequence

$$0 \to \mathrm{Pic}^0(Z) \to \mathrm{Pic}(Z) \to L'(|Z|) \to 0,$$

where $\mathrm{Pic}^0(Z) = H^1(Z, \mathscr{O}_Z)$, and $L(|Z|)$ is the sublattice of L generated by the base elements $E_v \subset |Z|$, and $L'(|Z|)$ is its dual lattice.

6.1.A $\mathbb{Q}$-Cartier Divisors and Canonical Coverings

We introduce two 'canonical' coverings. To any finite order element of $\mathrm{Cl}(X,o)$ we define an o-ramified cyclic covering of (X,o). If the class of $\Omega^2_{\widetilde{X}} = \mathscr{O}_{\widetilde{X}}(K_{\widetilde{X}})$ has finite order in $\mathrm{Cl}(X,o)$ (that is, $K_{\widetilde{X}}$ is $\mathbb{Q}$-Cartier), then the corresponding covering is called the 'analytic canonical covering' of (X,o). This is compared with the topological coverings discussed in 4.1.E associated with characters of L'/L. If L_X is a $\mathbb{Q}HS^3$, then the topological covering associated with $c_1(\Omega^2_{\widetilde{X}}) \in L'$ is called the 'topological canonical covering' of (X,o). ♣

Definition 6.1.9 A Weil divisor of (X,o) (or its class) is called $\mathbb{Q}$-*Cartier*, if its class in $\mathrm{Cl}(X,o)$ has finite order. Its order in $\mathrm{Cl}(X,o)$ is called its *index*.

In this subsection we associate a o-ramified covering $(X_D,o) \to (X,o)$ to any $\mathbb{Q}$-Cartier divisor D.

This is the analytic analogue of the topological construction from 4.1.E. Recall that in the topological situation 4.1.E we required $H_1(L_X,\mathbb{Q}) = 0$, we started with a cycle $l' \in L'$, such that the order of $[l'] \in H$ is N, and we considered the character $\alpha := \theta([l']) \in \widehat{H}$, which determined an o-ramified $\mathbb{Z}_N$-covering $(X_\alpha,o) \to (X,o)$.

In the present case, we start with a $\mathbb{Q}$-Cartier divisor D of index N, and we will construct a function f and the character $\mathrm{mod}_N \circ \mathrm{arg}_*(f) \in \widehat{H_1}$, which determines an o-ramified $\mathbb{Z}_N$-covering $(X_D,o) \to (X,o)$. If $H_1(L_X,\mathbb{Q}) = 0$, and $c_1(\mathscr{O}_{\widetilde{X}}(D)) = l'$, then the two coverings $(X_\alpha,o) \to (X,o)$ and $(X_D,o) \to (X,o)$ agree.

In general, if $c : (Y,o) \to (X,o)$ is a o-ramified covering then there is a morphism $\widetilde{Y} \to \widetilde{X}$ at the level of (convenient) resolutions (see below), and the pullback $\mathrm{Pic}(\widetilde{X}) \to \mathrm{Pic}(\widetilde{Y})$ induces a well-defined morphism $c^*_{\mathrm{Cl}} : \mathrm{Cl}(X,o) \to \mathrm{Cl}(Y,o)$.

Proposition 6.1.10 *Let D be a $\mathbb{Q}$-Cartier divisor of index N of (X,o). Then it determines a uniquely defined o-ramified Galois $\mathbb{Z}_N$-covering $c : (X_D,o) \to (X,o)$, where (X_D,o) is a normal surface singularity, and $c^*_{Cl}(D) = 0$ in $Cl(X_D,o)$. The covering $c : (X_D,o) \to (X,o)$ depends only on the class of D in $Cl(X,o)$.*

Proof Adding principal divisors to D we can assume that D is effective. Then $N \cdot D$ is an effective principal divisor of (X,o), hence $N \cdot D = \mathrm{div}(f)$ for some holomorphic function $f : (X,o) \to (\mathbb{C},0)$. Then X_D is defined as $X_{f,N}$, which is o-ramified by 4.1.32 (Case 1). Moreover, $c^*_{\mathrm{Cl}}(D)$ is the class of the (a priori rational) divisor $\mathrm{div}(f)/N$, but this equals $\mathrm{div}(z)$, an integral principal divisor of (X_D,o) (see 4.1.17).

Note that the added principal divisors do not alter the isomorphism class of $X_{f,N}$. Indeed, (the normalized) $X_{fg^N,N}$ and $X_{f,N}$ are isomorphic. □

The corresponding representation $H_1(L_X,\mathbb{Z}) \to \mathbb{Z}_N$ is described in 4.1.E, it is $\mathrm{mod}_N \circ \mathrm{arg}_*(f)$ (which is again not altered if we replace f by fg^N).

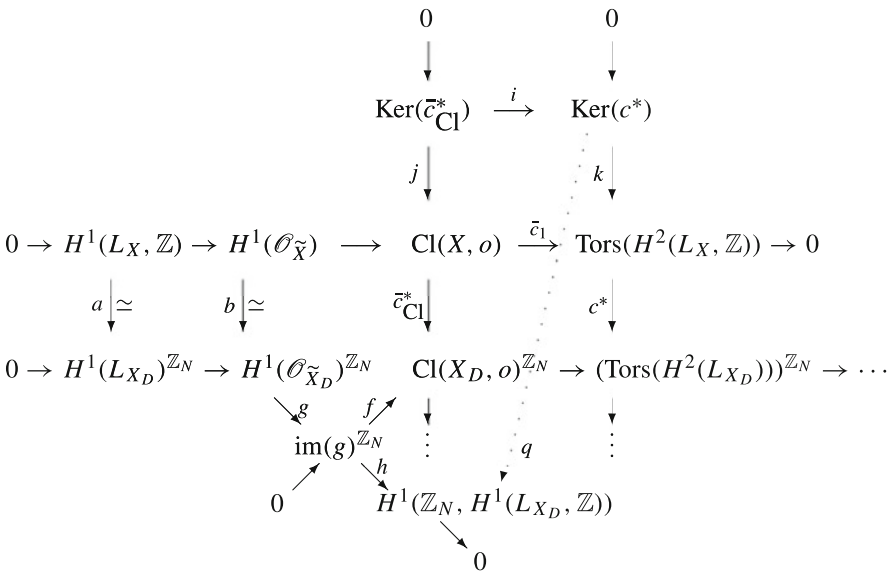
Next, fix a resolution $\phi : \widetilde{X} \to X$, and let $\widetilde{c} : \widetilde{X}_D \to \widetilde{X}$ be the normalized pullback of c via ϕ. The following statements can be compared with (4.1.7).

Proposition 6.1.11 (See e.g. [530, 692]) *The kernel of $c^*_{Cl} : Cl(X, o) \to Cl(X_D, o)$ is cyclic of order N and it is generated by the class of D. Furthermore, if L_X is a rational homology sphere then the image of c^*_{Cl} is exactly $Cl(X_D, o)^{\mathbb{Z}_N}$.*

Proof Consider the exact sequences (6.1.2) associated with $\widetilde{X}$ and $\widetilde{X}_D$:

$$\begin{array}{ccccccccc}
0 \to & H^1(L_X, \mathbb{Z}) & \to & H^1(\mathscr{O}_{\widetilde{X}}) & \longrightarrow & \mathrm{Cl}(X, o) & \to & \mathrm{Tors}(H^2(L_X, \mathbb{Z})) & \to 0 \\
& \downarrow & & \downarrow & & \downarrow{\scriptstyle c^*_{\mathrm{Cl}}} & & \downarrow{\scriptstyle c^*} & \\
0 \to & H^1(L_{X_D}, \mathbb{Z}) & \to & H^1(\mathscr{O}_{\widetilde{X}_D}) & \xrightarrow{g} & \mathrm{Cl}(X_D, o) & \to & \mathrm{Tors}(H^2(L_{X_D}, \mathbb{Z})) & \to 0
\end{array}$$

We wish to replace the second exact row by the corresponding $\mathbb{Z}_N$-invariants. Since the functor $A \mapsto A^{\mathbb{Z}_N}$ is not exact, we split this row into two short exact sequences $0 \to H^1(L_{X_D}, \mathbb{Z}) \to H^1(\mathscr{O}_{\widetilde{X}_D}) \to \mathrm{im}\,(g) \to 0$ and $0 \to \mathrm{im}(g) \to \mathrm{Cl}(X_D, o) \to \mathrm{Tors}(H^2(L_{X_D}, \mathbb{Z}))$, and then we consider the long exact sequences associated with the functor $A \mapsto A^{\mathbb{Z}_N}$. See below this new diagram completed with the first exact sequence from (4.1.7). In this diagram we also inserted a dash arrow, which corresponds to the morphism $q : \ker(c^*) \to H^1(\mathbb{Z}_N, H^1(L_{X_D}, \mathbb{Z}))$ as part of the second exact sequence from (4.1.7). In the diagram below it is defined as follows: take an element from $\ker(c^*)$, take its image by k, lift this element via $\bar{c}_1$, apply $\bar{c}^*_{\mathrm{Cl}}$, lift by f and apply h. One verifies that these sequence of operations provide a well-defined map, which is q. Hence its kernel is $\mathbb{Z}_N$ by (4.1.7).



The proof runs as follows. One verifies that $q|_{\mathrm{im}(i)} = 0$. Hence $\mathrm{im}(i) \subset \ker(q) = \mathbb{Z}_N$. Then by diagram chasing one verifies that $\mathrm{im}(i) = \mathbb{Z}_N$ and i is injective. Here we also need that both a and b are isomorphisms, cf. (4.1.6) and (4.1.9).

Finally one verifies that in the above diagram $\mathrm{coker}(\bar{c}^*_{\mathrm{Cl}})$ embeds into $\mathrm{coker}(c^*)$. (Use diagram chasing and the facts that $\bar{c}_1$, q and b are onto.) But if L_X is a $\mathbb{Q}HS^3$ then $\mathrm{coker}(c^*) = 0$ (cf. (4.1.6) and (4.1.7)) hence $\mathrm{coker}(\bar{c}^*_{\mathrm{Cl}}) = 0$ too. □

Example 6.1.12 (Continuation of Example 4.1.40) The reader is invited to compute all the terms of the above diagram for the example from 4.1.40 (as a continuation of the data from 4.1.40). Since $\mathrm{Cl}(X, o) = \mathbb{Z}_4 \oplus \mathbb{Z}_4$ one gets $\mathrm{Cl}(X_D, o)^{\mathbb{Z}_2} = \mathbb{Z}_4 \oplus \mathbb{Z}_2$, while $(\mathrm{Tors}(H^2(L_{X_D}, \mathbb{Z})))^{\mathbb{Z}_2} = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Hence $\mathrm{Cl}(X_D, o)$ has an element of order 4 (which is even invariant), but its Chern class has order 2.

This also shows that for the -4, -2 cusp, in the exact sequence (6.1.2) the epimorphism $\bar{c}_1$ does not split. (This should be compared with the splitting properties from 6.1.16*(b)* and 6.2.4 valid whenever the link is a rational homology sphere.)

$$\begin{array}{ccccccccc} 0 \to & H^1(\mathscr{O}_{\widetilde{X}_D})/H^1(L_{X_D}, \mathbb{Z}) & = & \mathbb{C}/\mathbb{Z} & \longrightarrow & \mathrm{Cl}(X_D, o) & \xrightarrow{\bar{c}_1} & \mathrm{Tors}(H^2(L_{X_D}, \mathbb{Z})) & \to 0 \\ & & & \uparrow & & \uparrow & & \uparrow = & \\ & & 0 \to & \langle [1/2] \rangle & \longrightarrow & \mathbb{Z}_4 \oplus \mathbb{Z}_2 & \longrightarrow & \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \to 0 \end{array}$$

The second row is $0 \to \mathrm{im}(g)^{\mathbb{Z}_2} \to \mathrm{Cl}(X_D, o)^{\mathbb{Z}_2} \to (\mathrm{Tors}(H^2(L_{X_D}, \mathbb{Z})))^{\mathbb{Z}_2} \to 0$. It injects into the first one (as invariants). $\langle [1/2] \rangle = \mathbb{Z}_2$ is generated by the class of $1/2$ in $\mathbb{C}/\mathbb{Z}$, which is invariant (though $\mathbb{C}$ and $\mathbb{Z}$ have nontrivial invariants). [Here we obtained this cusp as a double cover. However, since the cusps are taut, any cusp with this topology has a $\mathbb{Z}_2$ action and it is realized as a double covering as above.]

6.1.13 Additionally, we also have the **Riemann–Hurwitz (or, logarithmic ramification) formula**. Recall that $\widetilde{c} : \widetilde{X}_D \to \widetilde{X}$ is the normalized pullback of $c : (X_D, o) \to (X, o)$ via $\phi : \widetilde{X} \to X$. Let $r : \widetilde{Z} \to \widetilde{X}_D$ be the *minimal* resolution of the Hirzebruch–Jung singularities of $\widetilde{X}_D$, and set $p = \widetilde{c} \circ r$.

Proposition 6.1.14

(a) (See e.g. [327, 20.2]) $\widetilde{c}^*(K_{\widetilde{X}} + E_{\widetilde{X}}) = K_{\widetilde{X}_D} + E_{\widetilde{X}_D}$.
(b) $p^*(K_{\widetilde{X}} + E_{\widetilde{X}}) = K_{\widetilde{Z}} + E_{\widetilde{Z}}$.

The formulae are valid even if we replace the cyclic covering c by any finite o-ramified covering $(X', o) \to (X, o)$. Its verification is a local computation. In part *(a)* the point is that $\widetilde{c}$ is finite, both $\widetilde{X}$ and $\widetilde{X}_D$ are normal, and $\mathrm{red}(\widetilde{c}^{-1}(E_{\widetilde{X}})) = E_{\widetilde{X}_D}$.

However, in *(b)* p is not finite. Let us recall a statement valid in such a non-finite case (see e.g. [410, p. 117]). Assume that both A and B are smooth, D_A and D_B are reduced normal crossing divisors, $g : (A, D_A) \to (B, D_B)$ generically finite, regular covering over $B \setminus D_B$, $D_A = \mathrm{red}(g^{-1}D_B)$. Then $K_A + D_A = g^*(K_B + D_B) + R_{log}$, where the logarithmic ramification divisor R_{log} is effective and g-exceptional.

In our case *(b)* $R_{log} = 0$ by the following local argument. If $p : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, o)$ is given by $x = u^a y^b$, $y = u^c y^d$, $r = ad - bc \neq 0$, then $p^*(dx \wedge dy/xy) = r \cdot du \wedge dv/uv$.

6.1.15 Applications The above facts can be used to define a covering associated with any $l' \in L'$. The construction depends on a choice, but it has no ambiguity whenever the link is a rational homology sphere.

Though usually we identify the homology classes $l \in L$ and the integral divisors supported by E, sometimes—as an accentuation—, we might use for the corresponding divisor the notation $\mathrm{div}(l)$ as well.

First, we associate to any $l' \in L'$ a $\mathbb{Q}$-Cartier divisor as follows. For parts *(a)–(b)* see [530, 692, 693].

Proposition 6.1.16

(a) Fix a resolution $\phi : \widetilde{X} \to X$, $l' \in L'$, and let N be the order of its class in L'/L. Then there exists a divisor $D = D(l')$ on $\widetilde{X}$ such that one has a linear equivalence $N \cdot D \sim div(N \cdot l')$ and $c_1\mathscr{O}_{\widetilde{X}}(D) = l'$. In particular, $\phi_*(D)$ has finite order N in $Cl(X, o)$.

(b) If $H^1(\widetilde{X}, \mathbb{Z}) = 0$ then D is unique up to a linear equivalence. Hence, in this case, the correspondence $l' \mapsto \mathscr{O}_{\widetilde{X}}(D(l'))$ is a section of the exact sequence (6.1.1).

(c) If $H^1(\widetilde{X}, \mathbb{Z}) = 0$ then the covering associated with l' defined in 6.1.10 via $D(l')$ and the covering associated with l' defined in 4.1.E via $\theta([l'])$ agree.

Proof *(a)* Since in (6.1.1) c_1 is onto, there exists a divisor D_1 on $\widetilde{X}$ with $c_1\mathscr{O}_{\widetilde{X}}(D_1) = l'$. Hence $\mathscr{O}_{\widetilde{X}}(ND_1 - \mathrm{div}(Nl'))$ equals $\epsilon(\mathscr{L})$ for some $\mathscr{L} \in \mathrm{Pic}^0(\widetilde{X}) = \mathbb{C}^{p_g}/H^1(\widetilde{X}, \mathbb{Z})$. Define D_2 so that $\mathscr{O}_{\widetilde{X}}(D_2) := \epsilon(\frac{1}{N}\mathscr{L}) \in \mathrm{Pic}(\widetilde{X})$. Then $D := D_1 - D_2$ works. For *(b)* use the fact that $\mathrm{Pic}(\widetilde{X})$ is torsion free. For *(c)* use the definitions and 3.4. □

An addendum to Propositions 6.1.10 and 6.1.11 is the following statement:

Corollary 6.1.17 *Fix l' and one of the possible divisors D associated with it as in 6.1.16. Then the image of l' via the pullback $L'(\widetilde{X}) \to L'(\widetilde{X}_D)$ is an integral cycle.*

Example 6.1.18 Let (X, o) be the isolated homogeneous hypersurface singularity, the cone over $C \subset \mathbb{P}^2$ with $\deg(C) = 3$. Take its minimal resolution with exceptional curve E ($g(E) = 1$ and $E^2 = -3$, cf. 2.1.9) and set $l' := \frac{1}{3}E$. Then we wish to find D such that $-3D + E = \mathrm{div}(f)$. This is realized e.g. by a linear form f such that in $\mathbb{P}^2$ the line $f = 0$ is tangent of C at an inflection point. However, in general, for an arbitrary (X, o) and l' (even if the link is a rational homology sphere) it is very hard to find D explicitly. Explicit examples for superisolated germs are given in [665].

Definition 6.1.19

(a) Write $\Omega^2_{\widetilde{X}} = \mathscr{O}_{\widetilde{X}}(K_{\widetilde{X}})$ and assume that $K_X = \phi_*(K_{\widetilde{X}})$ is $\mathbb{Q}$-Cartier. Then the cyclic covering associated with K_X (as in 6.1.10) is called the 'analytic canonical covering' of (X, o).

(b) Assume that the link of (X, o) is a rational homology sphere. The well-defined cyclic covering associated with $c_1(\mathscr{O}_{\widetilde{X}}(K_{\widetilde{X}}))$, constructed in 6.1.16 and 4.1.E is called the *topological canonical covering* of (X, o). (Note that by 4.1.36 the coverings associated with $c_1(\mathscr{O}_{\widetilde{X}}(K_{\widetilde{X}}))$ or $-c_1(\mathscr{O}_{\widetilde{X}}(K_{\widetilde{X}}))$ agree.)

Note that if both assumptions are satisfied then the analytic and topological canonical coverings agree. Nevertheless, if $H_1(\partial\widetilde{X}, \mathbb{Q}) = 0$, the *topological canonical covering* is well-defined even if K_X is not $\mathbb{Q}$-Cartier.

Example 6.1.20 The $\mathbb{Z}_2$-covering of the quotient-cusp singularity constructed in 4.1.37 is the topological canonical covering (l' in loc. cit. is $Z_K = -c_1(\mathscr{O}_{\widetilde{X}}(K_{\widetilde{X}}))$). But, in fact, quotient-cusps are taut and $\mathbb{Q}$-Gorenstein, cf. 6.3.25, hence this is the analytic canonical covering as well.

6.2 Natural Line Bundles

Fix a resolution $\widetilde{X} \to X$ and assume that the link is a $\mathbb{Q}HS^3$. Then to any Chern class $l' \in L'$ we associate (in three different ways) a line bundle $\mathscr{O}_{\widetilde{X}}(l')$ with Chern class l' and finite order in $\mathrm{Cl}(X, o)$. This provides a group homomorphism splitting of the exponential exact sequence (which extends the 'classical' construction $L \ni l \mapsto \mathscr{O}_{\widetilde{X}}(l)$). The interpretation of some $\mathscr{O}_{\widetilde{X}}(l)$'s as eigensheaves associated with the universal abelian covering provides one of the main technical tools of the book in the equivariant setup. ♣

6.2.1 Let $\phi : (\widetilde{X}, E) \to (X, o)$ be a fixed good resolution of (X, o). For notations see 2.1.15. *In this subsection we assume that L_X is a rational homology sphere.*

In the exact sequence (6.1.1) c_1 admits a natural group section s_L over the integral cycles $L \subset L'$. Namely, for any $l \in L$ one takes $\mathscr{O}(l) := \mathscr{O}(\mathrm{div}(l)) \in \mathrm{Pic}(\widetilde{X})$ with $c_1(\mathscr{O}(l)) = l$. In the sequel we extend s_L in a unique way to a natural group section $s : L' \to \mathrm{Pic}(\widetilde{X})$. Its existence basically is guaranteed by the facts that $H = L'/L$ is finite, while $\mathrm{Pic}^0(\widetilde{X}) \simeq \mathbb{C}^{p_g}$ is torsion free. In the body of the section we present several constructions of s, which emphasize its different geometrical aspects.

6.2.2 The First Construction of s (via $\mathrm{Cl}(X, o)$) [530] For any $l' \in L'$ consider the divisor $D(l')$ provided by Lemma 6.1.16. Since $H^1(\widetilde{X}, \mathbb{Z}) = 0$, $D(l')$ is unique with the required properties of 6.1.16. Therefore one has a well-defined map $l' \mapsto s(l') = \mathscr{O}_{\widetilde{X}}(D(l'))$. By the uniqueness $D(l'_1 + l'_2) \sim D(l'_1) + D(l'_2)$, hence s is a homomorphism and a section of (6.1.1) as well.

Definition 6.2.3 The line bundles $s(l')$, indexed by $l' \in L'$, and denoted also by $\mathscr{O}_{\widetilde{X}}(l') := s(l')$, will be called *natural line bundles*.

Corollary 6.2.4 (Recall the Assumption $H^1(L_X, \mathbb{Z}) = 0$)

(a) A line bundle $\mathscr{L} \in Pic(\widetilde{X})$ is natural if and only if some power of it has the form $\mathscr{O}_{\widetilde{X}}(l)$ for an integral cycle $l \in L$. Equivalently, $\mathscr{L}$ is natural if and only if its 'projection' by $Pic(\widetilde{X}) \to Pic(\widetilde{X})/L = Cl(X, o)$ has finite order (i.e., if it is $\mathbb{Q}$-Cartier).

(b) One has a natural isomorphism $Pic(\widetilde{X}) \to Pic^0(\widetilde{X}) \oplus L'$ given by $\mathscr{L} \mapsto (\mathscr{L} \otimes s(c_1\mathscr{L})^{-1}, c_1\mathscr{L})$. This induces a natural isomorphism $Cl(X, o) \to Pic^0(\widetilde{X}) \oplus H$.

In particular (since $Pic^0(\widetilde{X})$ is torsion free), under this identification H is isomorphic with the group of $\mathbb{Q}$-Cartier divisor classes of (X, o).

Example 6.1.12 shows that for such statements the assumption $H^1(L_X, \mathbb{Z}) = 0$ is necessary.

6.2.5 Notations Regarding the Universal Abelian Covering Let $c : (X_a, o) \to (X, o)$ be the universal abelian covering of (X, o). Recall that (X_a, o) is the unique normal singular germ such that $X_a \setminus \{o\}$ is the regular covering of $X \setminus \{o\}$ associated with $\pi_1(X \setminus \{o\}) \to H$, cf. 12.3.3.

Since $\widetilde{X} \setminus E \approx X \setminus \{o\}$, $\pi_1(\widetilde{X} \setminus E) = \pi_1(X \setminus \{o\}) \to H$ defines a regular Galois covering of $\widetilde{X} \setminus E$ as well. This has a unique extension $\widetilde{c} : Z \to \widetilde{X}$ with Z normal and $\widetilde{c}$ finite [242]. (In other words, $\widetilde{c} : Z \to \widetilde{X}$ is the normalized pullback of c via ϕ.) The (reduced) branch locus of $\widetilde{c}$ is included in E, and the Galois action of H extends to Z as well. Since E is a normal crossing divisor, the only singularities what Z might have are cyclic quotient singularities. Let $r : \widetilde{Z} \to Z$ be a resolution of these singular points such that $(\widetilde{c} \circ r)^{-1}(E)$ is a normal crossing divisor. Set $p := \widetilde{c} \circ r$.

$$\begin{array}{ccccc} \widetilde{Z} & \xrightarrow{r} & Z & \longrightarrow & (X_a, o) \\ & & \downarrow{\scriptstyle \widetilde{c}} & & \downarrow{\scriptstyle c} \\ & & \widetilde{X} & \xrightarrow{\phi} & (X, o) \end{array} \tag{6.2.1}$$

6.2.6 The Second Construction of the Section s (via $p^* : \mathrm{Pic}(\widetilde{X}) \to \mathrm{Pic}(\widetilde{Z})$) [479] One has the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \to & L & \to & L' & \to & H & \to & 0 \\ & & \downarrow & & \downarrow{\scriptstyle p^*} & & \downarrow{\scriptstyle p^H} & & \\ 0 & \to & L_a & \to & L'_a & \to & H_a & \to & 0 \end{array} \tag{6.2.2}$$

where the vertical arrows are pullbacks associated with $p = \widetilde{c} \circ r$ (e.g., p^* is the cohomology morphism $H^2(\widetilde{X}, \mathbb{Z}) \to H^2(\widetilde{Z}, \mathbb{Z})$ and the first arrow is the relative

cohomology morphism), and the bottom line is the 'lattice exacts sequence' (3.4.1) associated with the resolution $\widetilde{Z} \to X_a$ of (X_a, o). First we claim that

$$p^H = 0, \quad \text{or,} \quad p^*(l') \in L_a \text{ for any } l' \in L'. \tag{6.2.3}$$

Indeed, fix some $l' \in L'$. Set $\alpha = \theta([l']) \in \widehat{H}$. Then the universal abelian covering factorises through $(X_\alpha, o) \to (X, o)$. By 6.1.17, or by the line after (4.1.7) we get that the pull back of l' via this cyclic covering is already an integral cycle.

Hence, considering the integral divisor $\mathrm{div}(p^*(l'))$, the element $\mathscr{O}_{\widetilde{Z}}(p^*(l')) := \mathscr{O}_{\widetilde{Z}}(\mathrm{div}(p^*(l'))) \in \mathrm{Pic}(\widetilde{Z})$ is well-defined.

Theorem 6.2.7 *The line bundle $\mathscr{O}_{\widetilde{Z}}(p^*(l'))$ is a pullback of a unique element $\mathscr{L}$ of $Pic(\widetilde{X})$. This universal property of the line bundle $\mathscr{L}$ redefines the section s: If $\mathscr{L}$ is denoted by $\mathscr{O}_{\widetilde{X}}(l')$ then $s : L' \to Pic(\widetilde{X})$, defined by $l' \mapsto \mathscr{O}_{\widetilde{X}}(l')$, is a group section of c_1 in (6.1.1), which extends s_L.*

Furthermore, the definition of $\mathscr{O}_{\widetilde{X}}(l')$ is independent of the choice of the resolution $r : \widetilde{Z} \to Z$.

Proof Let N be the order of $[l'] \in H$. Consider the divisor D in $\widetilde{X}$ (constructed in 6.1.16) with the property $N \cdot D \sim \mathrm{div}(Nl')$. Then the pullback to $\mathrm{Pic}(\widetilde{Z})$ of $\mathscr{O}_{\widetilde{X}}(D)$ is $\mathscr{O}_{\widetilde{Z}}(p^*(l'))$. On the other hand, one verifies that $p^* : \mathrm{Pic}(\widetilde{X}) \to \mathrm{Pic}(\widetilde{Z})$ is injective (either using the two exponential exact sequences, or by the identity $p_*(p^*\mathscr{L}) = (p_*\mathscr{O}_{\widetilde{Z}}) \otimes \mathscr{L}$, see also 6.2.9). □

6.2.8 The Third Construction of s (via $c_*\mathscr{O}_{X_a,o}$) [328, 479, 530, 533] Associated with the resolution $\phi : \widetilde{X} \to X$ we consider the 'unit closed-open cube' $Q := \{l' \in L' : \lfloor l' \rfloor = 0\}$. Obviously, for any $h \in H$ there is a unique element $r_h \in Q$, whose class is h. It is the minimal representative of h in the cone $L'_{\geq 0}$.

Theorem 6.2.9 ([479, 530, 533] (For the Cyclic Case see also [188–190])) *Assume, as above, that $H^1(\widetilde{X}, \mathbb{Z}) = 0$. Consider the finite covering $\widetilde{c} : Z \to \widetilde{X}$. Then $\widetilde{c}_*\mathscr{O}_Z$ is a vector bundle and its H-eigensheaf decomposition has the form:*

$$\widetilde{c}_*\mathscr{O}_Z \simeq \oplus_{\alpha \in \widehat{H}} \mathscr{L}_\alpha, \tag{6.2.4}$$

where $\mathscr{L}_{\theta(h)} = \mathscr{O}_{\widetilde{X}}(-r_h)$ for any $h \in H$. In particular, $\widetilde{c}_*\mathscr{O}_Z \simeq \oplus_{l' \in Q} \mathscr{O}_{\widetilde{X}}(-l')$.

More generally, for any $l' \in L'$ one has

$$\widetilde{c}_*\mathscr{O}_Z(-\widetilde{c}^*(l')) \simeq \oplus_{h \in H} \mathscr{O}_{\widetilde{X}}(-r_h + \lfloor r_h - l' \rfloor). \tag{6.2.5}$$

Proof We prove the first identity (6.2.4), the proof of the second one is similar. The proof is based on a reduction to a similar statement valid for cyclic Galois coverings, which is considered e.g. in [328, §9]. In this way one verifies that $\widetilde{c}_*\mathscr{O}_Z$ is free, $\mathrm{rank}\, \widetilde{c}_*\mathscr{O}_Z = |H|$, and $\widetilde{c}_*\mathscr{O}_Z$ has an eigenspace decomposition $\oplus_\alpha \mathscr{L}_\alpha$, where all the characters $\alpha \in \widehat{H}$ appear, and $\mathscr{L}_\alpha|_{\widetilde{X} \setminus E}$ is a line bundle. By reduction to the cyclic case, one obtained that, in fact, $\mathscr{L}_\alpha$ itself is a line bundle. Moreover, the eigensheaf corresponding to the trivial character equals $\mathscr{O}_{\widetilde{X}}$.

Next, we identify $\mathscr{L}_\alpha$ for any character. Fix $\alpha_0 \in \widehat{H}$. It generates the subgroup $\langle\alpha_0\rangle$ in $\widehat{H}$. The morphism $\pi_1(M) \to H \to \widehat{\langle\alpha_0\rangle}$ defines a normal space Y as a cyclic Galois $\widehat{\langle\alpha_0\rangle}$-covering of $\widetilde{X}$. Decompose $\widetilde{c}$ as $\widetilde{e} \circ \widetilde{f}$, $Z \xrightarrow{\widetilde{f}} Y \xrightarrow{\widetilde{e}} \widetilde{X}$. Then $\mathscr{L}_{\alpha_0}$ can also be recovered from the cyclic covering $\widetilde{e}$ as the α_0-eigenspace of $\widetilde{e}_*\mathscr{O}_Y$.

Let r be the order of α_0. Write $\alpha_0([\partial D_v]) = e^{2\pi i m_v/r}$ for some $0 \le m_v < r$. Set $l := \sum_v m_v E_v \in L$. Then $l/r \in L'$ and $\theta([l/r]) = \alpha_0$. Set $h = [l/r] \in H$; cf. 4.1.33.

On the other hand, see [328, 9.8], the α_0-eigensheaf of $\widetilde{e}_*\mathscr{O}_Y$ is a certain line bundle $\mathscr{L}^{-1}$ with the properties (i) $\mathscr{L}^r = \mathscr{O}_{\widetilde{X}}(l)$ and (ii) $\widetilde{e}^*\mathscr{L} = \mathscr{O}_Y(\widetilde{e}^*(l/r))$. From (i) follows that $c_1(\mathscr{L}) = l/r$. Hence (ii), via 6.2.7, reads as $\mathscr{L} = \mathscr{O}_{\widetilde{X}}(l/r)$. Therefore, $\mathscr{L}^{-1} = \mathscr{O}_{\widetilde{X}}(-l/r)$. But l/r is in the unit cube Q, thus $l/r = r_h$.

We support the above statements by a local computation, whose globalization gives the above proof. Write $\mathscr{O}_x$ for $\mathbb{C}[[x]]$ (which might serve also as the x-coordinate in the (x, y)-plane, where the ramification divisor D is $x = 0$ with multiplicity m).

Consider the ramified covering $C := \{z^r = x^m\} \subset (\mathbb{C}^2, 0) \xrightarrow{p} (\mathbb{C}, 0)$, $(x, z) \mapsto x$. Then $p_*\mathscr{O}_{C,0}$ as an $\mathscr{O}_x$ module has a direct sum decomposition $\oplus_{j=0}^{r-1} z^j \mathscr{O}_x$. The $\mathbb{Z}_r$-action is induced by the monodromy on the regular part, namely by the permutation of the pages induced by the loop $x(s) = \{e^{2\pi i s}\}_{0\le s\le 1}$. This is the multiplication of z by $\xi = e^{2\pi i m/r}$. Hence $z^j\mathscr{O}_x$ is the ξ^j-eigensheaf of $p_*\mathscr{O}_{C,0}$. If we globalize $z\mathscr{O}_x$ we get a line bundle, say $\mathscr{F}^{-1}$. Then the local representative of $\mathscr{F}^{-r}$ is $z^r\mathscr{O}_x = x^m\mathscr{O}_x$, whose global reading is $\mathscr{F}^{-r} = \mathscr{O}(-D)$. Hence, for global ramified coverings $p_*\mathscr{O}_C = \sum_{j=0}^{r-1} \mathscr{F}^{-j}$, where $\mathscr{F}^r = \mathscr{O}(D)$, and the $e^{2\pi i m/r}$-eigensheaf is $\mathscr{F}^{-1}$.

Next, we replace $(C, 0)$ by its normalization $\widehat{C}$. For simplicity we assume that $\gcd(r, m) = 1$. Set $m' \in \mathbb{Z}$ with $mm' \equiv 1 (\mathrm{mod}\, r)$. The normalization is $(\mathbb{C}, 0) = (\widehat{C}, 0) \to (C, 0)$, $(x(t), z(t)) = (t^r, t^m)$, the composed covering $\widehat{p}$ is $t \mapsto x(t) = t^r$. Hence $\widehat{p}_*\mathscr{O}_{\widehat{C},0} = \sum_{j=0}^{r-1} t^j\mathscr{O}_x$ and $t\mathscr{O}_x$ is the $\xi^{m'}$-eigensheaf. $\mathscr{F}^{-1}$ is included as $t^m\mathscr{O}_x$ in $\mathscr{L}^{-1} = t^{r\{m/r\}}\mathscr{O}_x$. With the choice $m < r$ one has $\mathscr{L} = \mathscr{F}$, and $\widehat{p}^*(\mathscr{L}^{-1}) = \mathscr{O}_t(-l)$, where l is the divisor $t = 0$ with multiplicity m. Hence $\mathscr{L}^{-1} = \mathscr{O}(-l/r)$. □

Corollary 6.2.10 *The set of natural line bundles on $\widetilde{X}$ coincides with the set of line bundles of type $\mathscr{L} \otimes \mathscr{O}(l)$, where $\mathscr{L}$ is an eigensheaf of $\widetilde{c}_*\mathscr{O}_Z$ and $l \in L$. Or, via (6.2.5), the set of natural line bundles coincides with the set of eigensheaf of bundles of type $\widetilde{c}_*\mathscr{O}_Z(-\widetilde{c}^*(l'))$, $l' \in L'$.*

Corollary 6.2.11

(a) Let $c : (X_a, o) \to (X, o)$ be the universal abelian covering. Then $c_*\mathscr{O}_{X_a,o}$ has an eigen-decomposition

$$c_*\mathscr{O}_{X_a,o} \simeq \oplus_{l'\in Q} (\phi_*(\mathscr{O}_{\widetilde{X}}(-l')))_o,$$

where the $\theta(h)$-eigenspace is indexed by $l' = r_h$.

(b) Fix $h \in H$, and consider the correspondence from part (a) between $\theta(h)$-eigenfunctions $f_a \in \mathcal{O}_{X_a,o}$ and global sections s_a of $\mathcal{O}_{\widetilde{X}}(-r_h)$ (via p^*). If the restriction of the divisor of s_a to E is $l \in L$, then the restriction of the divisor of f_a (lifted to $\widetilde{Z}$) to the exceptional locus sitting in $\widetilde{Z}$ is $p^*(l + r_h)$.

In particular, the divisor of any such f_a (lifted to $\widetilde{Z}$ and restricted to the exceptional set) is of type $p^*(l')$ for a certain $l' \in L'$, which satisfies $\lfloor l' \rfloor = div_E(s_a)$.

Proof Theorem 6.2.9 implies *(a)*. Next we verify *(b)*. First, $p^*(l)$ is the divisor of $p^*(s_a)$ in $\mathscr{L} := p^*(\mathcal{O}_{\widetilde{X}}(-r_h))$. Second, since $\mathscr{L} \subset \mathcal{O}_{\widetilde{Z}}$, the divisor of $p^*(s_a)$ in $\mathcal{O}_{\widetilde{Z}}$ (that is, of the lifted f_α) is the sum of the divisor of $p^*(s_a)$ in $\mathscr{L}$ with $p^*(r_h)$. □

Example 6.2.12 Let (X, o) be a weighted homogeneous singularity with a $\mathbb{Q}HS^3$ link, $c : (X_a, o) \to (X, o)$ be the universal abelian covering, $\widetilde{X} \to X$ the minimal good resolution and $\widetilde{c} : Z \to \widetilde{X}$ the normalized pullback of c, cf. 6.2.5. Let E_0 and $E_{a,0}$ be the corresponding central exceptional curves in $\widetilde{X}$ and Z respectively. Recall that we denoted the order of $[E_0^*]$ in H by $\mathfrak{o}$. This means that if γ is a small loop around E_0 in $\widetilde{X}$ then its homology class has order $\mathfrak{o}$ in H. Therefore, by covering theory, if $D \subset \widetilde{X}$ is a small disc transversal to E_0 (sitting in a generic fiber) then $\widetilde{c}^{-1}(D)$ consists of $H/\mathfrak{o}$ discs $\{D_{a,i}\}_i$ in Z (that is, $H/\mathfrak{o} = \deg(\widetilde{c}|_{E_{a,0}})$), and for each i the degree of $D_{a,i} \to D$ is $\mathfrak{o}$. Therefore, the $E_{a,0}$-coefficient satisfies $m_{E_{a,0}}(\widetilde{c}^*(E_0)) = \mathfrak{o}$.

Thus, if $f \in \mathcal{O}_{X_a,o}$ and $s \in H^0(\mathcal{O}_{\widetilde{X}}(-r_h))$ correspond by the correspondence 6.2.11*(b)*, and $m_{E_0}(\mathrm{div}(s)) = \ell$, then $m_{E_{a,0}}(\widetilde{c}^*(\ell E_0 + r_h)) = \mathfrak{o} \cdot (\ell + m_{E_0}(r_h))$ is the vanishing order of f along $E_{a,0}$. This is the homogeneous degree of f. This motivates to redefine the grading of the graded ring $\mathcal{O}_{X_a,o} = \oplus_{d \in \mathbb{Z}_{\geq 0}} R_d$ into $\oplus_{d/\mathfrak{o}} R_{d/\mathfrak{o}}$. (See e.g. the expressions from 8.1.B.)

Here Z and $\widetilde{c}$ can be replaced by $\widetilde{Z}$ and p providing identical formulae.

Example 6.2.13 Let X be a normal affine surface, which admits a good $\mathbb{C}^*$ action, and let its singular point be $o \in X$. Let $\widetilde{X}$ be the resolution of $o \in X$, and let $O \in \widetilde{X}$ be a lifted $\mathbb{C}^*$ orbit to $\widetilde{X}$. (For their position inside the 'special analytic plumbing' $\widetilde{X}$ see 5.1.36.) If we replace the resolution of the affine variety by a small neighbourhood of E, then let O be the intersection of the lifted $\mathbb{C}^*$ orbit with this local neighbourhood. Note that there is a special orbit corresponding to each singular Seifert fiber and a 1-parameter family of 'generic' orbits.

We claim that $\mathcal{O}_{\widetilde{X}}(O)$ is a natural line bundle in $\mathrm{Pic}(\widetilde{X})$.

For each O we need to find a local analytic function $f_O : (X, o) \to (\mathbb{C}, 0)$ such that the divisor in $\widetilde{X}$ of $f_O \circ \phi$ has the form $n_O O + \sum_v n_v E_v$ for some $n_O, n_v \in \mathbb{Z}_{>0}$. This implies that $\mathcal{O}_{\widetilde{X}}(n_O O) \simeq \mathcal{O}_{\widetilde{X}}(-\sum_v n_v E_v)$.

In the case of special (singular) orbits we use 5.1.33. Indeed some power of the coordinate functions serve as 'end curve functions' (see also 5.1.39), which verify the above needed requirements.

A similar argument clarifies the case of the generic orbits as well. Let the UAC complete intersection equations be $\sum_{i,j} a_{i,j} z^{\alpha_j} = 0$, $1 \leq j \leq \nu$, $1 \leq i \leq \nu - 2$,

and where $\{a_{i,j}\}_{i,j}$ has full rank, cf. 5.1.33. Then we add one more equation of type $\sum_i b_i z^{\alpha_i} + w = 0$, such that the new larger matrix has again full rank. The new system corresponds to a system of splice quotient equations of the graph Γ' obtained from Γ by blowing up the central vertex. The point is that the resolution with dual graph Γ' of this splice quotient singularity associated with Γ' can be obtained from $\widetilde{X}$ by blowing up a certain point $P \in E_{v_0} \setminus \cup_j P_j$. The point P is determined by the choice of the coefficients $\{b_i\}_i$, and modification of the $\{b_i\}_i$'s provides all the possible points P. By the theory of splice quotient singularities, the end curve function $w = 0$ cuts out an end curve in the UAC, which projected on (X, o) is irreducible. Hence $\sum_i b_i z^{\alpha_i}$ is a weighted homogeneous function on the UAC of (X, o), and one of its powers is a homogeneous function on (X, o) whose reduced zero set is irreducible. Its strict transform is a certain orbit O, where $O \cap E_{v_0}$ depends on the choice of $\{b_i\}_i$. In particular all such orbits define the same line bundle, the natural line bundle associated with E_0^*.

6.2.14 More on Coverings (The proof of) Theorem 6.2.9 can be exploited further. Recall that any element $h \in H$ determines a cyclic (o-ramified) covering of (X, o). Indeed, write $\alpha := \theta(h) \in \widehat{H}$, and assume that it has order r, and it generates the subgroup $\langle\alpha\rangle$. The epimorphism $H \to \widehat{\langle\alpha\rangle}$ defines a $\mathbb{Z}_r$ Galois covering $c_\alpha : X_\alpha \to X$, ramified only at o.

6.2.15 The Covering $c_\alpha : X_\alpha \to X$ and Natural Line Bundles Set $l \in L$ such that $l/r \in L'$ and $l/r = r_h$. Then, by Corollary 6.2.11,

$$(c_\alpha)_* \mathscr{O}_{X_\alpha, o} \simeq \oplus_{i=0}^{r-1} \phi_*(\mathscr{O}_{\widetilde{X}}(-\{il/r\}))_o, \tag{6.2.6}$$

where the α^i-eigenspace is indexed by i. Moreover, if $\widetilde{c}_\alpha : Z_\alpha \to \widetilde{X}$ is the normalized pullback of c_α via ϕ, then

$$(\widetilde{c}_\alpha)_* \mathscr{O}_{Z_\alpha} \simeq \oplus_{i=0}^{r-1} \mathscr{O}_{\widetilde{X}}(-\{il/r\}). \tag{6.2.7}$$

Furthermore, $\widetilde{c}_\alpha^*(l/r)$ is an integral cycle in Z_α supported on $\widetilde{c}_\alpha^{-1}(E)$, cf. 6.1.17.

Note that, a priori, the right hand side of (6.2.6) (similarly of (6.2.7) is a graded vector space (respectively graded sheaf), hence the isomorphism (6.2.6) is an isomorphism of graded vector spaces. The ring structure of the left hand side can be recovered in the right hand side as follows. We wish to define a morphism

$$\mathscr{O}_{\widetilde{X}}(-\{il/r\}) \otimes \mathscr{O}_{\widetilde{X}}(-\{jl/r\}) \to \mathscr{O}_{\widetilde{X}}(-\{(i+j)l/r\}), \quad 0 \le i, j < r. \tag{6.2.8}$$

First we reinterpret this in the language of 'genuine' line bundles of $\widetilde{X}$ identified by integral divisors. Set $D := D(l/r)$, cf. Lemma 6.1.16. Hence $rD - \mathrm{div}(l) = \mathrm{div}(f)$ for certain function f. Write il as $r\{il/r\} + r[il/r]$, hence $r(iD - \mathrm{div}[il/r]) \sim \mathrm{div}(r\{il/r\})$, that is $D(\{il/r\}) = iD - \mathrm{div}[il/r]$ and

$$\mathscr{O}_{\widetilde{X}}(-\{il/r\}) = \mathscr{O}_{\widetilde{X}}(-iD + \mathrm{div}[il/r]).$$

Since $[il/r] + [jl/r] \le [(i+j)l/r]$, multiplying sections (and taking a natural inclusion) we get a natural morphism

$$\mathscr{O}_{\widetilde{X}}(-iD+\mathrm{div}[il/r])\otimes\mathscr{O}_{\widetilde{X}}(-jD+\mathrm{div}[jl/r]) \to \mathscr{O}_{\widetilde{X}}(-(i+j)D+\mathrm{div}[(i+j)l/r]).$$

If $i+j<r$ then this provides the wished multiplication (6.2.8). If $i+j \ge r$, then we need to compose the above morphism with another one of type

$$\mathscr{O}_{\widetilde{X}}(-(i+j)D+\mathrm{div}[(i+j)l/r]) \to \mathscr{O}_{\widetilde{X}}(-(i+j-r)D+\mathrm{div}[(i+j-r)l/r]).$$

Hence, via a tensor product, we need a morphism of $\mathscr{O}_{\widetilde{X}}$-modules $\mathscr{O}_{\widetilde{X}} \to \mathscr{O}_{\widetilde{X}}(rD-\mathrm{div}(l))$. This is defined by multiplication by f^{-1}.

6.2.16 The Reconstruction of $c_\alpha : X_\alpha \to X$ from Natural Line Bundles [530] Conversely, having $\phi : \widetilde{X} \to X$, we can start to consider the graded vector space

$$\oplus_{i=0}^{r-1}\phi_*(\mathscr{O}_{\widetilde{X}}(-\{il/r\}))_o = \oplus_{i=0}^{r-1}\phi_*(\mathscr{O}_{\widetilde{X}}(-iD+\mathrm{div}[il/r]))_o = \oplus_{i=0}^{r-1}\mathscr{O}_{X,o}(-i\phi_*D).$$

We convert it to a graded ring $\mathscr{R} = \oplus_{i=0}^{r-1}\mathscr{R}_i$ by the above multiplication, such that $\mathscr{R}$ becomes an $\mathscr{R}_0 = \mathscr{O}_{X,o}$ module. Then one can define (X_α, o) as $\mathrm{Spec}_{X,o}(\mathscr{R})$ together with the natural projection $c_\alpha : (X_\alpha, o) \to (X, o)$.

Moreover, if we consider the graded sheaves of $\mathscr{O}_{\widetilde{X}}$ modules, $\oplus_{i=0}^{r-1}\mathscr{O}_{\widetilde{X}}(-\{il/r\})$ too, and we convert it to a sheaf of rings by the above multiplication, then its $\mathrm{Spec}_{\widetilde{X}}$ gives the partial resolution Z_α (cf. Proposition 3.1 or Theorem 3.4 of [530]).

6.2.17 The Universal Abelian Covering Revisited [530] We wish to repeat the above construction of the ring-multiplication for the universal abelian covering (X_a, o). In this case, the index set Q or $\widehat{H}$ of the grading in the right hand side of (6.2.4) is less convenient.

Let us fix $l'_1, \dots, l'_\kappa \in L'$ such that their classes in H generate subgroups $\{\mathbb{Z}_{r_k}\}_{k=1}^{\kappa}$ and H is a direct sum of these subgroups. Set $\alpha_k := \theta([l'_k]) \in \widehat{H}$. Then $A := (\alpha_1, \dots, \alpha_\kappa) : H \to \prod_k \mathbb{Z}_{r_k}$ is an isomorphism. For the combinatorial/topological construction of the resolution graph of (X_a, o) in terms of these data see 4.2.6.

Choose divisors $D_k := D(l'_k)$ of $\widetilde{X}$ as in Lemma 6.1.16, that is, $r_kD_k - \mathrm{div}(r_kl'_k) = \mathrm{div}(f_k)$ for some functions f_k. Then, by the very construction, $Q = \{\{\sum_k i_kl'_k\} : 0 \le i_k < r_k,\ 1 \le k \le \kappa\}$, and $\mathscr{O}_{\widetilde{X}}(-\{\sum_k i_kl'_k\}) = \mathscr{O}_{\widetilde{X}}(-\sum i_kD_k + \mathrm{div}[\sum_k i_kl'_k])$. Let us denote this line bundle by $\mathscr{O}(\mathbf{i})$ for every $\mathbf{i} = (i_1, \dots, i_\kappa)$.

We define a ring structure on $\oplus_{\mathbf{i}}\mathscr{O}(\mathbf{i})$. For any pair $(\mathbf{i}, \mathbf{j})$ and index k we set $\epsilon_k = 0$ if $i_k + j_k < r_k$ and $\epsilon_k = 1$ otherwise. Write $\boldsymbol{\epsilon} := (\epsilon_1r_1, \dots, \epsilon_\kappa r_\kappa)$. Then we need to define a multiplication $\mathscr{O}(\mathbf{i}) \otimes \mathscr{O}(\mathbf{j}) \to \mathscr{O}(\mathbf{i}+\mathbf{j}-\boldsymbol{\epsilon})$. This, similarly as in Sect. 6.2.15 reduces to the existence of a morphism of $\mathscr{O}_{\widetilde{X}}$ modules $\mathscr{O}_{\widetilde{X}} \to \mathscr{O}_{\widetilde{X}}(\sum_k \epsilon_kr_kD_k - \mathrm{div}(\sum_k \epsilon_kr_kl'_k))$. This is given by multiplication by $\prod_k f_k^{-\epsilon_k}$. Note that $\mathscr{O}(\mathbf{0}) = \mathscr{O}_{\widetilde{X}}$, hence $\oplus_{\mathbf{i}}\mathscr{O}(\mathbf{i})$ is an $\mathscr{O}_{\widetilde{X}}$ module.

In this way, there is a possibility to recover the universal abelian cover (X_a, o) from the natural line bundles. Indeed, one can take (the $\mathscr{O}_{X,o}$-module) $\mathscr{R}_a := \oplus_{l' \in Q} (\phi_*(\mathscr{O}_{\widetilde{X}}(-l')))_o$, (with the module and ring structure just defined) and $(X_a, o) = \mathrm{Spec}_{X,o}\mathscr{R}_a$. For more details, see [530].

6.3 The Canonical Cycle

We list several combinatorial properties of the canonical cycle associated with a resolution. In 6.3.A we provide a splice formula and several combinatorial expressions for the topological (resolution independent) invariant $Z_K^2 + |\mathscr{V}|$. ♣

6.3.1 Fix a resolution $\phi : \widetilde{X} \to X$ of a normal surface singularity (X, o). Let $K_{\widetilde{X}}$ be a *canonical divisor* of $\widetilde{X}$ (defined up to a linear equivalence), $\mathscr{O}_{\widetilde{X}}(K_{\widetilde{X}}) = \Omega^2_{\widetilde{X}}$, and let $K = -Z_K$ be $c_1(\Omega^2_{\widetilde{X}}) \in L'$, the *(anti)canonical cycle of the resolution* ϕ.

Z_K can be determined combinatorially from $(L', (\ ,\ ))$ via the adjunction formula 12.4.8, namely $(-Z_K + E_v, E_v) + 2\chi(E_v) = 0$ for all $v \in \mathscr{V}$. That is,

$$-Z_K + E = \sum_v (2\chi(E_v) - \kappa_v)E_v^*, \tag{6.3.1}$$

where $\kappa_v = (E_v, E - E_v)$ is the valency of the vertex v in Γ. The advantage of the (6.3.1) version is that the E_w-coefficients $(E - Z_K, E_w^*)$ of $Z_K - E$ can be expressed as $\sum_v (2\chi(E_v) - \kappa_v)(E_v^*, E_w^*)$, where each (E_v^*, E_w^*) can be read from the graph as a determinant of a certain subgraph, whenever Γ is a tree, cf. 2.2.5(5).

6.3.2 In algebraic geometry in the $\mathbb{Q}$-Cartier situation (when $K_{\widetilde{X}}$ in $\mathrm{Cl}(X, o) = \mathrm{Pic}(\widetilde{X})/L$ has finite order), and for a fixed resolution, one defines the *discrepancy* of an exceptional irreducible component, see e.g. [331]. Note that above we defined the rational cycle $K = -Z_K$ combinatorially without any kind of analytic assumption. Under the $\mathbb{Q}$-Cartier assumption the E_v-coefficients of $K \in L'$ are the discrepancies of $\{E_v\}_v$. Therefore, as a generalization to arbitrary germs, we can think about the coefficients of K as the *numerical discrepancies of* $\{E_v\}_v$. See also 6.3.33.

6.3.3 *Examples*

(1) If the resolution is minimal then $Z_K \in \mathscr{S}'$, cf. 2.1.10.
(2) If $(Z_K, E_v) = 0$ for some v, then necessarily $E_v \approx \mathbb{P}^1$ and $E_v^2 = -2$.
(3) In particular, if $Z_K = 0$ then $E_v \approx \mathbb{P}^1$ and $E_v^2 = -2$ for every v. In this case one shows that Γ is an ADE (negative definite) graph. (For their form see 7.1.5(c).)
(4) Assume that the resolution $\widetilde{X}'$ is obtained from the resolution $\widetilde{X}$ by the modification π, the blowing up of a point q. Set $\pi^{-1}(q) = E_{new}$. Then the canonical cycles Z'_K and Z_K satisfy $\pi_*(Z'_K) = Z_K$ and $Z'_K = \pi^*(Z_K) - E_{new}$.
(5) $Z_K^2 + |\mathscr{V}|$ is independent of the resolution, it depends only on L_X (use (4)).
(6) [367] If the resolution is minimal, and $Z_K \neq 0$, then all the coefficients of Z_K are positive. (Use (1) and 2.1.19.)

Example 6.3.4 (Z_K in a Minimal Good Resolution) We will improve the above statement 6.3.3(6) as follows. Assume that (X, o) is not of type ADE, and $\widetilde{X}$ is a *minimal good resolution*. Then we claim that all the coefficients of Z_K are positive. (For some other specific properties regarding the coefficients of Z_K see [579, 716].)

The proof runs as follows. We start with the minimal resolution. At that step we know that all the coefficients of Z_K are strictly positive, cf. 6.3.3(6). Then the minimal good resolution is obtained by consecutive blow ups of smooth points. We show that at each step the newly created divisor still has a positive coefficient. Indeed, let $\widetilde{X}$ be such an intermediate resolution and $\pi : \widetilde{X}' \to \widetilde{X}$ the blow up. Let $\{C_j\}_j$ be the set of exceptional curves in $\widetilde{X}$ which contain p with multiplicity $m_p(C_j)$ at p. Let g_j and δ_j be the geometric genus and the delta-invariant of C_j. Write Z_K of $\widetilde{X}$ as $\sum_{j\in J} z_j C_j + D$, where D is effective supported on the other exceptional curves. Then, by adjunction formula, for any $i \in J$ one has $C_i^2 + 2 - 2g_i - 2\delta_i = (Z_K, C_i) = \sum_j z_j (C_j, C_i) + p_i$ for some $p_i \geq 0$. Or, $\sum_j (z_j - 1)(C_j, C_i) = a_i$, where $a_i := 2 - 2g_i - 2\delta_i - p_i - (C_i, \cup_{j\neq i} C_j)$. Note that $a_i \leq 0$ in any situation when such a blow up is needed. Since the intersection matrix $(C_i, C_j)_{i,j}$ is negative definite, $z_j - 1 \geq 0$ for every j by 2.1.19.

Finally, since $Z'_K = \pi^*(Z_K) - E_{new}$, the coefficient z'_{new} of Z'_K along the newly created exceptional curve is $z'_{new} = \sum_j z_j m_p(C_j) - 1 \geq \sum_j m_p(C_j) - 1 > 0$.

Example 6.3.5 (Newton Non-degenerate Case) Assume that $f : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ is a Newton non-degenerate hypersurface singularity, cf. 5.5. Consider the canonical resolution graph given by Oka's algorithm [524, Th. 6.1] (see 5.5.23 here). Recall that to each vertex $v \in \mathcal{V}$ one defines a vector $\ell_v \in N$. Then, for any non-empty set S and $v \in \mathcal{V}$ one defines $\mathrm{wt}_v(S) := \min_{p\in S}\langle l_v, p\rangle$, and $\mathrm{wt}(S) := \sum_{v\in\mathcal{V}} \mathrm{wt}_v(S) E_v \in L$. Furthermore, for any $h \in \mathbb{C}\{x_1, x_2, x_3\}$ we also write $\mathrm{wt}(h) = \mathrm{wt}(\mathrm{supp}(h))$.

Then one has the following combinatorial expression for Z_K, cf. [524, Th. 9.1]:

$$Z_K - E = \mathrm{wt}(f) - \mathrm{wt}(x_1x_2x_3).$$

Let $(X, 0) \subset (Y, 0)$ be a Newton non-degenerate Weil divisor as in 5.5.B. Let us define $\mathrm{wt}(f)$ similarly as above. Then we have the following identity in L, cf. [499]:

$$Z_K - E = \mathrm{wt}(f) - \sum (m_n + 1) E_v^*, \tag{6.3.2}$$

where the sum runs over edges (n, v) of Γ^{ext} with $n \in \mathcal{V}(\Gamma^{ext}) \setminus \mathcal{V}(\Gamma)$ and $v \in \mathcal{V}(\Gamma)$, and $m_n = \mathrm{wt}_n(\mathrm{supp}(f))$ (which equals m_σ for the corresponding non-compact face of $\partial N\Gamma_+(f)$, cf. 5.5.B).

Theorem 6.3.6 (Riemann–Roch Formula) *Fix a line bundle $\mathscr{L} \in Pic(\widetilde{X})$ and set $c_1(\mathscr{L}) = l' \in L'$ and $k := -Z_K - 2l'$. For any $l \in L_{>0}$ we consider the sheaf $\mathscr{L} \otimes \mathscr{O}_l$ on l. Then its analytic Euler characteristic satisfies*

$$\chi(\mathscr{L} \otimes \mathscr{O}_l) = -(l, l + k)/2. \tag{6.3.3}$$

Proof Use decreasing induction on l. If $l > E_v$, then use as inductive step the exact sequence $0 \to \mathscr{O}_{E_v}(-l + E_v) \to \mathscr{O}_l \to \mathscr{O}_{l-E_v} \to 0$ (tensored with $\mathscr{L}$); while for $l = E_v$ the classical Riemann–Roch theorem for irreducible curves. □

We denote the combinatorial term from the right hand side of (6.3.3) by $\chi_k(l)$, or just by $\chi(l)$ if $k = -Z_K$. This expression motivates the following.

Definition 6.3.7 The set of characteristic elements are defined as

$$\mathrm{Char} = \mathrm{Char}(L) = \{k \in L' : (l, l + k) \in 2\mathbb{Z} \text{ for any } l \in L\}. \tag{6.3.4}$$

Note that $-Z_K$ is a characteristic element and $\mathrm{Char} = -Z_K + 2L'$.

The expression (6.3.3) can be extended to L', that is, for any $k \in \mathrm{Char}$ one defines $\chi_k : L' \to \mathbb{Q}$ by $\chi_k(l') := -(l', l' + k)/2$. If $k = -Z_K$ then we write $\chi := \chi_k$.

Remark 6.3.8 The real quadratic form $\chi_{\mathbb{R}}(x) := -(x, x - Z_K)/2$, $x \in L \otimes \mathbb{R}$, provides the following inequality:

$$\min_{l \in L} \chi(l) \geq \min_{x \in L \otimes \mathbb{R}} \chi_{\mathbb{R}}(x) = \chi_{\mathbb{R}}(Z_K/2) = Z_K^2/8.$$

Example 6.3.9 Assume that $f : (X, o) \to (\mathbb{C}, 0)$ defines an isolated curve singularity $(C, o) := (\{f = 0\}, o)$. Denote by r the number of irreducible components of C. Fix a good embedded resolution of $(C, o) \subset (X, o)$ (see 2.1.8 and 4.1.22). Then 6.3.1 and A'Campo's formula 4.1.22 imply $(\mu(f) + r - 1)/2 = \chi(-\mathrm{div}_E(f))$.

On the other hand, if (C, o) is any reduced abstract curve which admits a smoothing (say f), with Milnor number $\mu(f)$, then by a formula of Buchweitz and Greuel [90] (see also 5.2.2 here) $\delta(C) = (\mu(f) + r - 1)/2$. Hence, for an isolated plane curve singularity the abstract (non-embedded) analytic invariant $\delta(C)$ of C, the embedded topological invariant $(\mu(f) + r - 1)/2$ and the resolution (topological/combinatorial) invariant $\chi(-\mathrm{div}_E(f))$ agree.

The identity $\delta(C) = \chi(-\mathrm{div}_E(f))$ is typical for curve-germs cut out by holomorphic functions on normal surface singularities, i.e. for Cartier divisors. However, without the Cartier assumption the corresponding identity might fail. Indeed, consider an irreducible curve germ $(C, o) \subset (X, o)$, an embedded resolution ϕ, let $S(C)$ be the strict transform of C and $l'(C) \in L'$ defined in such a way that $(l'(C) + S(C), E_v) = 0$ for all v. Then, even if $l'(C) \in L$, the identity $\delta(C) = \chi(-l'(C))$ is not necessarily true. (E.g., let (X, o) be given by $x^2 + y^3 + z^7 = 0$, and $\{z = 0\}$ has $\delta = 1$, and its strict transform in the minimal good resolution is supported (as a special cut) on the -7 curve. However, the generic cut of the -7 curve projects into a smooth curve in (X, o) [481].) For more, see [116, 117].

6.3.A $Z_K^2 + |\mathscr{V}|$ *Formulae*

The invariant $Z_K^2 + |\mathscr{V}|$ of the link behaves like a characteristic class in many index formulae. Using (6.3.1) and $b_1(L_X) = c(\Gamma) + 2g(\Gamma)$ one obtains the following.

Proposition 6.3.10 ([489]) *$Z_K^2 + |\mathscr{V}|$ in terms of the graph has the expression*

$$Z_K^2+|\mathscr{V}| = 2-2b_1(L_X)+\sum_v (E_v^2+3)+\sum_{v,w}(2\chi(E_v)-\kappa_v)(2\chi(E_w)-\kappa_w)(E_v^*, E_w^*).$$

Example 6.3.11 ([273]) Using 2.2.5(5) and (2.2.7), for the cyclic quotient singularity $X_{n,q}$ one gets

$$Z_K^2 + |\mathscr{V}| = 2(n-1)/n - 12 \cdot \mathbf{s}(q, n).$$

Example 6.3.12 ([490]) For a star-shaped graph, with the notation of Sect. 5.1, one has

$$Z_K^2 + |\mathscr{V}| = e\mathfrak{r}^2 + e + 5 - 12 \cdot \sum_{j=1}^{v} \mathbf{s}(\omega_j, \alpha_j).$$

For the proof use (6.3.10), the identities (3.5.2), and (2.2.7).

Example 6.3.13 Consider the surgery 3-manifold $S^3_{-d}(\#_i K_i)$ (cf. 3.5.12), with $\mu = 2\delta$, $\delta = \sum_i \delta_i$ (the sum of delta-invariants of K_i). Then, associated with the minimal resolution, one has an irreducible exceptional curve E with $E^2 = -d$ and $\delta(E) = \delta$. Hence by adjunction formula $Z_K = ((d-2+\mu)/d)E$ and $Z_K^2 + |\mathscr{V}|$ (valid in any resolution) is $1 - (d-2+\mu)^2/d$. If $\mu = (d-1)(d-2)$ (as in the superisolated singularity case), then $Z_K^2 + |\mathscr{V}| = 1 - d(d-2)^2$, see also 9.7.B.

For the general case of surgery 3-manifolds see 11.3.23 and 11.3.25.

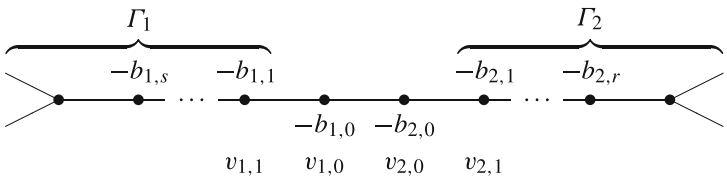
$Z_K^2 + |\mathscr{V}|$ for Brieskorn singularities or for suspensions can be computed via the formulae of Durfee and Laufer from 6.9, for some details see 6.9.14 and 6.9.15.

6.3.14 Splice Formula Assume that $\det(\Gamma) = 1$ and let $\mathfrak{G}$ be the splice diagram associated with the plumbing graph Γ (cf. 5.3.3). Assume that $\mathfrak{G}$ is obtained by splicing the diagrams $\mathfrak{G}_1$ and $\mathfrak{G}_2$ along the knots $K_1 \subset M(\mathfrak{G}_1)$, $K_2 \subset M(\mathfrak{G}_2)$ (cf. 5.3.7). Let Γ_i be the plumbing graphs, which correspond to $\mathfrak{G}_i$ by the correspondence 5.3.10. Recall also that $K_i \subset M(\mathfrak{G}_i)$ determines an open book decomposition, let μ_i be the first Betti number (Milnor number) of its fiber. Then one has the following.

Theorem 6.3.15 ([516])

$$\big(Z_K^2 + |\mathscr{V}|\big)(\Gamma) = \big(Z_K^2 + |\mathscr{V}|\big)(\Gamma_1) + \big(Z_K^2 + |\mathscr{V}|\big)(\Gamma_2) - 2 \cdot \mu_1 \cdot \mu_2.$$

Proof Consider the following graph $\widetilde{\Gamma}$ from the construction 5.3.10. The graph Γ is obtained from $\widetilde{\Gamma}$ by several (-1)-blow downs and one 0-chain absorption. Analysing the effect of these steps in the right hand side of the formula from 6.3.10 we obtain that $(Z_K^2 + |\mathscr{V}|)(\Gamma) = (Z_K^2 + |\mathscr{V}|)(\widetilde{\Gamma}) - 6$. On the other hand, if we delete from $\widetilde{\Gamma}$ the vertices $v_{1,0}$ and $v_{2,0}$ and adjacent edges, we get the graphs Γ_1 and Γ_2.



By A'Campo's theorem 4.1.22 in Γ_1 one has $\sum_{w \neq v_{1,1}} (2 - \kappa_w)(E_w^*, E_{v_{1,1}}^*) = \mu_1 - 1$, and also $(E_{v_{1,1}}^*, E_{v_{1,1}}^*) = -b_{1,0}$, cf. 5.3.12. One has similar facts for Γ_2 too. Furthermore, if $v \neq v_{1,1}$ is a node or end-vertex in Γ_1, and $u \neq v_{2,1}$ is a node or end-vertex in Γ_2, then by 5.3.17 the expression $-(E_v^*, E_u^*)$ computed in $\widetilde{\Gamma}$ equals the product of $(E_v^*, E_{v_{1,1}}^*)$ and $(E_u^*, E_{v_{2,1}}^*)$ (these last ones computed in Γ_1 and Γ_2 respectively). Then, the statement follows by the comparison of the expression from 6.3.10 applied for $\widetilde{\Gamma}$, Γ_1 and Γ_2 and the above identities (see also 5.3.13 and 5.3.17). □

For a more general formula valid for rational homology spheres see 9.3.11.

6.3.B *The Gorenstein Property*

We introduce the notion of Gorenstein and numerically Gorenstein singularities. Several criteria and examples are listed. ♣

Definition 6.3.16 The normal singularity (X, o) is called *Gorenstein* if $\Omega^2_{X\setminus\{o\}}$ is a holomorphically trivial line bundle, equivalently, if $Z_K \in L$ and one can choose $K_{\widetilde{X}} = -\mathrm{div}(Z_K)$. Analogously, (X, o) is called *numerically Gorenstein* if $\Omega^2_{X\setminus\{o\}}$ is a topologically trivial complex line bundle.

Though Gorenstein (local) rings can be defined even without the normality assumption, see e.g. [89], (e.g. complete intersections are Gorenstein even if they are not normal) in this book we discuss the Gorenstein property only for normal germs.

Example 6.3.17 Any isolated complete intersection $(f_1, \ldots, f_k) : (\mathbb{C}^{k+2}, o) \to (\mathbb{C}^k, o)$ is Gorenstein. Indeed, $\Omega^2_{X\setminus\{o\}}$ is trivialized by the restriction to $X \setminus \{o\}$ of the 2-form (on different charts given by)

$$\Big\{dx_i \wedge dx_j / \text{Jacobian determinant of } \frac{\partial (f_l)_{1\le l\le k}}{\partial (x_l)_{l\neq i,j}}\Big\}_{1\le i<j\le k+2}.$$

Lemma 6.3.18 ([162]) *(X, o) is numerically Gorenstein if and only if $Z_K \in L$.*

Proof $\Omega^2_{X\setminus\{o\}}$ is a topologically trivial complex line bundle if and only if its Chern class in $H^2(X \setminus \{o\}) \simeq L'/L$ is zero; see also the next remark. □

Remark 6.3.19 ([162])

(a) Let ξ be a rank two complex vector bundle over a CW-complex L with $H^i(L, \mathbb{Z}) = 0$ for $i > 3$. Then the following conditions are equivalent:

  (i) ξ is topologically trivial (as a complex bundle);
  (ii) the first Chern class $c_1(\xi) \in H^2(L, \mathbb{Z})$ is trivial;
  (iii) the second exterior power $\Lambda^2\xi$ is a topologically trivial complex line bundle.

  Let us check (ii)⇒(i). Isomorphism classes of rank two bundles over L are classified by homotopy classes $[L, BU_2]$. Since $H^i(L) = 0$ for $i > 3$, and $\pi_1(BU_2) = \pi_3(BU_2) = 0$, while $\pi_2(BU_2) = \mathbb{Z}$, $[L, BU_2]$ is identified with $H^2(L, \mathbb{Z})$ via c_1.

(b) In particular, the following conditions are equivalent: (i) (X, o) is numerically Gorenstein, (ii) $\Omega^1_{X\setminus\{o\}}$ (or $\Omega^1_{\widetilde{X}\setminus E}$) is topologically trivial complex vector bundle.

Remark 6.3.20

(a) The *real* tangent bundle of $X \setminus \{o\}$ as real vector bundle is always trivial (even if (X, o) is not numerically Gorenstein). Indeed, $X \setminus \{o\}$ is C^∞ diffeomorphic with $L_X \times (0, 1)$, and the tangent bundle of any orientable 3-manifold is trivial.
(b) On the other hand, the holomorphic triviality of the complex tangent bundle of $X \setminus \{o\}$ is a much harder problem: the famous conjecture of Zariski and Lipman predicts that the holomorphic tangent bundle $T_{X\setminus\{o\}}$ is holomorphically trivial if and only if (X, o) is smooth [388].
(c) Any numerically Gorenstein topological type carries a Gorenstein analytic structure [579].

Example 6.3.21 If $\det(I) = \pm 1$ then (X, o) is numerically Gorenstein.

Lemma 6.3.22

(a) Any o-ramified covering of a numerically Gorenstein singularity is numerically Gorenstein.
(b) Any singularity (X, o) has an o-ramified cyclic $\mathbb{Z}_r$ covering which is numerically Gorenstein. One can choose for r the order of the class of Z_K in H.
(c) The topological canonical covering and the universal abelian covering of a singularity with rational homology sphere link are numerically Gorenstein.

Proof

(a) Use the general Riemann–Hurwitz formula mentioned after 6.1.14.

(b)–(c) Consider a covering from 6.1.11 associated with $l' = Z_K$ (it is not unique if $H^1(L_X, \mathbb{Z}) \neq 0$). Since the pullback of Z_K is integral (cf. 6.1.17), by Riemann–Hurwitz formula 6.1.14 $Z_{\tilde{X}_D}$ is integral too. Next, the universal abelian covering factorizes through the cyclic covering from *(b)*, then use *(a)*. □

The following problem arises naturally.

Question 6.3.23 Can one classify/characterize the graphs of universal abelian coverings of rational homology sphere (connected, negative definite) plumbed 3-manifolds?

By the above lemma they are necessarily numerically Gorenstein.

6.3.24 $\mathbb{Q}$-Gorenstein Singularities Let K_X be the canonical divisor of (X, o), cf. 12.4. Note that (X, o) is Gorenstein if and only if K_X is Cartier (invertible) at $o \in X$, that is, K_X is zero in $\mathrm{Cl}(X, o)$. Furthermore, if (X, o) is Gorenstein then any o-ramified covering (X', o) of (X, o) is Gorenstein (pull back the nowhere vanishing 2-form, or use the Riemann–Hurwitz formula 6.1.14).

More generally, (X, o) is called $\mathbb{Q}$-Gorenstein, if there exists a positive integer r such that rK_X is a Cartier divisor at o (equivalently, if K_X has finite order in $\mathrm{Cl}(X, o)$). This happens if there exists an r-ple holomorphic 2-form, nowhere vanishing on $X \setminus \{o\}$. The smallest r with this property is called the *index of* (X, o). Again, if (X, o) is $\mathbb{Q}$-Gorenstein then any o-ramified covering (X', o) of (X, o) is $\mathbb{Q}$-Gorenstein (by the same argument as above).

If $H_1(L_X, \mathbb{Z})$ is finite, then

$$\text{any numerically Gorenstein,}\mathbb{Q}\text{-Gorenstein singularity is Gorenstein.} \tag{6.3.5}$$

Indeed, in (6.1.2), $\bar{c}_1(K_X) = 0$ and $rK_X = 0$ in $\mathrm{Cl}(X, o)$, hence $K_X = 0$ since $\mathbb{C}^{p_g}$ is torsion free. In fact, if $H_1(L_X, \mathbb{Z})$ is finite and (X, o) is $\mathbb{Q}$-Gorenstein then by 6.2.4 the index of (X, o) equals the order of $[Z_K]$ in H (otherwise it can be larger, it is a multiple of it).

More generally, one has the following.

Proposition 6.3.25 *If $H_1(L_X, \mathbb{Z})$ is finite then the following facts are equivalent:*

(a) *(X, o) is $\mathbb{Q}$-Gorenstein;*
(b) *$\Omega^2_{\tilde{X}} = \mathcal{O}_{\tilde{X}}(K_{\tilde{X}})$ is a natural line bundle (which automatically should agree with $\mathcal{O}_{\tilde{X}}(-Z_K)$);*
(c) *in Lemma 6.1.16 one can choose $K_{\tilde{X}}$ for $D = D(-Z_K)$ (hence $\phi_*(D) = K_X$);*
(d) *the topological canonical covering of (X, o) is Gorenstein;*
(e) *(X, o) has an o-ramified abelian covering, which is Gorenstein;*
(f) *the universal abelian covering of (X, o) is Gorenstein.*

Proof For $(a) \Leftrightarrow (b)$ use 6.2.4. $(a) \Leftrightarrow (c)$ follows from the definitions.

$(a) \Rightarrow (d)$ The topological canonical covering is numerically Gorenstein (cf. 6.3.22), and $\mathbb{Q}$-Gorenstein (since (X, o) is $\mathbb{Q}$-Gorenstein), hence Gorenstein by (6.3.5). $(d) \Rightarrow (e)$ is clear. $(e) \Rightarrow (f)$ uses the fact that any abelian covering is dominated by the universal abelian covering. Finally we prove $(f) \Rightarrow (b)$.

We will use the notations from 6.2.5. By the Riemann–Hurwitz $\widetilde{c}^*(K_{\widetilde{X}}) = -\widetilde{c}^*(E_{\widetilde{X}}) + K_Z + E_Z$ in $\mathrm{Pic}(Z)$. But in $\mathrm{Pic}(Z)$, K_Z can be replaced by the corresponding canonical cycle. Hence one has the linear equivalence $\widetilde{c}^*(K_{\widetilde{X}}) \sim \widetilde{c}^*(-Z_K)$. But one of the eigenspaces of $\widetilde{c}_*\mathscr{O}_Z(\widetilde{c}^*(K_{\widetilde{X}}))$ is $\mathscr{O}_{\widetilde{X}}(K_{\widetilde{X}})$, while all the eigenspaces of $\widetilde{c}_*\mathscr{O}_Z(\widetilde{c}^*(-Z_K))$ are natural, cf. (6.2.5). (Compare also with 6.8.20.) □

Example 6.3.26 Any rational singularity is $\mathbb{Q}$-Gorenstein.

Example 6.3.27 Any numerically Gorenstein splice quotient singularity is Gorenstein. Indeed, a splice quotient singularity is $\mathbb{Q}$-Gorenstein since its universal abelian covering is a complete intersection, cf. 5.4.12. Then use (6.3.5).

Remark 6.3.28

(a) If $H_1(L_X, \mathbb{Z})$ is not finite, then it can happen that (X, o) is numerically Gorenstein, it is $\mathbb{Q}$-Gorenstein, but it is not Gorenstein. (For an example combine Theorem 5.13 and Example 5.20 of [531].)
(b) By [579] any topological type admits a $\mathbb{Q}$-Gorenstein analytic structure.

Example 6.3.29 Assume that (X, o) **is a cone singularity**. Then, using the notations of 5.1.3 one has:

(a) [154, 748] (X, o) is Gorenstein if and only if there exists an integer n such that $\mathscr{L}^{-n} = K_C$. In such a case, by taking degrees, n is uniquely determined by $n \cdot C^2 = 2 - 2g(C)$. Therefore, n necessarily agrees with $\mathfrak{r} = \chi/e$ defined in 5.1.29.
(b) (X, o) is $\mathbb{Q}$-Gorenstein if and only if there exists $r'/r \in \mathbb{Q}$ such that $\mathscr{L}^{-r'} = K_C^r$. In this case necessarily $r'/r = \mathfrak{r} = \chi/e$ (by the same argument).

Indeed, let U be the complement of C in the total space of $\mathscr{L}$, and let $\pi : U \to C$ be the projection. Then, by 5.1.4(b) $\pi^* K_C = K_U$, hence $\pi_* K_U = K_C \otimes \pi_* \mathscr{O}_U = K_C \otimes (\oplus_{i \in \mathbb{Z}} \mathscr{L}^i)$. The Gorenstein property is equivalent to $K_U = \mathscr{O}_U$, that is, $K_C \otimes (\oplus_{i \in \mathbb{Z}} \mathscr{L}^i) = \oplus_{i \in \mathbb{Z}} \mathscr{L}^i$. In the $\mathbb{Q}$-Gorenstein case $\pi^* K_C^r = K_U^r = \mathscr{O}_U$, hence $K_C^r \otimes (\oplus_{i \in \mathbb{Z}} \mathscr{L}^i)$ should equal $\oplus_{i \in \mathbb{Z}} \mathscr{L}^i$ (compatibly with the weight decomposition).

Example 6.3.30 (Weighted Homogeneous Case)

(a) If a good resolution graph Γ is star-shaped then (via (6.3.1) and (3.5.2))

$$-(Z_K, E_0^*) = 1 + \mathfrak{r}, \quad -(Z_K, E_{js_j}^*) = 1 + (\mathfrak{r} - \omega_j')/\alpha_j, \tag{6.3.6}$$

where $\mathfrak{r} = \chi/e$, cf. 5.1.29. In particular, $-\mathfrak{r}$ is the 'log discrepancy' of E_0. Hence, if Γ is numerically Gorenstein then $\mathfrak{r} \in \omega_j' + \alpha_j\mathbb{Z} \subset \mathbb{Z}$ for all $1 \le j \le \nu$. The converse is also true. (In fact, in general, if a resolution graph is a tree, and $l' \in L'$ has integral E-coefficients on the end-vertices, then $l \in L$, cf. 3.4.3.) In this weighted homogeneous numerically Gorenstein case $\mathfrak{r} < 0$ can happen only in the ADE cases (cf. 5.1.21), when $Z_K = 0$ and $\mathfrak{r} = -1$. The identity $\mathfrak{r} = 0$ cannot happen by the above congruence, hence in any non-ADE case $\mathfrak{r} \ge 1$.

(b) [154] If (X, o) is weighted homogeneous, numerically Gorenstein, and it has $g = 0$, then (X, o) is Gorenstein. Indeed, consider the special analytic gluing 5.1.36. Then, if in the chart $U_{i,j}$ the cycle Z_K has multiplicity $n_{i,j}$ along $u_{i,j} = 0$ and multiplicity $m_{i,j}$ along $v_{i,j} = 0$, then the local forms $\omega_{i,j} := u_{i,j}^{n_{i,j}} v_{i,j}^{m_{i,j}} du_{i,j} \wedge dv_{i,j}$ glue together into a 2-form which has no poles and zeros along $\widetilde{X} \setminus E$. The statement follows also from 6.3.27.

(c) More generally, if (X, o) is weighted homogeneous with $g = 0$ then it is $\mathbb{Q}$-Gorenstein. Indeed, let r be the order of $[Z_K]$ in H. Then the local forms $\omega_{i,j}^{\otimes r}$ ($\omega_{i,j}$ from part (b)) glue together to a nowhere vanishing regular form on $\widetilde{X} \setminus E$. Or, differently, one can use the fact that the universal abelian covering is a complete intersection, cf. 5.1.33 and 6.3.25.

(d) Assume that $g = 0$. Recall that $h^1(\mathscr{O}_C(D^{(\ell)})) = 0$ for $\ell > \mathfrak{r}$ (and for $\ell = 0$), cf. 5.1.29. Assume next that (X, o) is Gorenstein and $p_g \neq 0$. Hence, by (a), necessarily, $\mathfrak{r} \ge 1$ and $\deg P_{H^1} \le \mathfrak{r}$ (cf. 5.1.26). With the notation of 5.1.22–5.1.26, we claim that for any ℓ with $0 \le \ell \le \mathfrak{r}$ in $\mathrm{Pic}(C)$ one has

$$D^{(\ell)} + D^{(\mathfrak{r}-\ell)} = K_C. \tag{6.3.7}$$

Indeed, using the congruences $\mathfrak{r} \equiv \omega_j'$ and $\omega_j\omega_j' \equiv 1 \pmod{\alpha_j}$, one shows that $D^{(\ell)} + D^{(\mathfrak{r}-\ell)} = \mathfrak{r}D + \sum_j \frac{1-\alpha_j}{\alpha_j} P_j$. But (via the notations of 5.1.23 and 5.1.24) $\rho^*(\mathfrak{r}D + \sum_j \frac{1-\alpha_i}{\alpha_j} P_j) = \mathfrak{r}D' - K_{C'}' + \rho^*(K_C) = \mathscr{L}'^{-\mathfrak{r}} - K_{C'}' + \rho^*(K_C) = \rho^*(K_C)$, since $\mathscr{L}'^{-\mathfrak{r}} = K_{C'}'$ by 6.3.29.

Therefore

$$P_{H^1}(t) = \sum_{0 \le \ell \le \mathfrak{r}} h^1(\mathscr{O}_C(D^{(\ell)}))t^\ell = \sum_{0 \le \ell \le \mathfrak{r}} h^0(\mathscr{O}_C(D^{(\mathfrak{r}-\ell)}))t^\ell, \tag{6.3.8}$$

which is $\sum_{\ell=0}^{\mathfrak{r}} \dim R_\ell t^{\mathfrak{r}-\ell}$, hence, up to a symmetry of coefficients, it is the $\mathfrak{r}$-truncation of $P_X(t)$. In particular, the coefficient of $t^{\mathfrak{r}}$ in $P_{H^1}(t)$ is 1.

Even if (X, o) is not necessarily weighted homogeneous, but Γ is star-shaped with $g = 0$ and numerically Gorenstein, then still (6.3.7) is valid at *degree level*: for any $\ell \in \mathbb{Z}$ one has

$$N(\ell) + N(\mathfrak{r} - \ell) = \deg D^{(\ell)} + \deg D^{(\mathfrak{r}-\ell)} = \deg K_C = 2g(C) - 2. \tag{6.3.9}$$

Example 6.3.31 Assume that (X, o) is a quotient singularity $(\mathbb{C}^2, 0)/G$, where G is a small subgroup of $\mathrm{GL}_2(\mathbb{C})$ acting linearly on $\mathbb{C}^2$ (whose coordinates are denoted by (z_1, z_2)). If $|G|$ is the order of G, then $(dz_1 \wedge dz_2)^{\otimes |G|}$ is G-invariant. Therefore, (X, o) is $\mathbb{Q}$-Gorenstein. Moreover, cf. [746], (X, o) is Gorenstein if and only if $dz_1 \wedge dz_2$ is G-invariant, hence, if and only if $G \subset \mathrm{SL}_2(\mathbb{C})$.

Example 6.3.32 Consider a **Newton non-degenerate (local analytic) Weil divisor** $(X, 0) \subset (Y, 0)$ as in 5.5.B, determined by f. We assume $\dim(X, 0) = 2$.

We say that $\Gamma_+(f)$, or f, is *Gorenstein-pointed* if there exists a $p \in M$ such that $\ell_\sigma(p) = m_\sigma + 1$ for all $\sigma \in \triangle_f^*$ with $\dim \sigma = 1$ and $\dim(F_\sigma \cap N\Gamma(f)) = 1$.

It is known that the affine toric variety $(Y, 0)$ is Gorenstein if and only if there is a $p \in M$ satisfying $\ell_\sigma(p) = 1$ for all $\sigma \in \triangle_\Sigma^{(1)}$, see e.g. [88, Th. 6.32]. If $(Y, 0)$ is Gorenstein, and additionally $(X, 0)$ is Cartier, and $\triangle_f^* = \triangle_\Sigma^*$, then f is Gorenstein pointed (since $m_\sigma = 0$ for $\sigma \in \triangle_\Sigma^{(1)}$). Furthermore, $(X, 0)$ is Gorenstein since $(Y, 0)$ is Gorenstein and f forms a regular sequence.

This fact extends even for non Cartier divisors. Indeed, in [499] is proved that the following conditions are equivalent:

(a) f is Gorenstein-pointed at some $p \in M$;
(b) there exists a $p \in M$ so that $\ell_v(p) = m_v + 1$ for all $v \in \mathscr{V}(\Gamma^{ext}) \setminus \mathscr{V}(\Gamma)$;
(c) there exists a $p \in M$ so that $\ell_v(p) = m_v + 1 - m_v(Z_K)$ for all $v \in \mathscr{V}(\Gamma)$;
(d) $(X, 0)$ is Gorenstein.

(Above the vectors ℓ_v are defined by Oka's algorithm 5.5.23, and $m_v = \min_{N\Gamma_+(f)} \ell_v$.)
When these conditions hold, they uniquely identify the very same point p.

The above statements have the following extension to the $\mathbb{Q}$-Gorenstein case.

We say that $\Gamma_+(f)$, or f, is *$\mathbb{Q}$-Gorenstein-pointed* if there exists a $p \in M_{\mathbb{Q}}$ such that $\ell_\sigma(p) = m_\sigma + 1$ for all $\sigma \in \triangle_f^*$ with $\dim \sigma = 1$ and $\dim(F_\sigma \cap N\Gamma(f)) = 1$.

An affine toric variety $(Y, 0)$ is $\mathbb{Q}$-Gorenstein if and only if there is a $p \in M_{\mathbb{Q}}$ satisfying $\ell_\sigma(p) = 1$ for all $\sigma \in \triangle_\Sigma^{(1)}$, see e.g. [16].

For the following properties the following implications (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) $\Rightarrow$ (d) hold, cf. [499].

(a) f is $\mathbb{Q}$-Gorenstein-pointed at some $p \in M$;
(b) there exists a $p \in M_{\mathbb{Q}}$ so that $\ell_v(p) = m_v + 1$ for all $v \in \mathscr{V}(\Gamma^{ext}) \setminus \mathscr{V}(\Gamma)$;
(c) there exists a $p \in M_{\mathbb{Q}}$ so that $\ell_v(p) = m_v + 1 - m_v(Z_K)$ for all $v \in \mathscr{V}(\Gamma)$;
(d) $(X, 0)$ is $\mathbb{Q}$-Gorenstein.

Again (a),(b),(c) uniquely identify the very same point $p \in M_{\mathbb{Q}}$.

Nevertheless, (d) $\Rightarrow$ (a) in general does not hold. A possible example is the following. Take $N = \mathbb{Z}^3$, Σ generated by $(1,0,0)$, $(0,1,0)$, $(1,0,1)$, $(0,1,1)$, and $f(x) = x_3^2 + x_1x_3 + x_2^2 + 2x_1x_2^2x_3^{-1}$. Then f is not $\mathbb{Q}$-Gorenstein pointed, however $(X,0)$ is the cyclic quotient $X_{3,1}$.

If (X,o) is $\mathbb{Q}$-Gorenstein-pointed then the expression of $Z_K \in L'$ from (6.3.2) transforms into

$$Z_K - E = \mathrm{wt}(f) - \mathrm{wt}(p). \tag{6.3.10}$$

Example 6.3.33 (Singularities of the Minimal Model Program (MMP)) Terminal, canonical, log terminal and log canonical singularities are important in the MMP. They usually are defined under the condition that K_X is $\mathbb{Q}$-Cartier using the discrepancies of the canonical divisor, for details see e.g. [307, 331, 410]. Nevertheless, specifically in the surface case, these notions have their combinatorial analogues via the cycle $Z_K \in L'$: naturally they are called *numerically* terminal, canonical, log terminal and log canonical singularities [331, p. 56]. Here we discuss the 'numerical' case (nevertheless the three cases discussed below,—the numerically terminal, canonical and log terminal—agree with the notion of terminal, canonical and log terminal); for more see [15, 307, 331, 410] and the references therein.

Fix the minimal ***good*** resolution graph Γ and write $Z_K = \sum_v a_v E_v \in L'$ ($a_v \in \mathbb{Q}$).

(a) (X,o) is called *numerically terminal* if $a_v < 0$ for all v. By 6.3.3(1) this can happen only if $\Gamma = \emptyset$, hence in this case (X,o) should be smooth.

(b) (X,o) is called *numerically canonical* if $a_v \le 0$ for all v. This in the minimal good resolution reads as $Z_K = 0$, cf. 6.3.4, hence by 6.3.3(3) (X,o) should be ADE.

(c) (X,o) is called *numerically log terminal* if $a_v < 1$ for every v. Consider the topological canonical covering $(X_D,o) \to (X,o)$; using the notations of 6.1.11 and 6.1.14, $\widetilde{c}^*(-Z_K + E) = -Z_{K(\widetilde{X}_D)} + E_{\widetilde{X}_D}$. Since the coefficients of $-Z_K + E$ are all positive, the same is true for $-Z_{K(\widetilde{X}_D)} + E_{\widetilde{X}_D}$, too. Since $Z_{K(\widetilde{X}_D)}$ is an integral cycle we must have $Z_{K(\widetilde{X}_D)} \le 0$. Hence in the minimal resolution of $\widetilde{X}_D$ the canonical cycle is zero, and (similarly as in case (b)), the topological canonical covering is an ADE. Since the fundamental group of the link of any ADE is finite, we get that the fundamental group of the link of any numerically log terminal singularity is finite. Conversely, if the fundamental group is finite, then the topological canonical covering is ADE (see 6.3.34), hence by the above Riemann–Hurwitz formula $\widetilde{c}^*(-Z_K + E) = -Z_{K(\widetilde{X}_D)} + E_{\widetilde{X}_D} \ge E_{\widetilde{X}_D}$, hence all the entries of $-Z_K + E$ are positive. This means that numerically log terminal singularities are exactly the quotient singularities, cf. 3.7.12. For their classification see 3.7.15.

(d) (X,o) is called *numerically log canonical* if $a_v \le 1$ for every v. In 7.1.5(h) we will prove that (X,o) is either (special) rational, or $Z_K = E$. This latter case

happens only if (X, o) is either cusp or simple elliptic (hence special minimally elliptic), cf. 7.2.21. The numerically log canonical rational case include e.g. the quotient-cusps.

The argument is the following. Using again the topological canonical covering as in case (c), one has $Z_{K(\widetilde{X}_D)} \leq E_{\widetilde{X}_D}$. Then this inequality holds in the minimal resolution $\widetilde{X}_{D,min}$ too. Since $Z_{K(\widetilde{X}_{D,min})}$ is an integral cycle, the support property 6.3.3(6) shows that either it is trivial (covering the case (c)) or it is $E_{\widetilde{X}_{D,min}}$. Hence, in this case, the topological canonical covering of (X, o) is either cusp or simple elliptic. For the complete list of numerically log canonical graphs see e.g. [331, 410]. (Note that the *minimal* resolution of the $(2, 3, 7)$ Brieskorn minimally elliptic singularity has a single vertex and $Z_K = E$, nevertheless it is not numerically log canonical.)

According to [720] (X, o) is numerically log canonical if and only if the fundamental group is either finite or solvable.

Remark 6.3.34 The argument from part (c) can be replaced by the next one as well: the topological canonical covering of a numerically log terminal singularity has finite fundamental group—hence it is rational—, but it is numerically Gorenstein as well. Hence it is ADE by 7.1.5(d). This property can also be used to classify links with finite fundamental group as well, replacing the argument from 3.7.15.

6.4 Vanishing Theorems

We prove the local analogue, valid for resolutions of normal surface singularities, of the General Kodaira Vanishing Theorem (or Kawamata–Viehweg Vanishing Theorem). The GKVT is valid for smooth complex projective varieties, for different versions see [306, 325, 717] and [190, 328, 329, 332] for applications and general overview. Some versions of the local case are called Grauert–Riemenschneider's, or Laufer's, or Ramanujam's Vanishing Theorem [229, 359, 584]; see also [210, 442, 593]. In these (local) theorems the required 'additional positivity' (compared with being an element of $-\mathscr{S}_{\mathbb{Q}}$) is distributed according to the coefficients of $-Z_K$ along all the exceptional components. We will prove the Unbalanced Vanishing Theorem too, see 6.4.12. In this case the required additional positivity is reorganised differently: one requires some stronger positivity along some of the components (indexed by $\mathscr{I} \subset \mathscr{V}$) and nothing along the others. This version will be used several times later when we wish to prove that certain properties of some objects indexed by $\mathscr{V}$ depend only on the indices from a (well-chosen) subset $\mathscr{I} \subset \mathscr{V}$. ♣

6.4.1 Fix a resolution $\phi : \widetilde{X} \to X$ as above. For $l_1, l_2 \in L_{>0}$ with $l_2 > l_1$ the morphisms $H^1(\mathscr{O}_{\widetilde{X}}) \to H^1(\mathscr{O}_{l_2})$ and $H^1(\mathscr{O}_{l_2}) \to H^1(\mathscr{O}_{l_1})$ are onto, and by the 'Theorem of formal functions' $H^1(\widetilde{X}, \mathscr{O}_{\widetilde{X}}) = (R^1\phi_*\mathscr{O}_{\widetilde{X}})_o = \varprojlim H^1(\mathscr{O}_l)$ (limit over $l \in L_{>0}$). Similarly, $H^1(\widetilde{X}, \mathscr{L}) = \varprojlim H^1(\mathscr{L} \otimes \mathscr{O}_l)$ for any $\mathscr{L} \in \mathrm{Pic}(\widetilde{X})$.

Theorem 6.4.2 (h^0-Vanishing Theorem) *Fix any $\Delta \in L \otimes \mathbb{Q}$ with $\lfloor \Delta \rfloor = 0$, $l \in L_{>0}$, and $\widetilde{\mathscr{L}} \in Pic(\widetilde{X})$ such that $(c_1(\widetilde{\mathscr{L}}) + \Delta, E_v) \leq 0$ for any $E_v \subset |l|$. Then*

$$h^0(l, \mathscr{O}_l(l) \otimes \widetilde{\mathscr{L}}) = 0.$$

Proof We proceed by induction on the coefficients of l. Since the intersection matrix is negative definite, there exists $E_v \subset |l|$ with $(E_v, l - \Delta) < 0$. (Take E_v with $(E_v, l - \Delta|_{|l|}) < 0$.) Then in the exact sequence

$$0 \to \mathscr{O}_{l-E_v}(l - E_v) \otimes \widetilde{\mathscr{L}} \to \mathscr{O}_l(l) \otimes \widetilde{\mathscr{L}} \to \mathscr{O}_{E_v}(l) \otimes \widetilde{\mathscr{L}} \to 0$$

$h^0(\widetilde{\mathscr{L}}(l)|_{E_v}) = 0$, hence the inductive step follows. □

Theorem 6.4.3 (Generalized Grauert–Riemenschneider Theorem [229, 359, 584])

Consider a line bundle $\mathscr{L} \in Pic(\widetilde{X})$ such that $c_1(\mathscr{L}(-K_{\widetilde{X}})) \in \Delta - \mathscr{S}_{\mathbb{Q}}$ for some $\Delta \in L \otimes \mathbb{Q}$ with $\lfloor \Delta \rfloor = 0$. Then for any $l \in L_{>0}$ one has the vanishing $h^1(l, \mathscr{L}|_l) = 0$. In particular, $h^1(\widetilde{X}, \mathscr{L}) = 0$.

Proof By Serre duality $h^1(l, \mathscr{L}|_l) = h^0(l, \mathscr{O}_l(l + K_{\widetilde{X}}) \otimes \mathscr{L}^{-1})$, hence we need

$$h^0(l, \mathscr{O}_l(l) \otimes \widetilde{\mathscr{L}}) = 0 \text{ whenever } l \in L_{>0} \text{ and } c_1(\widetilde{\mathscr{L}}) \in -\Delta + \mathscr{S}_{\mathbb{Q}}. \tag{6.4.1}$$

This follows from h^0-vanishing Theorem 6.4.2. □

Corollary 6.4.4 *Fix $\mathscr{L} \in Pic(\widetilde{X})$ and assume that $l \in L_{>0}$ satisfies $l \in \mathscr{S}' + Z_K + c_1(\mathscr{L})$. Then $H^1(\widetilde{X}, \mathscr{L}) = H^1(l, \mathscr{L}|_l)$.*

Proof $H^1(\widetilde{X}, \mathscr{L}(-l)) = 0$ by 6.4.3. □

Example 6.4.5 Let $(Z, o) \subset (X, o)$ be a Weil divisor on (X, o), and consider the divisorial sheaf $\mathscr{F} = j_*(\mathscr{O}_{X\setminus\{o\}}(Z|_{X\setminus\{o\}}))$ of (X, o), where j is the inclusion (cf. 12.4.2). In 12.4.2 we claim that $\mathscr{F} = \phi_*(\mathscr{O}_{\widetilde{X}}(\lfloor \phi^*(Z) \rfloor))$, where ϕ is a resolution, and $\phi^*(Z)$ is the Mumford pullback. [I.e., $\phi^*(Z) = D + S(Z)$ is numerically trivial, $D \in L'$ and $S(Z)$ is the strict transform of Z.]

Indeed, let $H^q_\infty(\widetilde{X}, \mathscr{F})$ denote the direct limit $\varinjlim H^q(\widetilde{X} \setminus \widetilde{X}_\alpha, \mathscr{F})$, where $\widetilde{X} = \cup_{\alpha \in A} \widetilde{X}_\alpha$, A is a directed set, each $\widetilde{X}_\alpha$ is compact, and $\widetilde{X}_\alpha$ is a subset of the interior of $\widetilde{X}_\beta$ whenever $\alpha < \beta$ (that is, the 'cohomology is supported near the boundary of

$\widetilde{X}$'). Then, for $\mathscr{F} = \mathscr{O}_{\widetilde{X}}(\lfloor \phi^*(Z) \rfloor)$, by the Laufer's exact sequence (cf. [359])

$$0 \to H^0(\widetilde{X}, \mathscr{F}) \xrightarrow{res} H^0_\infty(\widetilde{X}, \mathscr{F}) \to H^1_c(\widetilde{X}, \mathscr{F}).$$

By Serre duality and 6.4.3 $H^1_c(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(\lfloor \phi^*(Z) \rfloor)) = H^1(\widetilde{X}, \Omega^2_{\widetilde{X}}(-\lfloor \phi^*(Z) \rfloor)) = 0$. Hence res is an isomorphism. But it is the composition of two injective restrictions $H^0(\widetilde{X}, \mathscr{F}) \to H^0(\widetilde{X} \setminus E, \mathscr{F}) \to H^0_\infty(\widetilde{X}, \mathscr{F})$, hence the first restriction is an isomorphism as well.

6.4.6 Set $\lfloor Z_K \rfloor_+ := \max\{0, \lfloor Z_K \rfloor\}$. Note that if $\widetilde{X}$ is *minimal good* then $\lfloor Z_K \rfloor_+ = \lfloor Z_K \rfloor$, see 6.3.3(1) and 6.3.4. In general, write $\lfloor Z_K \rfloor$ as $\lfloor Z_K \rfloor_+ - \lfloor Z_K \rfloor_-$, where $\lfloor Z_K \rfloor_+$ and $\lfloor Z_K \rfloor_-$ have no common components.

Corollary 6.4.7

(a) If $\lfloor Z_K \rfloor_+ = 0$ *then* $p_g = 0$. *If* $\lfloor Z_K \rfloor_+ > 0$ *then for any* $Z \geq \lfloor Z_K \rfloor_+$, $Z \in L$, $p_g = h^1(\mathscr{O}_Z)$.

(b) If $l \in \mathscr{S}$ *and* $k \in \mathbb{Z}_{\geq 0}$ *such that* $kl + \lfloor Z_K \rfloor > 0$ *then*

$$\dim \frac{H^0(\widetilde{X}, \mathscr{O}_{\widetilde{X}})}{H^0(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(-\lfloor Z_K \rfloor - kl))} = \chi(\lfloor Z_K \rfloor + kl) + p_g. \tag{6.4.2}$$

Furthermore, for $l \in \mathscr{S} \setminus \{0\}$ *and* $k \gg 0$,

$$\dim \frac{H^0(\widetilde{X}, \mathscr{O}_{\widetilde{X}})}{H^0(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(-kl))} = -\frac{k^2 l^2}{2} + \textit{lower order terms in } k.$$

Proof

(a) We wish to show that $h^1(\mathscr{O}_{\widetilde{X}}(-\lfloor Z_K \rfloor_+)) = 0$. If $\lfloor Z_K \rfloor_- = 0$ then this follows from Theorem 6.4.3. Or else, consider the exact sequence $0 \to \mathscr{O}_{\widetilde{X}}(-\lfloor Z_K \rfloor_+) \to \mathscr{O}_{\widetilde{X}}(-\lfloor Z_K \rfloor) \to \mathscr{O}_{\lfloor Z_K \rfloor_-}(-\lfloor Z_K \rfloor) \to 0$. Then $h^1(\mathscr{O}_{\widetilde{X}}(-\lfloor Z_K \rfloor)) = 0$ by 6.4.3 and also $h^0(\mathscr{O}_{\lfloor Z_K \rfloor_-}(-\lfloor Z_K \rfloor)) = h^0(\mathscr{O}_{\lfloor Z_K \rfloor_-}(-\lfloor Z_K \rfloor_+ + \lfloor Z_K \rfloor_-)) = 0$ by 6.4.2.

(b) By 6.4.3 $h^1(\mathscr{O}_{\widetilde{X}}(-\lfloor Z_K \rfloor - kl) = 0$, hence (6.4.2) follows. By 2.1.19 $k_0 l \geq \lfloor Z_K \rfloor$ for some $k_0 > 0$, hence $\mathscr{O}_{\widetilde{X}}(-\lfloor Z_K \rfloor - kl) \subset \mathscr{O}_{\widetilde{X}}(-kl) \subset \mathscr{O}_{\widetilde{X}}(-\lfloor Z_K \rfloor - (k - k_0)l)$.

□

Remark 6.4.8 In 7.1.5(f) we will provide another proof of the vanishing $p_g = 0$ for the case $Z_K < E$ (compare also with 6.3.33(c)).

Example 6.4.9 In a minimal resolution of a cyclic quotient singularity $Z_K = E - E^*_{e_1} - E^*_{e_2}$ (see also (6.3.1)), where e_1, e_2 are the end vertices. Therefore, $\lfloor Z_K \rfloor = 0$ (this follows from 6.3.33(c) too). In particular, $\mathscr{O}_{\widetilde{X}}(-nE)(Z_K - \Delta) \in -\mathscr{S}'$ with $\Delta = Z_K$ and $\lfloor \Delta \rfloor = 0$. Hence, by Grauert–Riemenschneider Theorem $H^1(\mathscr{O}_E(-nE)) = 0$ for $n \geq 1$. Then, by induction, $H^1(\mathscr{O}_{nE}) = 0$ for $n \geq 1$, hence $h^1(\mathscr{O}_{\widetilde{X}}) = 0$.

For certain cycles the Grauert-Riemenschneider Theorem 6.4.3 can be improved.

Proposition 6.4.10 (Lipman's Vanishing Theorem [389, Theorem 11.1]) *Take $l \in L_{>0}$ with $h^1(\mathscr{O}_l) = 0$ and $\mathscr{L} \in Pic(\widetilde{X})$ for which $(c_1\mathscr{L}, E_v) \geq 0$ for any E_v in the support of l. Then $h^1(l, \mathscr{L}) = 0$.*

Proof Fix any $\bar{l}$ with $0 < \bar{l} \leq l$. Then $h^1(\mathscr{O}_{\bar{l}}) = 0$ too. Moreover, there exists $E_v \subset |\bar{l}|$ such that $(E_v, \bar{l} - Z_K) < 0$. Indeed, $(E_v, \bar{l} - Z_K) \geq 0$ for any v would imply $1 \leq h^0(\mathscr{O}_{\bar{l}}) = \chi(\bar{l}) = -(\bar{l}, \bar{l} - Z_K)/2 \leq 0$, which is a contradiction. Then, from the exact sequence $0 \to \mathscr{L}|_{E_v}(-\bar{l} + E_v) \to \mathscr{L}|_{\bar{l}} \to \mathscr{L}|_{\bar{l}-E_v} \to 0$ one gets $h^1(\mathscr{L}|_{\bar{l}}) = h^1(\mathscr{L}|_{\bar{l}-E_v})$. Hence, by induction, $h^1(\mathscr{L}|_l) = 0$. □

Remark 6.4.11 The assumption $h^1(\mathscr{O}_l) = 0$ regarding $l \in L_{\geq 0}$ in Proposition 6.4.10 is necessary. Indeed, the property $c_1(\mathscr{L}) \in -\mathscr{S}'$ alone does not guarantee the vanishing $h^1(\widetilde{X}, \mathscr{L}) = 0$. [E.g., $h^1(\widetilde{X}, \mathscr{O}_{\widetilde{X}})$ is non-zero in the non-rational case, but even with $-c_1(\mathscr{L}) \in \mathscr{S}' \setminus \{0\}$ one can find counterexamples, see 7.2.74*(b)*.]

Hence an 'additional positivity' is needed.

In Generalized Grauert–Riemenschneider Theorem 6.4.3, the 'additional positivity' is distributed according to the coefficients $\{(E_v, K)\}_v$ associated with $K = -Z_K$ along all the vertices of the graph. (We might call this case 'balanced distribution'.) The next result shows that this general statement has other versions too, when the 'additional positivity' is reorganized differently. That is, it is enough to impose a (stronger) positivity along some special vertices (and nothing along the other ones),—this is the case of the 'unbalanced distribution'.

Theorem 6.4.12 (Unbalanced Vanishing Theorem—Technical Version) *Let us fix a resolution with graph Γ, and assume that we can construct a sequence of cycles $\{x_i\}_{i=0}^t$ with the following properties:*

(a) $x_i \in L_{\geq 0}$ for all i; $x_0 = 0$, and $x_t \in (Z_K + \mathscr{S}') \cap L$;
(b) for every $0 \leq i < t$ one has $x_{i+1} = x_i + E_{v(i)}$ for a certain $v(i) \in \mathscr{V}$;

Define $\mathscr{I} := \{v(i) : (E_{v(i)}, x_i) \geq 2 - 2g(E_{v(i)})\}$, and for any $u \in \mathscr{I}$ set also $M_u = \max_{\{i\,:\,v(i)=u\}}\{(E_{v(i)}, x_i) + 2g(E_{v(i)})\}$.

Then, for any $\mathscr{L} \in Pic(\widetilde{X})$ with $c_1(\mathscr{L}) \in -\mathscr{S}'$, the 'additional unbalanced positivity' $(c_1(\mathscr{L}), E_u) \geq M_u - 1$ for all $u \in \mathscr{I}$ guarantees the vanishing $h^1(\widetilde{X}, \mathscr{L}) = 0$.

Proof Take the exact sequence $0 \to \mathscr{L}(-x_{i+1}) \to \mathscr{L}(-x_i) \to \mathscr{L}(-x_i)|_{E_{v(i)}} \to 0$. Using the assumptions we get that $H^1(\mathscr{L}(-x_{i+1})) \to H^1(\mathscr{L}(-x_i))$ is onto, hence by induction, $H^1(\mathscr{L}(-x_t)) \to H^1(\mathscr{L})$ is onto too. But $H^1(\mathscr{L}(-x_t)) = 0$ by 6.4.3. □

The above theorem is really 'unbalanced' only if the following holds:

$$(c) \qquad \mathscr{I} \text{ is a proper subset of } \mathscr{V}.$$

In this case, for $u \notin \mathscr{I}$ we require only $(c_1(\mathscr{L}), E_u) \geq 0$, but for $u \in \mathscr{I}$ we impose $(c_1(\mathscr{L}), E_u) \geq M_u - 1$.

Remark 6.4.13 Note that the existence of the sequence $\{x_i\}_{i=0}^t$ satisfying all *(a)–(b)–(c)*—and the induced bounds $\{M_u\}_u$—can be determined topologically.

Example 6.4.14

(a) It can happen that such a sequence (satisfying *(a)–(b)–(c)*) does not exist. (Take e.g. the graph with one vertex, Euler number -1 and genus 1.)

(b) Nevertheless, as we will see, such sequences in many cases do exist and (minimal) sequences with minimal $\mathscr{I}$ codify important information about the analytic and topological properties of the singularity. [E.g., if Γ is a minimal string, then by a direct verification such a sequence exists even with $\mathscr{I} = \emptyset$. More generally, later, based on more tools about rational singularities, we will be able to prove that such a sequence with $\mathscr{I} = \emptyset$ exists even for any rational graph Γ, showing that Proposition 6.4.10 is a particular case of the Unbalanced Vanishing Theorem. Several similar sequences will be constructed later modelled on Laufer computation sequence of 6.6.3, see 7.1.2 for the rational case. In the rational case the needed sequence can be constructed by decreasing induction as follows. Take $x_t \in (Z_K + \mathscr{S}') \cap L$ and construct x_i from x_{i+1} by taking $E_{v(i)}$ with $(x_i, E_{v(i)}) \leq 1$, whose existence follows as in the proof of 7.3.13 part (i). For the general case see 7.3.30 and 7.3.31.]

Let us assume that Γ is not a cyclic graph (in that case the graph is minimally elliptic, and for the vanishing theorem valid in that case see 7.2.31).

Lemma 6.4.15 *Let $\mathscr{N}$ denote the set of nodes ($v \in \mathscr{N}$ iff either $\kappa_v \geq 3$ or $g_v > 0$). For each loop (rational string supported by both ends on the same node) choose a vertex v_{loop} of the loop and let $\mathscr{N}^+$ be the union of $\mathscr{N}$ with all these vertices v_{loop}.*

Fix any $s \in \mathscr{S}$. Then there exists a sequence $\{x_i\}_{i=0}^t$ with the next properties:

(a) $x_i \in L_{\geq 0}$ for all i; $x_0 = 0$, and $x_t = s$;
(b) for every $0 \leq i < t$ one has $x_{i+1} = x_i + E_{v(i)}$ for a certain $v(i) \in \mathscr{V}$;
(c) $(E_{v(i)}, x_i) \leq 1$ whenever $v(i) \notin \mathscr{N}^+$.

Hence, a sequence required by Theorem 6.4.12 with $\mathscr{I} = \mathscr{N}^+$ always exists.

We postpone the proof until 7.1.8 (in the subsection of rational singularities).

Corollary 6.4.16 (Unbalanced Vanishing Theorem—First Conceptual Version) *Let Γ and $\mathcal{N}^+$ be as in Lemma 6.4.15. Then there exists a constant M (computable from Γ) such that $h^1(\mathscr{L}) = 0$ whenever $c_1(\mathscr{L}) \in -\mathscr{S}'$ and $(c_1(\mathscr{L}), E_n) \geq M$ for all $n \in \mathcal{N}^+$.*

Lemma 6.4.15 and Corollary 6.4.16 are the prototype of more general statements, cf. 7.3.30, 7.3.31.

6.4.A The Cohomological Cycle

If $p_g > 0$ then the cohomological cycle Z_{coh} is the unique minimal element $l \in L_{>0}$ with $h^1(\mathscr{O}_l) = p_g$. A numerically Gorenstein germ is Gorenstein if and only if $Z_{coh} = Z_K$ in the minimal or minimal good resolution. ♣

6.4.17 We fix a resolution and we assume that $p_g > 0$ (hence $\lfloor Z_K \rfloor_+ > 0$, cf. 6.4.7).

Proposition 6.4.18 ([587, 4.8]) *The set $L_{p_g} := \{l \in L_{>0} : h^1(\mathscr{O}_l) = p_g\}$ has a unique minimal element, denoted by Z_{coh}, and called the cohomological cycle of the resolution. It has the property that $h^1(\mathscr{O}_l) < p_g$ for any $l \not\geq Z_{coh}$ $(l > 0)$.*

Proof Assume that $h^1(\mathscr{O}_{l_1}) = h^1(\mathscr{O}_{l_2}) = p_g$ for $l_1 \neq l_2$, $l_1, l_2 \in L_{>0}$. Set $l = \min\{l_1, l_2\}$. If $l = 0$ then there is an exact sequence $0 \to \mathscr{O}_{l_1+l_2} \to \mathscr{O}_{l_1} \oplus \mathscr{O}_{l_2} \to A \to 0$, where A has zero-dimensional support, hence $H^1(\mathscr{O}_{l_1+l_2}) \to H^1(\mathscr{O}_{l_1}) \oplus H^1(\mathscr{O}_{l_2}) = \mathbb{C}^{2p_g}$ surjective, a fact which cannot happen. Hence $l \neq 0$. Set also $\bar{l} = \max\{l_1, l_2\}$ and $a_i = l_i - l$. Consider the diagram with exact rows and columns.

$$\begin{array}{ccccccc}
 & & H^1(\mathscr{O}_{a_2}(-l_1)) & \to & H^1(\mathscr{O}_{a_2}(-l)) & \to & 0 \\
 & & \downarrow & & \downarrow & & \\
H^1(\mathscr{O}_{a_1}(-l_2)) & \to & H^1(\mathscr{O}_{\bar{l}}) & \to & H^1(\mathscr{O}_{l_2}) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
H^1(\mathscr{O}_{a_1}(-l)) & \to & H^1(\mathscr{O}_{l_1}) & \to & H^1(\mathscr{O}_{l}) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}$$

The exactness of the first row follows from the exact sequence $\mathscr{O}_{a_2}(-l-a_1) \to \mathscr{O}_{a_2}(-l) \to \mathscr{O}_{a_1 \cap a_2} \to 0$, where the support of $\mathscr{O}_{a_1 \cap a_2}$ is 0-dimensional. From the diagram one gets that $H^1(\mathscr{O}_{\bar{l}}) \to H^1(\mathscr{O}_{l_1}) \oplus H^1(\mathscr{O}_{l_2}) \to H^1(\mathscr{O}_l) \to 0$ is exact, which implies $H^1(\mathscr{O}_l) = p_g$ too. Hence, whenever $l_1, l_2 \in L_{p_g}$ one also has $\min\{l_1, l_2\} \in L_{p_g}$. This implies the proposition. □

By convention, if $p_g = 0$ then we set $Z_{coh} = 0$.

For a reinterpretation of Z_{coh} see 6.8.10 as well.

By 6.4.7 p_g is already realized by $h^1(\mathscr{O}_{\lfloor Z_K \rfloor_+})$, hence $Z_{coh} \leq \lfloor Z_K \rfloor_+$. If $Z_K \in L$ then we write $Z_{K,\pm}$ for $\lfloor Z_K \rfloor_\pm$, cf. 6.4.6.

Lemma 6.4.19 ([587, 4.20]) *If (X, o) is Gorenstein then for any $0 < l < Z_{K,+}$ $(l \in L)$ the strict inequality $h^1(\mathscr{O}_l) < h^1(\mathscr{O}_{Z_{K,+}}) = p_g$ holds. In particular, in the Gorenstein case, $Z_{coh} = Z_{K,+}$ holds.*

Proof By Serre duality, we need to show that $h^0(\mathscr{O}_l(l - Z_K)) < h^0(\mathscr{O}_{Z_{K,+}}(Z_{K,-}))$. In the cohomology long exact sequence of $0 \to \mathscr{O}_l(l - Z_K) \to \mathscr{O}_{Z_{K,+}}(Z_{K,-}) \to \mathscr{O}_{Z_{K,+}-l}(Z_{K,-}) \to 0$ the morphism $H^0(\mathscr{O}_{Z_{K,+}}(Z_{K,-})) \to H^0(\mathscr{O}_{Z_{K,+}-l}(Z_{K,-}))$ is non-trivial since both terms contain the constants. □

The converse is also true in the following strong sense (see also 6.4.22(b)).

Proposition 6.4.20 ([587, 4.21], [336, §3]) *Assume that (X, o) is numerically Gorenstein. If there exists $E_v \subset |Z_{K,+}|$ with $h^1(\mathscr{O}_{Z_{K,+}}) > h^1(\mathscr{O}_{Z_{K,+}-E_v})$ then (X, o) is Gorenstein. In particular, if $Z_{coh} = Z_{K,+}$ then (X, o) is Gorenstein.*

Proof Fix E_v as above. Then $H^1(\mathscr{O}_{E_v}(E_v - Z_{K,+})) \to H^1(\mathscr{O}_{Z_{K,+}})$ is non-trivial. By Serre duality $H^0(\mathscr{O}_{Z_{K,+}}(Z_{K,+} + K_{\widetilde{X}})) \to H^0(\mathscr{O}_{E_v}(Z_{K,+} + K_{\widetilde{X}}))$ is non-trivial too. By the vanishing $h^1(\mathscr{O}_{\widetilde{X}}(K_{\widetilde{X}})) = 0$ the homomorphism $H^0(\mathscr{O}_{\widetilde{X}}(Z_{K,+} + K_{\widetilde{X}})) \to H^0(\mathscr{O}_{Z_{K,+}}(Z_{K,+} + K_{\widetilde{X}}))$ is surjective, hence $H^0(\mathscr{O}_{\widetilde{X}}(Z_{K,+} + K_{\widetilde{X}})) \to H^0(\mathscr{O}_{E_v}(Z_{K,+} + K_{\widetilde{X}}))$ is also non-trivial (†). Finally, notice that by 6.4.2 $h^0(\mathscr{O}_{Z_{K,-}}(Z_{K,+} + K_{\widetilde{X}})) = 0$, hence $H^0(\mathscr{O}_{\widetilde{X}}(Z_K + K_{\widetilde{X}})) \to H^0(\mathscr{O}_{\widetilde{X}}(Z_{K,+} + K_{\widetilde{X}}))$ is an isomorphism. Hence the homomorphism from (†) can be compared with $H^0(\mathscr{O}_{\widetilde{X}}(Z_K + K_{\widetilde{X}})) \to H^0(\mathscr{O}_{E_v}(Z_K + K_{\widetilde{X}}))$, and we obtain that this last homomorphism is non-trivial as well (‡).

Now we analyse the line bundle $\mathscr{O}_{\widetilde{X}}(Z_K + K_{\widetilde{X}})$. Firstly, $H^0(\mathscr{O}_{\widetilde{X}}(Z_K + K_{\widetilde{X}})) \neq 0$ by (‡). Let $s \in H^0(\mathscr{O}_{\widetilde{X}}(Z_K + K_{\widetilde{X}}))$ be a generic section, write its effective divisor $\mathrm{div}(s)$ as $\mathrm{div}_E(s) + S(s)$, where $\mathrm{div}_E(s)$ is supported on E. Since $\mathscr{O}_{\widetilde{X}}(Z_K + K_{\widetilde{X}})$ is numerically trivial, $(\mathrm{div}(s), E_u) = 0$ for all $u \in \mathscr{V}$, hence $\mathrm{div}_E(s) \in \mathscr{S}$. We claim that in fact $\mathrm{div}_E(s) = 0$. Indeed, otherwise, by $\mathrm{div}_E(s) \geq Z_{min} \geq E$, which would contradict (‡). Hence necessarily $\mathrm{div}_E(s) = S(s) = 0$, and s trivializes $\mathscr{O}_{\widetilde{X}}(Z_K + K_{\widetilde{X}})$.

For the second statement note that if $Z_{K,+} = 0$ then (X, o) is ADE (cf. 6.3.3(3)), hence it is Gorenstein. Otherwise use the first part. □

Remark 6.4.21

(a) The above proof can be compared with 7.2.17, which also guarantees that certain line bundles are trivial.

(b) In fact, in the proof of 6.4.18 we show the following. Let $l_1, l_2 \in L_{>0}$ be effective cycles, set $l = \min\{l_1, l_2\}$ and $\overline{l} = \max\{l_1, l_2\}$. Then $h^1(\mathscr{O}_{\overline{l}}) + h^1(\mathscr{O}_l) \geq h^1(\mathscr{O}_{l_1}) + h^1(\mathscr{O}_{l_2})$. (This is an 'opposite' matroid rank inequality.)

In particular, for any $l \in L_{>0}$ we have $h^1(\mathscr{O}_l) = h^1(\mathscr{O}_{\min\{l, Z_{coh}\}})$.

Remark 6.4.22

(a) Assume that $p_g \neq 0$. Then $h^1(\mathcal{O}_{Z_{coh}}) > h^1(\mathcal{O}_{Z_{coh}-E_v})$ for any $E_v \subset |Z_{coh}|$. From the exact sequence of $0 \to \mathcal{O}_{E_v}(E_v - Z_{coh}) \to \mathcal{O}_{Z_{coh}} \to \mathcal{O}_{Z_{coh}-E_v} \to 0$ we obtain $H^1(\mathcal{O}_{E_v}(-Z_{coh} + E_v)) \neq 0$; by duality $H^0(\mathcal{O}_{E_v}(Z_{coh} + K_{\widetilde{X}})) \neq 0$. Thus

$$(Z_K - Z_{coh}, E_v) \leq 0 \text{ for any} E_v \subset |Z_{coh}|.$$

E.g., if $|Z_{coh}| = E$ then $Z_K - Z_{coh} \in \mathscr{S}'$. (Compare with the inequality from 6.6.16.)

(b) If $\widetilde{X}$ is the minimal resolution, and $E_v \not\subset |Z_{coh}|$ then $(Z_K - Z_{coh}, E_v) \leq (Z_K, E_v) \leq 0$, hence (using (a) too) $Z_K - Z_{coh} \in \mathscr{S}'$. In particular, in the numerically Gorenstein case, either $Z_K = Z_{coh}$ (when (X, o) is Gorenstein), or $Z_K - Z_{coh} \geq Z_{min}$.

(c) Consider the modification $\pi : \widetilde{X}' \to \widetilde{X}$, the blowing up of a point $q \in E$. Set $\pi^{-1}(q) = E_{new}$. If $q \notin |Z_{coh}(\widetilde{X})|$ then $Z_{coh}(\widetilde{X}') = \pi^*(Z_{coh}(\widetilde{X}))$. Otherwise, $Z_{coh}(\widetilde{X}') \leq \pi^*(Z_{coh}(\widetilde{X})) - E_{new}$ (compare with the behaviour of Z_K, cf. 6.3.3(4)). (This can be deduced naturally e.g. via the following picture, cf. 6.8.10: Z_{coh} is the largest cycle which equals the pole cycle of a differential form on $\widetilde{X} \setminus E$, hence one has to analyse the pullback of this form.)

6.4.23 Proposition 6.4.18 has the following generalizations. The proof runs similarly as the proof of 6.4.18, and it is left to the reader.

Proposition 6.4.24

(a) Fix a line bundle $\mathscr{L} \in Pic(\widetilde{X})$ with $h^1(\widetilde{X}, \mathscr{L}) > 0$. The set $L_{\mathscr{L}} := \{l \in L_{>0} : h^1(l, \mathscr{L}) = h^1(\widetilde{X}, \mathscr{L})\}$ has a unique minimal element, denoted by $Z_{coh}(\mathscr{L})$, called the cohomological cycle of $\mathscr{L}$ (and of ϕ). It has the property that $h^1(l, \mathscr{L}) < h^1(\widetilde{X}, \mathscr{L})$ for any $l \not\geq Z_{coh}(\mathscr{L})$ $(l > 0)$.

(b) Fix $Z > 0$ and $\mathscr{L} \in Pic(Z)$ with $h^1(Z, \mathscr{L}) > 0$. The set $L_{Z,\mathscr{L}} := \{l \in L,\ 0 < l \leq Z : h^1(l, \mathscr{L}) = h^1(Z, \mathscr{L})\}$ has a unique minimal element, denoted by $Z_{coh}(Z, \mathscr{L})$, called the cohomological cycle of $(Z, \mathscr{L})$. It has the property that $h^1(l, \mathscr{L}) < h^1(Z, \mathscr{L})$ for any $l \not\geq Z_{coh}(Z, \mathscr{L})$ $(0 < l \leq Z)$.

6.5 Base Point Freeness

We provide several criteria, which guarantee for a line bundle on $\widetilde{X}$ to be base point free. Then we discuss Mumford's version of the 'Generalized lemma of Castelnuovo', whose output is the surjectivity of a morphism of type $H^0(\mathscr{F} \otimes \mathscr{L}^j) \otimes H^0(\mathscr{L}) \to H^0(\mathscr{F} \otimes \mathscr{L}^{j+1})$ in certain cases. [This is a discussion valid

for any singularity, later in the case of special (e.g. rational, elliptic) germs we will prove sharper characterizations as well.] ♣

6.5.1 We fix a resolution $\phi : \widetilde{X} \to X$ as above. Let $\mathscr{F}$ be a sheaf defined on a neighborhood of E in $\widetilde{X}$. The *base points* of $\mathscr{F}$ are the points $x \in E$ such that $s(x) = 0$ for all $s \in (\phi_*\mathscr{F})_o$. The exceptional component E_v is called a *fixed component* of $\mathscr{F}$ if any section $s \in (\phi_*\mathscr{F})_o$ vanishes along E_v.

In this section we restrict ourself to line bundles $\mathscr{F} = \mathscr{L} \in \mathrm{Pic}(\widetilde{X})$. Note that if $\mathscr{L}$ has no base points then it has no fixed components either, and if it has no fixed components then $c_1(\mathscr{L}) \in -\mathscr{S}'$. Indeed, since $H^0(\mathscr{L}(-E_v)) \to H^0(\mathscr{L})$ is not an isomorphism, $H^0(\mathscr{L}|_{E_v}) \neq 0$, hence $\deg_{E_v}(\mathscr{L}) \geq 0$.

Theorem 6.5.2 (Shepherd-Barron, Laufer [634, Th. 2], [365, Th. 3.1]) *Let $\widetilde{X}$ be the minimal resolution and $\mathscr{L} \in Pic(\widetilde{X})$. Assume that $c_1(\mathscr{L}(-2K_{\widetilde{X}})) \in -\mathscr{S}'$ then $\mathscr{L}$ is base point free.*

Proof Let $q \in E$ be a point. We need to show that $H^0(\mathfrak{m}_q \cdot \mathscr{L}) \to H^0(\mathscr{L})$ is not an isomorphism, which follows from $H^1(\mathfrak{m}_q \cdot \mathscr{L}) = 0$. Let $\pi : \widetilde{X}' \to \widetilde{X}$ be the blow up at q, and ℓ the new exceptional divisor. Then, by the Leray spectral sequence, it is enough to show $H^1(\widetilde{X}', \pi^*(\mathscr{L})(-\ell)) = 0$. Let $\{E_v\}_v$ be the irreducible exceptional curves on $\widetilde{X}$, and $\{E'_v\}_v$ their strict transforms. By the theorem of formal functions, we need to prove the vanishing of $h^1(\mathscr{O}_C(\pi^*(\mathscr{L})(-\ell))) = h^0(\mathscr{O}_C(\pi^*(\mathscr{L}^{-1})(\ell + C + K_{\widetilde{X}'})))$ for all effective curves $C = m\ell + \sum_v m_v E'_v$.

We will use induction on $s := m + \sum_v m_v$.

Assume $s = 1$. If $C = \ell$, then $\deg_\ell(\pi^*(\mathscr{L})(-\ell)) = 1$, hence the vanishing follows.

If $C = E'_v$, and E_v has multiplicity μ at q, then $E'_v = \pi^*(E_v) - \mu\ell$, $(C, \ell) = \mu$, $C^2 = E_v^2 - \mu^2$, $(C, K_{\widetilde{X}'}) = (E_v, K_{\widetilde{X}}) + \mu$. Hence $(C, \pi^*\mathscr{L}^{-1} + \ell + C + K_{\widetilde{X}'}) = -(E_v, \mathscr{L}) + E_v^2 + (E_v, K_{\widetilde{X}}) + 2\mu - \mu^2 \leq -\mu^2 + 2\mu - (E_v, K_{\widetilde{X}}) + E_v^2 < -(\mu - 1)^2$, hence the vanishing follows again.

Assume that $s > 1$ and the vanishing holds for all curves with smaller s. Let E' be a component of C (which will be determined later), and set $C' := C - E'$. Tensoring the exact sequence $0 \to \mathscr{O}_{E'}(-C') \to \mathscr{O}_C \to \mathscr{O}_{C'} \to 0$ by $\pi^*\mathscr{L}(-\ell)$, and via Serre duality, we need to show that $h^0(\mathscr{O}_{E'}(\pi^*\mathscr{L}^{-1}(\ell + C + K_{\widetilde{X}'}))) = 0$, or, $(E', \pi^*\mathscr{L}^{-1} + \ell + C + K_{\widetilde{X}'})$ is negative for a certain E'. Since $K_{\widetilde{X}'} = \pi^*K_{\widetilde{X}} + \ell$, this number is $(\pi_*(E'), \mathscr{L}^{-1}(K_{\widetilde{X}})) + (E', 2\ell + C)$. Finally, via the assumption about $\mathscr{L}$, we need a component E' in the support of C such that (‡) $-(\pi_*E', K_{\widetilde{X}}) + (E', 2\ell + C) < 0$.

Note also that $-(\pi_*E', K_{\widetilde{X}}) \leq 0$ by the minimality of the resolution.

If C is a multiple of ℓ, then $E' = \ell$ works. So, assume that C is not such a cycle.

Then, there exists v with $(C, \pi^*(E_v)) < 0$. Let again be the multiplicity of E_v at q be μ. Then $(E'_v, 2\ell + C) < 2\mu - (\ell, C)\mu$. Hence, if $(\ell, C) \geq 2$ then $E' = E'_v$ works.

Hence, assume that $(\ell, C) \leq 1$. We can assume that ℓ is not a component of C, otherwise $E' = \ell$ works in (‡). The case $(\ell, C) = 0$ is also easy: in this case $C = \pi^*(D)$ for some effective cycle D, take E_v with $(E_v, D) < 0$, then $E' = E'_v$ works again. Hence, in fact, we will assume that $(\ell, C) = 1$.

In this case $C = \pi^*(D) - \ell$ for some effective cycle D of $\widetilde{X}$. Let E_v be that component which contains q, hence $m_v = 1$ and $\mu = 1$ as well. Note that for any other component E_u of C ($u \neq v$), either (‡) holds or $(E_u, D) \geq 0$. But if $(E_u, D) \geq 0$ for every $u \neq v$, then necessarily $(E_v, D) < 0$.

Checking E_v in the wished inequality, we see that either (‡) holds or $(E_v, D) = -1$ and $(E_v, K_{\widetilde{X}}) = 0$. Since $m_v = 1$ and $D^2 < 0$, then necessarily we must have $(E_u, D) = 0$ for all $u \neq v$. Hence, (‡) holds again for E'_u whenever $(E_u, K_{\widetilde{X}}) > 0$.

Hence, the only remaining case is the following: all the components in the support of D are (-2)-curves (cf. 6.3.3(3)), $(E_v, D) = -1$, $m_v = 1$, $(E_u, D) = 0$ for $u \neq v$. But such an effective cycle does not exist (since $(D, D + K_{\widetilde{X}}) = D^2$ is even).

□

Corollary 6.5.3 *In a minimal resolution $\mathscr{O}_{\widetilde{X}}(nK_{\widetilde{X}})$ and $\mathscr{O}_{\widetilde{X}}(nK)$ are base point free for $n \geq 2$.*

The proof of Theorem 6.5.2 shows the following fact as well.

Theorem 6.5.4 ([634, Prop. 4]) *Let $\widetilde{X}$ be the minimal resolution and $\mathscr{L} \in Pic(\widetilde{X})$ such that $c_1(\mathscr{L}(-K_{\widetilde{X}})) \in -\mathscr{S}'$. Assume also that there exists no effective cycle $D = E_v + \sum_{u \in I} m_u E_u \in L$ (where $I \subset \mathscr{V} \setminus \{v\}$ with $m_u > 0$) such that $(E_v, D) = -1$ and $(E_u, D) = 0$ for $u \in I$. Then $\mathscr{L}$ is base point free.*

Example 6.5.5 In general, $\mathscr{O}_{\widetilde{X}}(K_{\widetilde{X}})$ can have base points, even base components. Take e.g. the hypersurface singularity $x^2 + y^3 + z^{12}$, whose minimal graph contains two vertices, one of them is a smooth elliptic curve E with $E^2 = -1$, and a (-2)-curve E'. One checks (e.g. by computing the divisors of the three coordinate functions) that $Z_K = 2E + E'$, and $H^0(\mathscr{O}_{\widetilde{X}}(-Z_K)) = H^0(\mathscr{O}_{\widetilde{X}}(-Z_K - E'))$. Similar fact can happen even if we impose for the link to be an integral homology sphere (take e.g. $x^2 + y^3 + z^{17} = 0$). Both examples show that the assumption in Theorem 6.5.4 cannot be (totally) dropped (in both cases such a D exists, it is the elliptic cycle, cf. Definition 7.2.7).

Remark 6.5.6 Base point freeness is valid in non-minimal good resolutions as well, if we impose for $(\mathscr{L}, E_v)$ slightly more positivity than in Theorem 6.5.2. For

example, by an adaptation of the above proof, $\mathscr{L}$ is base point free in any good resolution, whenever $(\mathscr{L}, E_v) \geq 2(K, E_v) + 2$ for any $v \in \mathscr{V}$.

The next theorem exemplifies a different type of imposed inequality; in this case not only the base point freeness is guaranteed but also the fact that the generic section intersects E transversally.

Theorem 6.5.7 ([107, 4.1]) *Let $\widetilde{X}$ be a good resolution and $\mathscr{L} \in Pic(\widetilde{X})$ such that $c_1(\mathscr{L}) \neq 0$ and $(\mathscr{L}, E_v) \geq (K + E, E_v) + 2$ for any $v \in \mathscr{V}$. Then $\mathscr{L}$ is base point free, and the generic section has a normal crossing divisor whose noncompact part intersects each irreducible exceptional curve.*

Proof Take the exact sequence $0 \to \mathscr{L}(-E) \to \mathscr{L} \to \mathscr{L}|_E \to 0$. Then the vanishing $H^1(\mathscr{L}(-E)) = 0$ holds from $c_1\mathscr{L}(-E - K) \in -\mathscr{S}'$ and 6.4.3, hence $H^0(\mathscr{L}) \to H^0(\mathscr{L}|_E)$ is onto. Next, we show that $\mathscr{L}|_E$ is generated by global sections.

Assume that q is a smooth point of E. Using the exact sequence $0 \to \mathscr{L}|_E \otimes \mathfrak{m}_q \to \mathscr{L}|_E \to \mathbb{C}_q \to 0$, it is enough to show that $H^1(E, \mathscr{L}|_E \otimes \mathfrak{m}_q) = 0$. Let C_q be a disc through q transversal to E. Then $H^1(E, \mathscr{L}|_E \otimes \mathfrak{m}_q) = H^1(E, \mathscr{L}(-C_q)|_E) = H^0(E, \mathscr{L}^{-1}(C_q + E + K_{\widetilde{X}})|_E)^*$. Since $(c_1\mathscr{L}^{-1}(C_q + E + K), E_v) < 0$ for any v, the vanishing of this group follows.

Next, assume that $q \in E_v \cap E_u$. Let C_q be again a generic disc in $\widetilde{X}$ through q, which intersects both E_v and E_u transversally. Similarly as above, $H^1(\mathscr{L}(-C_q)|_E) = 0$, hence $H^0(\mathscr{L}|_E) \to \mathscr{L}|_E \otimes \mathscr{O}_{C_q}$ is onto. The local presentation of the last group is $\mathbb{C}[x, y]/(xy, x + y)$, which admits a natural epimorphism to $\mathbb{C}[x, y]/(x, y) = \mathbb{C}_q$.

This finishes the proof that $\mathscr{L}$ is base point free.

Finally notice that the projective morphism $\iota : E \to \mathbb{P}^N$ induced by the globally generated $\mathscr{L}$ is finite. Indeed, assume that $\iota(E_u)$ is a point. Then the general hyperplane section of $\mathbb{P}^N$ misses this point, which implies $(\mathscr{L}, E_u) = 0$. But this contradicts the fact that $(\mathscr{L}, E_v) > 0$ for any v guaranteed by the assumptions. Hence, $s \in H^0(\widetilde{X}, \mathscr{L})$, which is generic in $H^0(E, \mathscr{L}|_E)$, has all the wished properties. □

Theorem 6.5.8 (Mumford's 'Generalized Lemma of Castelnuovo' [449]) *Let $\widetilde{X}$ be a resolution of (X, o), and $\mathscr{F}$ a coherent sheaf on $\widetilde{X}$. Let $\mathscr{L} \in Pic(\widetilde{X})$ be a base point free line bundle such that $H^1(\mathscr{F} \otimes \mathscr{L}^{-1}) = 0$. Then the following hold:*

(a) $H^1(\mathscr{F} \otimes \mathscr{L}^j) = 0$ for any integer $j \geq -1$;
(b) $H^0(\mathscr{F} \otimes \mathscr{L}^j) \otimes H^0(\mathscr{L}) \to H^0(\mathscr{F} \otimes \mathscr{L}^{j+1})$ is onto for any $j \geq 0$.

Proof The proof is based on induction over the dimension of the support of $\mathscr{F}$.

Assume $\dim \operatorname{supp}(\mathscr{F}) = 0$, and set $s \in H^0(\mathscr{L})$ such that $s(x) \neq 0$ for all $x \in \operatorname{supp}(\mathscr{F})$. Then *(a)* holds by support argument, and $H^0(\mathscr{F}) \otimes_{\mathbb{C}} \mathbb{C}\langle s^i \rangle \to H^0(\mathscr{F} \otimes \mathscr{L}^i)$ is an isomorphism, hence *(b)* follows too.

Suppose that we are given $\mathscr{F}$, and we already have proven the statement for any $\mathscr{F}'$ with (‡) $\dim \operatorname{supp}(\mathscr{F}') < \dim \operatorname{supp}(\mathscr{F})$. By the point freeness property

there exists a generic section $s \in H^0(\mathscr{L})$ such that for any $x \in \widetilde{X}$ (if we take an isomorphism $\mathscr{L}|_U \approx \mathscr{O}|_U$ near U so that s becomes a function on U) s is not a zero divisor in the stalk $\mathscr{F}_x$ of $\mathscr{F}$ at x. This means that $\mathscr{F} \to \mathscr{F} \otimes \mathscr{L}$, defined by $f \mapsto f \otimes s$, is injective. Tensorised by $\mathscr{L}^{-1}$ we get an injective morphism of sheaves $\mathscr{F} \otimes \mathscr{L}^{-1} \to \mathscr{F}$, whose cokernel will be denoted by $\mathscr{F}'$. Note that (‡) is satisfied.

For any $j \in \mathbb{Z}$ we consider the exact sequence

$$0 \to \mathscr{F} \otimes \mathscr{L}^{j-1} \xrightarrow{\otimes s} \mathscr{F} \otimes \mathscr{L}^{j} \to \mathscr{F}' \otimes \mathscr{L}^{j} \to 0. \tag{6.5.1}$$

This, for $j = -1$, induces an epimorphism $H^1(\mathscr{F} \otimes \mathscr{L}^{-1}) \to H^1(\mathscr{F}' \otimes \mathscr{L}^{-1})$, hence $H^1(\mathscr{F}' \otimes \mathscr{L}^{-1}) = 0$, and the induction step for $\mathscr{F}'$ can be applied. In particular $H^1(\mathscr{F}' \otimes \mathscr{L}^{j}) = 0$ for all $j \geq -1$, and *(a)* follows by induction on j from (6.5.1).

Next, consider the following diagram, whose lines are exact by part *(a)*:

$$\begin{array}{ccccccccc}
0 \to & H^0(\mathscr{F} \otimes \mathscr{L}^{j-1}) \otimes H^0(\mathscr{L}) & \to & H^0(\mathscr{F} \otimes \mathscr{L}^{j}) \otimes H^0(\mathscr{L}) & \to & H^0(\mathscr{F}' \otimes \mathscr{L}^{j}) \otimes H^0(\mathscr{L}) & \to 0 \\
& \downarrow m_{j-1} & & \downarrow m_j & & \downarrow m'_j & \\
0 \to & H^0(\mathscr{F} \otimes \mathscr{L}^{j}) & \xrightarrow{\otimes s} & H^0(\mathscr{F} \otimes \mathscr{L}^{j+1}) & \to & H^0(\mathscr{F}' \otimes \mathscr{L}^{j+1}) & \to 0
\end{array}$$

This induces an exact sequence $\operatorname{coker}(m_{j-1}) \xrightarrow{\bar{s}} \operatorname{coker}(m_j) \to \operatorname{coker}(m'_j) \to 0$. Let $\beta : H^0(\mathscr{F} \otimes \mathscr{L}^{j}) \to H^0(\mathscr{F} \otimes \mathscr{L}^{j}) \otimes H^0(\mathscr{L})$ defined by $f \mapsto f \otimes s$, then $m_j \circ \beta = \cdot \otimes s$, hence $\bar{s}$ is the zero map. Since $\operatorname{coker}(m'_j) = 0$ by induction on the dimension of the support, the vanishing $\operatorname{coker}(m_j)$ follows too. □

Remark 6.5.9 Consider the notations and the assumptions of Theorem 6.5.8. Then from its proof the next statement follows as well. Let $n := \dim \operatorname{supp}(\mathscr{F})$ and choose $n+1$ sufficiently generic sections $s_0, \ldots, s_n \in H^0(\mathscr{L})$. Then $H^0(\mathscr{F} \otimes \mathscr{L})$ is spanned by $H^0(\mathscr{F}) \otimes_{\mathbb{C}} \mathbb{C}\langle s_i \rangle$, $0 \leq i \leq n$.

Corollary 6.5.10 *Consider the situation of Theorem 6.5.8. Additionally, take $l \in L$ effective such that $H^1(\mathscr{O}(-l) \otimes \mathscr{F} \otimes \mathscr{L}^{j+1}) = 0$ for $j \geq 0$. Then $H^0(l, \mathscr{F} \otimes \mathscr{L}^{j}) \otimes H^0(l, \mathscr{L}) \to H^0(l, \mathscr{F} \otimes \mathscr{L}^{j+1})$ is onto for any $j \geq 0$.*

Proof Consider the diagram

$$\begin{array}{ccc}
H^0(\widetilde{X}, \mathscr{F} \otimes \mathscr{L}^{j}) \otimes H^0(\widetilde{X}, \mathscr{L}) & \to & H^0(\widetilde{X}, \mathscr{F} \otimes \mathscr{L}^{j+1}) \\
\downarrow & & \downarrow \\
H^0(l, \mathscr{F} \otimes \mathscr{L}^{j}) \otimes H^0(l, \mathscr{L}) & \to & H^0(l, \mathscr{F} \otimes \mathscr{L}^{j+1})
\end{array}$$

The first horizontal arrow is onto by the theorem, the second vertical arrow is onto by the assumption, hence the second horizontal arrow is onto too. □

The Generalized Grauert–Riemenschneider Theorem 6.4.3, the 'Base point freeness Theorem' 6.5.2 and the Castelnuovo–Mumford Theorem imply the following.

Example 6.5.11 Consider the minimal resolution $\phi : \widetilde{X} \to X$.

(a) Consider a line bundle $\mathscr{F} \in \mathrm{Pic}(\widetilde{X})$ such that $c_1(\mathscr{F}(-jK_{\widetilde{X}})) \in \Delta - \mathscr{S}_{\mathbb{Q}}$, where $j \geq 3$ and $\Delta \in L_{\mathbb{Q}}$ with $\lfloor \Delta \rfloor = 0$, and another one $\mathscr{L} \in \mathrm{Pic}(\widetilde{X})$ with $c_1(\mathscr{L}(-iK_{\widetilde{X}})) = 0$ for some $2 \leq i \leq j-1$. Then Theorems 6.5.2 and 6.5.8 apply for $\mathscr{F}$ and $\mathscr{L}$.

(b) Consider the *canonical ring*

$$\mathscr{R} := \oplus_{n \geq 0} (\phi_* \mathscr{O}_{\widetilde{X}}(nK_{\widetilde{X}}))_o.$$

Here, for $n = 0$, $(\phi_* \mathscr{O}_{\widetilde{X}})_o = \mathscr{O}_{X,o}$. Then $\mathscr{R}$ is generated in degrees ≤ 4.

If, additionally, $\mathscr{O}_{\widetilde{X}}(K_{\widetilde{X}})$ is base point free, then $\mathscr{R}$ is generated in degrees ≤ 2.

(c) Assume $Z_K \neq 0$. Then, by 6.5.10, the graded ring $\oplus_{n \geq 0} H^0(\lfloor Z_K \rfloor, \mathscr{O}_{\widetilde{X}}(nK_{\widetilde{X}}))$ is (finitely) generated in degrees ≤ 4.

6.6 The Monoids $\mathscr{S}$ and $\mathscr{S}'$

Partial information regarding the local ring $\mathscr{O}_{X,o}$ can be obtained via the monoid of divisors (the 'analytic monoid') $\mathscr{S}_{an} := \{\mathrm{div}_E(f)\}_{f \in \mathscr{O}_{X,o}}$ in the presence of a resolution $\widetilde{X} \to X$. Even if we fix the topological type and a possible resolution graph, in general it is hard to determine $\mathscr{S}_{an}$: it depends heavily on the analytic structure (see the next Sect. 6.7). This difficulty can partly overcome by introducing the combinatorial monoid $\mathscr{S} \subset L_{\geq 0}$ with $\mathscr{S}_{an} \subset \mathscr{S}$, its elements are the topological candidates for $\mathrm{div}_E(f)$'s (see 2.1.17–2.1.18). The extension to L' is the combinatorial monoid $\mathscr{S}'$. (Its elements play a key role in the vanishing theorems and base point freeness criteria as well, cf. 6.4–6.5.)

In this section we present some properties of $\mathscr{S}$ and $\mathscr{S}'$. E.g., we prove that for any $l' \in L'$ there exists a unique minimal $e(l') \in L_{\geq 0}$ such that $s(l') := l' + e(l') \in \mathscr{S}'$. This $s(l')$ can be found by a 'Laufer type' computation sequence, whose generalized versions will constitute the key tool in many proofs of the present book. Along such sequences one can run optimally inductive arguments regarding the behaviour of the cohomology groups.

As a particular case, we also define Artin's fundamental cycle Z_{min} and the algorithmic Laufer combinatorial sequence, which detects it. ♣

6.6.1 The monoids $\mathscr{S}$ and $\mathscr{S}'$ are combinatorial objects associated with a fixed resolution graph Γ, cf. 2.1.17. We list here some additional properties. First we prove that they are closed with respect to taking the minimum of elements in the

same H-class. (Note that for $l'_1, l'_2 \in L'$ it can happen that $\min\{l'_1, l'_2\}$, defined in $L \otimes \mathbb{Q}$, is not in L'.)

Lemma 6.6.2 *For any fixed $h \in H$ set $L'_h := \{l' \in L' : [l'] = h\}$.*

(a) If $l'_1, l'_2 \in L'_h$ then $l' := \min\{l'_1, l'_2\} \in L'_h$ too.
(b) If $l'_1, l'_2 \in \mathscr{S}' \cap L'_h$ then $l' := \min\{l'_1, l'_2\} \in \mathscr{S}' \cap L'_h$ too.

Proof

(a) $l' = r_h + \min\{l'_1 - r_h, l'_2 - r_h\}$.
(b) Write $l'_i = l' + l_i$ for some $l_i \in L_{\geq 0}$. Then for any fixed v, $E_v \not\subset |l_i|$ for at least one of the i's. Therefore, $(l', E_v) = (l'_i - l_i, E_v) \leq (-l_i, E_v) \leq 0$.

□

Proposition 6.6.3 *Let $\widetilde{X} \to X$ be a resolution of (X, o) as above.*

(a) For any $l' \in L'$ there exists a unique minimal element $e(l') \in L_{\geq 0}$ with $s(l') := l' + e(l') \in \mathscr{S}'$. (Hence $-l'$ is the sum of the integral effective cycle $e(l')$ and of the numerically effective cycle $-s(l')$, with $e(l')$ minimal.)
(b) $e(l')$ can be found by the following (generalized Laufer's) algorithm. One constructs a 'computation sequence' $z_0, z_1, \ldots, z_t \in L_{\geq 0}$ with $z_0 = 0$ and $z_{i+1} = z_i + E_{v(i)}$, where the index $v(i)$ is determined by the following principle. Assume that z_i is already constructed. Then, if $l' + z_i \in \mathscr{S}'$, then one stops, and $t = i$. Otherwise, there exists at least one $v \in \mathscr{V}$ with $(l' + z_i, E_v) > 0$. Take for $v(i)$ one of these v's. Then this algorithm stops after finitely many steps, and $z_t = e(l')$.

Proof

(a) Since I is negative definite, there exists at least one effective cycle $l \in L$ with $l' + l \in \mathscr{S}'$ (take e.g. a large multiple of some x with $(x, E_v) < 0$ for any v). The uniqueness of minimal l follows from 6.6.2.
(b) First we prove that $z_i \leq e(l')$ for any i. For $i = 0$ this is clear. Assume that it is true for a certain i but not for $i + 1$, i.e. $E_{v(i)} \not\subset |e(l') - z_i|$. But this would imply $(l' + z_i, E_{v(i)}) = (s(l'), E_{v(i)}) - (e(l') - z_i, E_{v(i)}) \leq 0$, a contradiction. Now, since $z_i \leq e(l')$ for any i, the algorithm must stop, and $z_t \leq e(l')$. By the minimality of $e(l')$ one gets $z_t = e(l')$ too.

□

The vanishing theorems 6.4.3 and 6.4.10 create a special interest in line bundles with $c_1(\mathscr{L}) \in -\mathscr{S}'$.

Corollary 6.6.4 *For any $\mathscr{L} \in \mathrm{Pic}(\widetilde{X})$ take $c_1 := c_1(\mathscr{L})$ and $e := e(-c_1)$. Then $c_1(\mathscr{L}(-e)) = -s(-c_1) \in -\mathscr{S}'$ and*

$$h^1(\mathscr{L}(-e)) - h^1(\mathscr{L}) = \chi(\mathscr{O}_e(c_1)) = \chi(e - c_1) - \chi(-c_1) \leq 0.$$

In particular, the computation of any $h^1(\mathscr{L})$ can be reduced, modulo the combinatorics of L, to the computation of some $h^1(\mathscr{L}')$ with $c_1(\mathscr{L}') \in -\mathscr{S}'$.

Proof Let $\{z_i\}_i$ be the computation sequence of $e(-c_1)$. For any $0 \le i < t$

$$0 \to \mathscr{L}(-z_{i+1}) \to \mathscr{L}(-z_i) \to \mathscr{L}(-z_i)|_{E_{v(i)}} \to 0$$

is exact with $(-c_1 + z_i, E_{v(i)}) > 0$, hence $h^0(\mathscr{L}(-z_i)|_{E_{v(i)}}) = 0$ and $h^1(\mathscr{L}(-z_{i+1})) - h^1(\mathscr{L}(-z_i)) = \chi(-c_1 + z_{i+1}) - \chi(-c_1 + z_i) \le 0$. □

Example 6.6.5 If $\mathscr{L} = \mathscr{O}_{\widetilde{X}}(-l')$ for some $l' \in L'$ (cf. 6.2) then 6.6.4 reads as

$$h^1(\mathscr{O}_{\widetilde{X}}(-s(l'))) - h^1(\mathscr{O}_{\widetilde{X}}(-l')) = \chi(\mathscr{O}_{e(l')}(-l')) = \chi(s(l')) - \chi(l') \le 0.$$

Remark 6.6.6 Fix any system of integers $\{a_v\}_{v\in\mathscr{V}}$ and set $L'\{a_v\}_v := \{l' \in L' : (l', E_v) \le a_v \text{ for any } v\}$. If each a_v is sufficiently negative then $L'\{a_v\}_v$ is in the Grauert–Riemenschneider vanishing zone $Z_K + \mathscr{S}'$, hence $h^1(\mathscr{L}) = 0$ for any $\mathscr{L}$ with $-c_1(\mathscr{L}) \in L'\{a_v\}_v$, cf. 6.4.3. More generally, for an arbitrary $\{a_v\}_{v\in\mathscr{V}}$, using similar sequences as in Proposition 6.6.3 connecting $L'\{a_v\}$ with $Z_K + \mathscr{S}'$, one proves that there exists a topological constant $c = c(\{a_v\}_{v\in\mathscr{V}})$ such that $h^1(\mathscr{L}) \le c$ for any $\mathscr{L}$ with $-c_1(\mathscr{L}) \in L'\{a_v\}_v$. See also [129, Lemma 2.4].

The next consequence of 6.6.3 is the existence of the fundamental cycle.

Corollary 6.6.7

(a) [27, 28] $\mathscr{S} \setminus \{0\}$ has a unique minimal element Z_{min}.

(b) [359] Z_{min} can be found by the following 'Laufer's algorithm'. One constructs a computation sequence $z_1, \dots, z_t$ with $z_1 = E_w$ (arbitrarily chosen), and $z_{i+1} = z_i + E_{v(i)}$, where the index $v(i)$ is determined as follows. Assume that z_i is already constructed. Then, if $z_i \in \mathscr{S}$, then one stops, and $t = i$. Otherwise, there exists at least one $v \in \mathscr{V}$ with $(z_i, E_v) > 0$. Take for $v(i)$ one of these v's. Then this algorithm stops after finitely many steps, and $z_t = Z_{min}$ (independently of all the choices).

(More generally, one can start with an arbitrary z_1 with $0 < z_1 \le Z_{min}$ and $|z_1|$ connected, then the very same algorithm gives $z_t = Z_{min}$, and 6.6.9(b) also holds.)

Definition 6.6.8 The cycle $Z_{min} \in L_{>0}$ has several names in the literature: *minimal, fundamental, or Artin cycle*. Here we will adopt the name 'fundamental cycle'.

The sequence from (b) is called the *Laufer's computation sequence for Z_{min}*.

Remark 6.6.9

(a) Since Γ is connected one has $Z_{min} \ge E$. The equality holds if and only if $\kappa_v \le -E_v^2$ for any $v \in \mathscr{V}$.

(b) Induction along the computation sequence of Z_{min} proves $H^0(\mathscr{O}_{Z_{min}}) = H^0(\mathscr{O}_{z_i}) = \{\text{constants}\} \approx \mathbb{C}$, and $h^1(\mathscr{O}_{Z_{min}}) = \sum_i h^1(\mathscr{O}_{E_{v(i)}}(-z_i))$. This last identity shows that $h^1(\mathscr{O}_{Z_{min}}) = 0$ if and only if Γ is a tree of smooth rational curves and $(E_{v(i)}, x_i) = 1$ for all $1 \le i < t$.
(c) Assume that the resolution is minimal and (X, o) is numerically Gorenstein. Then, by 6.3.3, $Z_K \in \mathscr{S}$. Hence, either $Z_K = 0$ or $Z_K \ge Z_{min}$.
(d) If $l_1, l_2 \in \mathscr{S}$ and $l_1 \ge l_2 > 0$ then $l_1^2 \le (l_1, l_2) \le l_2^2 < 0$. Hence $l_1^2 \le Z_{min}^2$.
(e) If the resolution $\widetilde{X}'$ dominates another resolution $\widetilde{X}$, $\pi : \widetilde{X}' \to \widetilde{X}$, then $\pi^*(Z_{min}) = Z'_{min}$.
(f) The integers Z_{min}^2, $\chi(Z_{min})$ and $\min_{l \in L_{>0}} \chi(l)$ are independent of the choice of the resolution, they depend only on L_X.

Remark 6.6.10

(a) Let $\mathfrak{m}_{X,o}$ be the maximal ideal of $\mathscr{O}_{\widetilde{X}}$. Since $\mathrm{div}_E(f) \ge Z_{min}$ for any $f \in \mathfrak{m}_{X,o}$, we have $\mathfrak{m}_{X,o} = \phi_*(\mathscr{O}_{\widetilde{X}}(-Z_{min}))_o$. Therefore, for any integer $k \ge 1$,

$$\mathfrak{m}_{X,o}^k \subset \phi_*(\mathscr{O}_{\widetilde{X}}(-kZ_{min}))_o. \tag{6.6.1}$$

(b) Assume that $\widetilde{X}$ is the minimal resolution, $Z_K \ne 0$ and fix an integer r such that $rZ_K \in L$. By 6.5.11(a), for any integer n,

$$\phi_*(\mathscr{O}_{\widetilde{X}}(-3rZ_K))_o \otimes \phi_*(\mathscr{O}_{\widetilde{X}}(-2rZ_K))_o^{\otimes n} \to \phi_*(\mathscr{O}_{\widetilde{X}}(-(3+2n)rZ_K))_o$$

is onto. Since $\phi_*(\mathscr{O}_{\widetilde{X}}(-arZ_K))_o \subset \mathfrak{m}_{X,o}$ for $a = 2, 3$, we get that $\phi_*(\mathscr{O}_{\widetilde{X}}(-(3+2n)rZ_K))_o \subset \mathfrak{m}_{X,o}^{n+1}$. Choose an affine function $n \mapsto k(n)$ such that $k(n)Z_{min} \ge (3+2n)rZ_K$. Then

$$\phi_*(\mathscr{O}_{\widetilde{X}}(-k(n)Z_{min}))_o \subset \mathfrak{m}_{X,o}^{n+1}. \tag{6.6.2}$$

(For the case $Z_K = 0$ see Theorem 7.1.17*(a)*.)
(c) In (6.6.1) and (6.6.2) Z_{min} can be replaced as follows. Fix any non-zero element $l \in \mathscr{S}$. Since there exist integers a and b such that $al \ge Z_{min}$ and $bZ_{min} \ge l$, we get that there exist two affine functions $k \mapsto a(k)$, $b(k)$ such that for any k

$$\mathfrak{m}_{X,o}^{a(k)} \subset \phi_*(\mathscr{O}_{\widetilde{X}}(-kl))_o, \quad \phi_*(\mathscr{O}_{\widetilde{X}}(-b(k)l))_o \subset \mathfrak{m}_{X,o}^k. \tag{6.6.3}$$

Lemma 6.6.11 ([661]) *Fix a resolution $\widetilde{X}$ of a normal surface singularity (X, o).*

(i) If $0 < l \le Z_{min}$ then $\chi(l) \ge \chi(Z_{min})$.
(ii) If $0 < l_1, l_2 \le Z_{min}$ and $\chi(l_1) = \chi(l_2) = \chi(Z_{min})$ then $l := \min\{l_1, l_2\}$ satisfies $\chi(l) \le \chi(Z_{min})$.

Proof For *(i)* use $h^0(\mathscr{O}_{Z_{min}}) = 1$ and $h^1(\mathscr{O}_l) \le h^1(\mathscr{O}_{Z_{min}})$. For *(ii)* apply *(i)* for l_1+l_2-l and use the identity $\chi(l_1+l_2-l) = \chi(l_1)+\chi(l_2)-\chi(l)-(l_1-l, l_2-l)$. □

With the notation of *(ii)* above, if $\chi(Z_{min}) < 0$ then $l \neq 0$, hence by *(i)–(ii)* $\chi(l) = \chi(Z_{min})$. The same property holds even if $\chi(Z_{min}) = 0$, see Proposition 7.2.8.

Therefore, whenever $\chi(Z_{min}) \le 0$ we get a unique positive cycle C with $C \le Z_{min}$ and $\chi(C) = \chi(Z_{min})$ such that $\chi(l) > \chi(C)$ for any $0 < l < C$.

Definition 6.6.12 ([661]) C is called the *characteristic cycle* of the resolution $\widetilde{X}$.

Characteristic cycles were first introduced by Karras in the context of Kodaira singularities (cf. Sect. 6.7.A), our presentation follows the thesis of J. Stevens [661] about Kulikov singularities. (Later Tomaru also considered them under the name 'minimal cycle' [694].)

Remark 6.6.13 Set $L_{\min\chi} := \{l \in L : \chi(l) = \min\chi\}$. Similar proof as of 6.6.11 shows that if $l_1, l_2 \in L_{\min\chi}$ then $\min\{l_1, l_2\} \in L_{\min\chi}$ as well. Hence $L_{\min\chi}$ has a unique minimal element.

6.6.A *The Representatives r_h and s_h*

Any $h \in H$ has unique minimal representatives $r_h \in L'_{\ge 0}$ and $s_h \in \mathscr{S}'$. The two representatives are involved in rather different geometric constructions: e.g. r_h is motivated by the eigensheaf decompositions of the universal abelian covering (see e.g. 6.2.9), while s_h is motivated by vanishing theorems (hence in h^1 formulae, e.g. as in 7.1.12) and also its appearance in the reduction of the lattice cohomology to the first quadrant (see 7.3.A and Proposition 11.1.26).

The rational cycles r_h and s_h are rather arithmetical, it is very hard to identify them concretely. In this subsection we determine them for strings and star-shaped graphs together with the values $\chi(r_h)$ and $\chi(s_h)$ (needed later e.g. in the verification of the Seiberg–Witten Invariant Conjecture). ♣

6.6.14 Recall that for any $h \in H$, $r_h \in L'$ is the minimal representative of h ($[r_h] = h$) in the cone $L'_{\ge 0}$. Replacing the cone $L'_{\ge 0}$ by $\mathscr{S}'$, by 6.6.2 we obtain the following.

Corollary 6.6.15 *For any $h \in H$ consider all the representatives $l' + L \in L'$ of h. Then $(l' + L) \cap \mathscr{S}'$ has a unique minimal element s_h.*

Clearly $s_0 = 0$, and $s_h \ge r_h$. Strict inequality might appear (take e.g. the lens space $L(8, 5)$). Moreover, $s_h = r_h$ if and only if $r_h \in \mathscr{S}'$, otherwise $s_h = s(r_h)$ in the

sense of 6.6.3. Using the algorithm from 6.6.3*(b)*, or 6.6.5, we obtain

$$\chi(s_h) \leq \chi(r_h). \tag{6.6.4}$$

Even at Euler-characteristic level, strict inequality can appear, see 6.8.19 and 7.1.21.

Example 6.6.16 Assume that $H^1(\widetilde{X}, \mathbb{Z}) = 0$ and $\widetilde{X}$ is minimal, hence $Z_K \geq 0$. Let $r_{[Z_K]}$ and $s_{[Z_K]}$ be the two representatives of $h = [Z_K] \in H$. Consider the computation sequence $\{z_i\}_{i=0}^t$, $z_0 = r_{[Z_K]}$, $z_t = s_{[Z_K]}$, which connects them, cf. 6.6.3. Then $h^1(\mathscr{O}_{Z_K - z_i})$ is constant along the sequence. In particular, $h^1(\mathscr{O}_{Z_K - s_{[Z_K]}}) = h^1(\mathscr{O}_{\lfloor Z_K \rfloor}) = p_g$, cf. 6.4.7. Hence $Z_{coh} \leq Z_K - s_{[Z_K]}$.

6.6.17 The two representatives r_h and s_h are involved in many geometric constructions, formulae (and their roles and appearances might depend on the geometrical situation). The next examples provide key information needed later, and also show the interplay between the two representatives.

Example 6.6.18 (s_h in the Minimal Resolution of a Cyclic Quotient [478])

We will use the notations of 2.2.8 and 2.3. In the verification of several statements the identities (2.2.4) will be very useful. Clearly $L'/L = H = \mathbb{Z}_n$, and $[E_s^*]$ is one of its generators. Indeed, $[E_i^*] = [n_{i+1,s} E_s^*]$ in H for $1 \leq i \leq s$.

For any $0 \leq a < n$ we write $s_{[aE_s^*]} = \sum_{i=1}^s a_i E_i^*$, where $a_i \geq 0$ for any i. Next, we clarify the relationship between a and the system $E(a) := (a_1, \ldots, a_s)$. We claim that the integers $a_1, \ldots, a_s$ satisfy

$$\sum_{t=k}^{i} n_{k,t-1} a_t < n_{ki} \quad \text{and} \quad \sum_{t=k}^{i} n_{t+1,i}\, a_t < n_{ki} \quad (k \leq i). \tag{6.6.5}$$

Let us verify the second inequality. We set $c_{ki} := n_{ki} - \sum_{t=k}^{i} n_{t+1,i} a_t$. Then $c_{k,k-1} = 1$, and we wish to prove that $c_{ki} > 0$. We run increasing induction on i starting with $k-1$. In the inductive step we assume $c_{kt} > 0$ for all $k-1 \leq t \leq i$. Let $l := \sum_{t=k}^{i+1} c_{k,t-1} E_t$. One verifies that $(s_{[aE_s^*]} - l, E_v) \leq 0$ for all $v \neq i+1$. By the minimality of $s_{[aE_s^*]}$ we must have $(s_{[aE_s^*]} - l, E_{i+1}) > 0$, showing, in fact, that $c_{k,i+1} > 0$.

We separate from this set of inequalities the following system:

$$(SI) \qquad \begin{cases} a_1 \geq 0, \ \ldots\ , \ a_s \geq 0 \\ n_{k+1,s} a_k + n_{k+2,s} a_{k+1} + \cdots + n_{ss} a_{s-1} + a_s < n_{ks} \text{ for any } 1 \leq k \leq s. \end{cases}$$

We claim that this system determines all the possible combinations $(a_1, \ldots, a_s)$. Indeed, the integer a can be recovered from $(a_1, \ldots, a_s)$ as

$$a = n_{2s} a_1 + n_{3s} a_2 + \cdots + n_{ss} a_{s-1} + a_s \tag{6.6.6}$$

(since $[E_k^*] = n_{k+1,s}[E_s^*]$); while any $0 \le a < n$ determines inductively the entries $a_1, \dots, a_s$ by

$$a_k = \left[\frac{a - \sum_{t=1}^{k-1} n_{t+1,s} a_t}{n_{k+1,s}}\right] \quad (1 \le k \le s). \tag{6.6.7}$$

It is a curiosity that the system (SI) can also be interpreted in the language of 'generalized' continued fractions. For any $1 \le k \le s$ write $r_k := n_{ks}/n_{k+1,s}$. Then

$$\frac{a}{n} = \cfrac{a_1 + \cfrac{\cdots + \cfrac{a_{s-1} + \cfrac{a_s}{r_s}}{\cdots}}{r_2}}{r_1} \qquad (0 \le a < n).$$

The inequalities (SI) imply that all the possible fractions $\frac{a_s}{r_s}$, $(a_{s-1} + \frac{a_s}{r_s})/r_{s-1}, \dots, a/n$ in the above expression are < 1; and this property guarantees the uniqueness of the entries $(a_1, \dots, a_s)$.

It is easy to identify some combinations $E(a)$. E.g.,

$$s_{[(n-1)E_s^*]} = r_{[(n-1)E_s^*]} = -E_s^* + \sum_{k=1}^{s} E_k = (b_1 - 1)E_1^* + \sum_{k=2}^{s} (b_k - 2)E_k^*.$$

Starting from this, the set $\{E(a) = (a_1, \dots, a_s),\ 0 \le a < n\}$ can be generated inductively as follows. Start with $E(n-1) = (b_1 - 1, b_2 - 2, \dots, b_s - 2)$. Assume that $E(a) = (a_1, \dots, a_s)$ is already determined. Then, if $a_s > 0$, then $E(a-1) = (a_1, \dots, a_{s-1}, a_s - 1)$. If $a_k = \cdots = a_s = 0$, but $a_{k-1} \ne 0$, then

$$E(a-1) = (a_1, \dots, a_{k-2}, a_{k-1} - 1, b_k - 1, b_{k+1} - 2, \dots, b_s - 2).$$

In particular,

$$s_{[(a-1)E_s^*]} - s_{[aE_s^*]} = -E_{k-1}^* + (b_k - 1)E_k^* + \cdots + (b_s - 2)E_s^* = -E_s^* + \sum_{t=k}^{s} E_t. \tag{6.6.8}$$

In order to prove this fact compare the system of identities (6.6.7) for a and $a-1$.

Next, we are interested in the computation of $\chi(s_{[aE_s^*]})$, a formula which is needed later. Note that $-Z_K = E_1^* + E_s^* - E$ (cf. (6.3.1)), hence $\chi(-E_s^*) = -(E_s^*, E_s^* + Z_K)/2 = -(E_s^*, E - E_1^*)/2 = (n-1)/2n$ (use 2.2.5(5)). Furthermore,

the inequalities also guarantee that the integral and fractional parts of aq'/n are

$$[aq'/n] = \sum_{t=1}^{s} a_t n_{t+1,s-1} \text{ and } \{aq'/n\} = (\sum_{t=1}^{s} a_t n_{1,t-1})/n. \tag{6.6.9}$$

This can be proved as follows. Using (6.6.6), for the first part we have to prove

$$\sum_{t=1}^{s} n_{t+1,s-1} a_t \le \frac{n_{1,s-1}}{n} \cdot \sum_{t=1}^{s} n_{t+1,s} a_t < \sum_{t=1}^{s} a_t n_{t+1,s-1} + 1.$$

This follows from the identities of type (†) $n_{t+1,s} \cdot n_{1,s-1} = n \cdot n_{t+1,s-1} + n_{1,t-1}$ (as particular cases of (2.2.4)) and the first inequality from (6.6.5). The second part follows from the expression of $[aq'/n]$ and the identities (†) used again.

Therefore, using $a_k = \cdots = a_s = 0$ and (6.6.8), we obtain

$$\chi(s_{[(a-1)E_s^*]}) - \chi(s_{[aE_s^*]}) = \frac{n-1}{2n} - \sum_{t=1}^{s} \frac{a_t n_{1,t-1}}{n} = \frac{n-1}{2n} - \left\{\frac{aq'}{n}\right\}. \tag{6.6.10}$$

By summation, for any $0 \le a < n$ we obtain

$$\chi(s_{[aE_s^*]}) = \frac{a(1-n)}{2n} + \sum_{i=1}^{a} \left\{\frac{iq'}{n}\right\} = \frac{a}{2n} + \sum_{i=1}^{a} \left(\!\!\left(\frac{iq'}{n}\right)\!\!\right). \tag{6.6.11}$$

(One gets a symmetric version by replacing E_s^* by E_1^* and q' by q.) By summation

$$\sum_h \chi(s_h) = \frac{n-1}{4} - n \cdot \mathbf{s}(q,n). \tag{6.6.12}$$

Example 6.6.19 (r_h in the Minimal Resolution of a Cyclic Quotient)

Clearly, $r_h = \{s_h\}$. Also, since $aE_s^* = \frac{a}{n}\sum_i n_{1,i-1}E_i$, we get

$$r_{[aE_s^*]} = \sum_i \left\{a \cdot n_{1,i-1}/n\right\} E_i \qquad (0 \le a < n). \tag{6.6.13}$$

(The interested reader can verify that the two formulae provide the same output.) Although the expression (6.6.13) looks simple, it also suggests some possible hidden arithmetical properties of the collection $\{r_i\}_{i=1}^s$ of the coefficients of $r_h = \sum_i r_i E_i$. We exemplify this by the computation of $H^0(\widetilde{X}, \mathscr{O}(-r_h))$.

First, let us see what kind of answer we expect. $H^0(\widetilde{X}, \mathscr{O}(-r_h))$ is the $e^{2\pi i a/n}$-eigenspace of $\mathbb{C}[z_1, z_2]$ under the action $\xi * (z_1, z_2) = (\xi z_1, \xi^q z_2)$, that is, the vector space $\mathbb{C}\langle z_1^k z_2^\ell \,:\, k \ge 0,\ \ell \ge 0,\ k + \ell q \equiv a \ (\mathrm{mod}\, n)\rangle$. If we set $k + lq = a + mn$,

then the parameter set of the monomials is

$$\{(m,\ell)\in\mathbb{Z}^2_{\geq 0},\ a+nm-\ell q\geq 0\}. \tag{6.6.14}$$

Next, we compute $H^0(\widetilde{X},\mathscr{O}(-r_h))$ using the affine charts $\{U_i\}_{i=0}^s$ of $\widetilde{X}$ from 2.3.1. The transition functions connecting the different trivializations $H^0(U_i,\mathscr{O}(-r_h))\simeq\mathbb{C}[u_i,v_i]$ can be determined as follows. We identify $H^0(U_i,\mathscr{O}(-r_h))$ formally with monomials (with rational exponents) of type $u_i^{r_i}v_i^{r_{i+1}}\mathbb{C}[u_i,v_i]$, where $r_0=r_{s+1}=0$, such that the coordinates (u_i,v_i) transform as in $\widetilde{X}$ (cf. 2.3). E.g., the section $v_0^{r_1+m}u_0^{\ell}$ transforms under the coordinate change $u_0=v_1^{-1}$, $v_0=u_1v_1^{b_1}$ into $v_1^{r_2}u_1^{r_1}v_1^{b_1r_1-r_2+b_1m-\ell}u_1^m$. Hence the transformation $\mathbb{C}[u_0,v_0]\mapsto\mathbb{C}[u_1,v_1]$ is $v_0^mu_0^{\ell}\mapsto v_1^{b_1r_1-r_2+b_1m-\ell}u_1^m$. More generally, $v_0^mu_0^{\ell}$ in U_i reads as

$$v_i^{n_{1,i}r_1-r_{i+1}+n_{1,i}m-n_{2,i}\ell}u_i^{n_{1,i-1}r_1-r_i+n_{1,i-1}m-n_{2,i-1}\ell}.$$

Since $r_1=a/n$ and by (6.6.13) $r_{i+1}=\{a\cdot n_{1,i}/n\}$, we obtain

$$n_{1,i}r_1-r_{i+1}=a\cdot n_{1,i}/n-\{a\cdot n_{1,i}/n\}=[a\cdot n_{1,i}/n]=[n_{1,i}r_1]\in\mathbb{Z}_{\geq 0}.$$

In particular, for $(m,\ell)\in\mathbb{Z}^2_{\geq 0}$, $v_0^mu_0^{\ell}$ extends to a regular section if and only if

$$(*_j)\qquad n_{1,i}r_1-r_{i+1}+n_{1,i}m-n_{2,i}\ell\geq 0$$

for all $i=0,1,\ldots,s$. This, for $i=s$ coincides with the previously obtained (6.6.14); hence we expect that the collection $(*_i)$ is the consequence of $(*_s)$. This is indeed the case, and in fact, $(*_i)$ implies $(*_{i-1})$, whose verification is left to the reader. (The slightly more complicated case of star-shaped graphs is detailed in 6.6.21.)

Example 6.6.20 (s_h in a Minimal Good Star-Shaped Graph)

We will use the notation of Sect. 5.1, and we assume $g=0$. For each leg j $(1\leq j\leq\nu)$ let n^j_{ik} be the substring determinant n_{ik} introduced in 2.2.8. We write

$$s_h=a_0E_0^*+\sum_{j=1}^{\nu}\sum_{t=1}^{s_j}a_{jt}E^*_{jt}. \tag{6.6.15}$$

To the collection $(a_{j1},\ldots,a_{js_j})$ we associate (similarly as in 6.6.6) the integer

$$a_j:=\sum_{t\geq 1}n^j_{t+1,s_j}a_{jt}. \tag{6.6.16}$$

We emphasize that

$$\sum_t a_{jt}E^*_{jt} - a_j E^*_{js_j} \in L, \text{ and it is supported on the leg } j. \tag{6.6.17}$$

The integers $\{a_0; a_{jt}\}_{j,t}$ satisfy the following inequalities:

$$(SI) \qquad \begin{cases} a_0 \geq 0; , a_{jk} \geq 0 & (1 \leq j \leq \nu,\ 1 \leq k \leq s_j); \\ \tilde{a}_{jk} := \sum_{t \geq k} n^j_{t+1,s_j} a_{jt} < n^j_{k,s_j} & (1 \leq j \leq \nu,\ 1 \leq k \leq s_j); \\ 1 + a_0 - \ell b_0 + \sum_j \left[\frac{\ell\omega_j + a_j}{\alpha_j}\right] \leq 0 \text{ for any } \ell > 0. \end{cases}$$

The first two sets of inequalities follow similarly as (SI) in 6.6.18. For the last one, we proceed as follows. For any $\ell > 0$ we define $y(\ell) := \ell E_0 + \sum_{j,k} u_{jk} E_{jk} \in L$, where

$$u_{j1} := \left[\frac{\ell n^j_{2s_j} + \tilde{a}_{j1}}{n^j_{1s_j}}\right],\ u_{j2} := \left[\frac{u_{j1} n^j_{3s_j} + \tilde{a}_{j2}}{n^j_{2s_j}}\right],\ \ldots,\ u_{j,s_j} := \left[\frac{u_{j,s_j-1} + \tilde{a}_{j,s_j}}{n^j_{s_j s_j}}\right]. \tag{6.6.18}$$

Consider $l' := s_h - y(\ell)$. Then $(l', E_{jk}) \leq 0$ for any $1 \leq j \leq \nu$ and $1 \leq k \leq s_j$.

Indeed, by the definition of u_{jk} one has $u_{jk} n^j_{ks_j} \leq u_{j,k-1} n^j_{k+1,s_j} + \tilde{a}_{jk}$. Using $n^j_{k,s_j} = b_{jk} n^j_{k+1,s_j} - n^j_{k+2,s_j}$ and $\tilde{a}_{jk} = n^j_{k+1,s_j} a_{jk} + \tilde{a}_{j,k+1}$ we get

$$-a_{jk} - u_{j,k-1} - b_{jk} u_{jk} \leq (u_{j,k} n^j_{k+2,s_j} + \tilde{a}_{j,k+1})/n^j_{k+1,s_j}.$$

Hence, the left hand side is less than or equal to the integral part of the right hand side, which is $u_{j,k+1}$. Therefore, since $l' = s_h - y(\ell) \in s_h + L$ and $y(\ell) > 0$, by the minimality of s_h we must have $(l', E_0) > 0$.

Note that from (SI) $a_j = \tilde{a}_{j1} < n^j_{1s_j} = \alpha_j$. Hence we obtain a 'reduced system':

$$(SI_{red}) \qquad \begin{cases} a_0 \geq 0; , \ \alpha_j > a_j \geq 0 & (1 \leq j \leq \nu); \\ 1 + a_0 - \ell b_0 + \sum_j \left[\frac{\ell\omega_j + a_j}{\alpha_j}\right] \leq 0 & \text{for any } \ell > 0. \end{cases}$$

The correspondence between the integral solutions of the two systems of inequalities (SI) and (SI_{red}) follows from the corresponding statements at the level of strings, cf. (6.6.6) and (6.6.7). (Note that the determinant of the j-leg is exactly α_j.)

In fact, the following sets are in one-to-one correspondence:

(1) the representatives $\{s_h\}_{h \in H}$;
(2) the set of integral solutions $\{a_0; a_{jk}\}_{jk}$ of (SI);
(3) the set of integral solutions $\{a_0; a_j, \ldots, a_\nu\}$ of (SI_{red}).

Till now we proved (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3). (3) $\Rightarrow$ (1) follows from the next statement.

Fact *For each* $c_0 \geq 0$ *and collection* $\{c_j\}_{j=1}^{v}$ *with* $\alpha_j > c_j \geq 0$ *consider the integers* $c_{jk} \geq 0$ *satisfying (6.6.16) (with 'c' instead of 'a') determined uniquely from* c_j *by similar identities as (6.6.7). Consider* $c_h := c_0 E_0^* + \sum_{j,t} c_{jt} E_{jt}^* \in \mathscr{S}'$. *If* c_h *is not minimal in* $(c_h + L) \cap \mathscr{S}'$, *then for some* $\ell > 0$ *one has*

$$1 + c_0 - \ell b_0 + \sum_j \left[\frac{\ell\omega_j + c_j}{\alpha_j} \right] > 0.$$

The proof runs as follows. Define the integers $\tilde{c}_{jk}$, and for any $\ell \geq 0$ the integers u_{jk} similarly as in (6.6.18), but now from the coefficients c_{jk} (instead of a_{jk}). Also, set $y(\ell) := \ell E_0 + \sum_{j,k} u_{jk} E_{jk}$. Then (a) the E_0-coefficient of $y(\ell)$ is ℓ and (b) $(c_h - y(\ell), E_{jk}) \leq 0$ for any j, k (for this repeat the previous proof). The point is that for any fixed ℓ, $y(\ell)$ is the unique maximal effective cycle which satisfies both (a) and (b). (This can be proved similarly, or using Proposition 11.3.13.)

Now, consider c_h and assume that it is not minimal in $(c_h + L) \cap \mathscr{S}'$; hence there exists $x \in L_{>0}$ with $c_h = s_h + x$. Let ℓ be the E_0-coefficient of x, and take $y(\ell)$ associated with c_h as discussed above. Then both x and $y(\ell)$ satisfy (a) and (b), hence by the maximality of $y(\ell)$ we get $x \leq y(\ell)$. Since $y(0) = 0$ and $x > 0$, we get that $\ell > 0$. Next, $(s_h + x - y(\ell), E_{jk}) = (c_h - y(\ell), E_{jk}) \leq 0$, and also $(s_h + x - y(\ell), E_0) \leq 0$ since $y(\ell) - x$ is effective supported on the legs. Therefore, $s_h + x - y(\ell) \in \mathscr{S}'$, and by the minimality of s_h we obtain $x = y(\ell)$. But then $0 \geq (E_0, s_h) = (E_0, c_h - y(\ell))$.

Next, we determine the values $\chi(s_h)$. Assume that s_h corresponds to the set $\{a_0; a_1, \ldots, a_v\}$, and $a_j > 0$ for some j. Let $s_{h_j^-}$ correspond to $\{a_0; a_1, \ldots, a_{j-1}, a_j - 1, a_{j+1}, \ldots, a_v\}$. Then, similarly as in 6.6.18 (see (6.6.8)) we obtain

$$s_{h_j^-} = s_h - E_{js_j}^* + \sum_{t=k_j}^{s_j} E_{jt}, \quad \text{and } a_{jt} = 0 \text{ for } k_j \leq t \leq s_j.$$

Hence

$$\chi(s_{h_j^-}) - \chi(s_h) = \chi(-E_{js_j}^*) + (s_h, E_{js_j}^*).$$

In the last sum replace s_h by the sum from (6.6.15). By (3.5.2) the $a_0 E_0^*$ term has contribution $a_0/(e\alpha_j)$. For a leg $j' \neq j$, $(\sum_t a_{j't} E_{j't}^*, E_{js_j}^*) = (a_{j'} E_{j's_{j'}}^*)$ because of (6.6.17), hence it contributes with $a_{j'}/(e\alpha_j\alpha_{j'})$. Finally, from 2.2.2) and (3.5.2)

$(E^*_{jt}, E^*_{js_j}) = n^j_{t+1,s_j}/(e\alpha_j^2) - n^j_{1,t-1}/\alpha_j$. Therefore, similarly as in (6.6.10),

$$\chi(s_{h_j^-}) - \chi(s_h) = \chi(-E^*_{j,s_j}) + \frac{1}{e\alpha_j}\Big(a_0 + \sum_{t=1}^{v}\frac{a_t}{\alpha_t}\Big) - \Big\{\frac{a_j\omega'_j}{\alpha_j}\Big\}.$$

By (3.5.2) and (6.3.6) we have

$$\chi(-E^*_{j,s_j}) = \frac{1}{2} + \frac{\mathfrak{r}}{2\alpha_j} - \frac{1}{2e\alpha_j^2}, \qquad \chi(a_0E_0^*) = -\frac{a_0^2}{2e} - \frac{a_0}{2}(1+\mathfrak{r}).$$

Hence, an induction by decreasing $(a_1, \dots, a_v)$ to $(0, \dots, 0)$, and via the notation $\mathfrak{a} := (a_0 + \sum_{j=1}^{v} a_j/\alpha_j)/|e|$, we obtain

$$-\chi(s_h) = \frac{a_0}{2} - \frac{e\mathfrak{r}\mathfrak{a}}{2} + \frac{e\mathfrak{a}^2}{2} - \sum_{j=1}^{v}\sum_{i_j=1}^{a_j}\Big(\Big(\frac{i_j\omega'_j}{\alpha_j}\Big)\Big). \tag{6.6.19}$$

Example 6.6.21 (r_h in a Minimal Good Star-Shaped Graph)

In general, $r_h = \{s_h\}$, or, it is the fractional part of any representative of h. Nevertheless, similarly as in the string case, we wish to emphasize some compatibility properties among the coefficients of r_h, which from these general statements cannot be seen. Let us write $r_h = r_0E_0 + \sum_{j,i} r_{ji}E_{ji}$, and we consider also the dual representation $r_h = a_0E_0^* + \sum_{j,i} a_{ji}E^*_{ji}$. Similarly as for s_h we define the reduced coefficients too by $a_j := \sum_i n^j_{i+1,s_j}a_{ji}$ $(1 \le j \le v)$.

Since for any j the following identity holds

$$-E^*_{j,i+1} + n^j_{1,i}E^*_{j,1} - n^j_{2,i}E_0^* = \sum_{k=1}^{i} n^j_{k+1,i}E_{jk}, \tag{6.6.20}$$

we get that $-r_{j,i+1} + n^j_{1,i}r_{j,1} - n^j_{2,i}r_0 \in \mathbb{Z}$, hence for any j and i

$$r_{j,i+1} = \{n^j_{1,i}r_{j,1} - n^j_{2,i}r_0\}. \tag{6.6.21}$$

In particular, all the coefficients are determined by r_0 and $\{r_{j,1}\}_j$. Furthermore, (6.6.20) for $i = s_j$ intersected with r_h (the left hand side with $r_0E_0 + \sum_{j,i} r_{ji}E_{ji}$, the right hand side with $a_0E_0^* + \sum_{j,i} a_{ji}E^*_{ji}$) gives

$$\alpha_j r_{j,1} - \omega_j r_0 = \sum_{k=1}^{s_j} n^j_{k+1,i}a_{jk} = a_j. \tag{6.6.22}$$

Next, assume that (X, o) is a weighted homogeneous singularity, $\widetilde{X}$ is its minimal good resolution. Let us compute $H^0(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(-r_h))$. This computation leads also to the equivariant Poincaré series of the universal abelian covering. (Such a description is not provided by the Dolgachev–Pinkham theory, and the next computation of the equivariant Poincaré series is different from the one given by Neumann in [506]. For a discussion in the spirit of [506] see 8.1.B.)

Consider the special analytic plumbing from 5.1.36. Consider non-negative integers ℓ, m_j, k, and the section

$$\varphi := u_0^{\ell} \prod_j (v_0 - p_j)^{m_j} v_0^k \in H^0(U_0, \mathscr{O}(-r_h)).$$

By similar computations as in 6.6.19 for the transitions function of the line bundle $\mathscr{O}(-r_h)$ we get that φ extends regularly to U_{-1} if and only if

$$k + \sum_j m_j \leq b_0\ell + (-r_h, E_0) = b_0\ell + a_0. \tag{6.6.23}$$

Similarly, along the j-leg, φ has a regular extension in $U_{j,i}$ if and only if

$$(*_{j,i}) \qquad n^j_{1,i}(m_j + r_{j,1}) - n^j_{2,i}(\ell + r_0) - r_{j,i+1} \geq 0. \tag{6.6.24}$$

We show that $(*_{j,i})$ implies $(*_{j,i-1})$. For this it is enough to show that for $\ell \geq 0$

$$\left\lceil \frac{n^j_{2,i}(\ell + r_0) - n^j_{1,i} r_{j,1}}{n^j_{1,i}} \right\rceil \geq \frac{n^j_{2,i-1}(\ell + r_0) - n^j_{1,i-1} r_{j,1} + r_{j,i}}{n^j_{1,i-1}}. \tag{6.6.25}$$

Note that by (6.6.21) one has $n^j_{2,i-1}r_0 - n^j_{1,i-1}r_{j,1} + r_{j,i} = -k$, where $k := [-n^j_{2,i-1}r_0 + n^j_{1,i-1}r_{j,1}]$. If one rewrites the inequality in terms of k and one uses $n^j_{2,i}/n^j_{1,i} > n^j_{2,i-1}/n^j_{1,i-1}$, the inequality (6.6.23) follows.

In particular, only the restrictions $(*_{j,s_j})$ are relevant, namely $\alpha_j(m_j + r_{j,1}) - \omega_j(\ell + r_0) \geq 0$. This, via (6.6.22) reads as $\alpha_j m_j \geq \omega_j \ell - a_j$. Using (6.6.22) we have $a_j < \alpha_j$, hence the restriction for m_j is

$$m_j \geq \lceil (\omega_j \ell - a_j)/\alpha_j \rceil \geq 0. \tag{6.6.26}$$

Let us fix ℓ, the exponent of u_0^{ℓ} in the expression of the section φ. The universal abelian covering has a natural $\mathbb{C}^*$ action, hence the eigenspace $H^0(\widetilde{X}, \mathscr{O}(-r_h))$ admits a $\mathbb{C}^*$ action too. The homogeneous elements of $\mathscr{O}_{X_a,o}$ are graded by their vanishing order along the central exceptional curve in the minimal good resolution of (X_a, o). Sections φ with fixed ℓ correspond exactly to the $\mathfrak{o}(\ell + r_0)$-homogeneous elements of $\mathscr{O}_{X_a,o}$, cf. 6.2.12 (see also 8.1.B and 8.6.13). In the next paragraphs we will use the index ℓ for them.

The inequality (6.6.26) shows that the minimal m_j is $\lceil(\omega_j\ell - a_j)/\alpha_j\rceil$, and (6.6.23) shows that the maximal v_0-degree is $b_0\ell + a_0$. Therefore, the dimension of $H^0(\widetilde{X}, \mathscr{O}(-r_h))_\ell$ is

$$\dim\, H^0(\widetilde{X}, \mathscr{O}(-r_h))_\ell = \max\{0,\ 1 + N_{r_h}(\ell)\} \quad (\ell \geq 0), \tag{6.6.27}$$

where

$$N_{r_h}(\ell) := b_0\ell + a_0 - \sum_j \left\lceil \frac{\omega_j\ell - a_j}{\alpha_j} \right\rceil.$$

Assume that $s_h > r_h$, set $\Delta := s_h - r_h$, and let Δ_0 be the E_0-multiplicity of Δ (i.e. $\Delta_0 = -(\Delta, E_0^*)$). We claim that

$$\dim\, H^0(\widetilde{X}, \mathscr{O}(-r_h))_\ell = \max\{0,\ 1 + N_{r_h}(\ell)\} = 0 \ \text{ for } \ell < \Delta_0. \tag{6.6.28}$$

This is compatible with the fact that the inclusion $H^0(\widetilde{X}, \mathscr{O}(-s_h)) \subset H^0(\widetilde{X}, \mathscr{O}(-r_h))$ is an isomorphism (by 6.6.4 and its proof, see also (8.2.8)).

This claim (6.6.28) can also be proved as follows.

Assume that $s_h := \bar{a}_0E_0^* + \sum_{j,i} \bar{a}_{ji}E_{ji}^*$ and set $N_{s_h}(\ell) = b_0\ell + \bar{a}_0 - \sum_j \left\lceil \frac{\omega_j\ell - \bar{a}_j}{\alpha_j} \right\rceil$. Since $s_h - r_h$ is an integral combination of elements of type $E_0^* - \alpha_j E_{js_j}^*$ and $b_0E_0^* - \sum_j \omega_j E_{js_j}^*$, one obtains that $N_{s_h}(\ell) = N_{r_h}(\ell + \Delta_0)$. Assume that $1 + N_{r_h}(\ell) > 0$ for some $\ell < \Delta_0$. Then $1 + N_{s_h}(\ell) > 0$ for some $\ell < 0$, a fact which contradicts (SI_{red}) from 6.6.20.

For the corresponding statement at h^1-level see 6.8.19.

6.6.B *The 'Local' Zariski Decomposition*

Besides $l' \mapsto s(l') \in \mathscr{S}'$ (or the decomposition $s(l') = l' + e(l')$) there is another universal cycle associated with any $l' \in L'_{\mathbb{Q}} = L \otimes \mathbb{Q}$ provided by the so-called 'Zariski decomposition' $s_Z(l') = l' + e_Z(l')$. Originally it was defined for cycles on a projective smooth surface [782]. For the local version see e.g. [210, 727]. For any fixed $l' \in L'$ the advantage of $s_Z(l')$ compared with $s(l')$ is an additional orthogonality property. On the other hand, its disadvantage is that in general $\mathscr{O}(-s_Z(l'))$ (as a natural line bundle) is not well-defined—since $s_Z(l')$ does not necessarily belong to L'—, but even if $s_Z(l') \in L'$, $\mathscr{O}(-l')$ and $\mathscr{O}(-s_Z(l'))$ can hardly be compared, since $s_Z(l') - l'$ in general is not an integral cycle. ♣

6.6.22 The local Zariski decomposition is defined as follows. Below $L_{\mathbb{Q},\geq 0} := \mathbb{Q}_{\geq 0}\langle E_v\rangle_{v\in\mathscr{V}}$ and $\mathscr{S}_{\mathbb{Q}}$ is as in 2.1.17 (though in this subsection one can replace $L_{\mathbb{Q}}$ by $L_{\mathbb{R}}$ with the same output).

Proposition 6.6.23 *Let $\widetilde{X} \to X$ be a resolution of (X, o) as above.*

(a) For any $l' \in L_{\mathbb{Q}}$ there exists a unique minimal element $e_Z(l') \in L_{\mathbb{Q},\geq 0}$ with $s_Z(l') := l' + e_Z(l') \in \mathscr{S}_{\mathbb{Q}}$. That is, $-l'$ is the sum of the rational effective cycle $e_Z(l')$ and of the numerically effective rational cycle $-s_Z(l')$, with $e_Z(l')$ minimal.

(b) The rational cycles $e_Z(l')$ and $s_Z(l')$ from (a) satisfy the following additional orthogonality property: $(E_v, s_Z(l')) = 0$ for any E_v from the support of $e_Z(l')$. In particular, $(e_Z(l'), s_Z(l')) = 0$.

(c) The rational cycles $e_Z(l')$ and $s_Z(l')$ from (a) satisfy the following universality condition as well: There exists a unique decomposition of $-l'$ into $\widetilde{e}_Z(l') + (-\widetilde{s}_Z(l'))$ with $\widetilde{e}_Z(l') \in L_{\mathbb{Q},\geq 0}$, $\widetilde{s}_Z(l') \in \mathscr{S}_{\mathbb{Q}}$, and $(E_v, \widetilde{s}_Z(l')) = 0$ for any $E_v \in |\widetilde{e}_Z(l')|$.

Proof Part *(a)* follows similarly as part *(a)* of 6.6.3. In fact, $s_Z(l')$ is the unique minimal element of the intersection of convex real cones $(l' + L_{\mathbb{R},\geq 0}) \cap \mathscr{S}_{\mathbb{R}}$.

For *(b)* fix some $E_v \in |e_Z(l')|$, and note that for $t \in \mathbb{Q}_{\geq 0}$, $0 < t \ll 1$, the rational cycle $e_Z(l') - tE_v$ is still effective, and $s_t := s_Z(l') - tE_v$ is still in $\mathscr{S}_{\mathbb{Q}}$ provided that $(E_v, s_Z(l')) \neq 0$. Hence $(E_v, s_Z(l')) = 0$ necessarily by the minimality of $s_Z(l')$.

(c) Take such a pair $\widetilde{e}_Z$ and $\widetilde{s}_Z$. Then by *(a)–(b)* $\widetilde{e}_Z \geq e_Z$. Choose $E_v \in |\widetilde{e}_Z - e_Z|$. Then $(E_v, \widetilde{e}_Z) = (E_v, -l') \geq (E_v, e_Z)$, hence $(E_v, \widetilde{e}_Z - e_Z) \geq 0$, which implies $(\widetilde{e}_Z - e_Z)^2 \geq 0$. This can happen only if $\widetilde{e}_Z = e_Z$. □

Corollary 6.6.24

(a) $l' \leq s_Z(l') \leq s(l')$.

(b) $e_Z(nl') = n \cdot e_Z(l')$ and $s_Z(nl') = n \cdot s_Z(l')$ for any $n \in \mathbb{Z}_{>0}$.

(c) For any $l \in L'$ choose $o \in Z_{>0}$ such that $ol' \in L$ and $o \cdot s_Z(l') \in L$. Then for $n = ok + q$, $k, q \in \mathbb{Z}_{\geq 0}$

$$nl' \leq ql' + ok \cdot s_Z(l') \leq s_Z(ql') + ok \cdot s_Z(l') = s_Z(nl') \leq s(nl').$$

Hence, for any $l' \in L'$ the inclusion induces an isomorphism

$$H^0(\widetilde{X}, \mathscr{O}(-ql' - ok \cdot s_Z(l'))) \to H^0(\widetilde{X}, \mathscr{O}(-nl')). \tag{6.6.29}$$

The isomorphism (6.6.29) can be used to determine the qualitative asymptotic behaviour of $n \mapsto h^0(\widetilde{X}, \mathscr{O}(-nl'))$. For this we need the following facts.

Lemma 6.6.25

(a) Consider $l'_2 \in L'$ and $l = \sum_{v\in\mathscr{J}} n_v E_v^* \in \mathscr{S}$ with $\mathscr{J} \subset \mathscr{V}$, $n_v > 0$.

If $\mathscr{I} = \mathscr{V}$ then $h^1(\widetilde{X}, \mathscr{O}(-l_2' - kl)) = 0$ for $k \gg 0$. If $\mathscr{I} \neq \mathscr{V}$ then there exists $x = \sum_{v \in \mathscr{V} \setminus \mathscr{I}} m_v E_v \in L_{>0}$, with $h^1(\widetilde{X}, \mathscr{O}(-l_2' - kl - x)) = 0$ (for $k \gg 0$). In particular,

$$h^1(\widetilde{X}, \mathscr{O}(-l_2' - kl)) = h^1(x, \mathscr{O}_x(-l_2' - kl)) = h^1(x, \mathscr{M}\mathscr{N}^{\otimes k}),$$

where $\mathscr{M}, \mathscr{N} \in Pic(x)$ with $\mathscr{N}$ numerically trivial.

(b) [130] Assume that $x \in L_{>0}$ and $\mathscr{M}, \mathscr{N} \in Pic(x)$ with $\mathscr{N}$ numerically trivial. Then $k \mapsto h^1(x, \mathscr{M}\mathscr{N}^{\otimes k})$ is periodic for $k \gg 0$.

Proof *(a)* The statement for $\mathscr{I} = \mathscr{V}$ follows from 6.4.3. In the other case choose a cycle x supported on $\mathscr{V} \setminus \mathscr{I}$ such that $(x + l_2' - Z_K, E_v) \leq 0$ for all $v \notin \mathscr{I}$. Then use 6.4.3 again. Part *(b)* is proved in [130, Theorems 8]. □

Example 6.6.26 With the above notations assume $l' \in L'$ and write $n = ok + q$ with $0 \leq q < o$. Then, from (6.6.29), by the cohomology exact sequence of a sequence of type $0 \to \mathscr{L}(-y) \to \mathscr{L} \to \mathscr{L}|_y \to 0$ and Riemann–Roch formula one gets

$$\dim \frac{H^0(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(-ql'))}{H^0(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(-nl'))} = \dim \frac{H^0(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(-ql'))}{H^0(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(-ql' - ok \cdot s_Z(l')))}$$

$$= \chi(ok \cdot s_Z(l')) - (ok \cdot s_Z(l'), ql') + h^1(\mathscr{O}_{\widetilde{X}}(-ql')) - h^1(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(-ql' - ok \cdot s_Z(l'))).$$

Note that for each fixed q the expression

$$\chi(ok \cdot s_Z(l')) - (ok \cdot s_Z(l'), ql') + h^1(\mathscr{O}_{\widetilde{X}}(-ql'))$$

is a polynomial of degree 2 with quadratic part $-n^2 s_Z(l')^2/2$. Furthermore, for each fixed q, the expression $h^1(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(-ql' - ok \cdot s_Z(l')))$ is periodic for $n \gg 0$.

This can be compared with a similar identity applied directly for nl', namely

$$\dim \frac{H^0(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(-ql'))}{H^0(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(-nl'))} = \chi(okl') - (okl', ql') + h^1(\mathscr{O}_{\widetilde{X}}(-ql')) - h^1(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(-nl')).$$

In $\chi(okl')$ the leading quadratic part is $-n^2 l'^2/2$, which equals $-n^2(s_Z(l')^2 + e_Z(l')^2)/2$. Therefore, if $l' \notin \mathscr{S}'$ then (by comparison of the above two asymptotic behaviours) $n \mapsto h^1(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(-nl'))$ has a quadratic asymptotic behaviour with leading quadratic term $-n^2 e_Z(l')^2/2$ (which is nonzero whenever $l' \notin \mathscr{S}'$).

If $l' \in \mathscr{S}'$ then $n \mapsto h^1(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(-nl'))$ is bounded, cf. 6.6.6.

For more regarding the local Zariski decomposition and asymptotic behaviours of cohomology groups see [727, §1], or [129, 130, 210, 442].

6.7 The Monoids $\mathscr{S}_{an}$ and $\mathscr{S}'_{an}$

In the presence of a resolution the divisors of analytic functions of (X, o) (resp. of the universal abelian covering (X_a, o)) determine the monoids $\mathscr{S}_{an}$ (resp. $\mathscr{S}'_{an}$), sub-monoids of $\mathscr{S}$ and $\mathscr{S}'$. The unique minimal element of $\mathscr{S}_{an} \setminus \{0\}$ is the 'maximal ideal cycle' Z_{max}. It will be compared with the multiplicity, embedding dimension and the Hilbert–Samuel function of (X, o). Some peculiar properties of $\mathscr{S}_{an}$ (e.g., 'jump'-phenomenon) are also discussed. ♣

6.7.1 Let us fix a singularity (X, o) and one of its resolutions ϕ. Then the monoid $\mathscr{S} \subset L$ is well-defined, cf. 2.1.17, and it depends only on the *topological type* of (X, o). A rather subtle information about the *analytic structure* is provided by the analytic counterpart $\mathscr{S}_{an}$, a submonoid of $\mathscr{S}$.

Definition 6.7.2 For each holomorphic function $f \in \mathscr{O}_{X,o}$ consider $\mathrm{div}_E(f) \in \mathscr{S}$, cf. 2.1.18. Then define $\mathscr{S}_{an}$ as $\{\mathrm{div}_E(f) : f \in \mathscr{O}_{X,o}\} \subset \mathscr{S}$.

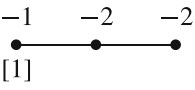
$0 \in \mathscr{S}_{an}$ since $0 = \mathrm{div}_E(1)$. If $s_1, s_2 \in \mathscr{S}_{an}$ then $s_1 + s_2 \in \mathscr{S}_{an}$ (take $f_1 f_2$) and $\min\{s_1, s_2\} \in \mathscr{S}_{an}$ (take the generic combination of f_1 and f_2). Hence $\mathscr{S}_{an}$ is a submonoid of $\mathscr{S}$ closed under the operator min (associated with its embedding in $L_{\geq 0}$). Hence, $\mathscr{S}_{an} \setminus \{0\}$ has a *unique minimal element*. Following S. S.-T. Yau, we call it the *maximal ideal cycle* of ϕ. We denote it by Z_{max}. Clearly $Z_{max} \geq Z_{min}$.

Lemma 6.7.3 *Set the topologically defined sub-semigroup* $\widetilde{\mathscr{S}} := \{l \in \mathscr{S} : (l + E - Z_K, E_v) \leq -2 \text{ for all } v\}$ *of* $\mathscr{S}$. *Then* $\widetilde{\mathscr{S}} \subset \mathscr{S}_{an}$. *Hence, a shifted copy of* $\mathscr{S}$ *inside* $\mathscr{S}$ *is always in* $\mathscr{S}_{an}$ *(independently of the analytic structure supported on the fixed topological type).*

Proof Apply 6.5.7 for $\mathscr{L} = \mathscr{O}_{\widetilde{X}}(-l)$. □

Remark 6.7.4 The above lemma and the vanishing theorem 6.4.3 can be compared with the next fact. Assume that $l \in \mathscr{S}_{an}$. Then $h^1(\mathscr{L}) \geq h^1(\mathscr{L}(-l))$ for any $\mathscr{L} \in \mathrm{Pic}(\widetilde{X})$. Indeed, if $l \in \mathscr{S}_{an}$ then $\mathscr{O}_{\widetilde{X}}(-l)$ has no fixed components. Moreover, if s is a general section from $H^0(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(-l))$, then multiplication by s gives a monomorphism of sheaves $\mathscr{L} \to \mathscr{L}(-l)$ whose cokernel is supported along the non-compact divisor of s, hence it is Stein, and its h^1 vanishes.

Example 6.7.5 In general $\mathscr{S}_{an} \neq \mathscr{S}$ and $Z_{min} \neq Z_{max}$. Consider the hypersurface singularities $(X_1, o) = \{x^2 + y^3 + z^{18} = 0\}$ and $(X_2, o) = \{z^2 + x(y^4 + x^6) = 0\}$. They share the same graph



In the case of (X_1, o) the equalities $Z_{min} = Z_{max} = E = \mathrm{div}_E(z)$ hold. Nevertheless, $Z_{min} < Z_{max}$ for (X_2, o). (The divisors of the coordinate functions

can be determined via the algorithm from 4.1.C, cf. 4.1.20. These cases can be compared also with the discussion from 5.1.30, which explains that in the second case there is no function with multiplicity 1 along the central elliptic curve.)

Example 6.7.6 $Z_{min} \neq Z_{max}$ can happen even if L_X is a rational homology sphere. E.g., for splice quotient singularities it happens frequently, cf. 8.5.27. But it can happen even for hypersurface weighted homogeneous (or Newton non-degenerate) singularities too, see e.g. $\{z^4 = x(y^5 + x^{16})\}$ (example given by Tomari).

Example 6.7.7 ([150]) Consider the minimal resolution of the double point singularity $(X, o) = \{f(x, y) = z^2\}$. If f is either locally irreducible, or it has even multiplicity, then $Z_{min} = Z_{max}$.

The terminology 'maximal ideal cycle' is motivated by the fact that Z_{max} is the divisorial part of the $\mathscr{O}_{\widetilde{X}}$-module $\phi^*\mathfrak{m}_{X,o}$; that is, $\phi^*\mathfrak{m}_{X,o} = \mathscr{I} \cdot \mathscr{O}_{\widetilde{X}}(-Z_{max})$ for a sheaf of ideals $\mathscr{I}$ of $\mathscr{O}_{\widetilde{X}}$, which is supported in finitely many points, the base points of $\mathscr{O}_{\widetilde{X}}(-Z_{max})$.

6.7.8 The multiplicity $\mathrm{mult}(X, o)$ of (X, o) can be introduced in many different equivalent ways. Let us start with the most geometric one: *we define it as the degree of the generic linear map* $(X, o) \to (\mathbb{C}^2, 0)$.

Theorem 6.7.9 ([719])

(a) If $\phi^*\mathfrak{m}_{X,o} = \mathscr{O}_{\widetilde{X}}(-Z_{max})$, *then* $mult(X, o) = -Z_{max}^2$.
(b) If $\phi^*\mathfrak{m}_{X,o} = \mathscr{I}\mathscr{O}_{\widetilde{X}}(-Z_{max})$ *for a proper ideal* $\mathscr{I}$, *then* $mult(X, o) > -Z_{max}^2$.

E.g., if $\phi^*\mathfrak{m}_{X,o} = \mathfrak{m}_P \cdot \mathscr{O}(-Z_{min})$, *where* $\mathfrak{m}_P$ *is the maximal ideal of a smooth point* P *of* E, *then* $mult(X, o) = -Z_{min}^2 + 1$.

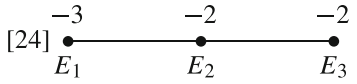
Proof If l_1 and l_2 are generic linear functions on (X, o), then $\mathrm{mult}(X, o)$ is the cardinality of $X \cap \{l_1 = \epsilon'\} \cap \{l_2 = \epsilon''\}$ for ϵ' and ϵ'' generic and small. This is the intersection multiplicity of the total transforms of l_1 and l_2 in $\widetilde{X}$. Since $(\mathrm{div}(\phi \circ l_i), E_v) = 0$, we have $\mathrm{mult}(X, o) = (S(\phi \circ l'), S(\phi \circ l'')) - Z_{max}^2$. But $(S(\phi \circ l'), S(\phi \circ l'')) \geq 0$, and it is zero if and only if there are no base points. □

Remark 6.7.10 If $\phi^*\mathfrak{m}_{X,o}$ has base points, then by additional blow-ups at these (infinitely near) points, one finds another resolution that has no base points. However, in general, the topological information (that is, the minimal resolution graph) is not sufficient to determine those resolutions that have no base points. Therefore, in general, the multiplicity of (X, o) cannot be determined from the resolution graph.

Example 6.7.11 Consider the normal hypersurface singularities $X_1 = \{x^2 + y^7 + z^{14} = 0\}$ and $X_2 = \{x^3 + y^4 + z^{12} = 0\}$. Then using the algorithm 4.1.16 one

checks that they share the same minimal resolution graph (one vertex with $g = 3$ and $E^2 = -1$). But their multiplicities are not the same.

Example 6.7.12 ([366]) Consider the family of weighted homogeneous isolated hypersurface singularities $(X_t, 0) := \{z^7 + x(y^9 + x^{27}) + tzy^8 = 0\}$. Since the weights are independent of t, the geometric genus, Milnor number, multiplicity and the link are constant along the deformation. In fact, for all t, $\mu_t = 1350$ and the minimal resolution graph is



Then $Z_{min} = E$. On the other hand, $Z_{max,t}$ is not constant: for $t = 0$ one has $Z_{max,0} = E_1 + 2E_2 + E_3$, but for $t \neq 0$ one gets $Z_{max,t} = Z_{min}$. (In fact, even the embedded components of the pullback of the corresponding maximal ideals jump.) The Milnor numbers of the generic linear sections are $\mu_0^{(2)} = 54$ and $\mu_t^{(2)} = 50$ for $t \neq 0$.

The family $\{z^5 + tzy^6 + xy^7 + x^{15} = 0\}$ given in [80] has similar properties.

See also 6.8.13 for an equisingular deformation (with constant p_g and constant rational homology sphere link) along which the embedding dimension jumps.

Example 6.7.13 Any normal surface singularity (X, o) of multiplicity two is analytically equivalent with a suspension of type $\{f(x, y) = z^2\}$, for a certain reduced plane curve singularity f. Indeed, consider a generic projection $\pi : (X, o) \to (\mathbb{C}^2, 0)$—which is a degree two branched covering—and let f be the reduced equation of the branched locus of π. Then the monodromy representations of the regular $\mathbb{Z}_2$ coverings over $(\mathbb{C}^2, 0) \setminus \{f = 0\}$ of π and of the natural suspension projection agree, hence the statement follows from Stein theorem 12.3.3.

Example 6.7.14 For any pair of analytic function (f, g), such that $(f, g) : (X, o) \to (\mathbb{C}^2, 0)$ is finite, we have $\mathrm{mult}(X, o) \leq \mathrm{degree}(f, g)$. Indeed, by flatness, $\mathrm{degree}(f, g) = \dim(\mathscr{O}_{X,x}/(f, g))$, which is bounded from below by the multiplicity (use e.g. a combination of Formula 14.4 and Theorem 14.10 from [411]).

Example 6.7.15 ([56]) As we mentioned e.g. in 6.7.11 $\mathrm{mult}(X, o)$ is not topological. However, we have the following topological upper bound. Fix a resolution and set $D \in L$ with $(D, E_v) \leq -(K_{\widetilde{X}} + E, E_v) - 2$ for every $v \in \mathscr{V}$. Then $\mathrm{mult}(X, o) \leq -D^2$. Indeed, by 6.5.7 applied for $\mathscr{L} = \mathscr{O}_{\widetilde{X}}(-D)$ we obtain two functions f and g with $\mathrm{div}_E(f) = \mathrm{div}_E(g) = D$ and disjoint strict transforms. Then an argument as in the proof of 6.7.9 shows that $\mathrm{degree}(f, g) = -D^2$. Then apply 6.7.14.

6.7.16 Some Terminology Regarding $\mathfrak{m} = \mathfrak{m}_{X,o}$ The vector space $\mathfrak{m}/\mathfrak{m}^2$ is called the *cotangent space of* $\mathscr{O}_{X,o}$. Its dimension $\mathrm{emb}(X, o)$ is called the *embedding dimension of* (X, o). It equals the minimal number of generators of the ideal $\mathfrak{m}$, or, it is the smallest dimension n of $(\mathbb{C}^n, o)$, where (X, o) can be embedded [395, (1.9)].

The *Hilbert–Samuel function* is defined as $f^{HS}(k) := \dim_{\mathbb{C}}(\mathscr{O}_{X,o}/\mathfrak{m}^k_{X,o})$ for any $k \geq 1$. The *Hilbert–Samuel polynomial* is the unique polynomial

$$P^{HS}(k) = \sum_{i=0}^{2} a_i \binom{k}{i} \quad (a_i \in \mathbb{Z}) \tag{6.7.1}$$

such that $P^{HS}(k) = f^{HS}(k)$ for every k sufficiently large.

Example 6.7.17 Usually, the embedding dimension cannot be determined from the topology either. For multiplicities and embedding dimensions of different analytic structures supported on the topological type identified by the graph of $S^3_{-1}(T_{2,3}\#T_{2,3})$ from 5.2.5 see [495]. For weighted homogeneous cases see e.g. 5.1.40 and 5.1.41.

Theorem 6.7.18 *Fix a resolution such that $\mathscr{O}_{\widetilde{X}}(-Z_{max})$ has no basepoints. Then*

(a) $\phi_*(\mathscr{O}(-\lfloor Z_K \rfloor - kZ_{max}))_o \subset \mathfrak{m}^k_{X,o} \subset \phi_*(\mathscr{O}(-kZ_{max}))_o$ for any $k > 0$;
(b) for $k \gg 0$ one has $\dim(\mathscr{O}_{X,o}/\mathfrak{m}^k_{X,0}) = -k^2 Z^2_{max}/2 +$ lower oder terms.

In particular, the first coefficient of the Hilbert–Samuel polynomial is mult$(X, o)/2$.

Proof *(a)* The first inequality follows from 6.5.8 applied for $\mathscr{L} = \mathscr{O}_{\widetilde{X}}(-Z_{max})$ and $\mathscr{F} = \mathscr{O}_{\widetilde{X}}(-\lfloor Z_K \rfloor - Z_{max})$. *(b)* follows from part *(a)* and 6.4.7. □

Remark 6.7.19 One has the following inequalities; the first one proved by Abhyankar in [3], for the second one we assume that $-Z^2_{max} = \mathrm{mult}(X, o)$:

$$1 \geq \mathrm{emb}(X, o) - \mathrm{mult}(X, o) \geq \chi(Z_{max}).$$

Indeed, let $\mathscr{I} \subset \mathfrak{m}$ be the ideal generated by two generic elements of $\mathfrak{m}$. Then $\mathrm{mult} = 1 + \dim(\mathfrak{m}/\mathscr{I}) \geq 1 + \dim(\mathfrak{m}/(\mathscr{I} + \mathfrak{m}^2)) = 1 + \dim(\mathfrak{m}/\mathfrak{m}^2) - \dim(\mathscr{I} + \mathfrak{m}^2)/\mathfrak{m}^2 = \mathrm{emb}(X, o) - 1$. Equality holds if and only if $\mathfrak{m}^2 \subset \mathscr{I}$, or, $\mathscr{I}\mathfrak{m} = \mathfrak{m}^2$ (see also [609]).

For the second one, use the exact sequence of $0 \to \mathscr{O}_{\widetilde{X}}(-2Z_{max}) \to \mathscr{O}_{\widetilde{X}}(-Z_{max}) \to \mathscr{O}_{Z_{max}}(-Z_{max}) \to 0$, and the inequalities $\mathrm{emb}(X, o) \geq \dim \mathfrak{m}/H^0(\mathscr{O}_{\widetilde{X}}(-2Z_{max}))$, and $h^1(\mathscr{O}_{\widetilde{X}}(-Z_{max})) \geq h^1(\mathscr{O}_{\widetilde{X}}(-2Z_{max}))$. This last inequality follows from 6.7.4.

Abhyankar's inequality is equality for singularities with multiplicity two (cf. 6.7.13) and e.g. for rational singularities (cf. 7.1.17). But it is strict e.g. for any Gorenstein (X, o), which is not a double point [610]. The identity

$1 = \mathrm{emb}(X, o) - \mathrm{mult}(X, o)$, or the 'minimal reduction' $\mathscr{I}\mathfrak{m} = \mathfrak{m}^2$ (or, similar identities close to this sharp one) imply several additional very special properties of the local ring $\mathscr{O}_{X,o}$, which can be proved merely ring-theoretically, see e.g. [609], or [724, (2.6)], or the corresponding literature in local ring theory starting with the work of Northcott.

6.7.20 The Analytic Monoid $\mathscr{S}'_{an}$ In order to define the submonoid $\mathscr{S}'_{an} \subset \mathscr{S}'$, we will use the notations of Sect. 6.2. In particular we assume that $H_1(L_X, \mathbb{Q}) = 0$.

Associated with the resolution $\widetilde{Z}$ of the universal abelian cover (X_a, o) of (X, o), we set its exceptional divisor E_a and its lattice L_a. Let $\mathscr{S}_{a,an}$ be the analytic monoid of (X_a, o) associated with the resolution $\widetilde{Z}$, that is, $\mathscr{S}_{a,an} = \{\mathrm{div}_{E_a}(f \circ \phi_a) : f \in \mathscr{O}_{X_a,o}\} \subset L_a$. Recall that $p^* : L' \to L'_a$ is the cohomological morphism induced by c, whose image is in L_a (cf. 6.2.6 and (6.2.3)).

Definition 6.7.21 We define the submonoid $\mathscr{S}'_{an}$ as $(p^*)^{-1}(\mathscr{S}_{a,an})$; in other words:

$$\mathscr{S}'_{an} = \{l' \in L' : p^*(l') = \mathrm{div}_{E_a}(f \circ \phi_a) \text{ for some } f \in \mathscr{O}_{X_a,o}\} \subset L'.$$

Lemma 6.7.22 $\mathscr{S}'_{an} \subset \mathscr{S}'$.

Proof If $p^*(l') = \mathrm{div}_{E_a}(f \circ \phi_a)$ then $|H|(l', E_v) = (p_* p^*(l'), E_v) = (p^*(l'), p^* E_v)_{L_a} = (\mathrm{div}_{E_a}(f \circ \phi_a), p^*(E_v))_{L_a} \le 0$. □

Note that $\mathscr{S}'$ can be written as a disjoint union of $\sqcup_{h \in H} \mathscr{S}'_h$, where $\mathscr{S}'_h = \{l' \in \mathscr{S}' : [l'] = h\}$. According to this decomposition, $\mathscr{S}'_{an}$ also decomposes as $\sqcup_{h \in H} \mathscr{S}'_{an,h}$, where $\mathscr{S}'_{an,h} = \mathscr{S}'_{an} \cap \mathscr{S}'_h$. Note that $\mathscr{S}'_{an,h_1} + \mathscr{S}'_{an,h_2} \subset \mathscr{S}'_{an,h_1+h_2}$.

Corollary 6.2.11 *(b)* implies the following (compare also with (8.2.11)).

Corollary 6.7.23

(a) $\mathscr{S}'_{an,0} = \mathscr{S}_{an}$. *More generally, for any* $h \in H$ *one has*

$$\mathscr{S}'_{an,h} = \{r_h + div_E(s), \text{ where } s \in \phi_*(\mathscr{O}_{\widetilde{X}}(-r_h))_o\}.$$

In particular, $\mathscr{S}'_{an}$ *does not depend on the choice of* $r : \widetilde{Z} \to Z$.

(b) If $l'_1, l'_2 \in \mathscr{S}'_{an,h}$, *then* $\min\{l'_1, l'_2\} \in \mathscr{S}'_{an,h}$.

Note also that Lemma 6.7.3 has its analogue for $\mathscr{S}'_{an}$ as well.

Example 6.7.24 Assume that (X, o) is a Hirzebruch–Jung singularity. We will use the notations of 2.3.1: (X, o) is the quotient of $(\mathbb{C}^2, 0)$ by $\mathbb{Z}_n$, where the action is $\xi * (z_1, z_2) = (\xi z_1, \xi^q z_2)$. Clearly $(X_a, o) = (\mathbb{C}^2, o)$.

Let $\{E_k\}_{k=1}^s$ be the irreducible exceptional curves of $\widetilde{X} \to X$, and introduce the integers n_{ij} as in (2.2.3). Recall also that the monoid $\mathscr{S}'$ is generated over $\mathbb{Z}_{\ge 0}$ by $\{E_k^*\}_k$. Then one verifies that $\min\{n_{1,k-1}E_1^*, n_{k+1,s}E_s^*\} = E_k^*$ for any $1 < k < s$.

At the level of divisors one gets $c^*(E_1^*) = \mathrm{div}_{E_a}(z_2)$, $c^*(E_s^*) = \mathrm{div}_{E_a}(z_1)$, and

$$c^*(E_k^*) = \mathrm{div}_{E_a}(z_2^{n_{1,k-1}} + z_1^{n_{k+1,s}}) = \min\{\mathrm{div}_{E_a}(z_2^{n_{1,k-1}}), \mathrm{div}_{E_a}(z_1^{n_{k+1,s}})\} \quad \text{for } 1 < k < s.$$

Hence each $E_k^* \in \mathscr{S}'_{an}$, which shows that $\mathscr{S}'_{an} = \mathscr{S}'$.

Remark 6.7.25

(a) The cases $\mathscr{S}'_{an} = \mathscr{S}'$, or $\mathscr{S}_{an} = \mathscr{S}$ are really remarkable. We will prove them for any rational singularity (and for an arbitrary resolution) in 7.1.13, and for minimally elliptic singularities (for the minimal and minimal good resolutions) in 7.2.31. But any further extension is seriously obstructed.

In general, it can happen that $|\mathscr{S} \setminus \mathscr{S}_{an}| = \infty$ (see 7.2.99 or 8.5.27). The existence of $s \in \mathscr{S} \setminus \mathscr{S}_{an}$ with $ns \in \mathscr{S}_{an}$ ($n \geq 2$) is also frequent (e.g. when $Z_{max} = 2Z_{min}$, or see 8.5.B). The identification of $\mathscr{S}'_{an}$ for a fixed analytic type can be very hard, and it is even harder to identify (for a fixed graph Γ, hence for a fixed $\mathscr{S}'$) the set of submonoids $\mathscr{S}'_{an}$ induced by all the possible analytic structures. E.g., the following problem is also open: for a fixed resolution graph determine the list of possible elements of $\mathscr{S}$, which can be realized as Z_{max} for different analytic structures.

(b) It is instructive to emphasise the following facts as well. Let us fix a topological type, a plumbing graph (hence L and $\mathscr{S}$). Then, by Winters–Pichon theorem [567] (see 2.5.8 here) for any $s \in \mathscr{S}$ there exists an analytic type (depending on s) for which $s \in \mathscr{S}_{an}$. Nevertheless, we have to be aware of the following phenomenon. Assume additionally that the link is an integral homology sphere, but it is not homeomorphic with the links of Brieskorn–Pham singularities of type $(2, 3, 5)$, $(2, 3, 7)$ and $(2, 3, 11)$. (These exceptions are exactly those integral homology sphere links which are provided either by rational or by minimally elliptic singularities, cf 7.2.29.) Then, by [488], there exists a (not necessarily minimal) resolution graph Γ, and a subset $I \subset \mathscr{V}(\Gamma)$, $|I| \geq 2$, such that for any analytic structure there exists at least one $v \in I$ with $E_v^* \notin \mathscr{S}_{an}$ (that is, there is no analytic structure in which they can be analytically and simultaneously realised). (Here, in this statement we can even assume that the corresponding knots in $M(\Gamma)$, associated with $\{E_v^*\}_{v \in I}$, are all topologically different.) We expect similar behaviour for arbitrary graphs too.

Here is a concrete example: take the graph from 5.2.5, and consider the cycles Z_{min} and E_v^*, where v is one of the (-3)-end-curves. Then they have no simultaneous analytic realisations. Indeed, since the corresponding links have linking number 2, in the case of analytic realization the pair of functions would define a covering $(X, o) \to \mathbb{C}^2$ of degree 2. This can happen only if (X, o) is of type $f(x, y) = z^2$ (cf. 6.7.13). Since all such integral homology sphere links are classified in 5.3.21, and our graph is not in this list, we get a contradiction.

For similar examples see also 7.2.33 and 7.2.100.

(c) Let us fix a resolution graph Γ such that $M(\Gamma)$ is a rational homology sphere. If $\mathscr{M}$ is a submonoid of $\mathscr{S}'$ denote its *saturation* $\{l' \in \mathscr{S}' : nl' \in \mathscr{M} \text{ for some } n \geq 1\}$ by $\overline{\mathscr{M}}$.

Recall that a resolution $\widetilde{X}$ satisfies the *end curve condition* (ECC) if $E_v^* \in \mathscr{S}'_{an}$ for every end-vertex v. In 8.5.23 we will show that if $\widetilde{X}$ satisfies ECC then $\overline{\mathscr{S}'_{an}} = \mathscr{S}'$.

Let us say (cf. [452]) that a resolution $\widetilde{X}$ satisfies the *weak end curve condition* (WECC) if $E_v^* \in \overline{\mathscr{S}'_{an}}$ for every end-vertex v. In fact, in [452, Prop. 9.2.2] it is proved that $\widetilde{X}$ satisfies the WECC if and only if $\overline{\mathscr{S}'_{an}} = \mathscr{S}'$. For some other characterisations of the WECC property see [456, §8] (and 8.5.37 below).

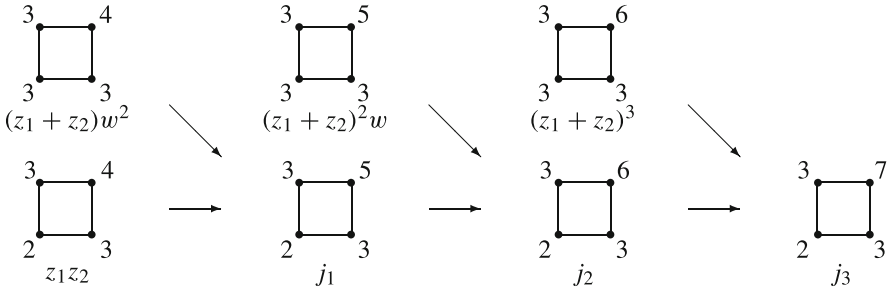
In 8.5.27 we present an example, which satisfies the ECC, hence $\overline{\mathscr{S}'_{an}} = \mathscr{S}'$, however $|\mathscr{S}' \setminus \mathscr{S}'_{an}| = \infty$. (In particular, [452, Lemma 9.1.2(b)] is unfortunately stated.)

6.7.26 The 'Jump'-Phenomenon In the identification of $\mathscr{S}_{an}$ it would be extremely helpful to find a finite set of 'generators' such that $\mathscr{S}_{an}$ is generated (as a 'sub-min-monoid' of $\mathscr{S}$) by these elements via semigroup addition of $\mathscr{S}$ and by 'min-operator' (induced by the embedding into L). Geometrically this means that we wish to find a set of 'distinguished' functions $\{g_i\}_i$ such that any element of $\mathscr{S}_{an}$ is realized as the divisor of a *generic combination* of monomials of $\{g_i\}_i$. Note that we do not require that for a certain *special combination* $f = \sum_\alpha c_\alpha \prod_i g_i^{\alpha_i}$ of the g_i-monomials $\mathrm{div}_E(f)$ is not strictly larger than the min of the divisors of the corresponding monomials. However, we require that if such a 'jump' appears for a certain special monomial combination, then $\mathrm{div}_E(f)$ can be realized as min of another monomial combination. In this way, in the determination of $\mathscr{S}_{an}$ we do not need to analyse the appearances of the possible jumps (in general a rather hard job).

Even if one has a candidate set $\{g_i\}_i$, the proof of the above property is not easy. Cases when one can list such a set of generators is remarkable: e.g., if (X, o) satisfies the end curve condition, that is, it is a splice quotient, then the set of 'end curve functions' serve as 'distinguished functions' in the above sense, see 8.5.B. (But even in this case, for special combinations of monomials a jump can occur. E.g., in the minimal good resolution of the germ $x^2 = y^3 + z^7$, the coordinate functions are end curve functions, and $\mathrm{div}(x^2) = \mathrm{div}(y^3 + z^7) > \min\{\mathrm{div}(y^3), \mathrm{div}(z^7)\}$.)

Next we present another example when certain elements of $\mathscr{S}_{an}$ appear as 'jumps'. Consider the complete intersection cusp singularity embedded in $(\mathbb{C}^4, 0)$, given by the equations $z_1z_3 = z_2z_4 = (z_1 + z_2 + z_3 + z_4)^3$. Its graph is a cyclic graph with 4 vertices, all of them decorated by -3 (in order to see this blow up the origin). Let us write $w := \sum_i z_i$. Then one verifies that (after convenient ordering of the vertices), the divisors of the functions z_1, z_2, $z_1 + z_2$, and w are

Here no jump is present yet. Note that the multiplicities of z_1z_2 are 3, 4, 3, 2. A computation shows that the cycles 3, n, 3, 2 for $n = 4, 5, 6, 7$ are elements of $\mathscr{S}$. The first one is realized by z_1z_2, but the others cannot be realized by the divisors of the coordinate functions via sum and min. However, we know (see 7.2.31 and 7.2.19) that they belong to $\mathscr{S}_{an}$, hence they must be realized by 'jumps'. Here is the picture of this 'non-min jump sequence' for a *special* combination of functions $j_1,\ j_2,\ j_3$:



where $j_1 = z_1z_2 - (z_1+z_2)w^2$, $j_2 = j_1 - (z_1+z_2)^2w$, $j_3 = j_2 - (z_1+z_2)^3$.

Example 6.7.27 Let $\mathscr{I} \subset \mathscr{V}$ be a subset of vertices, and consider the projection $pr : L \to \mathbb{Z}\langle E_v\rangle_{v\in\mathscr{I}}$, $\sum_{v\in\mathscr{V}} n_vE_v \to \sum_{v\in\mathscr{I}} n_vE_v$. Then $pr(\mathscr{S}_{an}) \subset \mathbb{Z}^{|\mathscr{I}|}$ is a sub-monoid, which might carry important information about the analytic type of (X, o).

E.g., if (X, o) is weighted homogeneous with rational homology sphere link with $\nu \geq 3$, and $\mathscr{I} = \{v_0\}$ consists of the central vertex, then by 5.1.26 $pr(\mathscr{S}_{an}) = \{\ell \in \mathbb{Z}_{\geq 0} : R_\ell \neq 0\} = \{\ell \in \mathbb{Z}_{\geq 0} : N(\ell) \geq 0\}$. For more on this numerical monoid see 8.5.28 or [351].

6.7.A Kulikov Singularities

Kulikov singularities were first introduced by Kulikov in his study of Arnold's uni- and bi-modal singularities [342] (see 7.2.24 here). They constitute a particular family of 'Kodaira singularities' of Karras [300, 301]. Later they were intensively studied by J. Stevens, see e.g. [661, 666]. The construction of the Kulikov analytic type connects a holomorphic (degenerating) family of curves with the resolution of surface singularities. The resolution graphs of Kulikov

singularities share a special property, they are called 'Kulikov graphs'. They appear frequently, e.g. the graphs of surgery manifolds $S^3_{p/q}(\#_i K_i)$ are Kulikov. Having this construction we will be able to compare different analytic structures (e.g. splice quotient or hypersurface analytic type versus Kulikov) on the same topological type. In this section—after the definitions—we determine the fundamental, maximal and the characteristic cycles. Later we will study with several details the rational and elliptic Kulikov singularities as well. ♣

6.7.28 Consider a holomorphic family $f : W \to D$ of genus g curves, cf. 2.1.14. We denote the divisor of f by $\mathrm{div}(f)$, and let $\{C_v\}_v$ be the irreducible components of $|\mathrm{div}(f)|$. We assume that (W, f) is 'minimal' (that is, W is a minimal smooth surface: it contains no (-1)-curves) and it 'has no multiple fibers' (that is, if $\mathrm{div}(f) = \sum_v n_v C_v$, then $\gcd_v\{n_v\} = 1$). Since f is flat, $\chi(\mathscr{O}_{\mathrm{div}(f)}) = \chi(\mathscr{O}_{f^{-1}(t)}) = 1 - g$.

Let $c_1 = c_1(\Omega^2_W) \in H^2(W, \mathbb{Z})$. It satisfies the usual adjunction relations $-c_1(C_v) = (C_v, C_v)_W + 2 - 2g(C_v) - 2\delta(C_v)$ and for an effective cycle supported on $|\mathrm{div}(f)|$ one has $\chi(\mathscr{O}_l) = -(l, l)_W/2 - c_1(l)/2$. We denote this expression by $\chi_W(l)$.

If $f : W \to D$ is minimal of genus $g = 0$ then f is a trivial fibration, that is $|\mathrm{div}(f)|$ contains only one irreducible component, which is isomorphic to $\mathbb{P}^1$. [Indeed, if $|\mathrm{div}(f)|$ has more than one component then by the semi-definiteness of the intersection form, cf. 2.1.14, we have $C_v^2 < 0$. Furthermore, the minimality and adjunction relations imply $c_1(C_v) \geq 0$. Therefore, $1 = 1 - g = \chi_W(\mathrm{div}(f)) = -(\mathrm{div}(f), \mathrm{div}(f))/2 - c_1(\mathrm{div}(f))/2 = -c_1(\mathrm{div}(f))/2 \leq 0$, a contradiction.]

By Castelnuovo's Contractibility Criterion any W can be contracted to a minimal surface W_{min} on which f induces a minimal family $f_{min} : W_{min} \to D$. By 2.4.4 this contraction is unique whenever $g > 0$ (since in a minimal family with $g > 0$ all the rational irreducible components have negative self-intersection, hence 2.4.4(3) applies). However, for genus 0 families the contraction to a minimal family is not unique (take e.g. two (-1)-curves with one transversal intersection, then one has two minimal representatives).

Now we describe the construction of Kulikov singularities (the name is given by J. Stevens, for the construction see [342, 660, 661], compare also with [300]). Start with a minimal family $f : W \to D$ of genus g. Perform successively a finite number of blowing ups such that at each time the center of the blow up sits on a unique component and along this component f has multiplicity one. (A center can be on a newly created component as well.) Suppose that the final family $\widetilde{f} : \widetilde{W} \to D$ contains r (-1)-curves, say $F_1, \ldots, F_r$; we denote their union by F. Let $\{E_v\}_v$ be the collection of the other irreducible components of the central fiber. Let $\widetilde{X}$ be a small neighbourhood of $E := \cup_v E_v$. Then E is

connected and $(E_v, E_u)_{\widetilde{W}}$ is negative definite, hence $\cup_v E_v$ in $\widetilde{X}$ can be contracted by Grauert Theorem [223]. In this way we get a normal surface singularity (X, o) with resolution $\widetilde{X} \to X$. Furthermore, the restriction of $\widetilde{f}$ to $\widetilde{X}$ factors to a holomorphic map $h : (X, o) \to (D, 0)$. The singularity (X, o), endowed with the map h, is called the Kulikov singularity associated with (W, f) and the sequence of blowing ups appearing in the construction.

Since in $\mathrm{div}(\widetilde{f})$ each F_i appears with multiplicity 1, $h : (X, o) \to (D, 0)$ defines an isolated singularity; $(h^{-1}(0), o)$ has r irreducible components (the images of $F_i \cap \widetilde{X}$). Moreover, (X, o) is a smoothing of the curve singularity $(h^{-1}(0), o)$ (the nearby fiber of h is the intersection of $\widetilde{X}$ with the generic smooth fiber of $\widetilde{f}$).

Write $\mathrm{div}(\widetilde{f})$ as $Z + \sum_i F_i$, where Z is supported on E. We claim that Z is the fundamental cycle Z_{min} in $\widetilde{X}$. Indeed, since $(Z, E_v) \leq (\mathrm{div}(\widetilde{f}), E_v)_{\widetilde{W}} = 0$, we get that $Z \in \mathscr{S}$ and $Z_{min} \leq Z$. Write $\Delta := Z - Z_{min}$. Since F intersects only multiplicity one E_v-components, hence the support ofΔ does not intersect F, one has $(F, \Delta)_{\widetilde{W}} = 0$. Therefore, $0 = (Z_{min}+\Delta+F)^2_{\widetilde{W}} = (Z_{min}+F)^2_{\widetilde{W}}+2(\Delta, Z_{min})+(\Delta, \Delta)$, with all terms on the right hand side non-positive. Therefore, $(\Delta, \Delta) = 0$, hence $\Delta = 0$.

In particular, $Z_{min} = \mathrm{div}_E(\widetilde{f}|_{\widetilde{X}})$ and $Z_{min} = Z_{max}$. Furthermore,

$$-Z^2_{min} = (Z_{min}, F)_{\widetilde{W}} = r. \tag{6.7.2}$$

Similarly, $\chi_{\widetilde{W}}(F) = \sum_{i=1}^r \chi(\mathscr{O}_{F_i}) = r$ too, hence

$$1-g = \chi_{\widetilde{W}}(Z_{min}+F) = \chi(Z_{min})+\chi_{\widetilde{W}}(F)-(Z_{min}, F)_{\widetilde{W}} = \chi(Z_{min}). \tag{6.7.3}$$

Note that the strict transform of $h^{-1}(0)$ in $\widetilde{X}$ intersects E transversally and along components with multiplicity one. Therefore, modifying h by a nearby generic hyperplane section (that is, by $h + sh' \in \mathscr{O}_{X,o}$, where $s \in (\mathbb{C}, 0)$ small parameter, $h' \in \mathscr{O}_{X,o}$, and the class of $h + sh'$ in $\mathfrak{m}_{X,o}/\mathfrak{m}^2_{X,o}$ is generic) provides the same topological picture. In particular, we obtain that for Kulikov singularities the strict transform of the generic hyperplane section is reduced, smooth, and it intersects the exceptional set transversally along components of multiplicity one. The converse is also true (see e.g. [300]): if a singularity has this property then its minimal resolution $\widetilde{X}$ (and the hyperplane section) can be extended to a holomorphic family $\widetilde{W} \to D$, hence (X, o) is Kulikov.

Moreover, a resolution graph (topological type) supports a Kulikov analytic structure if and only if its fundamental cycle has the following property: if $(Z_{min}, E_v) < 0$ for some v then the multiplicity of Z_{min} along E_v is one, cf. [300]. (Indeed, such a graph can be completed by (-1)-vertices to a semi-definite graph whose analytic realization is guaranteed by Winters theorem [753].) Graphs with this property will be called *Kulikov graphs*.

The Milnor fiber of the smoothing h is a genus g Riemann surface with r discs deleted, hence its Milnor number (first Betti number) satisfies $1-\mu = 2-2g-r$. By a formula of Buchweitz and Greuel [90] $\mu = 2\delta - r + 1$, where δ is the δ-invariant of $(h^{-1}(0), o)$. Therefore (for a direct proof see also [660]) $g = \delta - r + 1$. Hence

$$\chi(Z_{min}) = 1 - g = r - \delta \leq 1. \tag{6.7.4}$$

(In fact, $\chi(Z_{min}) \leq 1$ for any resolution graph, see 6.6.9(b), and the inequality $\delta \geq r - 1$ is valid for any curve-germ, see 12.4.6.)

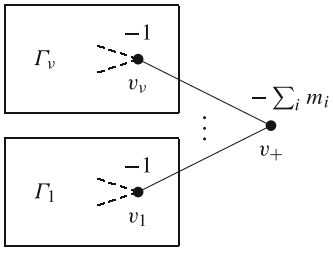
In general $\mathscr{O}_{\tilde{X}}(-Z_{min})$ is not free. E.g. $-Z_{min}^2 = r$ cannot equal $\mathrm{mult}(X, o)$ if $r = 1$.

Let us return to the Kulikov construction in the case $g > 0$. Let l be the strict transform of $\mathrm{div}(f)$ in $\widetilde{X}$, i.e., it is supported on $\cup_v C_v$ with the same coefficients. We claim that l is the characteristic cycle C (cf. 6.6.12).

Indeed, write Z_{min} as $l + \sum_\alpha E_\alpha$, where l and $\cup_\alpha E_\alpha$ have no common components, and if a component of l intersects a certain E_α then its multiplicity in l is 1. Therefore, there exists a computation sequence starting from l and ending with Z_{min} (cf. 6.6.7) such that at each step $(z_i, E_{v(i)}) = 1$ (cf. 6.6.9(b)), hence $\chi(l) = \chi(Z_{min})$. Next we show that there is no $0 < x < l$ such that $\chi(x) = \chi(l)$. For this we can assume that $\sum_\alpha E_\alpha = \emptyset$. (That is, all the regular points we blow up in the construction are situated on the central fiber of $f : W \to D$. In this case $l = Z_{min}$, hence for such case we prove that $C = Z_{min}$.) Assume that there exists $x < l = Z_{min}$ with $\chi(x) = \chi(l)$. Then, as above, there exists a computation sequence connecting x with Z_{min}, such that along the sequence χ is constant. Assume that at the very last step we add E_v. Then $(E_v, l - E_v) = 1$ (†). In particular, $(l, E_v) < 0$, hence E_v must intersect some $F_i's$, and the multiplicity of l along E_v is 1. Write $l = E_v + Y$. Then from (†) $(Y, E_v) = 1$, hence if we blow down all the F_i's, and we use $(\mathrm{div}(f), E_v)_W = 0$, in W we have $(E_v, E_v)_W = (-Y, E_v)_W = (-Y, E_v) = -1$. That is, W is not minimal.

Hence Z_{min} can be non-reduced only along the characteristic cycle C.

Example 6.7.29 We claim that the plumbing graph of $S^3_{-p/q}(\#_i K_i)$ (where K_i are algebraic knots, for the graph see 3.5.13) is Kulikov. Indeed, start with the graph:



It is negative semi-definite. Then one blows up k_1 generic points of the irreducible component corresponding to v_+. In this way one creates k_1 (-1)-curves. Then one blows up $k_2 - 1$ generic points of one of these new curves, and keep the other

$k_1 - 1$ unmodified; and continue this procedure. Then finally notice that the graph of all the irreducible components which are not (-1)-curves is exactly the graph of $S^3_{-p/q}(\#_i K_i)$.

6.8 The (Equivariant) Geometric Genus and Laufer's Duality

First we show that $h^1(\mathscr{O}_{\widetilde{X}})$ is independent of the choice of $\widetilde{X}$.
Then we present the following facts related to 'Laufer duality':

- duality of $H^1(\mathscr{O}_{\widetilde{X}})$ with the space of holomorphic 2-forms on $\widetilde{X} \setminus E$ modulo those which extend holomorphically over $\widetilde{X}$; we bound the possible poles too;
- we present similar interpretations for $h^1(\mathscr{O}_{\widetilde{X}}(-r_h))$ and $h^1(\mathscr{O}_{\widetilde{X}}(-s_h))$ as well, and we make their computation explicit for weighted homogeneous singularities (the output will be used later in the exemplification of the SWIC);
- using 'Laufer integration' we exemplify a phenomenon how specially constructed families of divisors can provide non-trivial families of bundles in $\mathrm{Pic}^0(\widetilde{X})$ (as an 'entrant' in the theory of Abel maps of surface singularities).

We also introduce the equivariant geometric genus via the universal abelian covering (whenever the link is a rational homology sphere).
In 6.8.A, for any $\mathscr{I} \subset \mathscr{V}$, we define a subspace in $H^0(\widetilde{X} \setminus E, \Omega^2_{\widetilde{X}})/H^0(\widetilde{X}, \Omega^2_{\widetilde{X}})$ generated by forms which have no poles along $\cup_{v\in\mathscr{I}} E_v$. The corresponding linear subspace arrangement (via p_g-surgery formulae) guides the 'p_g-distribution' along the sub-resolutions of $\widetilde{X}$ (given by subgraphs of Γ). ♣

6.8.1 Fix a resolution $\widetilde{X} \to X$. Recall that the geometric genus p_g of (X, o) is defined as $h^1(\widetilde{X}, \mathscr{O}_{\widetilde{X}})$. Moreover, cf. 6.4, for integral cycles $l_2 \geq l_1 > 0$, one has the surjectivity $H^1(\mathscr{O}_{l_2}) \to H^1(\mathscr{O}_{l_1})$, and $H^1(\mathscr{O}_{\widetilde{X}}) = \lim_{\leftarrow} H^1(\mathscr{O}_l)$. Furthermore, if $p_g \neq 0$ then $p_g = h^1(\mathscr{O}_{\lfloor Z_K \rfloor_+})$, cf. Corollary 6.4.7.

Lemma 6.8.2 *$h^1(\mathscr{O}_{\widetilde{X}})$ is independent of the resolution.*

We provide two proofs. The first one is a direct proof, while the other one in 6.8.7 uses a reinterpretation of p_g in terms of (X, o).

Proof Any two resolutions can be connected by a sequence of resolutions, such that consecutive pairs $(\widetilde{X}', \widetilde{X})$ have the property that $\widetilde{X}'$ is obtained from $\widetilde{X}$ by blowing up a point $q \in \widetilde{X}$ (cf. 2.4.3–2.4.5). Let $\psi : \widetilde{X}' \to \widetilde{X}$ be the blow up, $E_{new} := \psi^{-1}(q)$. Then by the Leray spectral sequence, $h^1(\mathscr{O}_{\widetilde{X}'}) = h^1(\mathscr{O}_{\widetilde{X}})$ whenever $\psi_*(\mathscr{O}_{\widetilde{X}'}) = \mathscr{O}_{\widetilde{X}}$ and $R^1\psi_*(\mathscr{O}_{\widetilde{X}'})_q = 0$. The first follows from the normality of $\widetilde{X}$ (see e.g. the proof in [258, 11.4]), the second from $R^1\psi_*(\mathscr{O}_{\widetilde{X}'})_q = \lim_{\leftarrow} H^1(\mathscr{O}_{nE_{new}}) = 0$, cf. 6.4.9. □

Example 6.8.3 In general, p_g is not topological. E.g., for the two examples from 6.7.5, $p_g(X_1, o) = 3$ while $p_g(X_2, o) = 2$. (Use e.g. formulae from 6.9.)

6.8.4 The p_g-Formula of Laufer Let us discuss a different realizations of the geometric genus $p_g = h^1(\widetilde{X}, \mathscr{O}_{\widetilde{X}})$, where $\widetilde{X} \to X$ is a resolution.

By Serre duality $H^1(\widetilde{X}, \mathscr{O}_{\widetilde{X}})^* \simeq H^1_c(\widetilde{X}, \Omega^2_{\widetilde{X}})$. Also, Laufer proved in [359, proof of Theorem 3.4] that the direct limit $H^0_\infty(\widetilde{X}, \Omega^2_{\widetilde{X}}) := \lim_{\to} H^0(\widetilde{X} \setminus \widetilde{X}_\alpha, \Omega^2_{\widetilde{X}})$ (see 6.4.5) is isomorphic to $H^0(\widetilde{X} \setminus E, \Omega^2_{\widetilde{X}})$, that is, any differential 2-from defined near the boundary of $\widetilde{X}$ has an analytic continuation to $\widetilde{X} \setminus E$. Hence, the Laufer's exact sequence $\cdots \to H^q_c(\widetilde{X}, \mathscr{F}) \to H^q(\widetilde{X}, \mathscr{F}) \to H^q_\infty(\widetilde{X}, \mathscr{F}) \to \cdots$ (cf. [359]) reads as

$$H^0_c(\widetilde{X}, \Omega^2_{\widetilde{X}}) \to H^0(\widetilde{X}, \Omega^2_{\widetilde{X}}) \to H^0(\widetilde{X} \setminus E, \Omega^2_{\widetilde{X}}) \to H^1_c(\widetilde{X}, \Omega^2_{\widetilde{X}}) \to H^1(\widetilde{X}, \Omega^2_{\widetilde{X}})$$

But $H^0_c(\widetilde{X}, \Omega^2_{\widetilde{X}}) = 0$ while $H^1(\widetilde{X}, \Omega^2_{\widetilde{X}}) = 0$ by 6.4.3. Hence,

Proposition 6.8.5 ([359])

$$H^1(\widetilde{X}, \mathscr{O}_{\widetilde{X}})^* \simeq H^1_c(\widetilde{X}, \Omega^2_{\widetilde{X}}) = H^0(\widetilde{X} \setminus E, \Omega^2_{\widetilde{X}})/H^0(\widetilde{X}, \Omega^2_{\widetilde{X}}), \tag{6.8.1}$$

where the last vector space is the space of global holomorphic 2-forms on $\widetilde{X} \setminus E$ up to those which can be extended holomorphically to the whole $\widetilde{X}$.

We denote the L^2-integrable forms of $H^0(X \setminus o, \Omega^2_{X\setminus o})$ by $H^0_{L^2}(X \setminus o, \Omega^2_{X\setminus o})$. These are 2-forms ω on $X \setminus 0$ with $\int_{X\setminus o} \omega \wedge \overline{\omega} < \infty$, where $\bar{\cdot}$ is the conjugation. Local computation at points of $E \subset \widetilde{X}$ shows that a form of $X \setminus 0$ is L^2-integrable if and only if its pullback has a holomorphic extension to $\widetilde{X}$.

Proposition 6.8.6 ([359, 606]) $H^0_{L^2}(X \setminus o, \Omega^2_{X\setminus o}) = H^0(\widetilde{X}, \Omega^2_{\widetilde{X}})$.

Corollary 6.8.7 $H^1(\widetilde{X}, \mathscr{O}_{\widetilde{X}})^* \simeq H^0(X \setminus o, \Omega^2_{X\setminus o})/H^0_{L^2}(X \setminus o, \Omega^2_{X\setminus o})$.

In particular, $H^1(\widetilde{X}, \mathscr{O}_{\widetilde{X}})$ is independent of the resolution $\widetilde{X}$.

Remark 6.8.8

(a) Assume that $c : (X', o) \to (X, o)$ is a finite morphism between two normal surface singularities. Then $p_g(X', o) \geq p_g(X, o)$. Indeed, the pullback injects $H^0(X\setminus o, \Omega^2_{X\setminus o})/H^0_{L^2}(X\setminus o, \Omega^2_{X\setminus o})$ into $H^0(X'\setminus o, \Omega^2_{X'\setminus o})/H^0_{L^2}(X'\setminus o, \Omega^2_{X'\setminus o})$.
[See also [332, 5.13] where (for any dimension) it is proved that if $c : (X', o) \to (X, o)$ is a finite and (X', o) rational then (X, o) is rational too.]

(b) Since K_X is divisorial, one also has $H^0(X\setminus o, \Omega^2_{X\setminus o}) = H^0(X, K_X)$ (cf. 12.4.1).

Remark 6.8.9 By definition, $H^0(\widetilde{X} \setminus E, \Omega^2_{\widetilde{X}})$ is the space of global 2-forms with poles along E_v's of arbitrary order. Nevertheless, the set of poles can be bounded. Indeed, for any $Z \in L_{>0}$ consider the exact sequence of sheaves

$$0 \to \Omega^2_{\widetilde{X}} \to \Omega^2_{\widetilde{X}}(Z) \to \mathscr{O}_Z(Z + K_{\widetilde{X}}) \to 0.$$

Since $h^1(\Omega^2_{\widetilde{X}}) = 0$ (cf. 6.4.3) we get that

$$H^0(\widetilde{X}, \Omega^2_{\widetilde{X}}(Z))/H^0(\widetilde{X}, \Omega^2_{\widetilde{X}}) = H^0(\mathscr{O}_Z(Z + K_{\widetilde{X}})) = H^1(\mathscr{O}_Z)^*. \tag{6.8.2}$$

Assume that $p_g \neq 0$. Then from 6.4.7*(a)* $h^1(\mathscr{O}_{\lfloor Z_K \rfloor_+}) = p_g$, hence

$$p_g = \dim\left(H^0(\widetilde{X}, \Omega^2_{\widetilde{X}}(\lfloor Z_K \rfloor_+))/H^0(\widetilde{X}, \Omega^2_{\widetilde{X}})\right). \tag{6.8.3}$$

This holds if $p_g = 0$ too. Since $H^0(\widetilde{X}, \Omega^2_{\widetilde{X}}) \subset H^0(\widetilde{X}, \Omega^2_{\widetilde{X}}(\lfloor Z_K \rfloor_+)) \subset H^0(\widetilde{X} \setminus E, \Omega^2_{\widetilde{X}})$, by (6.8.1) and (6.8.3) we get that $H^0(\widetilde{X}, \Omega^2_{\widetilde{X}}(\lfloor Z_K \rfloor_+)) = H^0(\widetilde{X} \setminus E, \Omega^2_{\widetilde{X}})$. Hence, the poles of forms from $H^0(\widetilde{X} \setminus E, \Omega^2_{\widetilde{X}})$ are bounded by $\lfloor Z_K \rfloor_+$.

If (X, o) is numerically Gorenstein and $Z_K > 0$ then $\chi(Z_K) = 0$ and $h^0(\mathscr{O}_{Z_K}) = h^1(\mathscr{O}_{Z_K}) = p_g$. Hence, from the vanishing $h^1(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(-Z_K)) = 0$ we obtain

$$p_g = \dim\left(H^0(\widetilde{X}, \mathscr{O}_{\widetilde{X}})/H^0(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(-Z_K))\right). \tag{6.8.4}$$

If (X, o) is Gorenstein and $Z_K \geq 0$, via the isomorphism $\Omega^2_{\widetilde{X}} = \mathscr{O}_{\widetilde{X}}(-Z_K)$ the p_g formulae from (6.8.3) and (6.8.4) agree.

Remark 6.8.10 In this spirit we also have the fact that Z_{coh} is the smallest cycle $C \in L_{\geq 0}$ such that $H^0(\widetilde{X} \setminus E, \Omega^2_{\widetilde{X}}) = H^0(\widetilde{X}, \Omega^2_{\widetilde{X}}(C))$. Otherwise, Z_{coh} is the largest cycle, which equals the pole cycle of a differential form on $\widetilde{X} \setminus E$.

6.8.11 The Geometric Genus of the Universal Abelian Covering Assume that the link of (X, o) is a rational homology sphere.

Let $(X_a, o) \to (X, o)$ be the universal abelian covering of (X, o), and consider the notations of the diagram (6.2.1). By definition, the geometric genus $p_g(X_a, o)$ of (X_a, o) is $h^1(\widetilde{Z}, \mathscr{O}_{\widetilde{Z}})$. Recall that $r : \widetilde{Z} \to Z$ is the resolution of the cyclic quotient singularities of Z. Note that $r_*(\mathscr{O}_{\widetilde{Z}}) = \mathscr{O}_Z$ (by the normality of Z), and $R^1 r_*(\mathscr{O}_{\widetilde{Z}}) = 0$, by 6.4.9. Therefore, by Leray spectral sequence $p_g(X_a, o) = h^1(\mathscr{O}_Z)$. Since $\widetilde{c}$ is finite $H^1(\mathscr{O}_Z)$ is isomorphic to $H^1(\widetilde{X}, \widetilde{c}_* \mathscr{O}_Z)$. This by Theorem 6.2.9 has an eigenspace decomposition $\oplus_{h \in H} H^1(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(-r_h))$. The dimension of the $\theta(h)$-eigenspace is

$$p_g(X_a, o)_{\theta(h)} := h^1(\widetilde{X}, (\widetilde{c}_* \mathscr{O}_Z)_{\theta(h)}) = h^1(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(-r_h)).$$

By summation:

$$p_g(X_a, o) = \sum_{h \in H} h^1(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(-r_h)).$$

Clearly, for $h = 0$ we get $p_g(X_a, o)_{\theta(0)} = p_g(X, o)$.

Definition 6.8.12 If $H_1(L_X, \mathbb{Q}) = 0$ we define the equivariant geometric genus of (X, o) associated with $h \in H$ by $p_g(X_a, o)_{\theta(h)} = h^1(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(-r_h))$.

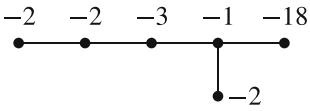
Via Proposition 6.6.4 it can also be expressed by s_h (compare also with (8.2.25)):

$$p_g(X_a, o)_{\theta(h)} = h^1(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(-s_h)) + \chi(r_h) - \chi(s_h). \tag{6.8.5}$$

Example 6.8.13 T. Okuma conjectured in [530, 5.3] that *if the link of* (X, o) *is a rational homology sphere, then* $p_g(X, o)$ *together with the resolution graph determine* $p_g(X_a, o)$. The next example (borrowed from [399, 4.5]) shows that, in general, this is not the case.

Consider the superisolated singularity (X, o), $f = (zy - x^2)^2 - xy^3 + z^5$, associated with a curve of $d = 4$ and one singular point of type $[2_3]$. The resolution graph Γ is

Γ :



Since the graph is star-shaped, it can be realized by a weighted homogeneous singularity $(X_w, 0)$ as well: it is an isolated complete intersection in $(\mathbb{C}^4, 0)$ with equations: $yz = x^2$, $z^5 + t^2 - xy^3 = 0$. (To verify this fact, use e.g. 5.1.33.) One verifies that both singularities have $p_g = 4$; in fact, (X, o) is an equisingular deformation of (X_w, o) by $yz - x^2 = \lambda t$, and $z^5 + t^2 - xy^3 = 0$. (Note that (X, o) and (X_w, o) have different embedding dimensions.)

The universal abelian covering $(X_{w,a}, o)$ of $(X_w, 0)$ is a hypersurface Brieskorn singularity $\{u^7 = v^{18} + w^2\}$, cf. 5.1.33. On the other hand, one verifies that (X_a, o) is not an equisingular deformation of $(X_{w,a}, o)$, see [399]. Since both singularities are Gorenstein, and $(X_{w,a}, o)$ is weighted homogeneous, if their geometric genera would be equal, then (X_a, o) would be an equisingular deformation of $(X_{w,a}, o)$ by [512, 3.3]. Hence $p_g(X_a, o) \neq p_g(X_{w,a}, o)$. On the other hand, for *any* analytic singularity (X', o) with the same topology as $(X_{w,a}, o)$ by (6.8.14) and 6.8.43 one has $p_g(X', o) \leq p_g(X_{w,a}, o)$. Hence $p_g(X_a, o) < p_g(X_{w,a}, o) = 10$.

6.8.14 Laufer's formula (6.8.1) has the following generalization.

Proposition 6.8.15 *Assume that the link of (X, o) is a rational homology sphere and fix $h \in H$. Let l'_h be either r_h or s_h. Then*

$$H^1(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(-l'_h))^* \simeq H^1_c(\widetilde{X}, \Omega^2_{\widetilde{X}}(l'_h)) = H^0(\widetilde{X} \setminus E, \Omega^2_{\widetilde{X}}(l'_h))/H^0(\widetilde{X}, \Omega^2_{\widetilde{X}}(l'_h)).$$

Proof Consider the exact sequence (analogously as in 6.8.4)

$$0 \to H^0(\widetilde{X}, \Omega^2_{\widetilde{X}}(l'_h)) \to H^0(\widetilde{X} \setminus E, \Omega^2_{\widetilde{X}}(l'_h)) \to H^1_c(\widetilde{X}, \Omega^2_{\widetilde{X}}(l'_h)) \to H^1(\widetilde{X}, \Omega^2_{\widetilde{X}}(l'_h)).$$

The first injectivity follows since $H^0_c(\widetilde{X}, \Omega^2_{\widetilde{X}}(l'_h)) = H^2(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(-l'_h))^* = 0$. Hence the statement follows once we show that $H^1(\widetilde{X}, \Omega^2_{\widetilde{X}}(l'_h)) = 0$. This, for $l'_h = r_h$ follows again from 6.4.3. Next, by 6.6.3, there exists a computation sequence $\{z_i\}_{i=0}^t$, $z_0 = r_h$, $z_t = s_h$, $z_{i+1} = z_i + E_{v(i)}$ and $(z_i, E_{v(i)}) > 0$. In the cohomology exact sequence of $0 \to \Omega^2_{\widetilde{X}}(z_i) \to \Omega^2_{\widetilde{X}}(z_{i+1}) \to \Omega^2_{\widetilde{X}}(z_{i+1})|_{E_{v(i)}} \to 0$ one has $h^1(\Omega^2_{\widetilde{X}}(z_{i+1})|_{E_{v(i)}}) = h^0(\mathscr{O}_{E_{v(i)}}(-z_i)) = 0$. Hence $h^1(\Omega^2_{\widetilde{X}}(z_i)) = 0$ by induction. □

Remark 6.8.16 Consider the exact sequence $0 \to \Omega^2_{\widetilde{X}}(r_h) \to \Omega^2_{\widetilde{X}}(s_h) \to \Omega^2_{\widetilde{X}}(s_h)|_\Delta \to 0$, where $\Delta := s_h - r_h$. From the proof of 6.8.15 we have the vanishings $H^1(\widetilde{X}, \Omega^2_{\widetilde{X}}(r_h)) = H^1(\widetilde{X}, \Omega^2_{\widetilde{X}}(s_h)) = H^1(\Omega^2_{\widetilde{X}}(s_h)|_\Delta) = 0$. Hence $\dim H^0(\Omega^2_{\widetilde{X}}(s_h))/H^0(\Omega^2_{\widetilde{X}}(r_h)) = \chi(\Omega^2_{\widetilde{X}}(s_h)|_\Delta) = \chi(r_h) - \chi(s_h)$. This combined with (6.8.5) gives

$$h^1(\mathscr{O}_{\widetilde{X}}(-r_h)) - h^1(\mathscr{O}_{\widetilde{X}}(-s_h)) = \dim H^0(\Omega^2_{\widetilde{X}}(s_h))/H^0(\Omega^2_{\widetilde{X}}(r_h)).$$

This and the two identities from 6.8.15 give $H^0(\widetilde{X} \setminus E, \Omega^2_{\widetilde{X}}(r_h)) = H^0(\widetilde{X} \setminus E, \Omega^2_{\widetilde{X}}(s_h))$ (a fact which can be proved using techniques from 6.2.6–6.2.7 as well).

6.8.17 In concrete computations it is always easier to find global sections than to determine higher cohomologies. This is one of the main advantages of the identity from 6.8.15. In several cases one can identify concrete basis for the vector space $H^0(\widetilde{X} \setminus E, \Omega^2_{\widetilde{X}}(l'_h))/H^0(\widetilde{X}, \Omega^2_{\widetilde{X}}(l'_h))$, for $l'_h = r_h$ or s_h. We exemplify this in the case of weighted homogeneous singularities (with $g = 0$).

Example 6.8.18 ($h^1(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(-r_h))$ for Weighted Homogeneous Singularities, $g = 0$)

We will use the notation and method of 6.6.21. Set $r_h = r_0 E_0 + \sum_{j,i} r_{ji} E_{ji} = a_0 E_0^* + \sum_{j,i} a_{ji} E^*_{ji}$. Fix ℓ, $\{m_j\}_j \in \mathbb{Z}$, $k \in \mathbb{Z}_{\geq 0}$, and let $u_0^{\ell - r_0} \prod_j (v_0 - p_j)^{m_j - r_{j1}} v_0^k dv_0 \wedge du_0$ be a section of $\Omega^2_{\widetilde{X}}(r_h)$ over $U_0 \setminus E$ (for notation see 6.6.19).

This under the transformation $v_0 = u_{-1}^{-1}, u_0 = u_{-1}^{b_0} v_{-1}$ transforms into the following section on U_{-1}:

$$\pm v_{-1}^{\ell - r_0} u_{-1}^{b_0(\ell+1) - \sum m_j - k + (r_h, E_0) - 2} \prod_j (1 - u_{-1} p_j)^{m_j - r_{j1}} du_{-1} \wedge dv_{-1}.$$

The regularity over $\widetilde{X} \setminus E$ requires that the exponent of u_{-1} should be non-negative:

$$k + \textstyle\sum_j m_j \le b_0(\ell+1) - 2 - a_0.$$

The same section in the chart U_{js_j} under the transformation $u_0 = u_j^{-\tau_j} v_j^{-\omega_j}$, $v_0 - p_j = u_j^{\omega_j'} v_j^{\alpha_j}$ transforms into a section whose v_j exponent is $-\alpha_j r_{j1} + \omega_j r_0 - \omega_j(\ell+1) + \alpha_j m_j + \alpha_j - 1$, which again should be non-negative. Using (6.6.22) this reads as

$$m_j \ge \lceil (\omega_j(\ell+1) + a_j - \alpha_j + 1)/\alpha_j \rceil \quad (1 \le j \le \nu). \tag{6.8.6}$$

The above two inequalities (with $\ell,\ m_j \in \mathbb{Z}$, $k \in \mathbb{Z}_{\ge 0}$) characterise sections from $H^0(\widetilde{X} \setminus E, \Omega^2_{\widetilde{X}}(r_h))$. They are regular over $\widetilde{X}$ whenever ℓ and m_j are non-negative. Since by (6.6.22) we have $\omega_j + a_j \ge 0$, the right hand side of (6.8.6) is automatically non-positive whenever $\ell \ge 0$. Hence the section is non-regular if and only if $\ell < 0$, hence $h^1(\mathcal{O}_{\widetilde{X}}(-r_h))$ equals

$$\sum_{\ell<0} \max\left\{0, -1 + b_0(\ell+1) - a_0 - \sum_j \lceil (\omega_j(\ell+1) + a_j - \alpha_j + 1)/\alpha_j \rceil\right\}.$$

We replace $\ell + 1$ by $-\ell$. Set $N_{r_h}(\ell) = b_0\ell + a_0 - \sum_j \left\lceil \frac{\omega_j \ell - a_j}{\alpha_j} \right\rceil$. Then

$$h^1(\mathcal{O}_{\widetilde{X}}(-r_h)) = \sum_{\ell \ge 0} \max\{0, -N_{r_h}(\ell) - 1\}. \tag{6.8.7}$$

Example 6.8.19 ($h^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-s_h))$ for Weighted Homogeneous Singularities, $g = 0$) Let r_h be $a_0 E_0^* + \sum_{j,i} a_{ji} E_{ji}^*$ as above, and set $s_h := \bar{a}_0 E_0^* + \sum_{j,i} \bar{a}_{ji} E_{ji}^*$. As usual, $\bar{a}_j := \sum_i n_{i+1,s_j}^j \bar{a}_{ji}$ $(1 \le j \le \nu)$. Then $\bar{a}_j,\ \bar{a}_0 \ge 0$, cf. 6.6.20. By similar computation as in the previous example 6.8.18 we get

$$h^1(\mathcal{O}_{\widetilde{X}}(-s_h)) = \sum_{\ell \ge 0} \max\{0, -N_{s_h}(\ell) - 1\}, \tag{6.8.8}$$

where $N_{s_h}(\ell) = b_0\ell + \bar{a}_0 - \sum_j \left\lceil \frac{\omega_j \ell - \bar{a}_j}{\alpha_j} \right\rceil$. Set $\Delta := s_h - r_h$ and let $\Delta_0 \in \mathbb{Z}_{\ge 0}$ be the E_0-coefficient of Δ. Then $N_{s_h}(\ell) = N_{r_h}(\ell + \Delta_0)$ (see also the end of 6.6.21),

hence

$$h^1(\mathscr{O}_{\widetilde{X}}(-s_h)) = \sum_{\ell \geq \Delta_0} \max\{0, -N_{r_h}(\ell) - 1\}. \tag{6.8.9}$$

In particular,

$$h^1(\mathscr{O}_{\widetilde{X}}(-r_h)) - h^1(\mathscr{O}_{\widetilde{X}}(-s_h)) = \chi(r_h) - \chi(s_h) = \sum_{0 \leq \ell < \Delta_0} \max\{0, -N_{r_h}(\ell) - 1\}.$$

This expression can be non-zero. Take e.g. the graph with $b_0 = 2$, and three legs all with invariants $(\alpha_j, \omega_j) = (3, 1)$. Then $s_h = \sum_{j=1}^{3} E^*_{js_j}$, $r_h = s_h - E_0$, $\chi(s_h) = h^1(\mathscr{O}_{\widetilde{X}}(-s_h)) = 0$, and $\chi(r_h) = h^1(\mathscr{O}_{\widetilde{X}}(-r_h)) = 1$.

6.8.20 Comparison with the 'Dual Eigensheaf Decomposition' of $\widetilde{c}_*\omega_Z$ Assume that $H_1(L_X, \mathbb{Q}) = 0$. Fix a resolution $\widetilde{X} \to X$ and consider the diagram (6.2.1). In this book the eigensheaf decomposition $\widetilde{c}_*\mathscr{O}_Z = \oplus_{h \in H} \mathscr{O}_{\widetilde{X}}(-r_h)$ is central and it is deeply exploited. It provides $\widetilde{c}_*(\mathscr{O}_Z)$ in terms of well-identified natural line bundles even when $\mathrm{Pic}^0(\widetilde{X}) \neq 0$ (when the Chern class does not identify a line bundle). Here $\mathscr{O}_{\widetilde{X}}(-r_h)$ stays for the $\theta(h)$-eigensheaf.

On the other hand, in the literature of (global) algebraic geometry in numerical invariant computations, rationality discussions, and duality statements the decomposition of the push-down of the dualizing sheaf is also considered (basically focusing only on cyclic coverings), see e.g. work of Viehweg, Esnault, Kawamata, Kollár, cf. e.g. [189, 190] and the references therein. Next we show that the two decompositions are equivalent. Set (as in the mentioned literature) $\omega_Z := \mathscr{O}_Z(K_Z)$ and $\omega_{\widetilde{X}} := \mathscr{O}_{\widetilde{X}}(K_{\widetilde{X}}) = \Omega^2_{\widetilde{X}}$. Then, by duality of finite maps (cf. [189, pages 441–442])

$$\begin{aligned} \widetilde{c}_*(\omega_Z) &= \mathscr{H}om_{\mathscr{O}_{\widetilde{X}}}(\widetilde{c}_*\mathscr{O}_Z, \omega_{\widetilde{X}}) = \mathscr{H}om_{\mathscr{O}_{\widetilde{X}}}(\oplus_h \mathscr{O}_{\widetilde{X}}(-r_h), \omega_{\widetilde{X}}) \\ &= \oplus_h \mathscr{O}_{\widetilde{X}}(r_h) \otimes_{\mathscr{O}_{\widetilde{X}}} \omega_{\widetilde{X}}. \end{aligned}$$

Here $\mathscr{O}_{\widetilde{X}}(r_h) \otimes_{\mathscr{O}_{\widetilde{X}}} \omega_{\widetilde{X}}$ is the $\theta(h)^{-1}$-eigensheaf of the H-action.

We warn the reader that usually the identification of $\omega_{\widetilde{X}}$ in $\mathrm{Pic}(\widetilde{X})$ can be hard: even though we know its Chern class c_1 by the adjunction relations, its precise position in $\mathrm{Pic}^{c_1}(\widetilde{X})$ depends on the analytic structure. However, there are certain special cases when $\widetilde{c}_*(\omega_Z)$ is a direct sum of 'known' line bundles, e.g. when (X, o) is $\mathbb{Q}$-Gorenstein. Under this assumption, $\omega_{\widetilde{X}}$ is the natural line bundle $\mathscr{O}_{\widetilde{X}}(-Z_K)$ (cf. 6.3.25), and $\widetilde{c}_*(\omega_Z) = \oplus_h \mathscr{O}_{\widetilde{X}}(r_h - Z_K)$ is a direct sum of natural line bundles.

6.8.21 Laufer's Duality (6.8.1) Revisited. Consequences [359], [363, p. 1281] Let us rewrite (6.8.1) in several steps using differential forms. The isomorphism

$$H^0(\widetilde{X} \setminus E, \Omega^2_{\widetilde{X}})/H^0(\widetilde{X}, \Omega^2_{\widetilde{X}}) \to H^1_c(\widetilde{X}, \Omega^2_{\widetilde{X}}) \tag{6.8.10}$$

is realized as follows. Fix a small tubular neighbourhood $N \subset \widetilde{X}$ of E such that its closure is compact in $\widetilde{X}$. Take any $\omega \in H^0(\widetilde{X} \setminus E, \Omega^2_{\widetilde{X}})$, and extend the restriction $\omega|_{\widetilde{X}\setminus N}$ to a $C^\infty(2,0)$-form $\tilde{\omega}$ on $\widetilde{X}$. Then $\bar{\partial}\tilde{\omega}$ is a compactly supported $C^\infty(2,1)$-form, hence determines a class in $H^1_c(\widetilde{X}, \Omega^2_{\widetilde{X}})$. If $\tilde{\omega}$ is a holomorphic extension then $\bar{\partial}\tilde{\omega} = 0$. The above isomorphism (6.8.10) is realized by $[\omega] \mapsto [\bar{\partial}\tilde{\omega}]$.

If λ is a $C^\infty(0,1)$ form and η is a $C^\infty(2,1)$ form with compact support in $\widetilde{X}$, then

$$(\lambda, \eta) \mapsto \int_{\widetilde{X}} \lambda \wedge \eta$$

realizes a perfect pairing (duality) $H^1(\widetilde{X}, \mathscr{O}_{\widetilde{X}}) \otimes H^1_c(\widetilde{X}, \Omega^2_{\widetilde{X}}) \to \mathbb{C}$. This combined with the first isomorphism (6.8.10) gives the perfect pairing

$$H^1(\widetilde{X}, \mathscr{O}_{\widetilde{X}}) \otimes H^0(\widetilde{X} \setminus E, \Omega^2_{\widetilde{X}})/H^0(\widetilde{X}, \Omega^2_{\widetilde{X}}) \to \mathbb{C}; \quad \langle [\lambda], [\omega] \rangle = \int_{\widetilde{X}} \lambda \wedge \bar{\partial}\tilde{\omega}. \tag{6.8.11}$$

Assume next that the class $[\lambda] \in H^1(\widetilde{X}, \mathscr{O}_{\widetilde{X}})$ is realized by a *Čech cocycle* $\lambda_{ij} \in \mathscr{O}(U_i \cap U_j)$, where $\{U_i\}_i$ is an open cover of E, $U_i \cap U_j \cap U_k = \emptyset$, and each connected component of the intersections $U_i \cap U_j$ is either a coordinate bidisc $B = \{|u| < 2\epsilon,\ |v| < 2\epsilon\}$ with coordinates (u, v), such that $E \cap B \subset \{uv = 0\}$ or a punctured coordinate bidisc $B = \{\epsilon/2 < |u| < 2\epsilon,\ |v| < 2\epsilon\}$ with coordinates (u, v), such that $E \cap B = \{v = 0\}$. Then λ is obtained as follows: one finds C^∞ functions λ_i on U_i such that $\lambda_i - \lambda_j = \lambda_{ij}$ on $U_i \cap U_j$, and one sets λ as $\bar{\partial}\lambda_i$ on U_i. Then, by Stokes theorem (see [363, 3.8]),

$$\langle [\lambda], [\omega] \rangle = \int_{\widetilde{X}} \lambda \wedge \bar{\partial}\tilde{\omega} = \sum_B \int_{|u|=\epsilon,\ |v|=\epsilon} \lambda_{ij}\omega. \tag{6.8.12}$$

This local formula has several consequences. We provide some prototypes, which can be altered (or even improved) by different other combinations and choices in the corresponding constructions.

Example 6.8.22 (Non-isomorphic Line Bundles (with Fixed Chern Class)) If $p_g > 0$ then the space of line bundles with zero Chern class can be identified with the image of $\epsilon : H^1(\widetilde{X}, \mathscr{O}_{\widetilde{X}}) \to \mathrm{Pic}(\widetilde{X})$, cf. (6.1.1). We wish to construct a 1-parameter family in $H^1(\widetilde{X}, \mathscr{O}_{\widetilde{X}}) = \mathbb{C}^{p_g}$ whose images in $\mathrm{Pic}(\widetilde{X})$ are all distinct.

By (6.8.11) there should exist a form $\omega \in H^0(\widetilde{X} \setminus E, \Omega^2)$, which has a pole $o \in \mathbb{Z}_{>0}$ along an irreducible exceptional curve, say E_v. Then for a generic point p on E_v there exists a local bidisc B with coordinates (u, v) such that $B \cap E_v = \{v = 0\}$ and ω has local equation α/v^o, where $\alpha \in H^0(B, \Omega^2_{\widetilde{X}})$ and $\alpha(p) \neq 0$.

Fix a divisor D on $\widetilde{X}$ whose local equation in B is $v^m u^n$. Let D_t be another divisor, which is the same as D in the complement of B and in B its local equation

is $v^m u^a (u + tv^{o-1})^b$, where $a \geq 0$, $b > 0$, $a + b = n$, and $t \in \mathbb{C}$ (with $|t| \ll 1$ whenever $o = 1$).

We claim that $\mathscr{O}(D)$ and $\mathscr{O}(D_t)$ are different in $\mathrm{Pic}(\widetilde{X})$. In fact, varying the parameter $t \in (\mathbb{C}, 0)$, all the line bundles $\mathscr{O}(D_t)$ are different. Indeed, assume that $\mathscr{O}(D_t - D)$ is trivial in $\mathrm{Pic}^0(\widetilde{X})$. It is represented by the divisor $D' = \mathrm{div}((u + tv^{o-1})/u)^b$, supported in $B = \{|u|, |v| < \epsilon\}$. We can fix ϵ such that the support of D' is in $\{|u| < \epsilon/2\}$, and set $B^* := \{\epsilon/2 < |u| < \epsilon, |v| < \epsilon\}$. Using the trivialization of $\mathscr{O}(D')$ in $\widetilde{X} \setminus \{|u| \leq \epsilon/2\}$ and the realization $\mathscr{O}(D')$ on B, we get that $\mathscr{O}(D')$ can be represented by the cocycle $g = ((u + tv^{o-1})/u)^b \in \mathscr{O}^*(B^*)$. Since this cocycle determines the trivial element of $H^1(\widetilde{X}, \mathscr{O}^*_{\widetilde{X}})$, $\log((u + tv^{o-1})/u)^b = b \log(1 + tv^{o-1}/u)$ is a cocycle in B^* representing the trivial element of $H^1(\widetilde{X}, \mathscr{O}_{\widetilde{X}})$. In particular, paired with ω via the formula (6.8.12) it should give zero. But for $t \neq 0$

$$\int_{|u|=\epsilon,\ |v|=\epsilon} \log(1 + t\frac{v^{o-1}}{u}) \cdot \frac{\alpha}{v^o} \neq 0.$$

Example 6.8.23 (Simultaneous Non-existence of Functions) Consider the situation of the previous example. Then both divisors D and D_t ($t \neq 0$ fixed) cannot be cut out simultaneously by holomorphic functions of $\widetilde{X}$. Note that even if we assume that D is cut out by a holomorphic function, and $\mathscr{O}(D)$ is base point free—hence the non-compact components of D can be 'moved' from the point p—, this move is restricted by constrains of the above construction.

Example 6.8.24 (Existence of Functions) We exemplify how this machinery helps to prove that a certain element $l \in \mathscr{S}$ is in fact in the analytic monoid $\mathscr{S}_{an}$. Let us fix $l \in \mathscr{S}$. Assume that there exists a component E_v of E such that

(a) $(l, E_v) < 0$,
(b) there exists a basis $\omega_1, \ldots, \omega_{p_g} \in H^0(\widetilde{X} \setminus E, \Omega^2_{\widetilde{X}})/H^0(\widetilde{X}, \Omega^2_{\widetilde{X}})$ such that their order of poles along E_v satisfy $o_1 > o_2 > \cdots > o_{p_g} \geq 2$.

Complete l arbitrarily with non-compact components to a divisor D on $\widetilde{X}$ such that (a) $(D, E_u) = 0$ for all $u \in \mathscr{V}$, and (b) in a bidisc neighbourhood B of a smooth point of E_v the divisor D has local equation $v^m u^n$ (where $\{v = 0\} = E_v \cap B$). Then consider the family D_t of divisors which is the same as D outside of B and in B it has local equation $v^m(u + \sum_{i=1}^{p_g} t_i v^{o_i - 1})^n$. Then if we vary $t = (t_i)_{i=1}^{p_g} \in \mathbb{C}^{p_g}$ the classes in $H^1(\widetilde{X}, \mathscr{O}_{\widetilde{X}})$ represented by the *Čech* cocycle $\log(D_t - D) := \log(1 + \sum_i t_i v^{o_i - 1}/u)$ fill in the whole $H^1(\widetilde{X}, \mathscr{O}_{\widetilde{X}})$. Hence, for convenient t, $\mathscr{O}(D_t)$ is trivial, hence D_t is a principal divisor cut out by a holomorphic function g of $\widetilde{X}$. [That is, $g = f \circ \pi$ for some continuous $f : X \to \mathbb{C}$, and the normality of (X, o) guarantees that f is holomorphic too.] In particular, $l \in \mathscr{S}_{an}$.

Remark 6.8.25 In a series of articles the author with J. Nagy defined and studied the Abel map associated with a resolution of a normal surface singularity with rational homology sphere link. The explicit description of the Abel map can be done via the Laufer integral (6.8.12), for details see [452].

6.8.26 The statements of the present Sect. 6.8 have their analogues for an effective cycle $Z \in L_{>0}$ (replacing $\widetilde{X}$). For some of them see 6.8.34 here (or [452]).

6.8.A *The Linear Subspace Arrangement* $\{\Omega_{\widetilde{X}}(\mathscr{I})\}_{\mathscr{I}\subset\mathscr{V}}$ *of Forms*

Fix a normal surface singularity (X, o) and one of its resolutions $\widetilde{X} \to X$.

Definition 6.8.27 For any $\mathscr{I} \subset \mathscr{V}$ let $\Omega_{\widetilde{X}}(\mathscr{I})$ (or $\Omega(\mathscr{I})$) be the subspace of $H^0(\widetilde{X} \setminus E, \Omega^2_{\widetilde{X}})/H^0(\widetilde{X}, \Omega^2_{\widetilde{X}})$ generated by differential forms $\omega \in H^0(\widetilde{X} \setminus E, \Omega^2_{\widetilde{X}})$, which have no poles along $E_{\mathscr{I}} \setminus \cup_{v\notin\mathscr{I}} E_v$.

Let $(\widetilde{X}/E_{\mathscr{V}\setminus\mathscr{I}}, o_{\mathscr{V}\setminus\mathscr{I}})$ denote the multi-germ obtained by contracting the connected components of $E_{\mathscr{V}\setminus\mathscr{I}}$ in $\widetilde{X}$. Let $\widetilde{X}(\mathscr{V} \setminus \mathscr{I})$ be a small neighbourhood of $E_{\mathscr{V}\setminus\mathscr{I}}$ in $\widetilde{X}$, which is the inverse image by ϕ of a small Stein neighbourhood of the multigerm $(\widetilde{X}/E_{\mathscr{V}\setminus\mathscr{I}}, o_{\mathscr{V}\setminus\mathscr{I}})$. We claim that

$$\dim \Omega_{\widetilde{X}}(\mathscr{I}) = p_g(\widetilde{X}/E_{\mathscr{V}\setminus\mathscr{I}}, o_{\mathscr{V}\setminus\mathscr{I}}). \tag{6.8.13}$$

Indeed, fix $Z = \sum_{v\in\mathscr{V}\setminus\mathscr{I}} n_v E_v$ with all $n_v \gg 0$. By (6.8.2) one has $\dim \Omega_{\widetilde{X}}(\mathscr{I}) = \dim H^0(\Omega^2_{\widetilde{X}}(Z))/H^0(\Omega^2_{\widetilde{X}}) = h^1(\mathscr{O}_Z)$, which equals $p_g(\widetilde{X}/E_{\mathscr{V}\setminus\mathscr{I}}, o_{\mathscr{V}\setminus\mathscr{I}})$ by the previous discussions (or by theorem of formal functions).

In this way, the 'distribution of the geometric genus along the sub-singularities' (indexed by full subgraphs) is replaced by a linear subspace arrangement of $\mathbb{C}^{p_g}$. It reflects in a subtle way the structure of the singularity.

This linear subspace arrangement is stable with respect to the restriction to sub-singularities in the following sense.

Proposition 6.8.28 *Fix $\mathscr{I} \subset \mathscr{V}$ as above.*

(a) Set $\overline{\Omega}(\emptyset) := H^0(\widetilde{X}(\mathscr{V} \setminus \mathscr{I}) \setminus E_{\mathscr{V}\setminus\mathscr{I}}, \Omega^2_{\widetilde{X}(\mathscr{V}\setminus\mathscr{I})})/H^0(\widetilde{X}(\mathscr{V} \setminus \mathscr{I}), \Omega^2_{\widetilde{X}(\mathscr{V}\setminus\mathscr{I})})$. Then the linear map $\rho : \Omega(\mathscr{I}) \to \overline{\Omega}(\emptyset)$, induced by restriction, is an isomorphism.

(b) Set $\mathscr{J} \subset \mathscr{V}$ with $\mathscr{J} \cap \mathscr{I} = \emptyset$. Let $\overline{\Omega}(\mathscr{J})$ be the subspace of $\overline{\Omega}(\emptyset)$ generated by forms from $H^0(\widetilde{X}(\mathscr{V} \setminus \mathscr{I}) \setminus E_{\mathscr{V}\setminus\mathscr{I}}, \Omega^2_{\widetilde{X}(\mathscr{V}\setminus\mathscr{I})})$ without pole along $E_{\mathscr{J}} \setminus \cup_{v\notin\mathscr{J}} E_v$. Then the restriction of ρ to $\Omega(\mathscr{J}) \cap \Omega(\mathscr{I})$ induces an isomorphism $\Omega(\mathscr{J}) \cap \Omega(\mathscr{I}) \to \overline{\Omega}(\mathscr{J})$.

In particular, for any $\mathscr{I}$, the subspace arrangement $\{\overline{\Omega}(\mathscr{J})\}_{\mathscr{J}\cap\mathscr{I}=\emptyset}$ of the multigerm $(\widetilde{X}/E_{\mathscr{V}\setminus\mathscr{I}}, o_{\mathscr{V}\setminus\mathscr{I}})$ and resolution $\widetilde{X}(\mathscr{V}\setminus\mathscr{I})$ can be recovered from the arrangement $\{\Omega(\mathscr{M})\}_{\mathscr{M}}$ from level $\widetilde{X}$ via $\{\Omega(\mathscr{I})\cap\Omega(\mathscr{J})\}_{\mathscr{J}\cap\mathscr{I}=\emptyset}$.

Proof

(a) If $[\omega]\in\ker(\rho)$, then ω has no pole along $E_{\mathscr{J}}$ (since $[\omega]\in\Omega(\mathscr{I})$), and has no pole along $E_{\mathscr{V}\setminus\mathscr{J}}$ either (since $\rho[\omega]=0$). Hence $[\omega]=0$, and ρ is injective. Since by (6.8.13) the dimension of the source and the target is the same, ρ is an isomorphism.

(b) By *(a)*, for any $\bar{\omega}\in\overline{\Omega}(\mathscr{J})$ there exists $\omega\in\Omega(\mathscr{I})$ with $\rho(\omega)=\bar{\omega}$. ω is necessarily in $\Omega(\mathscr{I}\cup\mathscr{J})$, hence $\Omega(\mathscr{J})\cap\Omega(\mathscr{I})\to\overline{\Omega}(\mathscr{J})$ is onto.

□

6.8.29 The identity (6.8.13) can be interpreted as an 'additivity formula' as well:

$$p_g(\widetilde{X}/E)-p_g(\widetilde{X}/E_{\mathscr{V}\setminus\mathscr{I}},o_{\mathscr{V}\setminus\mathscr{I}})=\operatorname{codim}\Omega_{\widetilde{X}}(\mathscr{I}).$$

This surgery formula is valid in any situation without any additional analytic or topological restriction, unlike the surgery formula from 8.7.A. (For the relationship between the two surgery formulae see [452].)

6.8.30 The subspace arrangement has the properties $\Omega(\emptyset)\simeq\mathbb{C}^{p_g}$, and $\Omega(I\cup J)=\Omega(I)\cap\Omega(J)$. In the next proposition we establish an interesting additional structure property of the arrangement, whenever the link is a rational homology sphere. It is the analytical analogue of topological convexity property Proposition 10.4.19.

6.8.31 Convexity Property of $\Omega(\{v\})$'s [452] Assume $H_1(L_X,\mathbb{Q})=0$. For simplicity write $\Omega_v:=\Omega(\{v\})$ for $v\in\mathscr{V}$, and define

$$\Pi(\mathscr{I}):=\begin{cases}\emptyset & \text{if } \mathscr{I}=\emptyset\\ \sum_{v\in\mathscr{I}}\Omega_v & \text{if } \mathscr{I}\neq\emptyset.\end{cases}$$

Proposition 6.8.32 *For any $\mathscr{I}\subset\mathscr{V}$ let $\Gamma_{\mathscr{I}}$ be the smallest connected full subtree of Γ whose set of vertices $\overline{\mathscr{I}}$ contains $\mathscr{I}$. Then $\Pi(\mathscr{J})=\Pi(\mathscr{I})$ for any $\mathscr{I}\subset\mathscr{J}\subset\overline{\mathscr{I}}$.*

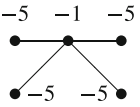
Proof By induction, it is enough to consider the case $\mathscr{J}=\mathscr{I}\cup\{u\}$, such that u is on the geodesic path connecting v,w with $v,w\in\mathscr{I}$. Moreover, it is enough to show that $\Omega_u\subset\Omega_v+\Omega_w$. Write the connected components of $\Gamma\setminus u$ as $\cup_{k=0}^{s}\Gamma_k$, and set $\mathscr{I}_k:=\mathscr{V}(\Gamma_k)$. Assume that $w\in\mathscr{I}_0$.

Choose an arbitrary $\omega\in\Omega_u$ and consider its restriction $\omega|_{\widetilde{X}(\mathscr{I}_0)}$ in $\overline{\Omega}(\emptyset):=H^0(\Omega^2(\widetilde{X}(\mathscr{I}_0)\setminus E_{\mathscr{I}_0}))/H^0(\Omega^2(\widetilde{X}(\mathscr{I}_0)))$. By Proposition 6.8.28*(a)* $\Omega(\mathscr{V}\setminus\mathscr{I}_0)\to$

$\overline{\Omega}(\emptyset)$ is bijective, hence there exists $\omega_v \in \Omega(\mathscr{V} \setminus \mathscr{I}_0)$ such that $\omega_v|_{\widetilde{X}(\mathscr{I}_0)} = \omega|_{\widetilde{X}(\mathscr{I}_0)}$. But $\Omega_v \supset \Omega(\mathscr{V} \setminus \mathscr{I}_0)$, hence $\omega_v \in \Omega_v$. On the other hand, $(\omega - \omega_v)|_{\widetilde{X}(\mathscr{I}_0)} = 0$, hence $\omega_w := \omega - \omega_v \in \Omega_w$. Thus $\omega = \omega_v + \omega_w \in \Omega_v + \Omega_w$. □

Example 6.8.33

(a) Consider the weighted homogeneous isolated hypersurface singularity $(X, o) = \{x^4 + y^4 + z^5 = 0\} \subset (\mathbb{C}^3, 0)$. One verifies that $p_g = 4$ (use either 5.5.4(c) or 6.9.14). We consider the minimal good resolution, whose graphs is



If ω is the Gorenstein form, then ω, $z\omega$, $x\omega$ and $y\omega$ generate $H^0(\widetilde{X} \setminus E, \Omega^2_{\widetilde{X}})/H^0(\Omega^2_{\widetilde{X}})$. The pole orders along the central curve E_0 are 7, 3, 2, 2. Let v_i $(1 \le i \le 4)$ be the end-vertices. Then for fixed i, $\mathscr{V} \setminus \{v_i\}$ represents a minimally elliptic singularity, cf. 7.2.B. Hence $\Omega_{v_i} \simeq \mathbb{C}$ by (6.8.13). If ξ_i are the roots of $\xi^4 + 1 = 0$, then $(x + \xi_i y)\omega$ generates Ω_{v_i} (notice again the 'jump-phenomenon', cf. 6.7.26). Furthermore, $\Omega_{v_0} = 0$ and for $i \neq j$ one has $\Omega_{v_i} \cap \Omega_{v_j} = 0$ and $\Omega_{v_i} + \Omega_{v_j} = \sum_{i=1}^4 \Omega_{v_i} \simeq \mathbb{C}^2 = \langle x\omega, y\omega \rangle$. In particular, the linear subspace arrangement $\{\Omega_v\}_v$ in $\mathbb{C}^{p_g} = \mathbb{C}^4$ is rather special, it is not generic at all.

(b) It is instructive to compute the subspace arrangements (at least) for elliptic singularities, or for the different analytic structures listed in 11.9.40.

6.8.34 Reduction to an Arbitrary $Z > 0$ The facts from this section and from 6.8, valid for $\widetilde{X}$ (or, for any $Z \gg 0$) can be generalized for any $Z \ge E$ as follows.

Regarding the Laufer duality, we have the following facts. Consider the above perfect pairing $\langle , \rangle : H^1(\widetilde{X}, \mathscr{O}_{\widetilde{X}}) \otimes H^0(\widetilde{X} \setminus E, \Omega^2_{\widetilde{X}})/H^0(\Omega^2_{\widetilde{X}}) \to \mathbb{C}$ given via integration of class representatives, cf. (6.8.11). In $H^1(\widetilde{X}, \mathscr{O}_{\widetilde{X}})$ let A be the image of $H^1(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(-Z)) \to H^1(\widetilde{X}, \mathscr{O}_{\widetilde{X}})$, hence $H^1(\widetilde{X}, \mathscr{O}_{\widetilde{X}})/A = H^1(\mathscr{O}_Z)$. On the other hand, in $H^0(\widetilde{X} \setminus E, \Omega^2_{\widetilde{X}})/H^0(\Omega^2_{\widetilde{X}})$ consider the subspace $B := H^0(\Omega^2_{\widetilde{X}}(Z))/H^0(\Omega^2_{\widetilde{X}})$ of dimension $h^1(\mathscr{O}_Z)$ (cf. (6.8.2). Since $\langle A, B \rangle = 0$, the pairing factorizes to a perfect pairing $H^1(\mathscr{O}_Z) \otimes H^0(\Omega^2_{\widetilde{X}}(Z))/H^0(\Omega^2_{\widetilde{X}}) \to \mathbb{C}$. It can be described by the very same integral form of the corresponding class representatives.

Next, for any $\mathscr{I} \subset \mathscr{V}$ define $\Omega_Z(\mathscr{I})$ as the subspace $H^0(\Omega^2_{\widetilde{X}}(Z|_{\mathscr{V}\setminus\mathscr{I}}))/H^0(\Omega^2_{\widetilde{X}})$ in $H^0(\Omega^2_{\widetilde{X}}(Z))/H^0(\Omega^2_{\widetilde{X}})$. By (6.8.2) $\dim \Omega_Z(\mathscr{I}) = h^1(\mathscr{O}_{Z|_{\mathscr{V}\setminus\mathscr{I}}})$, and $\{\Omega_Z(\mathscr{I})\}_{\mathscr{I}}$ is a linear subspace arrangement in $H^0(\Omega^2_{\widetilde{X}}(Z))/H^0(\Omega^2_{\widetilde{X}}) \simeq H^1(\mathscr{O}_Z)^*$.

Additionally to the linear subspace arrangement $\{\Omega_Z(\mathscr{I})\}_{\mathscr{I}}$ we can consider a more subtle object as well, a filtration indexed by $l \in L$, $0 \le l \le Z$, called the *multivariable divisorial filtration of forms*. Indeed, for any such l we define $\mathscr{G}_Z(l) := H^0(\Omega^2_{\widetilde{X}}(l))/H^0(\Omega^2_{\widetilde{X}}) \subset H^0(\Omega^2_{\widetilde{X}}(Z))/H^0(\Omega^2_{\widetilde{X}})$, equivalent to $H^1(\mathscr{O}_l)^* \hookrightarrow H^1(\mathscr{O}_Z)^*$, dual to the natural epimorphisms $H^1(\mathscr{O}_Z) \twoheadrightarrow H^1(\mathscr{O}_l)$. In particular, $\mathscr{G}_Z(l) \simeq H^1(\mathscr{O}_l)^*$. $\mathscr{G}_Z(l)$ is generated by forms with pole $\le l$. In particular, $\mathscr{G}_Z(0) = 0$ and $\mathscr{G}_Z(Z)$ is the total vector space, $\mathscr{G}_Z(l_1) \subset \mathscr{G}_Z(l_2)$ whenever $l_1 \le l_2$, and $\mathscr{G}_Z(l_1) \cap \mathscr{G}_Z(l_2) = \mathscr{G}_Z(\min\{l_1, l_2\})$ (†).

Note that (†) is dual with the following statement. Denote by $K(l) \subset H^1(\mathscr{O}_Z)$ the kernel of $H^1(\mathscr{O}_Z) \to H^1(\mathscr{O}_l)$. Then $K(l_1) + K(l_2) = K(\min\{l_1, l_2\})$ in $H^1(\mathscr{O}_Z)$.

Also, if $l = \sum_{v \notin \mathscr{I}} r_v E_v$ and all $r_v \gg 0$ then $\mathscr{G}_Z(\min(l, Z)) = \Omega_Z(\mathscr{I})$.

For $Z \gg 0$ we get a filtration $\mathscr{G}(l)$ of $H^0(\widetilde{X} \setminus E, \Omega^2_{\widetilde{X}})/H^0(\Omega^2_{\widetilde{X}})$ indexed by $l \in L_{\ge 0}$.

6.8.B A Topological Upper Bounds for p_g. The Invariant Path^i

We introduce the topological invariant Path^i associated with a fixed resolution graph. It serves as a bridge between analytic and topological invariants: it is an upper bound for p_g, but (as we will see in 11.6) it fits perfectly in the world of lattice-type invariants of the link as well. The invariant Path^i motivates the following question: classify those graphs for which the optimal equality $p_g =$ Path^i can be realized by some analytic structure, and in these cases describe or identify all such analytic structures. Furthermore, if $p_g =$ Path^i cannot be realized, one wishes to find the optimal topological bound replacing Path^i. (Both questions are open to the best of the author's knowledge.) ♣

6.8.35 We fix a topological type of normal surface singularities, and we wish to find an upper bound for the geometric genus of all analytic structures supported by this topological type. The upper bound will be given by the graph Γ of a fixed good resolution. Let $\mathscr{K}$ be the (topologically defined) set of cycles $\lfloor Z_K \rfloor_+ + L_{\ge 0}$. By 6.4.7 $h^1(\mathscr{O}_Z) = p_g$ for any $Z \in \mathscr{K}$ and for any analytic structure supported by Γ.

For any $g \in \mathbb{Z}_{\ge 0}$ and $d \in \mathbb{Z}$ let $B(g, d)$ be the maximum of $h^1(C, \mathscr{L})$, where C is any smooth projective curve with genus g and $\mathscr{L} \in \mathrm{Pic}(C)$ has degree d. Since in any such situation $h^0(C, \mathscr{L}) \le \max\{0, d+1\}$, one has $h^1(C, \mathscr{L}) = h^0(C, \mathscr{L}^{-1} \otimes K_C) \le \max\{0, 2g - d - 1\}$. In particular, $B(g, d)$ exists. In fact, for $g = 0$,

$$B(0, d) = \max\{0, -d - 1\} = h^1(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(d)).$$

A *path* with end-cycle $Z \in L_{\ge 0}$ is a sequence of integral cycles $\gamma := \{l_i\}_{i=0}^t$, $l_i \in L_{\ge 0}$ such that $l_0 = 0$, $l_t = Z$, and $l_{i+1} = l_i + \epsilon_i E_{v(i)}$ for some $v(i) \in \mathscr{V}$ and

$\epsilon_i \in \{+1, -1\}$. In the definition of a path we also require $l_i \neq l_j$ for $i \neq j$. (If this last property is not satisfied then we say that the path is 'non-injective'.)

Let $\mathscr{P}(Z)$ denote the set of paths with end-cycle Z, and let $\mathscr{P}$ denote the set of paths with arbitrary end-cycle Z from $\mathscr{K}$. A path is *increasing* if $\epsilon_i = +1$ for every i. Denote by $\mathscr{P}^i(Z) \subset \mathscr{P}(Z)$ and $\mathscr{P}^i \subset \mathscr{P}$ the subset of increasing paths.

Furthermore, for any Z, for any $\gamma \in \mathscr{P}(Z)$, $i < t$ and $\epsilon_i = 1$ define $B_i := B(g(E_{v(i)}), -(l_i, E_{v(i)}))$, $S^*(\gamma) := \sum_{i<t;\ \epsilon_i=1} B_i$, and $\mathrm{Path}^*(Z) := \min_{\gamma\in\mathscr{P}(Z)} S^*(\gamma)$. Similarly set $\mathrm{Path}^* := \min_{\gamma\in\mathscr{P}} S^*(\gamma)$ as well. It is convenient to define $S^*(\gamma)$ even for a non-injective path, by the very same expression $\sum_{i<t;\ \epsilon_i=1} B_i$.

6.8.36 Upper Bounds for $h^1(\mathscr{O}_Z)$ Fix some $Z \in L_{>0}$ and consider $\gamma \in \mathscr{P}(Z)$. If $\epsilon_i = -1$ then $h^1(\mathscr{O}_{l_{i+1}}) - h^1(\mathscr{O}_{l_i}) \leq 0$. Otherwise, from the exact sequence $0 \to \mathscr{O}_{E_{v(i)}}(-l_i) \to \mathscr{O}_{l_{i+1}} \to \mathscr{O}_{l_i} \to 0$ we get

$$h^1(\mathscr{O}_{l_{i+1}}) - h^1(\mathscr{O}_{l_i}) \leq h^1(\mathscr{O}_{E_{v(i)}}(-l_i)) \leq B_i \quad (0 \leq i < t,\ \epsilon_i = 1).$$

In particular, for any analytic structure with the fixed resolution graph Γ one has

$$h^1(\mathscr{O}_Z) \leq \mathrm{Path}^*(Z) \quad \text{and} \quad p_g \leq \mathrm{Path}^*. \tag{6.8.14}$$

6.8.37 The Case $g(E_v) = 0$ for all $v \in \mathscr{V}$ The inequalities from (6.8.14) are the starting point of a connection between the geometric genus (as analytic invariant) and the path lattice cohomology associated with the link, see 11.6. Since the path lattice cohomology is defined only for rational homology sphere links, in the sequel of this section *we will assume that* $g(E_v) = 0$ *for all* v. In this case, the sum $S^*(\gamma)$ is much nicer. Indeed, whenever $\epsilon_i = 1$ we have

$$B_i = \max\{0, (E_{v(i)}, l_i) - 1\} = \max\{0,\ \chi(l_i) - \chi(l_{i+1})\}. \tag{6.8.15}$$

Moreover, in (6.8.14) equality holds if and only if *for some* γ one has (i) $h^1(\mathscr{O}_{l_{i+1}}) = h^1(\mathscr{O}_{l_i})$ whenever $\epsilon_i = -1$, and (ii) the above cohomology exact sequences considered in 6.8.36 split for all i with $\epsilon_i = 1$.

Next, define B_i for $i < t$ and $\epsilon_i = -1$ by $B_i = \max\{0, \chi(l_i) - \chi(l_{i+1})\}$, and set $S(\gamma) := \sum_{i<t} B_i$ for any $\gamma \in \mathscr{P}(Z)$. Furthermore, set $\mathrm{Path}(Z) := \min_{\gamma\in\mathscr{P}(Z)} S(\gamma)$ and $\mathrm{Path}^i(Z) := \min_{\gamma\in\mathscr{P}^i(Z)} S(\gamma)$ as well. Consider also the versions Path and Path^i where the minimum is taken for $\gamma \in \mathscr{P}$ and $\gamma \in \mathscr{P}^i$ respectively.

The definition of the three Path-versions is motivated by the following facts: Path^* serves as the natural bound for $h^1(\mathscr{O}_Z)$, hence creates the bridge with the analytic exact sequences, Path will create the bridge with path lattice cohomology (see 11.6, especially the surjectivity statement from 11.6.7). In this case, along a path we measure the probable χ-decreases and we minimalize them. Finally, Path^i

is defined for increasing paths, which appear naturally in several geometric (Laufer type) constructions, see e.g. 7.3.D.

From definitions we obtain

$$\mathrm{Path}^*(Z) \le \mathrm{Path}(Z) \le \mathrm{Path}^i(Z) \quad \text{and} \quad \mathrm{Path}^* \le \mathrm{Path} \le \mathrm{Path}^i. \tag{6.8.16}$$

Lemma 6.8.38 *Assume that $g(E_v) = 0$ for all v. Then the minimum* $\mathrm{Path}^*(Z)$ *can be realized by an increasing path. In particular,* $\mathrm{Path}^*(Z) = \mathrm{Path}(Z) = \mathrm{Path}^i(Z)$ *and* $\mathrm{Path}^* = \mathrm{Path} = \mathrm{Path}^i$.

Proof Fix some non-increasing $\gamma = \{l_i\}_i$ and let j be the smallest index with $\epsilon_j = -1$. Write $v(j) = v$. Let k be the largest index with $k < j$, $v(k) = v$. This necessarily exists since each $l_i \ge 0$ and $l_0 = 0$. Then the path-segment $l_k, \dots, l_j$ of γ is increasing. For $i \in \{k+1, \dots, j\}$ write $l_i = \bar{l}_i + E_v$. Then $\bar{l}_{k+1} = l_k$ and $\bar{l}_j = l_{j+1}$. We will replace the path-segment $l_k, l_{k+1}, \dots, l_j,\ l_{j+1}$ of γ by the path-segment $\bar{l}_{k+1}, \dots, \bar{l}_j$ (all other elements will stay unmodified); let the new path be denoted by $\bar{\gamma}$. It might happen that it is a non-injective path. However, $E_{v(i)} \ne E_v$ for $k+1 \le i \le j-1$ and

$$\chi(l_i) - \chi(l_{i+1}) = \chi(\bar{l}_i) - \chi(\bar{l}_{i+1}) + (E_v, E_{v(i)}) \ge \chi(\bar{l}_i) - \chi(\bar{l}_{i+1}).$$

Hence $S^*(\bar{\gamma}) \le S^*(\gamma)$. If $\bar{\gamma}$ is injective, then we finished the inductive step.

Assume that $\bar{\gamma}$ is not injective. Let $\{\bar{l}_i\}_{i=0}^{\bar{t}}$ denote its terms. Since γ is injective, if $\bar{l}_n = \bar{l}_m$ for certain $n \ne m$ along $\bar{\gamma}$ then the lattice point $\bar{l}_n$ is realized exactly two times, exactly for the indices n and m. Then we shorten $\bar{\gamma}$ by eliminating (oriented) loops from it (they can be interpreted as Euler cycles in a directed graph). Start to travel along $\bar{\gamma}$, and let $\bar{l}_n$ be the first non-injectivity lattice point. Let $m > n$ be the index with $\bar{l}_m = \bar{l}_n$. Then replace the intermediate path $\bar{l}_n, \dots, \bar{l}_m$ by the lattice point $\bar{l}_n$. Then repeat the procedure with the shortened path. Finally we obtain an injective path $\bar{\gamma}'$ with less $\epsilon_k = -1$ and $S^*(\bar{\gamma}') \le S^*(\bar{\gamma}) \le S^*(\gamma)$.

By induction we find an increasing γ' with $S(\gamma') = S^*(\gamma') \le S^*(\gamma)$.

□

6.8.39 Paths with Fixed End-Cycles In the next lemma we analyse the Z-dependence of $\mathrm{Path}^i(Z)$ for arbitrary cycles $Z \in L_{>0}$. Regarding the assumption from its part *(b)*, note that if $\widetilde{X}$ is a minimal good resolution then $Z_K \ge 0$, cf. 6.3.4.

Lemma 6.8.40 *Assume that $g(E_v) = 0$ for all v. Then the following facts hold:*

(a) If $Z_1 \le Z_2$ then $\mathrm{Path}^i(Z_1) \le \mathrm{Path}^i(Z_2)$*;*

(b) Assume additionally that $Z_K \ge 0$. Then $\mathrm{Path}^i(Z) = \mathrm{Path}^i(\min\{Z, \lfloor Z_K \rfloor\})$. *Hence,* $\mathrm{Path}^i(\lfloor Z_K \rfloor) = \max_{Z>0} \mathrm{Path}^i(Z) = \mathrm{Path}^i$.

Proof

(a) Take $Z_2 = Z_1 + E_v$. Similarly as in the proof of 6.8.38, for any $\gamma \in \mathscr{P}^i(Z_2)$ (repeating the proof for the path γ completed with the segment $[Z_2, Z_1]$) one constructs $\bar{\gamma}' \in \mathscr{P}^i(Z_1)$ with $S(\bar{\gamma}') \leq S(\gamma)$.

(b) Assume that Z can be written as $\lfloor Z_K \rfloor + a - b$, with $a > 0$, $b \geq 0$, and a and b without common components. We claim that there exists $E_v \subset |a|$ such that $\chi(Z - E_v) \leq \chi(Z)$. Indeed, if not, then $(Z - Z_K, E_v) \geq 0$ for any $E_v \subset |a|$. Let $\{Z_K\}_1$ be the part of $\{Z_K\}$ supported on $|a|$, and $\{Z_K\}_2 = \{Z_K\} - \{Z_K\}_1$. Hence $(a - \{Z_K\}_1, E_v) \geq (b + \{Z_K\}_2, E_v) \geq 0$ for any $E_v \subset |a|$. By the negative definiteness of $(\ ,\ )$ this can happen only if $a - \{Z_K\}_1 \leq 0$, cf. 2.1.19. But $a > 0$ is integral, which leads to a contradiction. Therefore, there exists an increasing path connecting $Z - a$ with Z which is χ-nondecreasing. In particular, a path γ, which realizes $\mathrm{Path}^i(Z - a)$, can be completed to a longer path from 0 to Z with the same $S(\gamma)$. Hence $\mathrm{Path}^i(Z - a) \geq \mathrm{Path}^i(Z)$. This together with *(a)* gives $\mathrm{Path}^i(Z - a) = \mathrm{Path}^i(Z)$.

□

The next results analyse certain cases when the inequality $p_g \leq \mathrm{Path}^i$ from (6.8.14) is strict. Recall, cf. 6.4.19 and 6.4.20, that in the numerically Gorenstein context with $Z_K \geq 0$, $Z_K = Z_{coh}$ if and only if the analytic structure is Gorenstein.

Theorem 6.8.41 ([495]) *Assume that Γ is numerically Gorenstein and $g(E_v) = 0$ for all v. Assume also that $Z_K > Z_{coh}$ for some analytic structure (X, o). If one of the following properties hold:*

(1) either $\{\gamma \in \mathscr{P}^i(Z_K) : S(\gamma) = \mathrm{Path}^i\} \to \{E_v\}_{v \in \mathscr{V}}$, $\gamma \mapsto E_{v(t-1)}$, is surjective,

(2) or, the support $|Z_K - Z_{coh}|$ is E,

then $p_g(X, o) < \mathrm{Path}^i$.

Proof

(1) We prove that if $p_g = \mathrm{Path}^i$ and the surjectivity *(1)* holds then $Z_{coh} = Z_K$. Indeed, $p_g = \mathrm{Path}^i$ implies that along a path γ with $p_g = \mathrm{Path}^i = S(\gamma)$, whenever p_g can grow with $(E_{v(i)}, l_i) - 1 > 0$, it necessarily grows with this amount. On the other hand, for any choice of γ, $l_{t-1} = Z_K - E_{v(t-1)}$. Since $(E_{i(t-1)}, l_{t-1}) - 1 = 1$, necessarily $h^1(Z_K - E_{v(t-1)}) < h^1(Z_K) = p_g$. By the surjectivity *(1)* we get that this must be the case for any E_v, that is, $h^1(Z_K - E_v) < p_g$ for any v. Hence $Z_{coh} = Z_K$.

(2) Fix $\gamma \in \mathscr{P}^i(Z_K)$, $\gamma = \{l_i\}_{i=0}^t$, with $S(\gamma) = \mathrm{Path}^i$. Let γ' be the shorter path $\gamma' = \{l_i\}_{i=0}^{t-1}$. Then, as above, $S(\gamma') = S(\gamma) - 1$. Hence, via the assumption *(2)* and (6.8.14), $p_g = h^1(Z_{coh}) \leq h^1(\mathscr{O}_{Z_K - E_{v(t-1)}}) \leq S(\gamma') < S(\gamma) = \mathrm{Path}^i$.

□

By 6.4.22 assumption *(2)* is automatically satisfied in some cases.

Remark 6.8.42

(a) The definition of Path^i is natural and 'simple', however usually it is very hard to determine it explicitly: finding the minimalizing path can be rather challenging.
(b) Assume that the link is a rational homology sphere. The above inequality $p_g \leq \mathrm{Path}^i$ looks slightly artificial, even naive with questionable sharpness. Nevertheless, as we will prove, for rather important analytic structures along a well-chosen increasing path all the cohomological exact sequences from 6.8.36 split, and the equality $p_g = \mathrm{Path}^i$ holds (see below).

Example 6.8.43

(a) Assume that Γ is star-shaped with rational central vertex. (For notations see 5.1.26.) Then we will see in 11.3.A that $\mathrm{Path}^i = \sum_{\ell \geq 0} \max\{0, -N(\ell) - 1\}$, hence for weighted homogeneous singularities via (5.1.8) $p_g = \mathrm{Path}^i$.

The equality $p_g = \mathrm{Path}^i$ is realized (under the assumption that the link is a rational homology sphere) by the following families as well:
(b) rational singularities, cf. 7.1.4;
(c) maximally elliptic singularities (with $p_g = \ell_{seq}$), cf. 7.2.66;
(d) more generally: for AR singularities which satisfy the SWIC for the canonical spin^c structure, cf. 11.3.9 (see also 11.6.11);
(e) superisolated singularities, cf. 11.6.11;
(f) local Weil divisors in affine toric 3-varieties with non-degenerate Newton principal part, cf. 11.6.A.

Example 6.8.44 On the other hand, one can find topological types of singularities (even with integral homology sphere link) such that for *any* analytic structure the strict inequality $p_g < \mathrm{Path}^i$ holds. We follow [495], where the details can be found.

Consider the graph from 5.2.5. First one shows that $\mathrm{Path}^i = 4$. The inequality $\mathrm{Path}^i \leq 4$ can be checked by considering a concrete path $\gamma \in \mathscr{P}^i(Z_K)$ with $S(\gamma) = 4$. For $\mathrm{Path}^i > 3$ one argues as follows: there is a non-Gorenstein Kulikov analytic structure on Γ with $p_g = 3$ and $Z_{coh} \leq 2Z_{min} \leq Z_K - E$, hence 6.8.41 applies.

Next we claim that there is no analytic structure with $p_g = 4$, cf. [495]. (Under the assumption $p_g = 4$ one shows that (X, o) would be Gorenstein with multiplicity two, hence it would be a suspension type (cf. 6.7.13) with integral homology sphere link. In particular, its splice diagram would be as in 5.3.21, which is not the case.)

Then, for all analytic structures $h^1(\mathscr{O}_{Z_{min}}) = 2 \leq p_g \leq 3$. $p_g = 3$ can be realized by two rather different analytic structures. One of them is a non-Gorenstein Kulikov-type singularity, with $Z_{max} = Z_{min}$, $\mathrm{mult}(X, o) = 3$, $\mathrm{emb}(X, o) = 4$. The other one is a splice type complete intersection with $Z_{max} = 2Z_{min}$, $\mathrm{mult}(X, o) = 4$, $\mathrm{emb}(X, o) = 4$.

It is instructive to make the following comparison. Note that the exceptional curve of the minimal resolution in the previous example is a two-cuspidal rational curve with self-intersection -1, with 'graph' $-1 \bullet [\delta = 1 + 1]$. By analogy, consider singularities with topological type given by the genuine graph $-1 \bullet [g = 2]$ (with

similar adjunction relations). In this second case again one has $2 \leq p_g \leq 4$ for the possible analytic structures, however, in this case $p_g = 4$ can be realized by a Gorenstein Kulikov singularity (e.g. by $z : (\{x^2 + y^5 + z^{10} = 0\}, 0) \to (\mathbb{C}, 0)$). The difference between the two cases is explained by Stevens in [666] as follows. In both cases, in the Gorenstein case one must have $K_E = \mathscr{O}_E(-2E)$, hence $\mathscr{O}_E(-E)$ is a theta characteristic. In fact, by a cohomology computation it must be an odd theta characteristic. For a smooth $g = 2$ curve the divisor of a Weierstrass point is an odd theta characteristic. Hence if we perform the Kulikov construction by blowing up such a point then we get a Gorenstein Kulikov singularity, cf [661]. (Recall that the Kulikov construction here reads as follows: take $C \times \mathbb{C}$, where C is smooth of genus 2, blow up a point $(p, 0)$ with $p \in C$ and contract the strict transform E of C. The choice of the point p is crucial!) However, a two-cuspidal rational curve has only one theta characteristic, which is even. (The candidate Weierstrass points fall into the cusps.) Hence $-1 \bullet [\delta = 1 + 1]$ does not admit a Gorenstein Kulikov analytic structure (which is the 'expected' analytic type for a $(p_g = 4)$-analytic structure).

The above two-cuspidal example also shows that there exists a numerically Gorenstein topological type, even with integral homology sphere link, for which

(i) although an analytic type realizes the maximal p_g, it is not necessarily Gorenstein;
(ii) although an analytic type realizes the maximal p_g, its maximal ideal cycle is not necessarily the Artin cycle;
(iii) although the graph is Kulikov and numerically Gorenstein, it does not support any Gorenstein Kulikov analytic structure. (Compare with 6.3.20(c).)

Via 6.8.41 the (negative) statement (i) from above cannot happen if $p_g = \mathrm{Path}^i$ (and $\widetilde{X}$ is a resolution, where either $Z_K = Z_{coh}$ or $Z_K - Z_{coh} \geq E$, cf. 6.4.22).

Problem 6.8.45 Associate combinatorially an integer $\mathrm{MAX}(\Gamma)$ to any resolution graph Γ, such that for any analytic type supported by Γ one has $p_g \leq \mathrm{MAX}(\Gamma)$, and furthermore, for certain analytic structure one has equality.

The analogous integer $\mathrm{MIN}(\Gamma)$, targeting the optimal topological lower bound of p_g, is already known, cf. [453] and 6.8.48. The expectation is that for any fixed Γ any integer between $\mathrm{MIN}(\Gamma)$ and $\mathrm{MAX}(\Gamma)$ can be realized as the geometric genus of a certain analytic structure supported on Γ.

6.8.C A Topological Lower Bounds for p_g

We prove the inequality $p_g \geq 1 - \min \chi$ and several improvements of it. ♣

6.8.46 We consider the minimal resolution, hence $Z_K \in \mathscr{S}'$. Since $h^0(\mathscr{O}_{Z_{min}}) = 1$, cf. 6.6.9*(b)*, $p_g \geq h^1(Z_{min}) = 1 - \chi(Z_{min})$. In order to improve this inequality, we have to allocate a cycle which minimalizes χ. We denote $\min_{l \in L} \chi(l)$ by $\min \chi$.

Wagreich has proved that the expression $1 - \min \chi$ is independent of the resolution, and called it the 'arithmetical genus' $p_a(X, o)$ of (X, o) [719].

Proposition 6.8.47 ([719]) *If $p_g \neq 0$ then $p_g \geq 1 - \min \chi$.*

Proof We start with the following fact: There exists $l_m \in \mathcal{S}$ such that $\chi(l_m) = \min \chi$. Indeed, for any $l \in L$ we constructed in 6.6.3 a cycle $s(l) \in \mathcal{S}$, $s(l) \geq l$, with $\chi(s(l)) \leq \chi(l)$. Hence any l with $\chi(l) = \min \chi$ can be replaced by $s(l)$.

Since $\mathcal{S} \subset L_{\geq 0}$, $\min \chi$ is realized on $L_{\geq 0}$. Assume that $\min \chi$ is not realized on $L_{>0}$ (that is, $\min \chi = \chi(0) = 0$ and $\chi(l) > 0$ for any $l > 0$). For such singularities in 7.1.2 we will prove that $p_g = 0$. Since here $p_g > 0$, there exists $l_m \in L_{>0}$ with $\min \chi = \chi(l_m)$. But for any $l > 0$, from the cohomology exact sequence of $0 \to \mathcal{O}(-l) \to \mathcal{O} \to \mathcal{O}_l \to 0$ we get

$$p_g + \chi(l) = \dim H^0(\mathcal{O})/H^0(\mathcal{O}(-l)) + h^1(\mathcal{O}(-l)). \tag{6.8.17}$$

Since $H^0(\mathcal{O})/H^0(\mathcal{O}(-l))$ contains the class of constants, its dimension is ≥ 1. □

Remark 6.8.48 The statement and the proof of the above proposition might suggest that the inequality from 6.8.47 cannot be very sharp, and for a general graph Γ the smallest possible p_g (considered over all analytic structures supported on Γ) is much higher than $1 - \min \chi$. However, in [453] it is proved that the expression $1 - \min \chi$ is an optimal bound whenever $H_1(M(\Gamma), \mathbb{Q}) = 0$: the geometric genus of the generic analytic structure supported on Γ equals $1 - \min \chi$.

Once this said, still, our goal is to enrich the above inequality by introducing in it some other analytic invariants. We start with the improvement of the first statement of the proof of 6.8.47.

Lemma 6.8.49 *There exists $l_m \in L$ such that $\chi(l_m) = \min \chi$, $l_m \in \mathcal{S}$ and $l_m \leq Z_K$.*

Proof First note that in a minimal resolution (when $Z_K \in \mathcal{S}'$), $\chi(-a) > 0$ for any $a \in L'_{>0}$. (Analyse the real function $t \mapsto \chi(-ta)$, $t \in \mathbb{R}$.) In particular, for $a \in L'_{>0}$ and $b \in L'_{\geq 0}$ with $\dim |a| \cap |b| < 1$ one has $\chi(b - a) > \chi(b)$. Using the symmetry $x \leftrightarrow Z_K - x$ of χ, one also gets (for the same a and b) (†) $\chi(Z_K - b + a) > \chi(Z_K - b)$. Using these facts we obtain that there is $l \in L$ with $0 \leq l \leq Z_K$ such that $\chi(l) = \min \chi$. Then replace l by $s(l)$. Then $s(l) \in \mathcal{S}$ and $s(l) \leq Z_K$ still holds because of (†). □

If $l_m = Z_K$, then necessarily $Z_K \in L$ and $\min \chi = 0$. This is the case of elliptic singularities, whose detailed discussion is in Sect. 7.2. Moreover the inequalities from below can be verified for them directly (by substitution $\min \chi = 0$).

Proposition 6.8.50 ([334]) *Assume that one can choose $l_m \in \mathscr{S}$ and $0 < l_m < \lceil Z_K \rceil$ (hence $p_g > 0$). Let $e = emb(X, o)$ be the embedding dimension of (X, o). Then*

$$2^e(p_g + \min \chi) + 2^{e-1}\chi(\lceil Z_K \rceil) \geq \chi(\lfloor Z_K \rfloor) + p_g + 1. \tag{6.8.18}$$

This in the numerically Gorenstein case reads as $p_g + \min \chi \geq (p_g + 1)/2^e$.

If (X, o) is Gorenstein, then $p_g + \min \chi \geq (p_g + 1)/2^{e-1}$.

Proof In the proof we use the following algebraic lemma.

Lemma 6.8.51 *(Appendix of [334]) Let R be a local $\mathbb{C}$-algebra with embedding dimension $e \geq 2$. If $\{\mathfrak{a}_i\}_{i=1}^d$ ($d \geq 2$) are ideals of R with finite codimension, then*

$$d^{e-1}(\dim(R/\mathfrak{a}_1 + \cdots + \dim(R/\mathfrak{a}_d) > \dim(R/\mathfrak{a}_1 \cdots \mathfrak{a}_d).$$

Now we prove (6.8.18). Since $l_m \in \mathscr{S}$ by a computation one gets $\chi(\lceil Z_K \rceil - l_m) \leq \chi(\lceil Z_K \rceil) - \chi(l_m)$. Hence (6.8.17) written for $l = l_m$ and $l = \lceil Z_K \rceil - l_m$, summed, and applied 6.8.51, show that the left hand side of (6.8.18) satisfies

$$LHS \geq \dim \frac{H^0(\mathscr{O})}{H^0(\mathscr{O}(-\lceil Z_K \rceil))} + 1 \geq \dim \frac{H^0(\mathscr{O})}{H^0(\mathscr{O}(-\lfloor Z_K \rfloor))} + 1 = \chi(\lfloor Z_K \rfloor) + p_g + 1.$$

In the Gorenstein case we write $a = l_m$, $b = Z_K - l_m$ and we apply (6.8.17) for l_m. Then $h^1(\mathscr{O}(-a)) = h^1(\mathscr{O}_b(-a)) = h^0(\mathscr{O}_b) \geq \dim H^0(\mathscr{O})/H^0(\mathscr{O}(-b))$. Hence

$$p_g + \chi(l_m) \geq \dim \frac{H^0(\mathscr{O})}{H^0(\mathscr{O}(-a))} + \dim \frac{H^0(\mathscr{O})}{H^0(\mathscr{O}(-b))} \geq \frac{1}{2^{e-1}}\left(1 + \dim \frac{H^0(\mathscr{O})}{H^0(\mathscr{O}(-Z_K))}\right)$$

by 6.8.51. □

Remark 6.8.52 Equation (6.8.18) can be applied in two different ways. Let us assume that $Z_K \in L$. If e is fixed, then $p_g + \min \chi \geq (p_g + 1)/2^e$ (or $p_g + \min \chi \geq (p_g + 1)/2^{e-1}$ in the Gorenstein case), which is a considerable improvement of $p_g + \min \chi \geq 1$. However, as we have already mentioned, there are singularities when $p_g + \min \chi = 1$. For them we get the additional restriction $2^e \geq p_g + 1 = 2 - \min \chi$.

Surprisingly, in 6.8.51 even very 'naive' choices of the ideals might provide remarkable inequalities. (For an application see 6.9.31).

Proposition 6.8.53 ([334]) *Assume that (X, o) is numerically Gorenstein with $p_g > 0$. Then in the **minimal** resolution $(2^e - 1)p_g \geq 1 - Z_K^2$.*

Proof Apply 6.8.51 with $\mathfrak{a}_1 = \mathfrak{a}_2 = H^0(\mathscr{O}(-Z_K))$ and use the vanishing theorem 6.4.3 to get $\dim H^0(\mathscr{O})/H^0(\mathscr{O}(-2Z_K)) = p_g - Z_K^2$. □

Example 6.8.54 (Tomari's Bounds [690]) $H^1(\widetilde{X}, \mathscr{O}_{\widetilde{X}})$ is an $\mathscr{O}_{X,o}$-module; this is very transparent in the realization (6.8.1), where the module structure is given by multiplication of forms by functions. In particular, it has a natural filtration provided by the powers of the maximal ideal. Let us write $R := H^1(\mathscr{O}_{\widetilde{X}})$. Tomari's invariant T is the smallest positive non-negative integer with $\mathfrak{m}_{X,o}^T \cdot R = 0$. By Nakayama's Lemma, all the modules $R \supsetneq \mathfrak{m}R \supsetneq \cdots \supsetneq \mathfrak{m}^T R = 0$ are distinct modules, T is the number of non-zero ones.

One shows that for a generic element f of $\mathfrak{m}_{X,o}$ one has $\mathfrak{m}^k R = f^k \cdot R$ for every k, hence T is the order of nilpotency of multiplication by f. Tomari also shows that $\dim(R/fR) = h^1(\mathscr{O}_{Z_{max}})$. Therefore,

$$T - 1 + \dim(R/fR) \leq p_g \leq T \cdot \dim(R/fR), \quad \dim(R/fR) = h^1(\mathscr{O}_{Z_{max}}). \tag{6.8.19}$$

Using (6.8.1) one has the reinterpretation (where $i > 0$)

$$R/\mathfrak{m}^i R \simeq \{[\omega] \in H^0(\widetilde{X} \setminus E, \Omega^2)/H^0(\widetilde{X}, \Omega^2) \,:\, f^i[\omega] = 0\}; \tag{6.8.20}$$

$$T = \min\{t \in \mathbb{Z}_{\geq 0} \,:\, \mathfrak{m}^t \cdot H^0(\widetilde{X} \setminus E, \Omega^2)/H^0(\widetilde{X}, \Omega^2) = 0\}. \tag{6.8.21}$$

Hence, when (X, o) is Gorenstein, $T = \min\{t \in \mathbb{Z}_{\geq 0} \,:\, Z_K \leq t \cdot Z_{max}\}$ (use 6.8.9).

Tomari replaced $\dim(R/fR)$ in the first inequality of (6.8.19) by $1 - \min \chi$ by proving $T - \min \chi \leq p_g$. Also, for weighted homogeneous singularities, or for singularities with $Z_{min} = Z_{max}$ he verified that $\dim(R/fR) \leq 1 - \min \chi$. Therefore, in these cases, $p_g \leq T \cdot (1 - \min \chi)$ as well.

The inequality $p_g + \min \chi \geq T$ improves 6.8.47, and its proof is a slight modification of the proof of 6.8.47 using the multiplicative structure of the involved cohomology groups. Indeed, let us take the identity (6.8.17) applied for $l = l_m > 0$. Set $A := \mathfrak{m}/H^0(\mathscr{O}(-l_m))$ and $B := H^1(\mathscr{O}(-l_m))$, let a and b be their dimensions, hence $p_g + \min \chi \geq 1 + a + b$. Both A and B are $\mathscr{O}_{X,o}$ modules, and by Nakayama's Lemma $\mathfrak{m}^a A = 0$ (hence $\mathfrak{m}^{a+1} \subset H^0(\mathscr{O}(-l_m))$) and $\mathfrak{m}^b B = 0$. Consider now the natural morphisms $H^0(\mathscr{O}(-l_m)) \otimes R \xrightarrow{u} H^1(\mathscr{O}(-l_m)) \xrightarrow{v} R$, where the composition is the multiplication. Then $\mathfrak{m}^{a+b+1} \cdot R \subseteq v(u(\mathfrak{m}^b \cdot H^0(\mathscr{O}(-l_m)) \otimes R)) \subseteq v(\mathfrak{m}^b B) = v(0) = 0$. Hence $1 + a + b \geq T$.

6.8.D *Plurigenera*

Motivated by the Laufer duality, using 2-forms, we introduce the three families of plurigenera of Morales, Watanabe and Knöller and we provide their asymptotic behaviours as well. ♣

6.8.55 Some generalizations of the duality formula (6.8.1) are natural.

For any positive integer m we will consider the sheaf of m-ple holomorphic 2-forms. In local coordinates $z = (z_1, z_2)$ they have the form $\omega := \varphi(z)(dz_1 \wedge dz_2)^{\otimes m}$, where φ is holomorphic. One associates with ω a continuous $(2, 2)$ form $(\omega \wedge \bar{\omega})^{1/m}$, which in local coordinates is $\left(\frac{\sqrt{-1}}{2}\right)^2 \cdot |\varphi(z)|^{2/m} dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2$.

A global m-ple holomorphic form $\omega \in H^0(X \setminus o, \mathscr{O}_X(mK_X))$ is called $L^{2/m}$-integrable if for any sufficiently small relatively compact neighbourhood W of $o \in X$

$$\int_{W \setminus o} (\omega \wedge \bar{\omega})^{1/m} < \infty.$$

We denote the $L^{2/m}$-integrable forms of $H^0(X \setminus o, \mathscr{O}_X(mK_X))$ by $L^{2/m}(X \setminus o)$. Local computations at points of E show the following characterization.

Proposition 6.8.56 (Sakai [606, Th. 2.1]) *In a good resolution* $L^{2/m}(X \setminus o) \simeq H^0(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(mK_{\widetilde{X}} + (m-1)E))$ *holds.*

For any divisor D supported on E consider the isomorphism

$$\frac{H^0(\widetilde{X} \setminus E, \mathscr{O}_{\widetilde{X}}(mK_{\widetilde{X}}))}{H^0(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(mK_{\widetilde{X}} + D))} \simeq \frac{H^0(\widetilde{X} \setminus E, \mathscr{O}_{\widetilde{X}}(mK_{\widetilde{X}} + D))}{H^0(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(mK_{\widetilde{X}} + D))}.$$

This space embeds into $H^1_c(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(mK_{\widetilde{X}} + D))$, hence it is finite dimensional.

Definition 6.8.57 For each pair $(m, n) \in \mathbb{Z}_{>0} \times \mathbb{Z}$ we define

$$\Delta_{m,n}(X, o) := \dim_{\mathbb{C}} \frac{H^0(\widetilde{X} \setminus E, \mathscr{O}_{\widetilde{X}}(mK_{\widetilde{X}}))}{H^0(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(m(K_{\widetilde{X}} + E) + nE))}.$$

For some particular values of n the invariant $\Delta_{m,n}$ has already been considered:

(a) $\Delta_{m,0}$ is the m-th log-plurigenus $\lambda_m(X, o)$ of Morales [442],
(b) $\Delta_{m,-1}$ is the m-th L^2-plurigenus $\delta_m(X, o)$ of Watanabe [747],
(c) $\Delta_{m,-m}$ is the m-th plurigenus $\gamma_m(X, o)$ of Knöller [321].

E.g., $\Delta_{1,-1} = \delta_1(X,o) = \gamma_1(X,o) = p_g(X,o)$. Moreover, $\Delta_{m,n} \leq \Delta_{m,n-1}$, hence

$$\lambda_m(X,o) \leq \delta_m(X,o) \leq \gamma_m(X,o). \tag{6.8.22}$$

Let us give some examples. We start with cone singularities, for notations see 5.1.3.

Proposition 6.8.58 (For the Cases δ_m and λ_m See [528, 747])
Consider the algebraic cone singularity (X,o) associated with the ample $\mathscr{L} \in Pic(C)$ as in 5.1.3. Let $\widetilde{X}$ be the blow up $0 \in X$ and write E ($\simeq C$) for the exceptional curve. Set also $\mathscr{O}_E(K_E) = \Omega^1_E$. Then the following identity holds:

$$\Delta_{m,n} = \sum_{k>n} h^0(E, \mathscr{O}_E(mK_E + kE)). \tag{6.8.23}$$

Proof Consider the exact sequence

$$H^0(\mathscr{O}_{\widetilde{X}}(mK_{\widetilde{X}}(E)+(k-1)E)) \hookrightarrow H^0(\mathscr{O}_{\widetilde{X}}(mK_{\widetilde{X}}(E)+kE)) \xrightarrow{r} H^0(\mathscr{O}_E(mK_E+kE)).$$

For cone singularities the restriction r is onto. Indeed, if $s \in H^0(\mathscr{O}_E(mK_E + kE))$, then by Lemma 5.1.4 one has $r\pi^*(s) = s$. Finally note that $H^0(\mathscr{O}_{\widetilde{X}}(mK_{\widetilde{X}}(E) + kE)) = H^0(\widetilde{X} \setminus E, \mathscr{O}_{\widetilde{X}}(mK_{\widetilde{X}}))$ for $k \gg 0$ (because $\Delta_{m,n}$ is finite dimensional). □

Remark 6.8.59 (Topologically Cone Singularities) Assume that the exceptional curve of the minimal resolution $\widetilde{X} \to X$ of a singularity (X,o) consists of a unique irreducible nonsingular component E, that is, $\widetilde{X}$ topologically is a disc bundle over E (or, the link L_X is an S^1-bundle over E with negative Euler number). All the arguments regarding the formula (6.8.23) are valid except maybe the surjectivity of r. Therefore, for any analytic structure with this topological type one has

$$\Delta_{m,n} \leq \sum_{k>n} h^0(E, \mathscr{O}_E(mK_E + kE)).$$

Example 6.8.60 (The Homogeneous Hypersurface Singularity) Let $f : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ be a homogeneous degree d isolated hypersurface singularity, $(X,o) = \{f = 0\}$. We blow up $\Phi : B_0\mathbb{C}^3 \to \mathbb{C}^3$ the origin. The strict transform $\widetilde{X}$ of X is the minimal resolution of X. Let $\mathbb{P}^2$ be the exceptional plane of Φ, and $E := \widetilde{X} \cap \mathbb{P}^2$. Then E is a smooth curve in $\mathbb{P}^2$ of degree d, and $\widetilde{X}$ is the restriction of the tautological line bundle of $\mathbb{P}^2$ to E. Let H be the hyperplane in $\mathbb{P}^2$, and also its restriction to E. Then for this cone singularity $K_E = \mathscr{O}_E((d-3)H)$ and $\mathscr{O}_E(E) = \mathscr{O}_E(-H)$, hence

$$\Delta_{m,n} = \sum_{k>n} h^0(E, \mathscr{O}_E((m(d-3)-k)H).$$

But $H^0(E, \mathscr{O}_E(aH)) \simeq H^0(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(a))/H^0(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(a-d))$ for $H^1(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(a)) = 0$ for every $a \in \mathbb{Z}$. In particular, with the notation

$$\mathfrak{N}(a) := \sum_{k\geq 0} h^0(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(a-k)) = \#\{(a_1,a_2,a_3) \in \mathbb{Z}^3_{\geq 0} \ : \ \sum_i a_i \leq a\} = \binom{a+3}{3},$$

we get $\Delta_{m,n} = \mathfrak{N}(md - 3m - n - 1) - \mathfrak{N}(md - 3m - n - 1 - d)$. E.g., $p_g = \binom{d}{3}$.

Example 6.8.61 (The Quasi-Homogeneous Hypersurface Singularity) For $n = 0$ or $n = -1$ the following formulae were verified in [528, 747].

Let $f : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ be a quasi-homogeneous degree d isolated hypersurface singularity, $(X, o) = \{f = 0\}$. We denote the weights by $(w_1, w_2, w_3) \in \mathbb{Z}^3_{>0}$, that is f is a linear combination of monomials of type $z^a = \prod_i z_i^{a_i}$, where $\sum_i a_i w_i = d$. Set

$$\mathfrak{N}_w(a) := \#\{(a_1,a_2,a_3) \in \mathbb{Z}^3_{\geq 0} \ : \ \textstyle\sum_i a_i w_i \leq a\}.$$

Then $\Delta_{m,n} = \mathfrak{N}_w(m(d - \sum_i w_i) - n - 1) - \mathfrak{N}_w(m(d - \sum_i w_i) - n - 1 - d)$.

Example 6.8.62 ([444]) Assume that (X, o) is **weighted homogeneous and Gorenstein**. Let R be the local graded ring of (X, o) as in 5.1.26. Then $\delta_m = \sum_{\ell \leq m\mathfrak{r}} \dim R_\ell$ and $\lambda_m = \sum_{\ell < m\mathfrak{r}} \dim R_\ell$. (For $\delta_1 = p_g$ see 6.3.30(d).)

Example 6.8.63 ([747]) Assume that (X, o) is a **quotient singularity** $(\mathbb{C}^2, 0)/G$, where G is a small subgroup of $GL_2(\mathbb{C})$. Then $\delta_m(X, o) = 0$ for every $m > 0$. Indeed, using the covering $\mathbb{C}^2 \to X$ and the fact that any m-ple holomorphic form of $\mathbb{C}^2 \setminus \{0\}$ is $L^{2/m}$, by covering and integration we get the same for $X \setminus \{x\}$.

Example 6.8.64 Assume that (X, o) is Gorenstein and $\widetilde{X} \to X$ is minimal, hence $Z_K \in \mathscr{S}$. Then

$$\gamma_m = -(m^2 - m)Z_K^2/2 + p_g(X, o).$$

Indeed, similarly as in 6.8.4 (and by the vanishing $h^1(\mathscr{O}_{\widetilde{X}}(mK_{\widetilde{X}})) = 0$) one has

$$\gamma_m = \dim \frac{H^0(\widetilde{X} \setminus E, \mathscr{O}_{\widetilde{X}}(mK_{\widetilde{X}}))}{H^0(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(mK_{\widetilde{X}}))} = h^1_c(\mathscr{O}_{\widetilde{X}}(mK_{\widetilde{X}})) = h^1(\mathscr{O}_{\widetilde{X}}(-(m-1)K_{\widetilde{X}})).$$

This equals $h^1(\mathscr{O}_{\widetilde{X}}((m-1)Z_K))$, which can be computed from the cohomology exact sequence of $0 \to \mathscr{O}_{\widetilde{X}} \to \mathscr{O}_{\widetilde{X}}((m-1)Z_K) \to \mathscr{O}_{(m-1)Z_K}((m-1)Z_K) \to 0$. Here $h^0(\mathscr{O}_{(m-1)Z_K}((m-1)Z_K)) = 0$ by 6.4.2, hence $\gamma_m = p_g - \chi(\mathscr{O}_{(m-1)Z_K}((m-1)Z_K))$.

6.8.65 The next formulae show the **asymptotic behaviour** of the plurigenera.

In any resolution we apply the Zariski decomposition from 6.6.23 for $l' := Z_K - E$, namely we write $-l' = \widetilde{e}_Z(l') - \widetilde{s}_Z(l')$. Usually this is written in the form $K + E = N + P$ as well (hence $P = -\widetilde{s}_Z(l')$). Then one has the following facts:

If $\widetilde{X} \to X$ is a minimal resolution then (cf. [442, 528])

$$\gamma_m = -(m^2 - m)Z_K^2/2 + p_g(X, o) + p(m),$$

where $p(m)$ is periodic for $m \gg 0$. In a minimal good resolution (cf. [130, 528, 727])

$$\lambda_m = -m^2P^2/2 - m(Z_K, P)/2 + b(m),$$

where $b(m)$ is bounded, and it is periodic whenever (X, o) is Gorenstein. Finally, again, in a minimal good resolution and for certain bounded $b'(m)$ (cf. [528])

$$\delta_{m+1} = -m^2P^2/2 + m(Z_K, P)/2 + b'(m).$$

6.9 Relations with Smoothing Invariants

In 6.9.A we make a short presentation of some deformation theoretical results, which complement the invariants of the abstract germ (or the invariants read from a resolution) with the so-called 'smoothing invariants'. This subsection is more of an informative nature, though several of its statements will be used later. Similarly, in part 6.9.B we collect several concrete formulae and procedures, which provide different invariants from the 'Milnor package' for special hypersurface singularities; this also will help the reader to follow further computations and exemplifications. ♣

6.9.A The Formulae of Laufer, Durfee and Wahl

6.9.1 Fix a normal surface singularity (X, o). A deformations of (X, o) is a flat analytic germ $\lambda : (\mathcal{X}, o) \to (T, o)$ such that (X, o) is isomorphic to $(\lambda^{-1}(o), o)$ under a fixed isomorphism. (We assume that (T, o) is reduced.) By a theorem of Grauert, an isolated singularity (X, o) admits a *semi-universal deformation* $(\mathcal{X}, o) \to (B, o)$ [224, 616]. This has the property that any flat deformation over (T, o) is induced by a map $(T, o) \to (B, o)$, which is unique at the level of the tangent spaces. The space (B, o) is called the *base space of the semi-*

universal deformation. It is smooth if (X, o) is a complete intersection [706] or if $emb(X, o) = 4$ [613]. But, in general, (B, o) can be singular with many irreducible components which might have different dimensions.

A flat deformation $\lambda : (\mathscr{X}, o) \to (T, o)$ of (X, o) with $\dim T = 1$ is called a one parameter smoothing of (X, o) if $X_t = \lambda^{-1}(t)$ is smooth for $t \neq 0$. $F := X_t$ $(t \neq 0)$ is called the *Milnor fiber* of λ, it is a real connected 4-manifold with boundary L_X. Let $\mu := \operatorname{rank} H_2(F, \mathbb{Z})$ be its *Milnor number* and σ its signature, cf. 3.2.A. Since $H_1(F, \mathbb{Q}) = 0$ (a fact conjectured by Wahl [726], and proved by Greuel and Steenbrink [236]), the topological Euler characteristic $\chi_{top}(F)$ of F is $1 + \mu$. The Milnor fiber and all its invariants (e.g. μ and σ) might depend on the choice of the smoothing λ.

An irreducible component of B is called a smoothing component if the general fiber over it is smooth. Any 1-parameter smoothing lies on a certain smoothing component.

Example 6.9.2 Not every singularity (X, o) admits a smoothing, and, in general, it is an open problem how to decide if a specific singularity has any smoothing at all. For different obstructions see the papers of Laufer, Wahl, Looijenga, Pinkham [362, 394, 397, 572, 726, 730, 731] and the references listed in them.

For example, Dolgachev's triangle singularity $D_{p,q,r}$ (cf. 7.2.20) admits a smoothing if $p + q + r < 22$ [394], it cannot be smoothed if $p + q + r > 22$ [726], and the remaining cases $p + q + r = 22$ are decided in [572]: all but $D_{2,10,10}$ can be smoothed.

Example 6.9.3 For a rational singularity (i.e. $p_g(X, o) = 0$) any component of B is smoothing component. One of the components is distinguished. It can be constructed as follows [30]: let $\widetilde{X} \to X$ be the minimal resolution of (X, o). Then all the deformations of the analytic structures of $\widetilde{X}$ (hence, which preserve the topological type of $\widetilde{X}$) project (via a finite group action) to deformations of (X, o). These deformations provide a component, called the Artin component. It is smooth [725].

Example 6.9.4 The first example when B has two components (both of them smoothing components) was constructed by Pinkham [570]: (X, o) is the cone over the rational normal curve of degree 4 (the minimal graph has one vertex with $e = -4$ and $g = 0$). B has two components: the Artin component B_1 has dimension 3, the other, B_2, has dimension 1. Both components are smooth and intersect each other transversally. $\mu(B_1) = 1$, while $\mu(B_2) = 0$, hence $F(B_2)$ is a rational homology ball.

The smoothing above B_2 can be realized as follows, see e.g. [726]. Consider the hypersurface singularity $\{f = 0\} = \{xy - z^2 = 0\}$ in $(\mathbb{C}^3, 0)$ with the $\mathbb{Z}_2$ action $(\pm 1) * (x, y, z) = (\pm x, \pm y, \pm z)$. The quotient singularity $\{f = 0\}/\mathbb{Z}_2$ is (X, o), and the smoothing $\{f = t\}$ of f induces a smoothing of (X, o). The Milnor fiber F_f of f has the homotopy type of S^2, and $\mathbb{Z}_2$ acts on F_f freely; hence $\chi_{top}(F) = 1$.

6.9.5 Smoothing Versus Resolution In the presence of a smoothing the singularity (X, o) is 'approximated' by two smooth surfaces, its resolution and its smoothing. The point is that their invariants are related in a very subtle way. The next two relations connect the topological invariants $Z_K^2 + |\mathscr{V}|$, $b_1(L_X)$ of the link, the analytic invariant p_g and the invariants μ and σ of a fixed smoothing. For different levels of generality, see [162, 362, 397, 655, 726].

Theorem 6.9.6 (Laufer's Formula [362]) *Assume that λ is a smoothing of a Gorenstein normal surface singularity (X, o). Then $\mu = 12p_g + Z_K^2 + |\mathscr{V}| - b_1(L_X)$. In particular, for (X, o) Gorenstein, $\mu(F)$ is independent of the smoothing.*

Theorem 6.9.7 (Durfee's Formula [162]) *The Milnor fiber of a smoothing of a normal surface singularity (X, o) satisfies $4p_g = \mu + \sigma + b_1(L_X)$. This combined with Laufer's formula 6.9.6 for $(X, 0)$ Gorenstein gives $\sigma + 8p_g + Z_K^2 + |\mathscr{V}| = 0$.*

These show that modulo the link-invariants $Z_K^2 + |\mathscr{V}|$ and $b_1(L_X)$, there are two independent relations connecting p_g, μ and σ, provided that (X, o) is Gorenstein. In particular, if by some other argument one can recover one of them from the topology of L_X, then all of them can be determined topologically.

6.9.8 Consider the minimal resolution $\widetilde{X}$ of (X, o) with $|\mathscr{V}_{min}|$ irreducible exceptional components and canonical cycle $Z_{K,min}$. Let θ_X and $\theta_{\widetilde{X}}$ denote the corresponding sheaves of derivations. Set $\theta := h^1(\widetilde{X}, \theta_{\widetilde{X}})$.

Theorem 6.9.9 (Wahl's Formulae [726]) *Let $\lambda : (\mathscr{X}, 0) \to (T, 0)$ be a smoothing, $\theta_{\mathscr{X}/T}$ the relative derivations, and define β as the (finite) dimension of coker$(\theta_{\mathscr{X}/T} \otimes \mathscr{O}_X \to \theta_X)$. (Usually it depends on λ). Then, β equals the dimension of the irreducible component of B on which the smoothing λ occurs, and*

$$\beta - \theta + 14p_g = 2(\mu + b_1(L_X) - |\mathscr{V}_{min}|).$$

If (X, o) is Gorenstein, then this and 6.9.6 imply $\beta = \theta + 10p_g + 2Z_{K,min}^2$.

Notice that β depends only on (X, o) and the Milnor number of the smoothing λ; in fact, $\beta - 2\mu$ is independent of the smoothing. If (X, o) is Gorenstein, then β itself is independent of the smoothing.

Consult also [726] for several similar formulae. For a review, and for several new formulas and conjectures see also [730, 731].

Example 6.9.10 If $p_g(X, o) = 0$ then $b_1(L_X) = 0$ too, hence for any smoothing $\sigma = -\mu$, $0 \le \mu \le |\mathscr{V}_{min}|$, and $\beta = \theta - 2(|\mathscr{V}_{min}| - \mu)$. For smoothings supported by the Artin component B_1 one has $\mu = |\mathscr{V}_{min}|$ and $\dim B_1 = \beta = \theta$.

Example 6.9.11 Consider a 'rational homology disc smoothing', that is, a smoothing with $\mu = 0$. Then the link is a rational homology sphere and by Durfee formula $p_g = 0$. A complete list of resolution graphs of weighted homogeneous complex surface singularities admitting a rational homology disk smoothing is given

in [57, 667], see also [728, 729]. In [728] Wahl conjectures that *the only complex surface singularities admitting a rational homology disc smoothing are the weighted homogeneous examples of this list.*

Remark 6.9.12 A singularity is called *rigid* if and only if all its deformations are trivial. It is conjectured that *there exists no rigid non-smooth reduced curve singularity, and no rigid non-smooth normal surface singularity*. (Note that the higher dimensional quotient singularities are rigid [617].) For more see [240].

6.9.B Thom–Sebastiani Type Results and Suspension-Formulae

For Milnor fibrations we will use the notations of 3.2.A, if it is needed we indicate the corresponding function in the subscript.

6.9.13 Some Thom–Sebastiani Type Results [623] Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ and $g : (\mathbb{C}^{m+1}, 0) \to (\mathbb{C}, 0)$ be isolated hypersurface singularities. Define $f \oplus g : (\mathbb{C}^{n+1} \times \mathbb{C}^{m+1}, 0) \to (\mathbb{C}, 0)$ by $(f \oplus g)(x, y) := f(x) + g(y)$. Then $F_{f\oplus g}$ is homotopically equivalent to the join space $F_f * F_g$. Furthermore, the geometric monodromies also satisfy $h_{geo,f\oplus g} \sim h_{geo,f} * h_{geo,g}$. In particular, $H_{n+m+1}(F_{f\oplus g}) = \tilde{H}_n(F_f) \otimes \tilde{H}_m(F_g)$ and $h_{f\oplus g} = h_f \otimes h_g$. Hence if $\mathrm{Div} = \sum_k (\xi_k) \in \mathbb{Z}[S^1]$ denotes the divisor of $\Delta = \det(1 - th) = \prod_k (\xi_k - t)$, then $\mathrm{Div}_{f\oplus g} = \mathrm{Div}_f \cdot \mathrm{Div}_g$. Furthermore, by a theorem proved independently by Deligne and Sakamoto [608], the Seifert forms satisfy $S_{f\oplus g} = (-1)^{(n+1)(m+1)} S_f \otimes S_g$ as well.

Example 6.9.14 (Invariants of Brieskorn Singularities) Since for $(\mathbb{C}, 0) \to (\mathbb{C}, 0)$, given by $z \mapsto z^a$ $(a \geq 2)$, the Seifert form (in a special base) is:

$$S(z \mapsto z^a) = \begin{pmatrix} -1 & 0 & 0 & \dots & 0 \\ 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -1 \end{pmatrix} \quad (a-1) \times (a-1) \text{ matrix},$$

the variation map, the intersection form and the monodromy operator can be written explicitly (use 3.2.15). Indeed, the pair (h, V) associated with $z \mapsto z^a$ has

the following monodromy eigenvalue decomposition $\oplus_{k=1}^{a-1}(e^{2\pi ik/a}, e^{2\pi ik/a}-1)$. Hence, the pair (h, V) for $f=\sum_{j=1}^{3} z_j^{a_j}$ is

$$(h,V)=\oplus_{k_1=1}^{a_1-1}\oplus_{k_2=1}^{a_2-1}\oplus_{k_3=1}^{a_3-1}\Big(\prod_j e^{2\pi ik_j/a_j}, \prod_j(1-e^{2\pi ik_j/a_j})\Big).$$

Since $b=V^{-1}(h-1)$, a computation gives Brieskorn's formula [82]

$$\sigma(a_1,a_2,a_3):=\sigma(\sum_{j=1}^{3} z_j^{a_j})=\sum_{k_1=1}^{a_1-1}\sum_{k_2=1}^{a_2-1}\sum_{k_3=1}^{a_3-1}\operatorname{sign}\sin(\pi\sum_{j=1}^{3}k_j/a_j). \tag{6.9.1}$$

In fact, $\mu_0=b_1(L_X)=|\{1\le k_j\le a_j-1,\ j=1,2,3,\ \sum_j k_j/a_j\in\mathbb{Z}\}|$, and if $S_t:=|\{1\le k_j\le a_j-1,\ j=1,2,3,\ t<\sum_j k_j/a_j<t+1\}|$, then $S_0=S_2$, $\mu_+=S_0+S_2=2S_0$, $\mu_-=S_1$, $\sigma=\sum_t(-1)^t S_t$, $\mu=\sum_j S_j+\mu_0=\prod_j(a_j-1)$. Also,

$$\mu_0(a_1,a_2,a_3)=\frac{(a_1,a_2)(a_2,a_3)(a_3,a_1)}{(a_1,a_2,a_3)}-(a_1,a_2)-(a_2,a_3)-(a_3,a_1)+2.$$

Using Durfee's formula 6.9.7 one gets $p_g=S_0+\mu_0/2=|\{1\le k_j\le a_j-1,\ j=1,2,3,\ \sum_j k_j/a_j\le 1\}|$ (compare also with 5.5.4(c)). Hence, by 6.9.6, $Z_K^2+|\mathcal{V}|$ is also determined.

Note that S_0 counts the lattice points in the interior of a tetrahedron. This number was obtained by Mordell in terms of Dedekind sums [445], see also [276, 577]. Since $\sigma=4S_0-\mu$, we obtain that if the integers a_j are relative prime then $\sigma(a_1,a_2,a_3)$ in terms of Dedekind sums is

$$-1+\frac{1+a_1^2a_2^2+a_2^2a_3^2+a_3^2a_1^2-a_1^2a_2^2a_3^2}{3a_1a_2a_3}-4\Big(\mathbf{s}(a_1a_2,a_3)+\mathbf{s}(a_2a_3,a_1)+\mathbf{s}(a_3a_1,a_2)\Big).$$

6.9.15 Suspensions Assume that $f'(x,y,z)=f(x,y)+z^N$, where f is an isolated plane curve singularity. Let Γ_f be the embedded resolution graph of f as in 2.2, 4.1.C; see also 4.1.28 for some additional notations. We wish to recover invariants of f' from Γ_f and N.

The Milnor number and the characteristic polynomial of f is determined by A'Campo's theorem 4.1.22. Hence, by Thom–Sebastiani theorem 6.9.13 we get the corresponding invariants for f' as well. In 4.1.28 we described the Jordan block structure of h_f. Hence we obtain $h_{f'}$ as well, and μ_0 for f' can be deduced from $\nu_0=b_1(L_{f'=0})=\dim\ker(h_{f'}-1)$ (cf. 3.2.2). In fact, an expression for $b_1(L_{f'=0})$ can be deduced also from the $b_1(L_{f'=0})=c(\Gamma(X_{f,N}))+2g(\Gamma(X_{f,N}))$, where $c(\Gamma(X_{f,N}))$ is given in (4.1.3), and $g(\Gamma(X_{f,N}))$ in the algorithms 4.1.16 (or 4.1.18). The very same algorithm provides the link of f' as well. Using the notations of 4.1.10 and 4.1.28, for the signature of f' we have the formula [466]:

Proposition 6.9.16 *Define the 'eta-invariant'* $\mathbb{Z} \ni \ell \mapsto \eta(f;\ell)$ *in terms of the graph* Γ_f *(with nodes* $\mathcal{N}$*) by*

$$\eta(f;\ell) = |\mathscr{A}| - 1 + \sum_{e\in\mathscr{E}}((\ell,\mathfrak{n}_e)-1) - \sum_{v\in\mathscr{V}}((\ell,\mathfrak{n}_v)-1) + 4\cdot\sum_{v\in\mathcal{N}}\sum_{w\in\mathscr{W}_v}\mathbf{s}(m_w,\ell;m_v).$$

Then

$$\sigma(f(x,y)+z^N) = \eta(f;N) - N\cdot\eta(f;1). \tag{6.9.2}$$

If f is irreducible, then 4.1.D shows that $\eta(f;\ell) = 4\cdot\sum_{v\in\mathcal{N}}\sum_{w\in\mathscr{W}_v}\mathbf{s}(m_w,\ell;m_v)$.

In fact, for irreducible f one can express $\sigma(f')$ in terms of signatures of Brieskorn singularities as well. Indeed, let $(p_i,q_i)_{i=1}^s$ be the Newton pairs of f (cf. 3.3.10), and define the splice diagram decorations $\{a_i\}_{i=1}^s$ as in 5.3.20. Set also $d_i := (N, p_{i+1}\cdots p_s)$ for $i < s$ and $d_s = 1$. Then [466]

$$\sigma(f') = \sum_{i=1}^{s} d_i\cdot\sigma(a_i,p_i,N/d_i). \tag{6.9.3}$$

This can be compared with the following consequences of (5.3.3)

$$\mu_0(f') = \sum_{i=1}^{s} d_i\cdot\mu_0(a_i,p_i,N/d_i) \quad\text{and}\quad \mu(f') = \sum_{i=1}^{s} p_{i+1}\cdots p_s\cdot\mu(a_i,p_i,N). \tag{6.9.4}$$

For the geometric genus (for f not necessarily irreducible) we have [472]:

$$\begin{aligned} 4p_g(f'=0) =& N(2-|\mathscr{A}|) + \sum_{v\in\mathscr{V}}\Big(m_v(N-1)+(N,m_v)\Big)(\kappa_v(\Gamma_f)-2) \\ &+ 4\sum_{v\in\mathcal{N}}\sum_{w\in\mathscr{W}_v}\Big(\mathbf{s}(m_w,N;m_v) - N\cdot\mathbf{s}(m_w,1;m_v)\Big). \end{aligned}$$

For the equivariant signature see [463]. For a different definition of the eta-invariant and other connections and comments see 9.1.21.

6.9.17 A different type of formula is proved in [64]: it uses the cycle $\mathrm{div}_E(f)$ in the lattice L of the embedded resolution of f.

$$p_g(f(x,y)+z^N=0) = \sum_{k=1}^{N-1}\chi(-\lfloor k\cdot\mathrm{div}_E(f)/N\rfloor).$$

6.9.C *Some Open Problems for Hypersurface Singularities*

Though the main focus of the book is not on hypersurface singularities, we still mention some guiding open problems, conjectures (or just certain difficulties and obstructions in their classifications). In this way we can highlight several results regarding normal surface singularities from a different perspective (embedding dimension three). Furthermore, the presentation emphasizes the parallelism with the C^∞ embedded classifications (imposed by the links), and their anomalies which might appear exactly in this peculiar case of complex surfaces. ♣

6.9.18 Analytic Classification of Isolated Hypersurface Singularities The classification might run over several different equivalence relations; we mention only two of them. We say that $f, g : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ are *right equivalent*, $f \overset{r}{\sim} g$, if there exists a biholomorphism germ $\phi : (\mathbb{C}^{n+1}, 0) \circlearrowleft$ satisfying $f = g \circ \phi$. The germs f and g are *contact equivalent*, or *biholomorphic equivalent*, $f \overset{c}{\sim} g$, if $\mathscr{O}_{V(f),0} \simeq \mathscr{O}_{V(g),0}$. In both classifications it is hard to identify the corresponding stratifications and orbits. For the simple, 1- and 2-modal germs, with respect to the right equivalence, see [21]. It would be helpful to have several (partial) characterizations of the equivalences involving certain geometrically defined invariants.

Let us mention here one such theorem. First notice that if $f \overset{r}{\sim} g$, then their Milnor algebras are isomorphic, $\mathbb{C}\{z\}/J(f) \simeq \mathbb{C}\{z\}/J(g)$. Furthermore, if $f \overset{c}{\sim} g$, then their Tjurina (or, moduli) algebras are isomorphic, $\mathbb{C}\{z\}/(f, J(f)) \simeq \mathbb{C}\{z\}/(g, J(g))$. Now, by a theorem of Mather and Yau, $\mathbb{C}\{z\}/(f, J(f)) \simeq \mathbb{C}\{z\}/(g, J(g))$ implies $f \overset{c}{\sim} g$ [409] (hence $\overset{c}{\sim}$ is characterized by the finite dimensional Tjurina algebra).

6.9.19 Topological Classification of Isolated Hypersurface Singularities Again, we might run different equivalence relations. We say that the holomorphic germs f and g are *topologically right equivalent*, $f \overset{tr}{\sim} g$, if there exists a homeomorphism germ $\phi : (\mathbb{C}^{n+1}, 0) \circlearrowleft$ such that $f = g \circ \phi$. We say that f and g are *topologically V-equivalent*, $f \overset{tv}{\sim} g$, if there exists a homeomorphism germ $\phi : (\mathbb{C}^{n+1}, 0) \circlearrowleft$ satisfying $\phi(V(f)) = V(g)$. Finally, f and g are *link equivalent*, $f \overset{l}{\sim} g$, if the pairs $(S^{2n+1}_\epsilon, S^{2n+1}_\epsilon \cap V(f))$ and $(S^{2n+1}_{\epsilon'}, S^{2n+1}_{\epsilon'} \cap V(g))$ are homeomorphic for all $0 < \epsilon, \epsilon' \ll 1$.

By the local conic structure 3.1.2 $f \overset{l}{\sim} g$ implies $f \overset{tv}{\sim} g$. The converse was proved by Saeki [596]. Here the proof for $n = 2$ needs special attention (since the h-cobordism can not be applied). However, in the case $n = 2$ Saeki proves that the links of f and g are even isotopic whenever $f \overset{l}{\sim} g$ [597].

Clearly $f \overset{tr}{\sim} g$ implies $f \overset{tv}{\sim} g$. The converse was proved by King [315] for any dimension except $n = 2$ and by Perron for $n = 2$ [562] as follows: the topologically V-equivalence implies topologically right-left equivalence. Moreover, if f and g

are topologically right-left equivalent then g is topologically right equivalent with either to f or to $\bar{f}$, the complex conjugate of f.

6.9.20 Topological/Smooth Classification of Knots and Open Books In parallel, we might try to classify (nonembedded, abstract) smooth oriented $(2n-1)$-manifolds, or higher-dimensional knots (closed oriented $(2n-1)$-manifolds smoothly of topologically embedded into S^{2n+1}); here we can even assume that their normal bundles are trivial. Or, we can plan to understand the classification of the open book decompositions of S^{2n+1} (up to smooth isomorphism, or homeomorphism, or isotopy). For $n = 1$ the possible relationships were discussed in 3.4.10: in S^3 the oriented binding of open books determine completely the open book up to an isotopy. This is the case for higher dimensions $n \geq 3$ as well by the work of Durfee and Kato (independently) [160, 303]. In fact, in this case the open book decompositions are classified by the conjugacy classes of unimodular integral bilinear forms (via their Seifert forms), see also [403]. However, for $n = 2$ according to Saeki we only have the equivalences of the following objects: smoothly equivalent bindings, topologically equivalent bindings (that is, equivalence up to homeomorphism) and topologically equivalent open books [597, 599]. In particular, the smooth binding does not determine the open book up to smooth isomorphism. The problem appears already regarding the smooth structure of the 4-dimensional page: Saeki constructed a binding which admits two different open book decompositions with non-isomorphic pages [597, Prop. 5.5]. For $n = 2$ the Durfee–Kato correspondence also fails: the Seifert form does not determine the open book, not even in the context of Milnor open books of holomorphic germs [23]. For more see also [419, 423].

Related with these abstract topological/smooth discussions we can ask: which objects (knots or open books) are realized by complex hypersurface singularities. This basically is an open problem for any dimension $n \geq 2$. Even for $n \geq 3$ the possible 'algebraic' Seifert forms are not classified (though Durfee showed that they have an upper triangular representative [160].) (Similarly, regarding the above example of Saeki, one can ask for the existence of two link equivalent singularities with different Milnor fibrations.)

Hence, for $n = 2$ the classifications are really challenging (and even the notion of 'topological equivalent germs' should be made precise in different contexts).

There exists another point of view as well: one classifies embeddings/immersions of 3-manifolds into S^5 up to *regular homotopy*, see e.g. [600] and references therein.

6.9.21 Around Zariski's Conjecture Assume that (X, o) is an isolated hypersurface singularity in $(\mathbb{C}^3, 0)$. One wishes to relate the abstract analytic type of (X, o) with the embedded topological type (say, with smooth link equivalence). The meeting point is the abstract link L_X: the first theory aims to understand the invariants of the hypersurface analytic structure supported by L_X, while the second one focuses on the possible embeddings of L_X into S^5, which can be realized by a hypersurface equation.

From the embedded topological type one can extract some immediate information. For example, in [678] Teissier proved that the Milnor number μ can be recovered from (S^5, L_X). In particular, via 6.9.6, the geometric genus can also be deduced from the embedded topological type. Lê extended Teissier's result for the characteristic polynomial $\Delta(t)$ [371]. In fact, using Milnor fibration, the universal covering of the link complement $S^5 \setminus L_X$ (which has $\pi_1(S^5 \setminus L_X) = \mathbb{Z}$) is homotopically F and the covering action is the geometric monodromy. However, it is rather hard to extract any other invariant from the embedding $L_X \subset S^5$, which goes beyond the homological package of the Milnor fibration 3.2.A. In fact, even from the variation map it is rather hard to extract any additional information more than the monodromy or the equivariant signature. [In fact, Xu and S. S.-T. Yau—after they proved that the embedded topological type of a weighted homogeneous singularity can be recovered from L_X and $\Delta(t)$—conjectured that this is true in general [758]. However, Artal Bartolo found a pair of superisolated germs (given by a 'Zariski pair') with the same L_X and $\Delta(t)$, but different monodromies [23].]

The next conjecture targets another analytic invariant, independent of p_g.

Conjecture 6.9.22 ([783] Zariski's Conjecture) The multiplicity of an isolated hypersurface singularity can be recovered from the embedded topological type (Zariski formulated it for topological V-equivalence).

This has been verified so far only for quasi-homogeneous singularities (see [232, 544, 758, 768], and for suspension singularities (with f irreducible) [417], and some sporadic cases.

In all the cases when Zariski's conjecture is proved, in fact, one proves substantially more, the most optimal result: one recovers the equisingularity type from the embedded topological type. Indeed, in the weighted homogeneous case one can recover the weights, and in the case of suspension $f(x, y) + z^N$, with f irreducible, from L_X and μ one recovers the Newton pairs of f and the integer N [417].

Here we wish to emphasize the following fact (much in the spirit of the present book). For the family of suspensions (with $f(x, y)$ irreducible), if the link is a rational homology sphere, merely the abstract link L_X alone determines the Newton pairs of f and the covering integer N [417]. This means that L_X determines not only the multiplicity of (X, o) but also several other discrete invariants which depend only on these data. Furthermore, L_X determines completely the embedding $L_X \subset S^5$ as well. This is really remarkable: though in the case of plane curves the abstract link S^1 can be embedded in many different ways into S^3 as (even suspension type) algebraic knot, in the case of surfaces if two suspension singularities have isomorphic $\mathbb{Q}HS^3$ links then the links embed into S^5 in the same way. The very same statement is valid for suspensions with $b_1(L_X) = 0$, f arbitrary but $N = 2$ fixed [364].

Finally, the very same statement is valid for the family of Newton non-degenerate singularities (including the weighted homogeneous case) with $\mathbb{Q}HS^3$ links as well

[76]: L_X determines the Newton boundary (up to a natural equivalence, cf. 5.5.7). For a short proof in the weighted homogeneous case see 5.1.27.

A rather optimistic conjecture was formulated by Mendris and the author.

Conjecture 6.9.23 ([417]) Assume that (X, o) is an isolated hypersurface singularity with $\mathbb{Q}HS^3$ link. Then L_X characterizes completely the embedded topological type $L_X \subset S^5$, the geometric genus (more generally, the equivariant Hodge numbers) and the multiplicity.

Regarding Conjecture 6.9.23 it would be rather helpful to understand what is really specific in the links of hypersurface singularities. The following problem/question of Laufer is still unanswered.

Problem 6.9.24 ([259]) Let Γ be the graph of a normal surface singularity (X, o). Then (a) give necessary conditions on Γ for (X, o) to be necessarily a hypersurface singularity, and (b) give sufficient conditions on a graph Γ for a hypersurface singularity to exist with graph Γ.

In the case $b_1(M(\Gamma)) = 0$ the proofs in [417] (for suspensions, f irreducible) and [76] (Newton non-degenerate case) are algorithmic, that is, in these cases one can test in principle whether a certain graph belongs to these families or not.

We wish to add here also the following question related with the Zariski's Conjecture. One can try to find some intermediate analytic invariant between the embedded topological type and the multiplicity. A possible candidate is the Hodge spectrum of Steenbrink and Varchenko [654, 655, 657, 714]. It is known that it is constant along μ-constant deformation and it behaves semicontinuously [656, 712, 713]. Hence, the following question of Steenbrink is very natural:

Question 6.9.25 ([259]) Let (X, o) be an isolated hypersurface singularity.

(a) Is the spectrum an invariant of its embedded topological type?
(b) Does the spectrum determine the multiplicity?

In the case of quasi-homogeneous singularities the answer to both questions is yes: (a) is answered positively in [598], while [618] shows that from the spectrum one can recover the weights, which clearly provide the multiplicity. (See also [465] how the variation map determines the mod 2 spectral pairs.)

6.9.26 μ-Constant Deformations Regarding Zariski's Conjecture, its 'local version' was formulated by Teissier [677], which aims to prove that μ-constant deformations of hypersurfaces are equimultiple. For verification in the special cases of weighted homogeneous germs see e.g. [209, 232, 544] and for Newton non-degenerate ones [133, 523] (in these cases even the embedded topological type is constant). For a deformation of type $f(x) + tg(x)$ it was verified by Trotman [700].

But, naturally, one can investigate along a μ-constant deformation the stability of other invariants as well. Probably the most famous question is the following.

Conjecture 6.9.27 Along a μ-constant deformation the embedded topological type of a complex hypersurface is constant as well.

The statement is proved for $n \neq 2$ by Lê and Ramanujam [379], see also [688]. (In higher dimension the h-cobordism theorem is used.) For $n = 2$ Parusiński proved it for a deformation of type $f(x) + tg(x)$ [558] (and for the $\overset{tr}{\sim}$-equivalence).

In the above conjecture ($n = 2$ case), even the constancy of the abstract link is open.

On the other hand, along a μ-constant deformation of hypersurfaces the geometric genus stays constant. [Indeed, along μ-constant family the spectrum is constant [656], while p_g is the number of spectral numbers in $(-1, 0]$ [602].] Furthermore, as part of the proof of Lê–Ramanujam, one also has the following homotopical statement, even for $n = 2$: the monodromy fibrations of f_0 and $f_{\neq 0}$ are of the same fiber homotopy. But then the algebraic monodormies are the same, and by 3.2.14 the integral homologies of the links are the same, and by the Laufer formula 6.9.6 $Z^2_K + |\mathcal{V}|$ is the same (and by 6.9.7 the signatures are the same).

The importance of $Z^2_K + |\mathcal{V}|$ is motivated by the discussion from the next paragraph, though what would be really needed is the stability of $Z^2_{K,min}$ (Z^2_K computed in the minimal resolution), which is not guaranteed by the above known statements.

The constancy of the link, or of $Z^2_{K,min}$ (though looks topological) would imply several strong analytic statements too. In the next discussion the invariants are computed in the minimal resolution. Along a deformation $\{X_t\}_t$ the integers $Z^2_{K,min}$ and p_g behave semicontinuously: $Z^2_{K,min}(X_0) \leq Z^2_{K,min}(X_t)$ [710] and $p_g(X_0) \geq p_g(X_t)$ [184]. In the Gorenstein case (e.g. for hypersurfaces) the constancy of $Z^2_{K,min}(X_t)$ is equivalent to the existence of a *simultaneous RDP resolution*, which implies the constancy of $p_g(X_t)$ too [365]. The existence of a simultaneous RDP resolution implies (possibly after a base change) the existence of a *very weak simultaneous resolution*. The existence of a simultaneous RDP resolution is weaker than the existence of a *weak simultaneous resolution*, which is equivalent to the constancy of the links L_{X_t} [365].

6.9.28 Around Durfee's Conjecture Examples show that for a hypersurface singularity p_g is substantially smaller than μ, and σ tends to be rather negative. These observations led to the formulation of **Durfee's Conjectures** [162]:

Strong inequality: If (X, o) is an isolated complete intersection surface singularity then $6p_g \leq \mu$.

Weak inequality: If (X, o) is a normal surface singularity, then for any smoothing $4p_g \leq \mu + \mu_0$. Equivalently, $\sigma \leq 0$.

Semicontinuity of σ: If $\{(X_t, o)\}_{t\in(\mathbb{C},0)}$ is a flat family of isolated surface singularities (with smoothings) then $\sigma(X_{t=0}) \leq \sigma(X_{t\neq 0})$.

If (X, o) is Gorenstein, then by formulae of Laufer and Durfee an inequality of type $C \cdot p_g \leq \mu + \mu_0$ (for some constant C) transforms into $-\sigma \geq (C - 4)p_g$.

A counterexample to the *weak inequality* was given by Wahl [726, page 240]; it is a minimally elliptic singularity (not ICIS). If one combines the results from [726, 2.2(d)] with [438] examples with arbitrary large positive σ can be constructed.

A counterexample to the *semicontinuity* of the signature was found by Kerner and the author in [309]: the semicontinuity already fails for some degenerations of hypersurfaces with non-degenerate Newton principal part. This excludes degeneration arguments in possible proofs of Durfee's inequalities.

Example 6.9.29 ([310, 311]) Assume that (X, o) is a homogeneous ICIS of codimension $r = e - 2$ ($e = \mathrm{emb}(X, o)$) and multidegree $(d, \dots, d)$. If $r = 1$ then $6p_g = \mu + 1 - \mathrm{mult}(X, o)$. For any r the inequality $4p_g \leq \mu + 1 - \mathrm{mult}(X, o)$ is valid. Moreover, if $r \geq 2$ is fixed, then $\frac{\mu}{p_g}$ asymptotically tends to $C_{2,r} := \frac{4(r+1)}{r+1/3}$, although $C_{2,r} \cdot p_g \leq \mu + 1$ does not hold in general. (The constant 4 is the best bound valid for any d and r.) For precise formulae see [loc.cit.].

Hence, [310, 311] show that the *strong inequality* also fails for some non-hypersurface isolated complete intersection singularities, and without other restrictions, the best that we can expect is the weak inequality. This is proved 'asymptotically' in the following sense [334]: Consider the set of strict ICIS of *fixed* embedding dimension e. Then $-\sigma$ tends to infinity whenever the multiplicity tends to infinity.

Example 6.9.30 For **hypersurfaces** we have the following 'positive' results:

- $8p_g < \mu$ for $(X, 0)$ of multiplicity 2, Tomari [691],
- $6p_g \leq \mu - 2$ for $(X, 0)$ of multiplicity 3, Ashikaga [32],
- $6p_g \leq \mu - \nu + 1$ for quasi-homogeneous singularities, Xu and S. S.-T. Yau [759],
- $6p_g \leq \mu$ for suspension singularities $\{g(x, y) + z^k = 0\}$, the author [466, 468],
- $6p_g \leq \mu$ for absolutely isolated singularities, Melle–Hernández [416].

For a short proof of $\sigma \leq 0$ in the suspension case see [467].

Example 6.9.31 The following cases are studied by Kollár end the author in [334]: For smoothings of Gorenstein singularities with embedding dimension e one has:

1. If $L = L'$ then $-\sigma \geq 2^{4-e}(p_g + 1)$. This for $e = 3$ reads as $6p_g \leq \mu + \mu_0$.
2. If $e = 3$ and $p_g > 0$ then $-\sigma \geq p_g + |\mathscr{V}_{min}| + 1$, where $|\mathscr{V}_{min}|$ is the number of irreducible exceptional curves in the minimal resolution.

The proofs are rather conceptual. Indeed, by a result of Elkies [183, Theorem, page 4], there is a $\xi \in L$ such that $\xi^2 + |\mathscr{V}| \geq 0$ and $Z_K - \xi = 2l \in 2L$. Then $8 \min \chi \leq 8\chi(l) = Z_K^2 - \xi^2 \leq Z_K^2 + |\mathscr{V}|$. This via 6.8.50 reads as $8p_g + Z_K^2 + |\mathscr{V}| \geq 2^{4-e}(p_g + 1)$. Then use Durfee's theorem 6.9.7 for the left hand side. For (2) use Durfee's formula and 6.8.53: $-\sigma = 8p_g + Z_{K,min}^2 + |\mathscr{V}_{min}| \geq p_g + |\mathscr{V}_{min}| + 1$.

Durfee's Conjecture $6p_g \leq \mu$ for hypersurface singularities is still open. For the new conjectured bounds for complete intersections see [310, 311]. For recent development see [186].

6.10 Spin and Spin^c Structures

We introduce (in several equivalent ways) and classify spin and spin^c structures. A special focus is on rational homology sphere three manifolds and four dimensional manifolds with such boundaries. The basic application is the link L_X and the space of a resolution $\widetilde{X}$ with boundary $\partial\widetilde{X} \simeq L_X$. If L_X is a rational homology sphere then Spin^c is an $H = L'/L$ torsor. In a similar way as the analytic invariants of the universal abelian covering decompose into H-eigen-invariants according to the covering transformation, the topological invariants will have a decomposition according to the Spin^c structures of the link. ♣

6.10.1 In this section M is an oriented connected 3-manifold, which is a rational homology sphere. In the main application M is the link L_X of a singularity. We set $H_1 := H_1(M, \mathbb{Z})$. In the next general discussions we also refer to an arbitrary oriented 4-manifold N, with $\partial N = M$, $H_1(N, \mathbb{Z}) = 0$ and $H_2(N, \mathbb{Z})$ torsion free. If $M = L_X$ then a possible N is $\widetilde{X}$, where $\phi : (\widetilde{X}, E) \to (X, o)$ is a fixed resolution.

6.10.2 The Linking Form M carries a symmetric non-singular bilinear form

$$b_M : H_1 \times H_1 \to \mathbb{Q}/\mathbb{Z},$$

called the *linking form* of M. If $[v_1]$ and $[v_2] \in H_1$ are represented by the disjoint 1-cycles v_1 and v_2, and $nv_1 = \partial w$ for some integer n, then $b_M([v_1], [v_2]) = (w \cdot v_2)/n \pmod{\mathbb{Z}}$, where $(w \cdot v_2)$ is the algebraic intersection of the two cycles.

6.10.3 b_M **as a Discriminant Form** Assume that $L \approx \mathbb{Z}^s$ carries a symmetric non-degenerate bilinear form $(\ ,\ ) : L \times L \to \mathbb{Z}$. Set $L' := \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$, the inclusion $i_L : L \to L'$ determined by the form $(\ ,\ )$, and the natural extension $(\ ,\ )_{\mathbb{Q}}$ of the form to L'. One defines the *discriminant space* $D(L)$ by $\mathrm{coker}(i_L) = L'/L$, and the *discriminant bilinear form*

$$b_{D(L)} : D(L) \times D(L) \to \mathbb{Q}/\mathbb{Z}$$

by $b_{D(L)}([l'_1], [l'_2]) = (l'_1, l'_2)_{\mathbb{Q}} \pmod{\mathbb{Z}}$.

Assume that M is the boundary of an oriented 4-manifold N with $H_1(N, \mathbb{Z}) = 0$ and with $H_2(N, \mathbb{Z})$ torsion-free. Let L be the non-degenerate intersection lattice $(H_2(N, \mathbb{Z}), (\ ,\ ))$. Then $L' \approx H_2(N, \partial N, \mathbb{Z})$ and $(H_1, b_M) = (D(L), -b_{D(L)})$, cf. [397].

6.10.4 Quadratic Functions and Forms Associated with b_M A map $q : H_1 \to \mathbb{Q}/\mathbb{Z}$ is called *quadratic function* if $b(x, y) := q(x + y) - q(x) - q(y)$ is a bilinear form on H_1. If in addition $q(nx) = n^2 q(x)$ for any $x \in H_1$ and $n \in \mathbb{Z}$ then q is called *quadratic form*. In these cases we say that the quadratic function, respectively form, is associated with b. If $b = b_M$ for some M, then we denote by $Q^c(M)$ (resp.

by $Q(M)$) the set of quadratic functions (resp. forms) associated with b_M. Clearly, $Q(M) \subset Q^c(M)$.

Quadratic functions appear in a natural way. Indeed, let N be as in 6.10.3. Pick a characteristic element $k \in L'$ (that is, $(l, l+k) \in 2\mathbb{Z}$ for any $l \in L$) and define

$$q_{D(L),k}([l']) := \frac{1}{2}(l'+k, l')_{\mathbb{Q}} \pmod{\mathbb{Z}}.$$

Then $-q_{D(L),k}$ is a quadratic function associated with $b_M = -b_{D(L)}$. If $k \in \mathrm{Im}(i_L)$, then $-q_{D(L),k}$ is a quadratic form.

The set $Q(M)$ is non-empty (later we will identify it with $\mathrm{Spin}(M)$, whose non-emptiness will be proved). It is a $G := H^1(M, \mathbb{Z}_2)$ torsor, i.e. G acts freely and transitively on $Q(M)$. To see the action, first identify G with $\mathrm{Hom}(H_1, \mathbb{Z}_2)$ and regard $\mathbb{Z}_2$ as $(\frac{1}{2}\mathbb{Z})/\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$. Then, the difference of any two quadratic forms is an element of G, which provides the natural action $(\chi, q) \mapsto \chi + q$.

Similarly, the set $Q^c(M)$ is a $\widehat{H_1} := \mathrm{Hom}(H_1, \mathbb{Q}/\mathbb{Z})$ torsor. The free and transitive action $\widehat{H_1} \times Q^c(M) \to Q^c(M)$ is again $(\chi, q) \mapsto \chi + q$. In particular, the inclusion $Q(M) \subset Q^c(M)$ is G-equivariant via the natural monomorphism $G \hookrightarrow \widehat{H_1}$. One might replace the $\widehat{H_1}$ action on $Q^c(M)$ by an action of H_1, $H_1 \times Q^c(M) \to Q^c(M)$, defined by $(h, q) \mapsto q + b_M(h, \cdot)$. Then the natural monomorphism $G \hookrightarrow \widehat{H_1}$ is replaced by the Bockstein-homomorphism $G = H^1(M, \mathbb{Z}_2) \to H^2(M, \mathbb{Z}) \approx H_1$.

6.10.5 In the sequel we will use the **Stiefel–Whitney classes** of the tangent bundle TP of a manifold P, for details see e.g. [433]. Recall that a connected manifold P is orientable if and only if its first Stiefel–Whitney class $w_1 \in H^1(P, \mathbb{Z}_2)$ is zero. Moreover, if $w_1 = 0$ then the distinct orientations of P are in one-to-one correspondence with elements of $H^0(P, \mathbb{Z}_2) = \mathbb{Z}_2$.

6.10.6 Some Properties of w_2 Let P be an oriented 4-manifold. Then its second Stiefel–Whitney class $w_2 \in H^2(P, \mathbb{Z}_2)$ satisfies the following properties.

(a) **Integrality** [274]: w_2 is the mod 2 reduction of a class from $H^2(P, \mathbb{Z})$.
(b) **Wu's formula**: [368, p. 86] $w_2(l) = (l, l)$ in $\mathbb{Z}_2$ for any $l \in H_2(P, \mathbb{Z}_2)$.
(c) [368, App. D] If P is an *almost-complex* manifold, then the mod 2 reduction of its first Chern class is w_2.

Therefore, if in the above discussion, N admits an almost complex structure, then $k = c_1(TN) \in L'$ is a characteristic element. Hence $-q_{D(L),k}$ is a quadratic function associated with b_M, which is a quadratic form whenever $c_1(TN) \in \mathrm{Im}(i_L)$.

6.10.7 Generalities on the Group Spin_n Fix an integer $n \geq 3$. The spin group Spin_n is the non-trivial twofold covering of the special orthogonal group SO_n. Since $\pi_1(\mathrm{SO}_n) = \mathbb{Z}_2$, this is exactly the universal covering $\eta : \mathrm{Spin}_n \to \mathrm{SO}_n$.

The structure group of any oriented and connected n-manifold P can be reduced to SO_n (via an oriented Riemannian structure), let $\mathrm{SO}_n(TP) \to P$ be the SO_n principal bundle of positive oriented orthonormal frames on the tangent bundle TP.

A *spin structure* on P is a Spin_n-principal bundle $\mathrm{Spin}_n(TP) \to P$ together with a twofold covering $\eta_P : \mathrm{Spin}_n(TP) \to \mathrm{SO}_n(TP)$ compatible with the corresponding group actions. Double covers of $SO_n(TP)$ are classified by $H^1(SO_n(TP), \mathbb{Z}_2)$, and the filtration $SO_n \to SO_n(TP) \to P$ induces the exact sequence

$$0 \to H^1(P, \mathbb{Z}_2) \to H^1(SO_n(TP), \mathbb{Z}_2) \xrightarrow{\xi} H^1(SO_n, \mathbb{Z}_2) \xrightarrow{w} H^2(P, \mathbb{Z}_2)$$

Here $H^1(SO_n, \mathbb{Z}_2) = \mathbb{Z}_2$, and w sends its generator into $w_2(TP)$, the *second Stiefel–Whitney class* of TP. Therefore, from this sequence, one has the following.

Proposition 6.10.8 *An oriented manifold P is spin if and only if $w_2(TP) = 0$. In this case, the set Spin(P) of the possible spin structures is an $H^1(P, \mathbb{Z}_2)$ torsor.*

This can be compared with the following exact sequence of Čech cohomologies:

$$H^1(P, \mathbb{Z}_2) \to H^1(P, \mathrm{Spin}_n) \xrightarrow{\xi} H^1(P, \mathrm{SO}_n) \xrightarrow{w_2} H^2(P, \mathbb{Z}_2)$$

Here $H^1(P, \mathrm{SO}_n)$ (resp. $H^1(P, \mathrm{Spin}_n)$) classifies the isomorphism classes of abstract principle bundles over P with structure group SO_n (resp. Spin_n). The condition to be spin means that $\mathrm{SO}_n(TP) \in H^1(P, \mathrm{SO}_n)$ is in the image of ξ. Furthermore, $w_2(\mathrm{SO}_n(TP))$ is the *second Stiefel–Whitney* class of TP. Therefore, we see again that being spin is equivalent to $w_2 = 0$. (However, $H^1(P, \mathbb{Z}_2) \to H^1(P, \mathrm{Spin}_n)$ in general is not injective: distinct spin structures might be equivalent as abstract principle bundles.)

Example 6.10.9 Let N be an oriented 4-manifold with $H_1(N, \mathbb{Z}) = 0$. If N admits an almost complex structure then its first Chern class is a characteristic element, and its mod 2 reduction is $w_2(TN)$. Hence, N is spin if and only if the intersection form L is even, and in this case N carries exactly one spin structure.

E.g., if N is the Milnor fiber of an isolated hypersurface singularity, then its intersection form is even. This can be seen even in an elementary way from 3.2.13, since $b = -V^{-1} - (\theta^{-1} \circ V^*)^{-1}$. The same is true for isolated complete intersection singularities, since for their Milnor fiber N the line bundle $\Lambda^2 T^*N$ is automatically holomorphically trivial (note that $\mathrm{Pic}^0(N) = H^1(\mathscr{O}_N) = 0$). In fact, the same fact remains valid for any smoothing of a Gorenstein singularity [621]. Hence in all these cases $w_2(TN) = 0$ from 6.10.6(c) (hence N is spin) and the intersection form is even by Wu's formula 6.10.6(b). Additionally, $H^1(N, \mathbb{Z}_2) = 0$ in the case of an ICIS, hence the spin structure on N is unique.

Example 6.10.10 Assume $n = 3$. Then any oriented 3-manifold is automatically spin. This follows from the fact that $w_2(TP) = w_1^2(TP)$ (cf. [368, p. 86] or [433]), and $w_1(TP) = 0$ since P is oriented.

Example 6.10.11 If the 3-manifold in 6.10.10 is the link L_X (under the assumption $H_1(L_X, \mathbb{Q}) = 0$), then the vanishing of $w_2(TL_X)$ can be verified directly. Consider $N = \widetilde{X}$ and the lattices L and L' associated with $\widetilde{X}$, and the Ext-exact sequence

$$0 \to \mathrm{Hom}(H, \mathbb{Z}_2) \to \mathrm{Hom}(L', \mathbb{Z}_2) \to \mathrm{Hom}(L, \mathbb{Z}_2) \xrightarrow{b} \mathrm{Ext}^1(H, \mathbb{Z}_2) \to 0.$$

Let $\hat{K} \in \mathrm{Hom}(L, \mathbb{Z}_2)$ be the mod 2 reduction of the first Chern class c_1 of $-K_{\widetilde{X}}$, hence $\hat{K}(E_v) \equiv E_v^2 \pmod 2$. Moreover, $H^2(L_X, \mathbb{Z}_2) = \mathrm{Ext}^1(H, \mathbb{Z}_2)$, and $b(\hat{K}) = w_2(TL_X)$. Hence, the vanishing of $w_2(TL_X)$ is equivalent to the fact that $\hat{K}$ can be lifted to $\mathrm{Hom}(L', \mathbb{Z}_2)$. Let $e \in \mathbb{Z}_2^s$ be the column vector $(E_v^2)_v$, which agrees with the diagonal of the intersection matrix I. Since $-E_v = \sum_w I_{vw} E_w^*$, a possible extension $\hat{K}(E_v^*) = m_v$ provides a column vector $m = (m_v)_v$ with (†) $Im \equiv e$ (mod 2). We prove that for any intersection matrix I associated with a resolution graph (tree) the equation (†) has a solution m. In fact, the negative definiteness of I is not even needed. (In particular, the boundary of any complex normal crossing curve configuration in a smooth complex surface is spin too, whenever the graph is a tree). The proof runs over induction on the number of vertices of the graph Γ. Let v be an end, w its neighbour, Γ' (respectively Γ'') be the graph obtained from Γ by deleting v and e_w replaced by $e_w + 1$ (respectively by deleting v and w). In the case $e_v \equiv 1$ if m' is a solution for Γ', then $(m', m_w + 1)$ is a solution for Γ. If $e_v = 0$, and m'' is a solution for Γ'', then $(m'', 0, a)$ is a solution for Γ for a convenient $a \in \mathbb{Z}_2$.

6.10.12 The Identification $\mathrm{Spin}(M)$ **and** $Q(M)$ There is a natural G-equivariant identification of $\mathfrak{q} : \mathrm{Spin}(M) \to Q(M)$. Indeed, for any spin structure $\epsilon \in \mathrm{Spin}(M)$ there exists a simple connected oriented spin 4-manifold N with $\partial N = M$, whose induced spin structure on M is exactly ϵ (see, e.g. [213, 5.7.14]). Then $w_2(TN) = 0$, hence by Wu's formula the form L on $H_2(N, \mathbb{Z})$ is even. Hence $-q_{D(L),0}$ is a quadratic form of b_M, and it depends only on the spin structure ϵ and not on the choice of N. The correspondence $\epsilon \mapsto -q_{D(L),0}$ realizes the above identification.

6.10.13 Generalities on the Group Spin_n^c Let $\mathbb{Z}_2 = \{\pm \mathbf{1}\}$ be the kernel of $\eta : \mathrm{Spin}_n \to \mathrm{SO}_n$, and consider the subgroup $\mathbb{Z}_2 = \{\pm 1\} \subset \mathrm{U}_1$ too. Define Spin_n^c as $\mathrm{Spin}_n \times \mathrm{U}_1 / \pm (\mathbf{1}, 1)$. This gives the exact sequence

$$1 \to \mathbb{Z}_2 \to \mathrm{Spin}_n \times \mathrm{U}_1 \xrightarrow{p} \mathrm{Spin}_n^c \to 1.$$

Moreover, the map $\mathrm{Spin}_n \times \mathrm{U}_1 \to \mathrm{SO}_n \times \mathrm{U}_1$, $(a, \lambda) \mapsto (\eta(a), \lambda^2)$ factorizes through p giving rise to another exact sequence

$$1 \to \mathbb{Z}_2 \to \mathrm{Spin}_n^c \xrightarrow{\eta^c} \mathrm{SO}_n \times \mathrm{U}_1 \to 1. \tag{6.10.1}$$

The surjection $\mathrm{Spin}^c_n \xrightarrow{\eta^c} \mathrm{SO}_n \times \mathrm{U}_1 \xrightarrow{pr_1} \mathrm{SO}_n$ belongs to the exact sequence

$$1 \to \mathrm{U}_1 \to \mathrm{Spin}^c_n \to \mathrm{SO}_n \to 1, \tag{6.10.2}$$

where the composition $\mathrm{U}_1 \to \mathrm{Spin}^c_n \xrightarrow{\eta^c} \mathrm{SO}_n \times \mathrm{U}_1 \xrightarrow{pr_2} \mathrm{U}_1$ is the double cover.

For an oriented n-manifold P, a spin^c structure consists of a principal Spin^c_n-bundle $\mathrm{Spin}^c_n(TP) \to P$ and an isomorphism of $\mathrm{Spin}^c_n(TP)/\mathrm{U}_1 \to P$ and $\mathrm{SO}_n(TP) \to P$ as principal SO_n-bundles.

Having such a principal Spin^c_n bundle $\mathrm{Spin}^c_n(TP) \to P$, one automatically has a principle $\mathrm{SO}_n \times \mathrm{U}_1$ bundle $\mathrm{Spin}^c_n(TP)/\mathbb{Z}_2 \to P$ (cf. (6.10.1).

This provides a possible alternative definition: a spin^c structure consists of a pair $(\mathrm{Spin}^c_n(TP), \mathscr{L})$, where $\mathrm{Spin}^c_n(TP) \to P$ is a Spin^c_n-principal bundle, and $\mathscr{L} \to P$ is a U_1-principal bundle, together with a twofold covering $\eta^c : \mathrm{Spin}^c_n(TP) \to \mathrm{SO}_n(TP) \times \mathscr{L}$, which is compatible with the corresponding group actions.

The group morphisms provide the Čech cohomology exact sequence

$$H^1(P, \mathbb{Z}_2) \to H^1(P, \mathrm{Spin}^c_n) \xrightarrow{\xi^c} H^1(P, \mathrm{SO}_n) \times H^1(P, \mathrm{U}_1) \xrightarrow{w_2^c} H^2(P, \mathbb{Z}_2).$$

Let $c_1 : H^1(P, \mathrm{U}_1) \to H^2(P, \mathbb{Z})$ be the *isomorphism* given by the first Chern class of the corresponding line bundle; and let r denote the mod 2 reduction $H^2(P, \mathbb{Z}) \to H^2(P, \mathbb{Z}_2)$. Then $w_2^c = w_2 + r \circ c_1$. The condition P to be spin^c means that there exists a U_1-bundle $\mathscr{L} \to P$ such that $(\mathrm{Spin}^c_n(TP), \mathscr{L})$ lies in the image of ξ^c, that is $w_2(TP) = r(c_1(\mathscr{L}))$. Hence we proved the following.

Proposition 6.10.14 *An oriented manifold P is spin^c if and only if its second Stiefel–Whitney class is the mod 2 reduction of an integral class.*

If P is spin^c, then the set of possible spin^c structures is a $H^2(P, \mathbb{Z})$ torsor. Indeed, (6.10.2) provides the fibration $\mathrm{BU}_1 \to \mathrm{BSpin}^c_n \to \mathrm{BSO}_n$. The tangent bundle corresponds to a homotopy class $\tau \in [P, \mathrm{BSO}_n]$ and all the supported spin^c structures correspond to liftings $[P, \mathrm{BSpin}^c_n]$ of τ. They are parametrized by $[P, \mathrm{BU}_1]$.

Example 6.10.15 The inclusion $\mathrm{Spin}_n \times \{1\} \hookrightarrow \mathrm{Spin}_n \times \mathrm{U}_1$ composed with p provides a monomorphism $\mathrm{Spin}_n \hookrightarrow \mathrm{Spin}^c_n$. Furthermore (although the inclusion $\mathrm{U}_k \hookrightarrow \mathrm{SO}_{2k}, a \mapsto a^{\mathbb{R}}$, does not lift to a map $\mathrm{U}_k \to \mathrm{Spin}_{2k}$), the map $\mathrm{U}_k \hookrightarrow \mathrm{SO}_{2k} \times \mathrm{U}_1$, $(a \mapsto (a^{\mathbb{R}}, \det(a))$ admits a η^c-lifting, and monomorphism $\mathrm{U}_k \hookrightarrow \mathrm{Spin}^c_{2k}$.

The inclusions $\mathrm{Spin}_n \hookrightarrow \mathrm{Spin}^c_n$ and $\mathrm{U}_k \hookrightarrow \mathrm{Spin}^c_{2k}$ show that any oriented manifold with a spin or almost complex structure is spin^c.

Note also that for $n = 3$ one has $\mathrm{Spin}_3 = \mathrm{SU}_2$ and $\mathrm{Spin}^c_3 = \mathrm{U}_2$.

Example 6.10.16 Consider an oriented 4-manifold N as above. Then N is automatically spin^c, cf. [756] (or 6.10.6(a) here). Furthermore, assume that $H_1(N, \mathbb{Z}) = 0$. From the fibration $\mathrm{B}\mathbb{Z}_2 \to \mathrm{BSpin}^c_n \to \mathrm{BSO}_n \times \mathrm{BU}_1$, any lifting of $[N, \mathrm{BSO}_n]$ to

$[N, \mathrm{BSpin}^c_n]$ is uniquely determined by the line bundle $\mathscr{L}$. On the other hand, this line bundle is uniquely determined by its first Chern class $c_1(\mathscr{L}) \in H^2(N, \mathbb{Z}) = L'$. The identity $r \circ c_1 = w_2$ combined with Wu's formula shows that $c_1(\mathscr{L})$ belongs to the set of characteristic elements $\mathrm{Char} \subset L'$. Hence, the characteristic elements parametrize the spin^c structures $\widetilde{\sigma} \in \mathrm{Spin}^c(N)$ under the correspondence $\widetilde{\sigma} \mapsto c_1(\widetilde{\sigma}) := c_1(\mathscr{L})$. The set $\mathrm{Spin}^c(N)$ is an L' torsor, the action satisfies $c_1(l' * \widetilde{\sigma}) = 2l' + c_1(\widetilde{\sigma})$. (Char is also an L' torsor via $(l', k) \mapsto 2l' + k$.) There is a natural involution $\widetilde{\sigma} \mapsto \overline{\widetilde{\sigma}}$ such that $c_1(\overline{\widetilde{\sigma}}) = -c_1(\widetilde{\sigma})$.

Example 6.10.17 Fix a 3-manifold M with $H_1(M, \mathbb{Q}) = 0$. Then it is spin^c (since it is already spin). The set $\mathrm{Spin}^c(M)$ is an $H^2(M, \mathbb{Z})$ torsor.

Let N be a 4-manifolds as above with $\partial N = M$ and $H_1(N, \mathbb{Z}) = 0$. We claim that the restriction $R : \mathrm{Spin}^c(N) \to \mathrm{Spin}^c(M)$ is onto. Indeed, take any $\widetilde{\sigma} \in \mathrm{Spin}^c(N)$. Then $R(l' * \widetilde{\sigma}) = [l'] * R(\widetilde{\sigma})$. Since $\{[l'] * R(\widetilde{\sigma})\}_{[l']} = \mathrm{Spin}^c(M)$, the claim follows.

The natural cohomological morphism $H^2(N, \mathbb{Z}) \to H^2(M, \mathbb{Z})$ is the factorization $L' \to L'/L$, $l' \mapsto [l']$. This projects Char onto Char$/L$. Then $c_1 : \mathrm{Spin}^c(N) \twoheadrightarrow \mathrm{Char} \subset L'$ induces a map $c : \mathrm{Spin}^c(M) \twoheadrightarrow \mathrm{Char/L} \subset L'/L$ such that $c(R(\widetilde{\sigma})) = [c_1(\widetilde{\sigma})]$.

Moreover, $c([l'] * \sigma) = 2[l'] + c(\sigma)$ for any $[l'] \in L'/L$ and $\sigma \in \mathrm{Spin}^c(M)$.

While c_1 is injective, c in general is not. Its fibers are $H^1(M, \mathbb{Z}_2)$ torsors; $c^{-1}(0) \simeq \mathrm{Spin}(M)$. These facts will be explained next.

We consider the action of L on Char defined by $l * k := k + 2l$. Let Char$/2L$ be its orbit space. Then Char$/2L$ is an L'/L torsor by the action induced by $l' * k = k + 2l'$.

Moreover, the composition $R \circ c_1^{-1} : \mathrm{Char} \to \mathrm{Spin}^c(N) \to \mathrm{Spin}^c(M)$ factorizes to $\mathrm{Char}/2L \to \mathrm{Spin}^c(M)$. This map is a bijection of L'/L torsors. In the sequel we identify $\mathrm{Spin}^c(M)$ by this bijection. Then $c : \mathrm{Spin}^c(M) \to \mathrm{Char}/L$ transformed into $c : \mathrm{Char}/2L \to \mathrm{Char}/L$. Its fibers are $\{l' \in L' : 2l' \in L\}/L \simeq H^1(M, \mathbb{Z}_2)$ torsors. The trivial element 0 of L'/L is in Char$/L$ (since $\mathrm{Spin}(M) \neq \emptyset$), and

$$c^{-1}(0) = (\mathrm{Char} \cap L)/2L \simeq \mathrm{Spin}(M).$$

Corollary 6.10.18 *With the notations of 6.10.17 there is a canonical L'/L-equivariant identification*

$$\mathfrak{q}^c : \mathit{Spin}^c(M) \to Q^c(M).$$

Moreover, this identification is compatible with the G-equivariant identification $\mathfrak{q} : \mathit{Spin}(M) \to Q(M)$ via the inclusions $\mathit{Spin}(M) \subset \mathit{Spin}^c(M)$ and $Q(M) \subset Q^c(M)$.

Proof For any $\sigma = R(\widetilde{\sigma})$ set the quadratic function $-q_{D(L),k}$, where $k = c_1(\widetilde{\sigma})$. □

Lemma 6.10.19 *There is an involution $\sigma \mapsto \overline{\sigma}$ of $Spin^c(M)$ which satisfies:*

(a) $c(\overline{\sigma}) = -c(\sigma)$,
(b) $\overline{[l'] * \sigma} = [-l'] * \overline{\sigma}$,
(c) $Spin(M) = \{\sigma \in Spin^c(M) : \sigma = \overline{\sigma}\}$.

Proof In Char$/2L$, $\sigma = k$ and $\overline{\sigma} = -k$ realizes the involution. □

Note also the symmetry

$$q_{D(L),k}(x) = q_{D(L),-k}(-x) \quad \text{for any } x \in D(L), \tag{6.10.3}$$

which makes a dictionary between properties of involutive quadratic functions.

6.10.20 The Case of Singularities Assume that $M = L_X$, the link of a normal surface singularity, and $N = \widetilde{X}$ is a fixed good resolution. We use the standard notations L an L' as usual. One has a distinguished characteristic element $K = c_1(\mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}})) \in$ Char. The quadratic function $-q_{D(L),K}$ is a form if and only if (X, o) is numerically Gorenstein.

Definition 6.10.21 For any $k \in$ Char we write $\widetilde{\sigma}(k)$ for that spin^c structure of $\widetilde{X}$ for which $c_1(\widetilde{\sigma}(k)) = k$. Similarly, let $\sigma[k] \in \text{Spin}^c(M)$ be the restriction of $\widetilde{\sigma}(k)$ to L_X.

6.10.22 Definition of k_r Assume that the link is a rational homology sphere. Then $\text{Spin}^c(\widetilde{X})$ is identified with the set of characteristic elements k on L', and if k and k' induces the same Spin^c structure on the link, then $k' = k + 2l$ for a certain $l \in L$. In this case $\chi_{k'}(x - l) = \chi_k(x) - \chi_k(l)$ for any $x \in L$, hence the two functions χ_k and $\chi_{k'}$ can be easily compared, and they have identical qualitative properties. Therefore, for each class $[k] = k + 2L$ (that is, for each Spin^c structure $\sigma[k]$ of L_X), we choose a representative of $[k]$. Since the set of classes is indexed by H; we define the set of representatives by $k_r := K + 2s_h$, for each $h \in H$. Since $s_0 = 0$, for the trivial class $h = 0$ we get $\chi_{k_r} = \chi$.

Since for any $x \in L$ one has $\chi_{k_r}(x) = \chi(s_h + x) - \chi(s_h)$, the function χ_{k_r} defined on the integral lattice L (up to an additive constant $\chi(s_h)$) can be identified with χ acting on the (rationally) shifted lattice $s_h + L = \{l' \in L' : [l'] = h\}$.

For some motivation regarding the definition of k_r see 7.3.11.

Remark 6.10.23 Assume that the intersection form L is even, hence $\widetilde{X}$ has a unique spin structure $\widetilde{\epsilon}$. The point is that, in general, $\widetilde{\sigma}(K) \neq \widetilde{\epsilon}$ (and $\widetilde{\sigma}(-K) \neq \widetilde{\epsilon}$ too), and their restrictions can be different as well, even if the restriction of $\widetilde{\sigma}(K)$ is spin.

More precisely: the form on $\widetilde{X}$ is even if and only if $K \in 2L'$; $\sigma[K] \in \text{Spin}(M)$ if and only if $K \in L$ (in this case $\sigma[K] = \sigma[-K]$); and finally, $\sigma[K] = R(\widetilde{\epsilon})$ if and only if $K \in 2L$. (The last fact follows from the equality $q_{D(L),0} = q_{D(L),K}$.)

6.10.A Turaev's Euler Structures

6.10.24 Combinatorial Euler Structures [705, I.2] Let Y be a finite connected CW-complex. Write $H_1 := H_1(Y, \mathbb{Z})$. An *Euler chain* in Y is a 1-dimensional singular chain θ with $\partial\theta = \sum_c (-1)^{\dim c} p_c$, where c runs over all open cells of Y and p_c is a point of c. If such a chain exists, then necessarily $\chi(Y) = \sum_c (-1)^{\dim c} = \partial^2\theta = 0$. Conversely, if the Euler number $\chi(Y) = 0$, then such an Euler chain exists. (In our applications Y will be an triangulated oriented 3-manifold.)

Fix two Euler chains θ and η with $\partial\theta = \sum_c (-1)^{\dim c} p_c$ and $\partial\eta = \sum_c (-1)^{\dim c} q_c$ ($p_c,\ q_c \in c$). Then θ and η determine an element $\theta/\eta \in H_1$ as follows. Let $\alpha_c : [0, 1] \to c$ be a path from p_c to q_c in c. Then θ/η is the class of $\theta - \eta + \sum_c (-1)^{\dim c} \alpha_c$. We say that θ and η defines the same combinatorial Euler structure on Y if $\theta/\eta = 0$.

The set of Euler structures is denoted by $\mathrm{Eul}(Y)$, the class of θ by $[\theta] \in \mathrm{Eul}(Y)$.

H_1 acts on $\mathrm{Eul}(Y)$ by $([h], [\theta]) \mapsto [h+\theta]$, where h is an arbitrary closed 1-chain. The action is free and transitive, hence $\mathrm{Eul}(Y)$ becomes an H_1-torsor.

Let $Y^a \to Y$ denote the universal abelian covering of Y with the natural CW-structure. A set of cells is called *fundamental* if over each cell of Y there exists exactly one cell from this set. A set of fundamental cells $\{c^a\}_c$ determines an Euler structure as follows. Fix a base point $* \in Y^a$, points $p_c^a \in c^a$, paths from $*$ to p_c^a oriented towards $*$ (resp. out of $*$) whenever c^a is odd (resp. even) dimensional. The projection of their sum is an Euler structure whose class does not depend on the choice of $*$ and p_c^a. By taking all possible sets of fundamental cells we get $\mathrm{Eul}(Y)$.

For cellular subdivision Y' of Y there is a canonical H_1-equivariant bijection $\mathrm{Eul}(Y) \to \mathrm{Eul}(Y')$ (for a set of fundamental cells $\{c^a\}_c$ of Y in Y^a, take the set of cells in Y'^a sitting in $\cup_c c^a$ as a fundamental set for Y').

6.10.25 Smooth Euler Structures [705, I.4] Let M be a compact, connected and oriented 3-manifold ($\partial M = \emptyset$). A *regular vector field* on M is a nowhere vanishing tangent vector field. Two regular vector fields are called *homologous* if there exists a point $p \in M$ such that the restrictions of the vector fields to $M \setminus \{p\}$ are homotopic as non-vanishing vector fields on $M \setminus \{p\}$. (In the presence of a triangulation this means that the restrictions to the 2-skeleton of the two vector fields are homotopic through non-vanishing vector fields of the 2-skeleton.) We call the set of classes of regular vector fields *smooth Euler structures of* M and we denote it by $\mathrm{Vect}(M)$; $[u]$ denotes the class of u.

The group $H_1 = H_1(M, \mathbb{Z})$ acts on $\mathrm{Vect}(M)$ freely and transitively. One possible way to define the action is the following. Fix a regular vector field u on M, and assume that $h \in H_1$ is represented by an oriented simple closed curve. Let T be a tubular neighbourhood with coordinates $(\alpha, z) \in S^1 \times \{z \in \mathbb{C} : |z| \le \pi\}$ such that u in T is given by $-\partial/\partial\alpha$. Then $h[u]$ coincides with u outside T and it is $\cos(|z|)\partial/\partial\alpha + \sin(|z|)\partial/\partial|z|$ in T (glued smoothly).

Let us fix a triangulation Y of M. Then one shows that there is a canonical H_1-equivariant bijection $\mathrm{Eul}(Y) \to \mathrm{Vect}(M)$. Since any two triangulations have

a common subdivision (cf. [437], see also 9.4.7), this canonical identification is independent of the choice of the triangulation. The sketch of a possible geometric identification is the following. There exists a canonical singular vector field u_{sing} on Y defined by barycentric coordinates with singularities at the barycenters. On the other hand, any Euler structure $[\theta]$ can be represented by a 'spider-like' Euler chain consisting of oriented arcs connecting a point $* \in M$ to the barycenters. The restriction of u_{sing} outside of a ball-neighbourhood of the spider is regular, and it has a regular extension u to the whole M. The correspondence is $[\theta] \mapsto [u]$.

For any $[u] \in \mathrm{Vect}(M)$ there exists a unique $c([u]) \in H$ such that $c([u])[-u] = [u]$. By definition $c(h[u]) = 2h + c([u])$.

6.10.26 Smooth Euler and $\mathrm{Spin}^c(M)$ Structures [705, XI.1] Let M be as in 6.10.25. Then there exists an H_1-equivariant isomorphism $\mathrm{Vect}(M) \to \mathrm{Spin}^c(M)$. Indeed, in the presence of a regular vector field u on M the tangent bundle TM of M splits as $\mathbb{R}u \oplus u^\perp$, and the oriented 2-dimensional vector bundle $u^\perp$ has the structure of an U_1-bundle. Hence one has the reduction of the structure group of TM to $\mathrm{U}_1 = (1) \oplus \mathrm{U}_1 \subset \mathrm{U}_2 = \mathrm{Spin}^c_3$.

By this identification the maps $c : \mathrm{Spin}^c(M) \to H_1$ and $c : \mathrm{Vect}(M) \to H_1$ can be identified as well.

6.10.27 An Alternative Homological Description Let M be an oriented 3-manifold ($\partial M = \emptyset$) with frame bundle $\mathrm{SO}_3(TM)$. In this part all homologies are over $\mathbb{Z}$.

A spin^c structure on M can also be interpreted as an element of $H^2(\mathrm{SO}_3(TM))$, whose restriction to every fiber is the nonzero element of $H^2(\mathrm{SO}_3) = \mathbb{Z}_2$. Indeed, if the spin^c structure is represented by the principal bundle $\mathrm{Spin}^c_3(TM) \to M$ and by the isomorphism $\mathrm{Spin}^c_3(TM)/\mathrm{U}_1 \to \mathrm{SO}_3(TM)$, then $\mathrm{Spin}^c_3(TM) \to \mathrm{Spin}^c_3(TM)/\mathrm{U}_1 \to \mathrm{SO}_3(TM)$ is a U_1-bundle, whose first Chern class is an element of $H^2(\mathrm{SO}_3(TM))$. Since M is parallelizable, $H^2(\mathrm{SO}_3(TM)) = H^2(M) \oplus H^2(\mathrm{SO}_3) = H^2(M) \oplus \mathbb{Z}_2$. Hence the spin^c structures are represented by elements of type $H^2(M) \times \{\hat{1}\}$.

This is an $H^2(M)$ torsor via the pullback homomorphism $H^2(M) \to H^2(\mathrm{SO}_3(TM))$ and addition. The Chern class c of a structure $\sigma = (h, \hat{1})$ is $c(\sigma) = 2h \in H^2(M)$.

Let $SM \to M$ be the S^2-fiber bundle of the unit tangent vectors of TM. An Euler structure is an element of $H^2(SM)$, whose restriction to every fiber is the (oriented) generator of $H^2(S^2) = \mathbb{Z}$. Hence it is represented in $H^2(SM) = H^2(M) \oplus \mathbb{Z}$ by elements of type $(h, 1)$. The equivalence is realized by the Poincaré dual of the image of the (unit) vector field. There elements form an $H^2(M)$ torsor via the pullback homomorphism $H^2(M) \to H^2(SM)$ and addition.

Finally, let $\mathrm{SO}_3(TM) \to SM$ be the map which sends the frame (e_1, e_2, e_3) to e_1. Then $H^2(SM) \to H^2(\mathrm{SO}_3(TM))$ is an $H^2(M)$ equivariant identification of $\mathrm{Vect}(M)$ with $\mathrm{Spin}^c(M)$.

The involution $\sigma \mapsto \overline{\sigma}$ of $\mathrm{Spin}^c(M)$ in these languages is realized by $(h, \hat{1}) \mapsto (-h, \hat{1})$ in $H^2(M) \times \{\hat{1}\}$, and by $(h, 1) \mapsto (-h, 1)$ in $H^2(M) \times \{1\} = \mathrm{Vect}(M)$.

Chapter 7
The Artin–Laufer Program

7.1 Rational Singularities

The geometric genus is one of the key motifs of the present book. The rational singularities are those singularities, which are invisible from the point of view of the geometric genus (or, from the point of view of the operation $X \rightsquigarrow R^1\phi_*\mathscr{O}_{\widetilde{X}}$). We provide several of their topological characterisations, and we list some examples as well. ♣

Definition 7.1.1 A singularity (X, o) is called rational if $p_g = 0$.

In other words, (X, o) is rational if there exists a resolution $\phi : \widetilde{X} \to X$ such that $R^1\phi_*\mathscr{O}_{\widetilde{X}} = 0$. By 6.8.2 the vanishing $R^1\phi_*\mathscr{O}_{\widetilde{X}} = 0$ is independent of the resolution.

In the sequel we fix an arbitrary good resolution (in fact, in 7.1.3 we will prove that any resolution of a rational singularity is automatically good).

By the theorem of formal functions, $p_g = 0$ if and only if $h^1(\mathscr{O}_l) = 0$ for all cycles $l \in L_{>0}$. The next theorem characterizes rational singularities topologically.

Theorem 7.1.2 (See [27, 28] for (i)-(iii)) *The following facts are equivalent ((i)⇔(ii)⇔ (iii) are known as **Artin's Rationality Criteria**):*

(i) $p_g = 0$,
(ii) $\chi(l) \geq 1$ *for all* $l \in L_{>0}$,
(iii) $\chi(Z_{min}) = 1$ *(or even (iii')* $\chi(Z_{min}) \geq 1$*)*,
(iv) ***(Laufer's Rationality Criterion** [359]) Γ is a tree of smooth rational curves and* $(E_{v(i)}, z_i) = 1$ *for all* $1 \leq i < t$ *along a computation sequence* $\{z_i\}_i$ *of* Z_{min} *(for the definition of the computation sequence see 6.6.7).*

Proof $(i) \Rightarrow (ii)$ $p_g = \dim \varprojlim H^1(\mathscr{O}_l) \geq h^1(\mathscr{O}_l) = h^0(\mathscr{O}_l) - \chi(\mathscr{O}_l) \geq 1 - \chi(l)$.

$(ii) \Rightarrow (i)$ Assume that $p_g \neq 0$. Then there exists an effective cycle $l > 0$ with $h^1(\mathscr{O}_l) \neq 0$. Let l be the minimal with this property. Note that $\chi(\mathscr{O}_{E_v}) \geq 1$

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implies $h^1(\mathscr{O}_{E_v}) = 0$, and $l - E_v \neq 0$ for any $E_v \subset |l|$. Thus $h^1(\mathscr{O}_l) \neq 0$, but $h^1(\mathscr{O}_{l-E_v}) = 0$ for any $E_v \subset |l|$. Consider the cohomology exact sequence of $0 \to \mathscr{O}_{E_v}(-l + E_v) \to \mathscr{O}_l \to \mathscr{O}_{l-E_v} \to 0$. Then the minimality assumption implies $h^1(\mathscr{O}_{E_v}(-l + E_v)) \neq 0$, or, by duality, $h^0(\mathscr{O}_{E_v}(l + K)) \neq 0$. Hence $(E_v, l + K) \geq 0$ for all $E_v \subset |l|$. By summation $(l, l + K) \geq 0$, or $\chi(l) \leq 0$, which leads to a contradiction.

$(ii) \Rightarrow (iii) \Leftrightarrow (iii') \Leftrightarrow (iv)$ follows from 6.6.9(b).

$(iv) \Rightarrow (i)$ Let $\{z_i\}_i$ be a computation sequence for Z_{min}, cf. 6.6.7, which satisfies the assumptions of part *(iv)*. Consider the infinite sequence: $z_1, \dots, z_i, \dots, Z_{min}, Z_{min}+z_1, \dots, Z_{min}+z_i, \dots, 2Z_{min}, \dots, nZ_{min}, nZ_{min}+z_1, \dots$. Induction proves $h^1(\mathscr{O}_{nZ_{min}+z_i}) = 0$ for all n and i. Since $Z_{min} \geq E$, we obtain $p_g = \dim \varprojlim H^1(\mathscr{O}_l) = 0$. □

If a resolution graph satisfies one of the above conditions *(ii)–(iii)–(iv)*, then it is called a *rational graph*.

Remark 7.1.3 One can construct a computation sequence of Z_{min} such that one of its terms is E. Since for an arbitrary graph, $(E, E_v) = E_v^2 + \kappa_v$ is at least 2, whenever $E_v^2 = -1$ and $\kappa_v \geq 3$, we obtain that such a situation cannot occur, hence the minimal good resolution of a rational singularity is automatically minimal.

Remark 7.1.4 In Sect. 6.8.B we considered combinatorial invariants $\mathrm{Path}^* = \mathrm{Path}^i$ associated with a graph of a singularity with rational homology sphere link, and we proved that $p_g \leq \mathrm{Path}^i$. Note that the infinite sequence from the proof of 7.1.2 shows that for a rational graph $\mathrm{Path}^i = 0$. *This provides a new topological criterion for rationality: (X, o) is rational if and only if* $\mathrm{Path}^i = 0$ *(in any graph).*

Next, we provide another proof for the vanishing $\mathrm{Path}^i = 0$. Note that we need to construct an increasing sequence $\{l_i\}_i$ connecting 0 with $\lfloor Z_K \rfloor$ such that $\chi(l_{i+1}) \geq \chi(l_i)$ for all i. In order to do this, we will construct the sequence in a reversed order. (This will serve as a prototype for further constructions, it is like a 'χ–non-increasing flow into the direction of the origin', an analogue of a homotopical contraction.) For this it is enough to prove that for any effective non-zero cycle $l \in L$ there exists $E_v \subset |l|$ such that $\chi(l - E_v) \leq \chi(l)$. In this way we get inductively the sequence in a reversed order starting from $\lfloor Z_K \rfloor$.

Assume that the above claim is not true for certain l. Then $\chi(l - E_v) > \chi(l)$ for all $E_v \subset |l|$. Since each E_v is rational, this reads as $(E_v, l - Z_K) \geq 0$, hence by summation $(l, l - Z_K) \geq 0$, which contradicts $\chi(l) > 0$ (cf. Theorem 7.1.2*(ii)*).

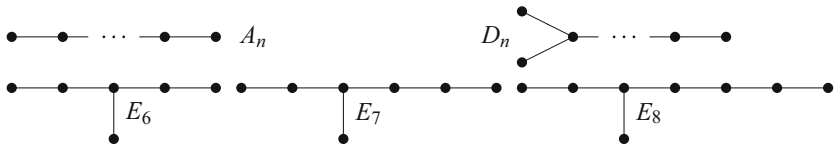
Example 7.1.5

(a) The rationality condition 7.1.2*(iv)* is guaranteed if Γ is a tree of $\mathbb{P}^1$'s and $\kappa_v \leq -E_v^2$ for any v. In this case $Z_{min} = E$. Singularities with $Z_{min} = E$ are called *minimal rational* [335]. E.g., the Hirzebruch–Jung singularities are rational.

On the other hand, if there exists a vertex with $-E_v^2 \leq \kappa_v - 2$ then the condition 7.1.2*(iv)* fails, hence Γ is not rational.

(b) The geometric genus of a Brieskorn–Pham hypersurface singularity $\{x^a + y^b + z^c = 0\}$ is $p_g = \#\{(i, j, k) | i, j, k \in \mathbb{Z}_{>0},\ \frac{i}{a} + \frac{j}{b} + \frac{k}{c} \leq 1\}$, cf. 6.8.61 or 6.9.14. Hence, the rational ones are characterized by $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} > 1$. The only possible values (a, b, c) are: $(2, 2, n)$ (A_{n-1}), $(2, 3, 3)$ (D_4), $(2, 3, 4)$ (E_6) and $(2, 3, 5)$ (E_8).

(c) Hypersurface singularities with modularity zero are called *simple*. They are: A_n $(n \geq 1)$, D_n $(n \geq 4)$, E_n $(n = 6, 7, 8)$. Only D_n $(n \geq 5)$ and E_7 cannot be represented by Brieskorn–Pham equation. Their equations are $x^2y + y^{n-1} + z^2 = 0$ (for D_n) and $x^3 + xy^3 + z^2 = 0$ (for E_7) [21]. The minimal resolution graphs of simple singularities are the well-known (negative definite) ADE graphs (with only (-2) curves):



Simple singularities are called also *Du Val singularities*. They are all rational.

(d) If (X, o) is rational and numerically Gorenstein then its minimal good graph is minimal (see 7.1.3), hence $Z_K \in \mathscr{S} \subset L_{\geq 0}$ by 6.3.3(1). Since $\chi(Z_K) = 0$, by Artin's criterion we must have $Z_K = 0$; hence all the curves are rational (-2)-curves. By a combinatorial argument, such a negative definite graph is necessarily of type ADE.

In particular, if (X, o) is (non-smooth) rational with $\det(\Gamma) = 1$ then it is E_8.

(e) Any connected subgraph of a rational graph is rational. Decreasing the self-intersection numbers of a rational graph we obtain a rational graph. (Use 7.1.2*(iv)*.)

(f) Assume that the graph is minimal and $Z_K < Z_{min}$. Then the graph is rational. Indeed, $-2\chi(Z_{min}) = (Z_{min}, Z_{min} - Z_K) = (Z_{min} - Z_K)^2 + (Z_K, Z_{min} - Z_K) < 0$. (Although for rational graphs the coefficients of Z_K usually are small, one can find even minimal rational graphs with $Z_K \not\leq Z_{min}$.)

(g) A numerically log terminal singularity (cf. 6.3.33(c)) is rational. Indeed, in this case $0 \leq Z_K < E \leq Z_{min}$, hence (f) applies. Hence all quotient singularities are rational. This last statement follows from 6.8.8(a) too.

(h) Assume that (X, o) is numerically log canonical (cf. 6.3.33(d). Then in a good minimal resolution $Z_K \leq E \leq Z_{min}$. If $Z_K < Z_{min}$ then, again by (f), (X, o) is rational. If $Z_K = E = Z_{min}$ then $0 = \chi(Z_K) = \chi(E) = 1 - h^1(\mathscr{O}_E)$ shows that the link is not a rational homology sphere. In this $Z_K = E = Z_{min}$ case (X, o) is a minimally elliptic singularity, for the possible graphs see 7.2.21.

(i) Assume that the graph Γ has the following property: if we glue to some of the vertices of Γ (via new edges) new (rational) (-1)-vertices then we obtain a 'smooth graph' (which blows down to the empty graph). Such graphs are

rational by (e), they are called *sandwiched graphs*, and the singularities with such resolution graphs are called *sandwiched*. The terminology is motivated by the following fact (which follows from the previous definition): the singularity (X, o) is sandwiched if and only if it is analytically isomorphic to a germ of algebraic surface that admits a birational map $X \to \mathbb{C}^2$. Therefore, if we choose a resolution $\widetilde{X} \to X$, then X is sandwiched between two regular birational maps $\widetilde{X} \to X \to \mathbb{C}^2$. For more see [141, 378, 646].

(j) A quotient-cusp (cf. 4.1.37) is rational (use 7.1.2*(iv)*).

Remark 7.1.6 By 6.8.8(a), if $c : (X', o) \to (X, o)$ is a finite morphism between two normal surface singularities, and (X', o) is rational then (X, o) is rational too.

In particular, if c is an o-ramified covering (cf. 4.1.E), then the rationality of (X', o) implies the rationality of (X, o). [Note that the converse is not true, see the end of 6.8.19 and 7.1.21 for a rational singularity whose universal abelian covering is not rational.] Moreover, if either (X, o) or (X', o) is quotient, then the other one is also quotient. [Use 3.7.12, and the fact that the index $\pi_1(L_X) : \pi_1(L_{X'})$ is finite.] Furthermore, we have the following additional characterization of quotient singularities [188]: (X, o) *is quotient if and only if any (cyclic) o-ramified covering of it is rational.* Indeed, assume that any cyclic o-ramified covering of (X, o) is rational. Then (X, o) is rational. Consider its (topological) canonical covering (X', o), cf. 6.1.19. By 6.3.22 it is numerically Gorenstein. Since it is also rational, it is ADE, cf. 7.1.5(d). In particular, (X', o) is quotient (cf. 3.7.15), hence (X, o) is quotient too.

Remark 7.1.7

(a) Let Γ be a rational graph such that for any v with $(E_v, Z_{min}) < 0$ the multiplicity of E_v in Z_{min} is one. (For a motivation of this property see 6.7.A.) Then $Z_{min} = E$. (Indeed, assume that $Z_{min} > E$ and consider a computation sequence connecting E with Z_{min}. Let E_0 be the component we add in the last step. From $(Z_{min} - E_0, E_0) = 1$ we get $(Z_{min}, E_0) < 0$, hence by the assumption $E_0 \not\leq |Z_{min} - E|$, a contradiction.)

(b) [219, 382] Let Γ be a rational graph, $v \in \mathcal{V}$ and let $\Gamma_v(d)$ be the graph obtained from Γ by attaching an edge to v a new vertex with decoration d. Then $\Gamma_v(d)$ is negative definite for $d \ll 0$. Moreover, $\Gamma_v(d)$ is rational for $d \ll 0$ if and only if the E_v–multiplicity of $Z_{min}(\Gamma)$ is 1.

(c) [218, 261] Let Γ be the embedded resolution graph of a plane curve singularity with link $\cup_i K_i$. Along each arrow component (that is, along $K_i \subset S^3$) make a surgery with coefficient d_i. If each $d_i \gg 0$ then the graph is not negative definite, however it can be transformed by calculus into a rational (negative definite) graph.

(d) 3-manifolds with vanishing reduced Heegaard Floer homology are called L-spaces, and they are key fundamental objects in recent developments of topology [548, 549]. In [485] it is proved that a negative definite graph Γ is rational if and only if $M(\Gamma)$ is an L-space. This happens if and only if $\pi_1(M(\Gamma))$ is

not left-orderable. (The starting point is the characterization 11.1.28 via lattice cohomology, see also 11.2.22.) For further connections with Heegaard Floer theory see 11.7.

7.1.8 *The proof of Lemma 6.4.15.*

Proof $\{x_i\}_{i=0}^t$ is constructed as a concatenated sequence from several portions. Such a portion, connecting 'special' members x^* and x^{**} is constructed as follows.

Assume that a term x^* of the sequence is already constructed, and it satisfies the properties (i) $x^* \le s$ and (ii) $(x^*, E_v) \le 0$ for any $v \in \mathcal{V} \setminus \mathcal{N}^+$. If $x^* = s$ then we stop. If $x^* < s$ then we will construct from x^* another cycle x^{**} with $x^* < x^{**}$ (the 'portion' starting from x^* and ending with x^{**}) such that x^{**} will satisfy both (i) and (ii). The steps are the following. First choose $E_v \subset |s - x^*|$. Then the next term of the sequence will be $x^* + E_v$. Note that this step is not obstructed: if $v \in \mathcal{N}^+$ then (E_v, x^*) is unessential, while if $v \notin \mathcal{N}^+$ then $(E_v, x^*) \le 0$ by (ii). Then, complete the sequence adding at each step to the term x_i a certain $E_{v(i)}$ with $v(i) \notin \mathcal{N}^+$ and with $(E_{v(i)}, x_i) > 0$ (similarly as in the algorithm 6.6.3 or 6.6.7, but at this time we test and add only vertices $v \notin \mathcal{N}^+$). Note that along this procedure $x_i \le s$ always (indeed, if $x < s$, $(x, E_w) = 1$ then $x + E_w \le s$ too). Hence, the procedure stops and the output cycle x^{**} satisfies properties (i) and (ii). However, we need to verify that at each step when we add $E_{v(i)}$ one has $(E_{v(i)}, x_i) = 1$. Note that this algorithm runs independently on each connected string of $\Gamma \setminus \mathcal{N}^+$. Furthermore, the procedure can be regarded as the first part of the algorithm, which provides Z_{min} for a conveniently chosen rational singularity. Hence the statement follows from Laufer's criterion. (If the string has no (-1)-vertex, then an immediate elementary argument also works.)

Then we repeat the procedure replacing x^{**} with the new x^*. By induction, starting from $x^* = 0$, after finitely many steps, we get $x^{**} = s$. □

7.1.A Some Analytic Invariants Described Topologically

We show that $\mathrm{Pic}(\widetilde{X})$, $h^1(\mathscr{L})$ for $\mathscr{L} \in \mathrm{Pic}(\widetilde{X})$, $\mathscr{S}_{an}$, $\mathscr{S}'_{an}$, and the Hilbert–Samuel function of $\mathfrak{m}_{X,o}$ can be recovered topologically. ♣

In this subsections $\widetilde{X}$ is an arbitrary resolution of a rational singularity.

7.1.9 Line Bundles, Sections

Proposition 7.1.10 *If (X, o) is rational then $c_1 : \mathrm{Pic}(\widetilde{X}) \to L' = H^2(\widetilde{X}, \mathbb{Z})$ is an isomorphism. Hence all the line bundles are natural. Moreover, $\mathscr{L}$ is holomorphically trivial if and only if $\mathscr{L}$ is topologically trivial if and only if $(E_v, c_1\mathscr{L}) = 0$ for all v.*

Proof Use the exponential exact sequence (6.1.1). □

Next, for any $\mathscr{L} \in \mathrm{Pic}(\widetilde{X})$, we provide an explicit formula for $h^1(\widetilde{X}, \mathscr{L})$ in terms of $c_1(\mathscr{L})$. We start with the following property, which in the rational case improves the Grauert–Riemenschneider vanishing theorem 6.4.3 and the Base point freeness Theorem 6.5.2.

Theorem 7.1.11 *Assume that (X, o) is rational and $\mathscr{L} \in \mathrm{Pic}(\widetilde{X})$ with $c_1(\mathscr{L}) \in -\mathscr{S}'$. Then the following facts hold.*

(a) [389, Th. 12.1] $h^1(l, \mathscr{L}) = 0$ for any $l \in L_{>0}$. In particular, $h^1(\widetilde{X}, \mathscr{L}) = 0$.
(b) $h^1(\widetilde{X}, \mathscr{L}(-E_v)) = 0$ for any $v \in \mathscr{V}$. More generally, if z_i is any member of a computation sequence of Z_{min}, then $h^1(\widetilde{X}, \mathscr{L}(-z_i)) = 0$.
(c) $\mathscr{L}$ is base point free.

Proof For *(a)* use 6.4.10. For *(b)* consider the infinite sequence $\{y_j\}_{j\geq 0}$ of cycles used in the proof of 7.1.1*(iv)*⇒*(i)* (where $y_0 = 0$ and $y_1 = E_v$). Then, by increasing induction on j, one proves that $H^1(\mathscr{L}(-y_j)) \to H^1(\mathscr{L}(-y_i))$ is onto for all $j \geq i$. In particular, $H^1(\mathscr{L}(-nZ_{min} - E_v) \to H^1(\mathscr{L}(-E_v))$ is onto, hence $H^1(\mathscr{L}(-E_v)|_{nZ_{min}}) = 0$ for any $n \in \mathbb{Z}_{>0}$. In the proof of the general statement one starts with z_i instead of E_v. (Similarly, one sees that $H^1(\mathscr{L}(-nZ_{min}) \to H^1(\mathscr{L})$ is onto, hence $H^1(\mathscr{L}|_{nZ_{min}}) = 0$ for any $n \in \mathbb{Z}_{>0}$, which reproves the vanishing $H^1(\mathscr{L}) = 0$ of part *(a)* too.)

(c) Fix a point $q \in E$. We show that $H^1(\mathfrak{m}_q \cdot \mathscr{L}) = 0$. Let $\pi : \widetilde{X}' \to \widetilde{X}$ be the blow up at q with new exceptional divisor E_{new}. Then, by the Leray spectral sequence, it is enough to show that $H^1(\widetilde{X}', \pi^*(\mathscr{L})(-E_{new})) = 0$. This follows from part *(b)*. □

Corollary 7.1.12 *Using the notations of Proposition 6.6.3 one has*

$$h^1(\mathscr{O}_{\widetilde{X}}(-l')) = \chi(l') - \chi(s(l')).$$

In particular, $h^1(\mathscr{L})$ depends only on the topological data and it is independent of the analytic structure of (X, o).

Proof Combine 6.6.5 and 7.1.11. □

Corollary 7.1.13

(a) $\mathscr{S}'_{an} = \mathscr{S}'$, $\mathscr{S}_{an} = \mathscr{S}$, hence $Z_{min} = Z_{max}$ too.
(b) $\phi^*(\mathfrak{m}^k_{X,o}) = \mathscr{O}_{\widetilde{X}}(-kZ_{min})$ for any positive integer k.

Proof

(a) Consider the correspondence from Corollary 6.2.11. We show that for every $h \in H$ and $l \in L$ such that $l' := l + r_h \in \mathscr{S}'$, there exists $s \in H^0(\mathscr{O}_{\widetilde{X}}(-r_h))$ whose divisor is l. This is guaranteed if for every v the morphism $H^0(\mathscr{O}_{\widetilde{X}}(-l' - E_v)) \to H^0(\mathscr{O}_{\widetilde{X}}(-l'))$ is not onto. Since $h^0(\mathscr{O}_{E_v}(-l')) \neq 0$, it is enough to show that $h^1(\mathscr{O}_{\widetilde{X}}(-l' - E_v)) = 0$. This follows from Theorem 7.1.11*(b)*.
(b) follows from the fact that $\mathscr{O}_{\widetilde{X}}(-Z_{min})$ has no base points, cf. 7.1.11*(c)*. □

Remark 7.1.14 7.1.13*(a)* follows from the freeness property 7.1.11*(c)* too.

There is another direct argument for $\mathscr{S} = \mathscr{S}_{an}$ based on the fact that $\mathrm{Pic}^0(\widetilde{X}) = 0$. Indeed, fix $0 < l \in \mathscr{S}$. Then $l = \sum n_v E_v$ with all $n_v \geq 0$, cf. 2.1.19. A 'cut' associated with a fixed E_v is a (non-compact) divisor A_v of $\widetilde{X}$ such that $(A_v, E_v) = 1$ and $(A_v, E_w) = 0$ for all $w \neq v$. If $(l, E_v) = a_v \leq 0$, then the divisor $A := l - \sum_v a_v A_v$ is effective, and $(A, E_v) = 0$ for all v. Hence $\mathscr{O}_{\widetilde{X}}(A)$ is trivial, and there exists a holomorphic function g on $\widetilde{X}$ whose zero set is A and $g|_E = 0$. Hence $g = f \circ \pi$ for some *continuous* function $f : X \to \mathbb{C}$. But the normality of (X, o) guarantees that f, in fact, is holomorphic (see also 2.1.11). Hence $l \in \mathscr{S}_{an}$.

Corollary 7.1.15 (Cf. 6.7.9) *For a rational singularity* (X, o) *one has* $\mathrm{mult}(X, o) = -Z_{min}^2$. *In particular,* $\mathrm{mult}(X, o)$ *is topological.*

7.1.16 The Hilbert–Samuel Function For any normal surface singularity and $k \geq 1$ one has the inclusion $\mathfrak{m}_{X,o}^k \subset \phi_*(\mathscr{O}_{\widetilde{X}}(-kZ_{min}))_o$. It is an isomorphism for $k = 1$, but usually is not.

Theorem 7.1.17 ([27, 28]) *For rational singularities the following facts hold:*

(a) $\mathfrak{m}_{X,o}^k \simeq \phi_*(\mathscr{O}_{\widetilde{X}}(-kZ_{min}))_o \simeq$ *for any* $k \geq 1$,

(b) $\dim(\mathfrak{m}_{X,o}^k / \mathfrak{m}_{X,o}^{k+1}) = -kZ_{min}^2 + 1$.
In particular, the embedding dimension of (X, o) *is* $\mathrm{emb}(X, o) = -Z_{min}^2 + 1$.

(c) The Hilbert-Samuel function is

$$f^{HS}(k) = -k(k-1)/2 \cdot Z_{min}^2 + k \quad (k \geq 1).$$

Proof

(a) We prove the inclusion $\mathfrak{m}_{X,o}^k \supset \phi_*(\mathscr{O}_{\widetilde{X}}(-kZ_{min}))_o$. This follows from Mumford's Theorem 6.5.8 too if we take $\mathscr{F} = \mathscr{L} = \mathscr{O}_{\widetilde{X}}(-Z_{min})$, however, for the convenience of the reader we write down a direct proof suited to this situation (in the spirit of 'classical arguments'). In this proof we will replace $\mathscr{O}_{\widetilde{X}}(-Z_{min})$ by $\mathscr{L} := \mathscr{O}_{Z_{min}}(-Z_{min})$. Since $\dim \mathrm{supp}(\mathscr{L}) = 1$, according to Remark 6.5.9 we will need two generic sections to run the argument (the 'Castelnuovo's free pencil trick'). We will write Z for Z_{min} and $d := -Z^2$.

Complete Z with some cuts A such that $(Z+A, E_v) = 0$ for all v as in 7.1.14. Then $Z + A$ is the divisor of an analytic function g (cf. 7.1.14). The middle term of the exact sequence $0 \to \mathscr{O}_{\widetilde{X}} \to \mathscr{O}_{\widetilde{X}}(A) \to \mathscr{O}_A(A) \to 0$ can be replaced by $\mathscr{O}_{\widetilde{X}}(-Z)$ via multiplication by g, hence this sequence restricted to Z provides

$$0 \to \mathscr{O}_Z \xrightarrow{s_0} \mathscr{L} \xrightarrow{/s} \mathscr{O}_{A \cap Z} \to 0, \tag{7.1.1}$$

where s_0 means multiplication by a global section s_0 of $\mathscr{L}$ (class of g), and $/s$ means division by a section s, which is a local basis of the line bundle near A (it exists since $\mathscr{L}$ is free). Note that $\mathscr{O}_{A\cap Z}$ is an Artinian algebra of dimension d. This provides the cohomology exact sequence:

$$0 \to H^0(\mathscr{O}_Z) \xrightarrow{s_0} H^0(\mathscr{L}) \xrightarrow{/s} \mathscr{O}_{A\cap Z} \to 0$$

(since $h^1(\mathscr{O}_Z) = 0$). Take $s_1, \dots, s_d \in H^0(\mathscr{L})$ such that $s_1/s, \dots, s_d/s$ is a base of $\mathscr{O}_{A\cap Z}$. Since $h^0(\mathscr{O}_Z) = 1$, cf. 6.6.9(b), $H^0(\mathscr{L})$, as a vector space, is $\mathbb{C}\langle s_0, s_1, \dots, s_d\rangle$. Now, we tensor the exact sequence (7.1.1) by $\mathscr{L}$. It provides

$$0 \to H^0(\mathscr{L}) \xrightarrow{s_0} H^0(\mathscr{L}^2) \xrightarrow{/s^2} \mathscr{O}_{A\cap Z} \to 0$$

(since $h^1(\mathscr{L}^j) = 0$ by 7.1.11). Then the sections $s_0s_0, s_0s_1, \dots, s_0s_d, ss_1, \dots, ss_d$ form a base in $H^0(\mathscr{L}^2)$. By induction $H^0(\mathscr{L}^k)$ is freely generated by s_0^k, and $s_0^{k-l}s^{l-1}s_i$ for $l = 1, \dots, k$ and $i = 1, \dots, d$. In particular,

$$H^0(\mathscr{L})^{\otimes k} \to H^0(\mathscr{L}^k) \text{ is onto for any } k. \tag{7.1.2}$$

Consider the diagram:

$$\begin{array}{ccccc} & & \mathfrak{m}_{X,o}^{\otimes k} = [\phi_*(\mathscr{O}_{\widetilde{X}}(-Z))_o]^{\otimes k} & \xrightarrow{(*)} & [H^0(\mathscr{O}_Z(-Z))]^{\otimes k} \\ & & \downarrow & & \downarrow (**) \\ \phi_*(\mathscr{O}_{\widetilde{X}}(-kZ-Z))_o & \to & \phi_*(\mathscr{O}_{\widetilde{X}}(-kZ))_o & \longrightarrow & H^0(\mathscr{O}_Z(-kZ)) \end{array}$$

The arrow $(*)$ is onto by 7.1.11, while $(**)$ is onto by (7.1.2). The diagram shows that $\phi_*(\mathscr{O}_{\widetilde{X}}(-kZ))_o \subset \mathfrak{m}_{X,o}^k + \phi_*(\mathscr{O}_{\widetilde{X}}(-(k+1)Z))_o$. By induction one has $\phi_*(\mathscr{O}_{\widetilde{X}}(-kZ))_o \subset \mathfrak{m}_{X,o}^k + \phi_*(\mathscr{O}_{\widetilde{X}}(-lZ))_o$ for any $l > k$. Since $\phi_*(\mathscr{O}_{\widetilde{X}}(-lZ))_o \subset \mathfrak{m}_{X,o}^k$ for $l \gg k$, cf. 6.6.10, the wished inclusion follows.

(b) The exact sequence $0 \to \mathscr{O}_{\widetilde{X}}(-kZ-Z) \to \mathscr{O}_{\widetilde{X}}(-kZ) \to \mathscr{O}_Z(-kZ) \to 0$, 7.1.11, and *(a)* imply $\dim(\mathfrak{m}_{X,o}^k / \mathfrak{m}_{X,o}^{k+1}) = h^0(\mathscr{O}_Z(-kZ)) = \chi(\mathscr{O}_Z(-kZ))$□

Example 7.1.18 Fix a rational (X, o). If $Z_{min}^2 = -1$ then (X, o) is smooth (use 7.1.15 or 7.1.17). This reproves *Castelnuovo's Contractibility Criterion*, cf. 2.4.4.1.

If $Z_{min}^2 = -2$ then $\operatorname{emb}(X, o) = 3$, that is, (X, o) is a hypersurface, hence Gorenstein. Therefore, (X, o) is ADE, cf. 7.1.5(d).

Remark 7.1.19 Although for rational singularities several analytic invariants can be computed from their graphs, still the analytic structure might contain several key information, which are not topological. For example, rational germs usually are not

taut, the topological type might support several non–isomorphic analytic structures. Take e.g. a rational graph which has a vertex of valency four. For more on taut singularities see [361]. For several interesting properties and conjectures regarding deformations and equisingular moduli of rational singularities see [111, 326, 731].

Example 7.1.20 (Rational Kulikov Singularities) For the definition and several properties of Kulikov singularities see 6.7.A. If the Kulikov singularity (X, o) is rational, then (6.7.3) implies that $g = 0$, that is, the minimal surface W contains only one irreducible curve, a $\mathbb{P}^1$, with multiplicity 1. Hence in this case Z_{min} is reduced (see the last sentence in 6.7.A), that is, (X, o) is minimal rational, and h is the generic hyperplane section. (7.1.7(a) also proves that $Z_{min} = E$.) Conversely, any minimal resolution graph of a minimal rational singularity can be completed with a function to get a Kulikov singularity structure (use $\mathrm{Pic}^0(\widetilde{X}) = 0$, or see [300]).

Since $-Z_{min}^2 = r = \mathrm{mult}(X, o)$, we can assume that $r \geq 2$.

Let l be a *reduced* connected curve on a smooth surface such that for any $E_v \subset |l|$ one has $(l, E_v) = 0$ (in our case l is either $\mathrm{div}(\widetilde{f})$ in $\widetilde{W}$ or it is obtained from it by blowing down (-1) curves). Then the (-1) curves in $|l|$ are the 'ends' of $|l|$.

This applies in our case to the F_i's. In particular, if we fix one of the (-1)-curves F_i, $\mathrm{div}(\widetilde{f})$ can be contracted by successively blowing down the other curves but never this F_i. Hence all the irreducible components of $(h^{-1}(0), o)$ are smooth. The delta-invariant δ is the extreme value $r - 1$, cf. (6.7.4). Since h is the generic linear section, the embedding dimension of $(h^{-1}(0), o)$ is one less than the embedding dimension of (X, o), hence it is r. Hence $(h^{-1}(0), o)$ is a curve singularity in $(\mathbb{C}^r, 0)$ having r smooth branches in general position and $\delta = r - 1$. (Let us denote by R_r^r the curve singularity in $(\mathbb{C}^r, 0)$ consisting of the union of the germs of the coordinate axes. In fact, any curve singularity in $(\mathbb{C}^r, 0)$ having r smooth branches in general position and $\delta = r - 1$ is isomorphic to R_r^r, cf. 12.4.6. Hence $(h^{-1}(0), o)$ is isomorphic to R_r^r.) For more see e.g. [661].

7.1.B Some Invariants of the Universal Abelian Covering

Let $(X_a, o) \to (X, o)$ be the universal abelian covering of a rational singularity (X, o).

7.1.21 The Geometric Genus $p_g(X_a, o) = \sum_h p_g(X_a, o)_{\theta(h)}$ can be determined completely from the resolution graph of (X, o). Indeed, by 7.1.12 for any h one has

$$p_g(X_a, o)_{\theta(h)} = \chi(r_h) - \chi(s_h) = \frac{(K + 2s_h)^2 - (K + 2r_h)^2}{8}.$$

Therefore, (X_a, o) is rational if and only if $\chi(r_h) = \chi(s_h)$ for every $h \in H$. This happens e.g. if (X, o) is a quotient singularity (since $(X_a, o) \simeq (\mathbb{C}^2, 0)$).

It is easy to find examples with $r_h \neq s_h$ for some h: take e.g. the A_4 singularity (however, in this case, $\chi(r_h) = \chi(s_h)$). Or, take the star-shaped graph with $b_0 = 2$ and with three legs all with invariants $(\alpha_j, \omega_j) = (3, 1)$. In this case (X_a, o) is not rational, its graph consists of one elliptic (-3)-vertex. (See 6.8.19 too.)

7.1.22 The Multiplicity of (X_a, o) [530] As usual, regarding the universal abelian covering, we will use the notations of paragraph 6.2.5.

Fix a good resolution of a normal surface singularity with irreducible exceptional curves $\{F_v\}_v$ (below it will be either a well–chosen $\widetilde{X}$ or the associated $\widetilde{Z}$). Let $\mathscr{D}$ be a certain finite set of rational cycles supported $\cup_v F_v$. We denote by $\mathscr{D}^m$ *the set of minimal elements of* $\mathscr{D}$ with respect to the partial ordering $\leq$. Furthermore, we say that $\mathscr{D}$ is locally ordered if for any nonempty intersection point $F_u \cap F_v$, and for any two elelements $d_i = \sum_w m_{iw} F_w$ $(i = 1, 2)$ of $\mathscr{D}$, one has $\Delta^{u,v}_{1,2} := (m_{1u} - m_{2u})(m_{1v} - m_{2v}) \geq 0$. We also write $\min \mathscr{D}$ for the cycle $\sum_w m_w F_w$, where $m_w = \min_{d \in \mathscr{D}}\{m_{dw}\}$, $\sum_w m_{dw} F_w = d \in \mathscr{D}$.

The motivation behind the definition is the following. Assume that we have a finite collection of functions $\{f_d\}_d$ in $\mathscr{O}_{X,o}$ with divisors $\mathrm{div}_d = \mathrm{div}_E(f_d)$, and set $\mathscr{D} = \{\mathrm{div}_d\}_d$. Then the ideal sheaf $I(\mathscr{D})$ generated by $\{f_d \circ \phi\}_d$ has a divisorial part $\mathscr{O}_{\widetilde{X}}(-\min \mathscr{D})$, and $I(\mathscr{D}) = \mathscr{O}_{\widetilde{X}}(-\min \mathscr{D})$ whenever $\mathscr{D}$ is locally ordered. Otherwise at the intersection points $E_u \cap E_v$ an additional ideal sheaf might appear.

Fix a singularity (X, o). We wish to find a resolution $\widetilde{X}$ such that $(\{E^*_v\}_{v \in \mathscr{V}})^m$ is locally ordered. This does not happen for an arbitrary resolution, however we have the following fact.

Lemma 7.1.23 *For any fixed resolution $\widetilde{X}$ with exceptional curves $\{E_v\}_v$, there exists a modification $\gamma : \widetilde{X}' \to \widetilde{X}$, $\widetilde{X}'$ having exceptional curves $\{E'_v\}_v$, such that $(\{E'^*_v\}_{v \in \mathscr{V}(\widetilde{X}')})^m = (\{E'^*_v\}_{v \in \mathscr{V}(\widetilde{X})})^m = \gamma^*((\{E^*_v\}_{v \in \mathscr{V}(\widetilde{X})})^m)$ and $(\{E'^*_v\}_v)^m$ is locally ordered. (Here, by definition, $\gamma^*(\mathscr{D}) := \{\gamma^*(d) \mid d \in \mathscr{D}\}$.)*

Proof γ is a composition of blowing ups of infinitesimal closed points of intersection of exceptional curves. Along a single blowing up at a point the behaviour of the cycles can be analyzed. Hence, it is enough to follow the behaviour of the cycles $\gamma^*((\{E^*_v\}_v)^m)$ along the sequence of blowing ups. If $\Delta^{u,v}_{1,2} < 0$, then blowing up $E_u \cap E_v$ the newly created Δ-values will be larger (and they are all in $\mathbb{Z}/|H|^2$), hence after finitely many steps the collection of pullbacks become locally ordered. □

Hence, there exists a resolution $\widetilde{X} \to X$ such that $(\{E^*_v\}_{v \in \mathscr{V}})^m$ is locally ordered. We show that the multiplicity of (X_a, o) can be read from the graph of this resolution.

Theorem 7.1.24 ((Okuma) [530]) *Assume that (X, o) is rational and $\widetilde{X} \to X$ is a resolution such that $\mathscr{D}^a := (\{E^*_v\}_{v \in \mathscr{V}})^m$ is locally ordered. Set $Z^a_{min} := \min \mathscr{D}^a$. Then $p^*(Z^a_{min})$ is the maximal ideal cycle of the resolution $\widetilde{Z} \to X_a$, $\mathfrak{m}_{X_a,o}\mathscr{O}_{\widetilde{Z}} = \mathscr{O}_{\widetilde{Z}}(-p^*(Z^a_{min}))$, and*

$$\mathrm{mult}(X_a, o) = -(p^*(Z^a_{min}))^2 = -|H| \cdot (Z^a_{min})^2.$$

Proof By the eigen-decomposition of $c_*\mathscr{O}_{X_a,o}$ (cf. 6.2.11) we have

$$c_*\mathfrak{m}_{X_a,o} = \mathfrak{m}_{X,o} \oplus \oplus_{h\neq 0}(\phi_*(\mathscr{O}_{\widetilde{X}}(-r_h)))_o.$$

Moreover, by 6.2.11 and 6.7.21 the divisor of any $\theta(h)$-eigenfunctions f_a (lifted to $\widetilde{Z}$ and restricted to the exceptional set) is of type $p^*(l')$ for a certain $l' \in \mathscr{S}'_{an}$. Therefore, $(\phi_*(\mathscr{O}_{\widetilde{X}}(-r_h)))_o = (\phi_*(\mathscr{O}_{\widetilde{X}}(-s_h)))_o$ (compare with (8.2.8)). By 7.1.13 $\mathscr{S}'_{an} = \mathscr{S}'$, and by 7.1.11 all the line bundles $\mathscr{O}_{\widetilde{X}}(-s_h)$ are free. Furthermore note that $\mathscr{S}'$ is generated over $\mathbb{Z}_{\geq 0}$ by the generators $\{E_v^*\}_v$, hence $\min p^*\mathscr{D}^a$ is the divisorial part of $\mathfrak{m}_{X_a,o}\mathscr{O}_{\widetilde{Z}}$.

Finally, one verifies that if $\mathscr{D}$ is locally ordered then $p^*\mathscr{D}$ is locally ordered and $\min p^*\mathscr{D} = p^* \min \mathscr{D}$. Therefore, $\mathfrak{m}_{X_a,o}\mathscr{O}_{\widetilde{Z}} = \mathscr{O}_{\widetilde{Z}}(-p^*(Z^a_{min}))$. Then, by 6.7.9, $\mathrm{mult}(X_a, o) = -(p^*(Z^a_{min}))^2 = -(p_*p^*Z^a_{min}, Z^a_{min}) = -|H| \cdot (Z^a_{min})^2$. □

Remark 7.1.25 The same argument can be used to define the multiplicity of any cyclic o-ramified covering (or even for any abelian o-ramified cover) of (X, o).

Indeed, let $(X_\alpha, o) \to (X, o)$ be the $\mathbb{Z}_r$–cyclic covering associated with $\alpha = \theta(h)$, cf. 6.2.14, and set $\mathscr{D}^\alpha := \{l' \in \mathscr{S}',\ l' \neq 0,\ [l'] \in \langle h\rangle\}^m$. We fix a resolution in which $\mathscr{D}^\alpha$ is locally ordered, and we define $Z^\alpha_{min} := \min \mathscr{D}^\alpha$. Then $\mathfrak{m}_{X_\alpha,o}\mathscr{O}_{\widetilde{Z}} = \mathscr{O}_{\widetilde{Z}}(-p^*(Z^\alpha_{min}))$, and $\mathrm{mult}(X_\alpha, o) = -r \cdot (Z^\alpha_{min})^2$.

Remark 7.1.26 In 8.5 we will study singularities which satisfy the 'end curve condition'. By the proof of 7.1.14 rational singularities satisfy this condition, see also 8.5.4. Furthermore, by the End Curve Theorem 8.5.34 if a singularity satisfies the 'end curve condition' then it is of splice quotient (cf. 5.4). In particular, the universal abelian covering of a rational singularity is an isolated complete intersection singularity with embedding dimension less than or equal to the number of end vertices of Γ. (The inequality can be strict, see [530, 7.7].) For more see [530].

7.2 Elliptic Singularities

The simplest non-rational singularities are the elliptic ones. They will serve as key non-trivial testing examples for several theorems and conjectures.
Elliptic singularities are defined topologically via their graphs. The study of the combinatorial properties of the elliptic graphs will run over several subsections. In this first part we define and list some properties of the minimally elliptic cycle, another important cycle besides the canonical and fundamental cycle. ♣

7.2.1 For an arbitrary singularity and arbitrary resolution $h^0(Z_{min}) = 1$ (cf. 6.6.9(b)), hence $\chi(Z_{min}) \leq 1$. Theorem 7.1.2 characterizes the realization of the upper bound:

$$(X, o) \text{ is rational} \iff \min_{l\in L_{>0}} \chi(l) \geq 1 \iff \chi(Z_{min}) = 1.$$

Next, assume that $\min_{l\in L_{>0}} \chi(l) = 0$. Then $\chi(Z_{min}) \geq 0$; but $\chi(Z_{min}) = 1$ is impossible by the above characterization, hence $\chi(Z_{min}) = 0$. Conversely, if $\chi(Z_{min}) = 0$, then $\min_{l\in L_{>0}} \chi(l) = 0$ too. The first proof of this last statement was given by Laufer in [363, p. 1281] using analytic deformations. His argument was the following. Taking $\chi(Z_{min}) = 0$ as the definition of the ellipticity, one builds up the theory of elliptic singularities, and one proves that for a fixed elliptic topological type the generic analytic structure has geometric genus 1 (for a proof of this last statement in this book see 7.2.98). Therefore, for this analytic structure and for any $l \in L_{>0}$, $\chi(l) \geq 1 - h^1(\mathscr{O}_l) \geq 1 - p_g = 0$.

The next proof is purely combinatorial.

Proposition 7.2.2 $\chi(Z_{min}) = 0 \iff \min_{l\in L_{>0}} \chi(l) = 0$.

Proof We show the $\Rightarrow$ part. Set $r := \min_{l\in L_{>0}} \chi(l)$, and $L_r := \{l \in L_{>0} \mid \chi(l) = r\}$. Then L_r is finite, hence it has a maximal element, say l_1. We show that $l_1 \in \mathscr{S}$. Indeed, $\chi(l_1) < \chi(l_1 + E_v)$ implies $(l_1, E_v) < \chi(E_v) \leq 1$.

Notice that the statement we wish to prove depends only on the graph Γ. Assume that the statement is not true for a certain graph. Let Γ be a minimal graph with this property. That is, (a) $\chi(Z_{min}(\Gamma)) = 0$ (b) $r(\Gamma) < 0$, and (c) for any proper subgaph $\Gamma' \subset \Gamma$ with $\chi(Z_{min}(\Gamma')) = 0$ the equality $r(\Gamma') = 0$ holds.

Take an arbitrary element l of $\mathscr{S} \cap L_r$, where $r = r(\Gamma)$. By the above discussion such an l exists. $l \in \mathscr{S}$ implies $l \geq Z_{min}(\Gamma)$. Obviously (by (a) and (b)), $l \neq Z_{min}(\Gamma)$. Set $l' := l - Z_{min}(\Gamma)$. Then $r = \chi(l) = \chi(l') + \chi(Z_{min}(\Gamma)) - (l', Z_{min}(\Gamma))$. Hence $\chi(l') = (l', Z_{min}(\Gamma)) + r \leq r$. But $\chi(l') \geq r$ from the definition of r, hence $\chi(l') = r$ and $(l', Z_{min}(\Gamma)) = 0$. Hence, the support Γ' of l' is strictly smaller than Γ. Next, we verify that Γ' satisfies conditions (a) and (b), a fact which leads to a contradiction with (c). Indeed, since $Z_{min}(\Gamma') \leq Z_{min}(\Gamma)$ we have $h^1(Z_{min}(\Gamma')) \leq h^1(Z_{min}(\Gamma)) = 1$. But $h^1(Z_{min}(\Gamma'))$ cannot be zero, since this would imply that Γ' supports a rational singularity, which contradicts $\chi(l') = r \leq 0$. Hence, $\chi(Z_{min}(\Gamma')) = 0$. Moreover, $r(\Gamma') \geq r(\Gamma)$, in general; but $\chi(l') = r(\Gamma)$; therefore, in fact $r(\Gamma') = r(\Gamma) < 0$. □

Definition 7.2.3 Fix a resolution graph Γ of a normal surface singularity. The graph is called *elliptic* if $\min_{l\in L_{>0}} \chi(l) = 0$, or equivalently, if $\chi(Z_{min}) = 0$. The singularity is called *elliptic* if its graph is elliptic. The definition is independent of the choice of the resolution, cf. 6.6.9(f).

Elliptic singularities were introduced by Ph. Wagreich (via the first criterion, $\min_{l\in L_{>0}} \chi(l) = 0$) [719] and by H. Laufer (via the second one, $\chi(Z_{min}) = 0$) [363]. We warn the reader that in the literature the terminology 'elliptic' is not used uniformly: sometimes it covers the singularities characterized as in the above definition 7.2.3 (these germs are also called 'weakly' elliptic), sometimes it covers singularities with $p_g = 1$, or sometimes Gorenstein singularities with $p_g = 1$ (following Laufer, this last family in this book will be called 'minimally elliptic').

Ellipticity is characterized by the resolution graph. Taking connected subgraphs or decreasing the self-intersection numbers of an elliptic graph we obtain either an

elliptic graph or a rational graph. (For a subgraph Γ' of Γ one has $Z_{min}(\Gamma') \leq Z_{min}(\Gamma)$, hence $h^1(\mathscr{O}_{Z_{min}(\Gamma')}) \leq h^1(\mathscr{O}_{Z_{min}(\Gamma)}) = 1$, or $\chi(Z_{min}(\Gamma')) \geq 0$.)

Example 7.2.4 A singularity is elliptic provided that $p_g = 1$ (use $p_g \geq h^1(\mathscr{O}_{Z_{min}}) = 1 - \chi(Z_{min})$), or provided that it is Gorenstein with $p_g = 2$ (in a minimal resolution $2 = h^1(\mathscr{O}_{Z_K}) = h^0(\mathscr{O}_{Z_K}) \neq h^0(\mathscr{O}_{Z_{min}}) = 1$, hence $Z_{min} < Z_K$ and 6.4.19 applies).

Example 7.2.5 Assume that $p_g > 0$ and there exist $\omega \in H^0(\widetilde{X} \setminus E, \Omega^2)$ and $f \in H^0(\mathscr{O}_{\widetilde{X}})$ such that $\{f^i\omega\}_{i=0}^{p_g-1}$ is a basis of $H^0(\widetilde{X} \setminus E, \Omega^2)/H^0(\widetilde{X}, \Omega^2)$. Then (X, o) is elliptic. Indeed, by (6.8.21) $T \geq p_g$, hence by 6.8.54 $\min \chi \geq T - p_g \geq 0$.

Example 7.2.6 p_g of an elliptic singularity can be arbitrary large, see e.g. 7.2.89.

The geometric genus of elliptic singularities cannot be characterized topologically, see e.g. 6.8.3.

Definition 7.2.7 ([363]) Fix an arbitrary resolution. A nonzero effective cycle C is called *minimally elliptic* if $\chi(C) = 0$ and for any $0 < l < C$, $l \in L$, one has $\chi(l) > 0$.

If $\chi(Z_{min}) = 0$ then a minimally elliptic cycle exists: take any minimal element of the set $\{l > 0 \ : \ \chi(l) = 0\}$. But, in fact, there is more to it than that.

Proposition 7.2.8 ([363]) *Fix an arbitrary resolution. If $\chi(Z_{min}) = 0$ then there is a unique minimally elliptic cycle, denoted by C. It satisfies $C \leq Z_{min}$.*

Proof First we prove that if $0 < A, B$, and $\chi(A) = \chi(B) = 0$, and $M := \min\{A, B\}$, then $\chi(M) = 0$ too. Indeed, $0 \leq \chi(A + B - M) = \chi(A) + \chi(B) - \chi(M) - (B - M, A - M) \leq -\chi(M) \leq 0$, cf. 7.2.2, hence $\chi(M) = 0$.

Next, consider an arbitrary cycle $A > 0$ with $\chi(A) = 0$, and take $M = \min\{A, Z_{min}\}$. Then $M \neq 0$ since $|Z_{min}| = E$, and by the above argument $\chi(M) = 0$. Hence all the minimal elements C satisfy $C \leq Z_{min}$.

Finally, take two cycles $0 < A, B \leq Z_{min}$ with $\chi(A) = \chi(B) = 0$. Take $M := \min\{A, B\}$. We already know that $\chi(M) = 0$, and we will show $M \neq 0$ too. Indeed, assume $M = 0$. Then $(A, B) \geq 0$ and 7.2.2 gives $0 \leq \chi(A + B) = -(A, B) \leq 0$. This implies $(A, B) = 0$, hence the supports of A and B do not intersect. Hence $h^1(\mathscr{O}_{A+B}) = h^1(\mathscr{O}_A) + h^1(\mathscr{O}_B) \geq 2$. But $A + B \leq Z_{min}$, hence $h^1(A + B) \leq h^1(Z_{min}) = 1$, which is a contradiction. □

Remark 7.2.9 Let $\delta(E_v)$ (resp. $g(E_v)$) be the sum of delta-invariants of singularities (resp. genus) of E_v in a (not necessarily good) resolution. For an elliptic graph the following facts hold:

(a) $\sum_{v \in \mathscr{V}} h^1(\mathscr{O}_{E_v}) = \sum_{v \in \mathscr{V}} \delta(E_v) + g(E_v) \leq 1$.
(b) $\sum_{E_v \subset |C|} \delta(E_v) + g(E_v) \leq 1$. If $\delta(E_v) + g(E_v) = 1$ for $E_v \subset |C|$ then $E_v = C$.
(c) all the connected components of $\cup_{E_v \not\subset |C|} E_v$ support rational singularities. (It follows from the uniqueness of the minimally elliptic cyle.)

(d) the restriction $H^1(E,\mathbb{Z}) \to H^1(|C|,\mathbb{Z})$ is an isomorphism.
(e) C is the characteristic cycle in the sense of 6.6.12.

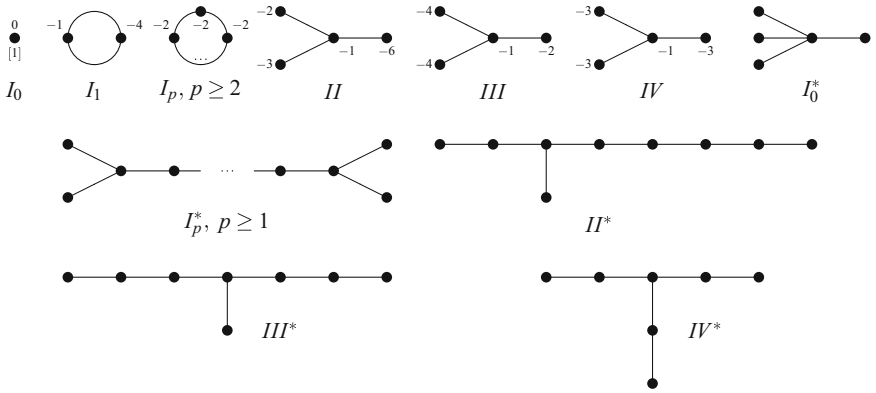
Lemma 7.2.10 ([363]) *Fix an elliptic graph. If C is a minimally elliptic cycle then C is the canonical cycle on its support $|C|$.*

Proof By 7.2.9(b) we can assume that each E_v is smooth and rational. Take an arbitrary $E_v \subset |C|$. Then $\chi(C-E_v) > 0$. This and $\chi(C) = 0$ imply $(E_v, C-Z_K) \geq 0$. By summation $(C, C-Z_K) = -2\chi(C) \geq 0$, but we already know that $\chi(C) = 0$. Hence all the inequalities must be equalities, that is, $(E_v, C-Z_K) = 0$ for all v. □

7.2.A *Elliptic Kulikov Singularities*

We prove that a Kulikov singularity is elliptic if and only if $g = 1$. Then we recall (reprove) Kodaira's classification of families of curves $f : W \to D$ with $g = 1$. We prove that an elliptic Kulikov singular is numerically Gorenstein if and only if it is Gorenstein, and we provide a simple combinatorial recognition property of the (numerically) Gorenstein property. ♣

7.2.11 By (6.7.3) a Kulikov singularity is elliptic if and only if the genus of the holomorphic minimal family of curves we start with has genus $g = 1$. Such families were classified by Kodaira [324]. The possible *minimal good dual graphs* of the families are the following (where the unmarked vertices represent (-2) curves):



From this one reads the *minimal* families as well. In the *minimal* family in the cases I_0, I_1 and II the central fiber is irreducible (either smooth elliptic, or rational with a node, or rational with a cusp); in the cases I_p $(p \geq 2)$, III and IV the central fiber is reducible but reduced; while in all other cases the fiber is non-reduced (hence only some of the vertices can be blown up along the Kulikov construction).

The above classification of the *minimal* families is based on the next observation.

From 6.7.A we have $0 = 1 - g = \chi_W(\mathrm{div}(f)) = -c_1(\mathrm{div}(f))/2$. But, by minimality and adjunction relations $c_1(C_v) \geq 0$ for all v, hence necessarily $c_1(C_v) = 0$ for all v. Here $c_1 = c_1(\Omega^2_W)$. That is, $(C_v, C_v)+2-2g(C_v)-2\delta(C_v) = 0$. By semi-definitness, if there exists C_v with $(C_v, C_v) = 0$ then necessarily C_v is the whole $\mathrm{div}(f)$, and $2 - 2g(C_v) - 2\delta(C_v) = 0$. This corresponds to the cases I_0, I_1 and II. Otherwise, C_v is a smooth rational (-2)-curve. Then we note that the set of semi-definite lattices with 1-dimensional radical, having the property that it is spanned by (-2)-vertices $v \in \mathscr{V}$ and that any proper subset of $\mathscr{V}$ is negative definite, coincides with the set of extended root systems $\widetilde{A}_n$, $\widetilde{D}_n$, $\widetilde{E}_6$, $\widetilde{E}_7$ and $\widetilde{E}_8$. The point is that the (minimal) lattice $\widetilde{A}_1$ has two geometric realizations as intersection matrix of certain curves, and the two minimal good plumbing graphs are I_2 and III. Similarly, $\widetilde{A}_2$ has two realizations I_3 and IV; while $\widetilde{A}_p$ is I_{p+1} $(p \geq 3)$, $\widetilde{D}$ is I^*, $\widetilde{E}_8 = II^*$, $\widetilde{E}_7 = III^*$, $\widetilde{E}_6 = IV^*$.

7.2.12 Recall that if (X, o) is an elliptic Kulikov singularity, then by the end of 6.7.A, the elliptic cycle is the strict transform of $\mathrm{div}(f)$. Additionally, we will also need the following fact as well.

Lemma 7.2.13 *Let $f : W \to D$ be a minimal family of $g = 1$ curves. Then Ω^2_W is holomorphically trivial line bundle. In particular, there exists a nowhere vanishing 2-form on W.*

Proof See e.g. [41, Theorem (12.1), page 161]. □

We will use the following terminology. Each exceptional curve F_i is the end of a chain of c_i rational curves leading from the elliptic cycle to F_i (including F_i). If F_i intersects C then $c_i = 1$, otherwise $c_i > 1$. We call c_i the *chain length* of F_i.

Corollary 7.2.14 ([301, 660]) *Let (X, o) be a Kulikov singularity associated with an elliptic family ($g = 1$) with chain lengths c_i, $i = 1, \ldots, r$; cf. 6.7.A. Then one has:*

(a) If $c_1 = \cdots = c_r$ then (X, o) is Gorenstein and $p_g(X, o) \geq c_1$ (later in 7.2.67 we will show that, in fact, $p_g(X, o) = c_1$).
(b) If an elliptic Kulikov graph is numerically Gorenstein then $c_1 = \cdots = c_r$.

Hence, an elliptic Kulikov singularity is Gorenstein if and only if it is numerically Gorenstein. (But a numerically Gorenstein elliptic Kulikov graph with $c_1 > 1$ supports a non-Gorenstein analytic type as well, e.g. the 'generic' one, but that analytic structure—by the previous statement—is not Kulikov.)

Proof

(a) Fix a nowhere vanishing holomorphic 2-form ω on W, let $\pi : \widetilde{W} \to W$ be the sequence of blowing ups provided by the Kulikov construction. Then $\pi^*\omega$ has vanishing order c_i along each F_i. Hence, if $c := c_1 = \cdots = c_r$, the form $h^{-c}\pi^*\omega$ is nowhere zero on $\widetilde{X} \setminus E$. Moreover, the forms $h^{-n}\pi^*\omega$, $n =$

$0, \dots, c-1$, project on independent elements of $H^0(\widetilde{X} \setminus E, \Omega^2_{\widetilde{X}})/H^0(\widetilde{X}, \Omega^2_{\widetilde{X}})$, cf. 6.8.

(b) Since $\widetilde{X}$ is minimal and $\Gamma(\widetilde{X})$ is elliptic and numerically Gorenstein, we have $Z_K \geq Z_{min}$ (†). Next, by $0 = 2\chi(Z_{min}) = (Z_{min}, Z_K - Z_{min})$, $\mathrm{div}(\widetilde{f}) = Z_{min} + \sum_i F_i$, and $(\mathrm{div}(\widetilde{f}), Z_K - Z_{min})_{\widetilde{W}} = 0$ we get that $(\sum_i F_i, Z_K - Z_{min})_{\widetilde{W}} = 0$ too. This combined with (†) for each i implies $(F_i, Z_K - Z_{min})_{\widetilde{W}} = 0$, or $(F_i, Z_K)_{\widetilde{W}} = (F_i, Z_{min})_{\widetilde{W}} = 1$. On the other hand, by adjunction relations on $\widetilde{W}$ we have $(\mathrm{div}(\pi^*\omega), F_i) = -F_i^2 - 2 = -1$. Therefore, $(\mathrm{div}(\pi^*\omega) + Z_K, F_i)_{\widetilde{W}} = 0$. On the other hand $(\mathrm{div}(\pi^*\omega) + Z_K, E_v)_{\widetilde{W}} = 0$ by adjunction relations for all other components. Hence $\mathrm{div}(\pi^*\omega) + Z_K$ is the kernel of the intersection form of $(\ ,\ )_{\widetilde{W}}$, hence $\mathrm{div}(\pi^*\omega) + Z_K = a(Z_{min} + \sum_i F_i)$ for some $a \in \mathbb{Q}$. Finally notice that ω has a trivial divisor, and a single blow up increases the vanishing order of the corresponding components by one. Hence, $\mathrm{div}(\pi^*\omega) = l + \sum_i c_i F_i$ for some cycle l supported on $L(\widetilde{X})$. Hence, necessarily, $c_1 = \ldots = c_r = a$. □

For a characterization of the generic hyperplane section $(h^{-1}(0), o)$—as an abstract curve germ—, for a Gorenstein elliptic Kulikov singularity, see [661].

For more on elliptic Kulikov and Kodaira singularities see [300, 301, 660].

7.2.B *Minimally Elliptic Singularities*

The simplest elliptic singularities are called 'minimally elliptic'; we provide several of their topological characterizations. Then we present Laufer's results: topological characterizations of several analytic invariants. Compared with rational singularities, in this case some of the proofs are harder (due to the fact that $\mathrm{Pic}^0(\widetilde{X})$ is not zero). ♣

Theorem 7.2.15 ([363]) *In the* minimal *resolution the following facts are equivalent. If a singularity satisfies them, it is called* minimally elliptic.

1. (X, o) *is numerically Gorenstein and* $Z_K = Z_{min}$.
2. $\chi(Z_{min}) = 0$ *and any proper subgraph of* Γ *supports a rational singularity.*
3. (X, o) *is numerically Gorenstein and* Z_K *is the minimally elliptic cycle.*
4. Z_{min} *is the minimally elliptic cycle.*
5. $p_g = 1$ *and* (X, o) *is Gorenstein.*

Therefore, for any minimally elliptic singularity in the minimal resolution one has $C = Z_{min} = Z_K = Z_{coh}$.

Theorem 7.2.15 gives a topological characterization of Gorenstein singularities with $p_g = 1$ (The second condition provides the name of the family.)

Proof First we check $1 \Leftrightarrow 2 \Leftrightarrow 3 \Leftrightarrow 4$. One can verify, that if there is some E_v with $\delta(E_v) + g(E_v) > 0$, then in fact $\delta(E_v) + g(E_v) = 1$, and $E = E_v = C = Z_{min} = Z_K$. Hence, in the sequel we assume that every E_v is smooth and rational.
$1 \Rightarrow 2$. Since $\chi(Z_K) = 0$ we obtain $\chi(Z_{min}) = 0$ too. Next, consider a proper subgraph $S \subset \Gamma$, fix a computation sequence of $Z_{min}(S)$, and complete it to a computation sequence of Z_{min}. Since $1 = h^1(Z_{min}) = \sum_v (E_{v(i)}, z_i) - 1$, there is exactly one index i for which $(E_{v(i)}, z_i) = 2$, otherwise $(E_{v(i)}, z_i) = 1$. $(E_v, Z_{min} - E_v) = (E_v, Z_K - E_v) = 2$ shows that $(E_{v(i)}, z_i) = 2$ in the very last step. Since in the previous steps the contributions equal 1, $h^1(Z_{min}(S)) = 0$.
$2 \Rightarrow 3$. Since $\chi(Z_{min}) = 0$, the minimally elliptic cycle C exists, cf. 7.2.8. If $|C| < E$ then C supports a rational singularity which contradicts $\chi(C) = 0$. Therefore $|C| = E$ and $C = Z_K$ by 7.2.10.
$3 \Rightarrow 4$. Since the resolution is minimal $Z_{min} \leq Z_K$, cf. 6.6.9(c). If $Z_{min} < Z_K$, by the definition of the minimally elliptic cycle, $\chi(Z_{min}) > 0$. This implies that the singularity is rational. Being rational and numerically Gorenstein we get $Z_K = 0$ (cf. 7.1.5), hence Z_K cannot be the minimally elliptic cycle. Therefore $Z_{min} = Z_K$.
$4 \Rightarrow 1$ follows from 7.2.10.

Hence $1 \Leftrightarrow 2 \Leftrightarrow 3 \Leftrightarrow 4$, and if any of these conditions is valid then $C = Z_{min} = Z_K$.
Next we verify $1, 2, 3, 4 \Rightarrow 5$.

By 6.4.7 $p_g = h^1(\mathscr{O}_{Z_K})$, hence $p_g = h^1(\mathscr{O}_{Z_K}) = h^1(\mathscr{O}_{Z_{min}}) = 1$.

In order to prove that (X, o) is Gorenstein we have to verify that $\mathscr{O}_{\widetilde{X}}(K_{\widetilde{X}} + Z_K) \approx \mathscr{O}_{\widetilde{X}}$. First we prove that $\mathscr{L} := \mathscr{O}_{Z_K}(K_{\widetilde{X}} + Z_K)$ is trivial. This follows from the next Lemma 7.2.17, since $h^0(\mathscr{L}) = h^1(\mathscr{O}_{Z_K}) = 1$ by duality.

Definition 7.2.16 An effective cycle D is k–connected, if for any $0 < A < D$ one has $(A, D - A) \geq k$.

Lemma 7.2.17

(a) [587, (3.11)] Fix some $D \in L_{>0}$. Assume that the restriction $\mathscr{L}$ of a topologically trivial line bundle of $\widetilde{X}$ to $\mathrm{Pic}(D)$ has $h^0(\mathscr{L}) > 0$. Then $\mathscr{L}$ is analytically trivial in $\mathrm{Pic}(D)$ provided that D is 1-connected.
(b) If Z_K coincides with the minimally elliptic cycle then it is 2-connected.
(c) If $Z_K \in L_{>0}$ then the restriction map $\mathrm{Pic}^0(\widetilde{X}) \to \mathrm{Pic}^0(Z_K)$ is an isomorphism.

Proof

(a) Fix $s \in H^0(D, \mathscr{L})$, $s \neq 0$, and let A be the maximal cycle along which s vanishes. Set $B = D - A$. From the exact sequence $0 \to \mathscr{L} \otimes \mathscr{O}_B(-A) \to \mathscr{L} \to \mathscr{L} \otimes \mathscr{O}_A \to 0$, one gets that $s \in H^0(B, \mathscr{L} \otimes \mathscr{O}_B(-A))$, and it has no fixed components. In particular, s restricted on each $E_v \subset |B|$ is non-zero, hence $(E_v, \mathscr{L}) + (E_v, -A) \geq 0$. Since $(E_v, \mathscr{L}) = 0$, we obtain $(A, B) \leq 0$. By the connectedness of D this can happen only if $A = 0$ ($A \neq D$ since $s \neq 0$).

But then the restriction of s to any E_v is a non-zero constant, hence s is never vanishing, and gives a trivialization of $\mathscr{L}$.

(b) For any $0 < A < Z_K$, by the definition of the minimally elliptic cycle we get $\chi(A) \geq 1$, or $(A, Z_K - A) \geq 2$.

(c) Use the isomorphism $H^1(\mathscr{O}_{\widetilde{X}}) \to H^1(\mathscr{O}_{Z_K})$. □

Set $\mathscr{L} := \mathscr{O}_{Z_K}(K_{\widetilde{X}} + Z_K)$. Since $h^0(\mathscr{L}) > 0$, by 7.2.17 $\mathscr{L}$ is trivial. By the vanishing 6.4.3, $H^0(\mathscr{O}_{\widetilde{X}}(K_{\widetilde{X}} + Z_K)) \to H^0(\mathscr{L}) = H^0(\mathscr{O}_{Z_K})$ is onto, which shows that $H^0(\mathscr{O}_{\widetilde{X}}(K_{\widetilde{X}} + Z_K))$ has a section whose image in $H^0(\mathscr{O}_{Z_K})$ is a non-zero constant, hence $K_{\widetilde{X}} + Z_K$ is a principal divisor of $\widetilde{X}$.

$5 \Rightarrow 3$. A Gorenstein singularity is numerically Gorenstein, hence we have to prove only that Z_K is a minimally elliptic cycle. Fix any $0 < l < Z_K$. By 6.4.19 $h^1(\mathscr{O}_l) < h^1(\mathscr{O}_{Z_K}) = p_g = 1$, hence $\chi(l) = h^0(\mathscr{O}_l) > 0$.

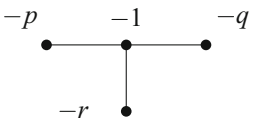
This ends the proof of Theorem 7.2.15. □

Example 7.2.18 ((Generalized) Simple Elliptic Singularities) Assume that Γ consists of a unique vertex E with $g(E) = 1$, $\delta(E) = 0$, and $E^2 = -d$. Then the graph is minimally elliptic with $C = Z_{min} = Z_K = E$.

The germ (X, o) is a complete intersection if and only if $d \leq 4$ (cf. 7.2.36 and 7.2.95); these germs are called 'simple elliptic'. Their equations are: $x^6 + y^3 + z^2 + \lambda xyz = 0$ $(d = 1)$; $x^4 + y^4 + z^2 + \lambda xyz = 0$ $(d = 2)$; $x^3 + y^3 + z^3 + \lambda xyz = 0$ $(d = 3)$; $x^2 + y^2 + \lambda zw = xy + z^2 + w^2 = 0$ $(d = 4)$. ($\lambda \in \mathbb{C}$ is such that the germ defines an isolated singularity.) The first three are deformations of exactly those Brieskorn-Pham singularities $(\{x^a + y^b + z^c = 0\}, 0) \subset (\mathbb{C}^3, 0)$ for which $1/a + 1/b + 1/c = 1$.

Example 7.2.19 (Cusp Singularities) By definition, the minimal resolution graph of a cusp singularity is a cyclic graph (with $g(E_v) = \delta(E_v) = 0$), cf. 3.5.B. Note that $Z_K = E = Z_{min}$. For combinatorial and topological properties see 3.5.B.

Example 7.2.20 (Triangle Singularities. [152, 153]) (Compare also with 3.7.6 and 5.1.35.) The *minimal good* resolution of the *Dolgachev's triangle singularity* $D_{p,q,r}$ has the form (with all genera zero):



In this resolution $Z_{min} \neq Z_K$, however in the *minimal resolution* $Z_{min} = Z_K$.

Example 7.2.21 Assume that (X, o) is a numerically log canonical singularity, and in the minimal good resolution $Z_K = E = Z_{min}$ (cf. 6.3.33 and 7.1.5 (h)). Then (X, o) is minimally elliptic and $\chi(E) = 0$ in this resolution. Therefore, either the

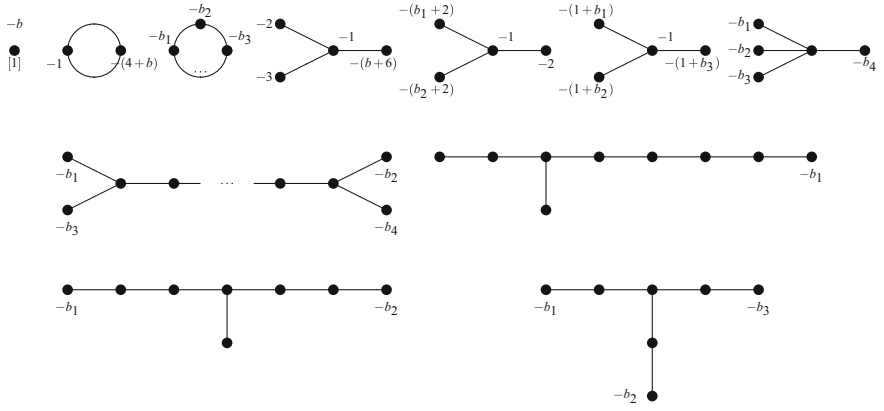
dual graph is cyclic, hence (X, o) is a cusp singularity, or E is irreducible with $g(E) = 1$.

Example 7.2.22 Assume that (X, o) is elliptic, and C is the minimally elliptic cycle in the minimal resolution. Then $|C|$ supports a minimally elliptic singularity (use Lemma 7.2.10 and Theorem 7.2.15 (3)).

Example 7.2.23 In [363] Laufer lists all the graphs and equations of hypersurface minimally elliptic singularities. E.g., with $Z^2_{min} = -1$ there are eight graphs and two more infinite families of graphs.

Example 7.2.24 (Unimodal and Bimodal Hypersurface Singularities) Kulikov in [342], see also [300], proved that all uni- and bimodal hypersurface singularities (in the sense of Arnold, cf. [21, Vol.1, 15.1]) are minimally elliptic and can be obtained by Kulikov construction with all $c_i = 1$, and ≤ 3 blow ups.

Here is the list of elliptic Kulikov singularities with all $c_i = 1$, cf. [300] (compare also with the list of Kodaira from 7.2.A). Below $b > 0$, respectively $b_j \geq 2$ with at least one of them strictly > 2. The unmarked vertices are -2. By 7.2.14 the corresponding singularities are all minimally elliptic. By the end of 6.7.A $Z_{min} = C$ is the (strict transform of) $\mathrm{div}(f)$.



These diagrams are realized by hypersurfaces if and only if $-Z^2_{min} \leq 3$ (cf. 7.2.36). But $-Z^2_{min}$ is exactly the number of blow ups in the Kulikov construction.

The above discussions also show that (the analytic type of) simple elliptic germs, cusps and triangle singularities are Kulikov.

7.2.25 Next, we prove the **topological characterization of certain analytic invariants**. In the proofs we will need the following technical lemma regarding computation sequences, which will be used in the general elliptic case as well.

Lemma 7.2.26 *Consider the minimal resolution of a minimally elliptic singularity.*

(a) Let $\{z_i\}_{i=1}^{t}$ be a computation sequence for Z_{min}, cf. 6.6.7. Then $\chi(z_i) = 1$ for all $i < t$, $(z_i, E_{v(i)}) = 1$ for all $i < t-1$, and in the last step: $(z_{t-1}, E_{v(t-1)}) = 2$.

(b) Fix any pair E_0 and E_1 ($E_0 \neq E_1$) of irreducible exceptional divisors. Then there exists a computation sequence for Z_{min} which starts with E_0 (i.e. $z_1 = E_0$) and ends with E_1 (i.e. $E_{v(t-1)} = E_1$). Moreover, let E_0 be an irreducible component whose coefficient in Z_{min} is greater than one. Then there exists a computation sequence for Z_{min} which starts and ends with E_0.

Proof

(a) See the proof of part 1⇒2, Theorem 7.2.15.

(b) Start with a computation sequence with $z_1 = E_0$. Suppose that E_1 may not be chosen to be the last. Then the computation sequence reaches a cycle D when we are forced to use the 'last E_1-component of Z_{min}', but still $D + E_1 < Z_{min}$. This implies additionally that: $(D, E_v) \leq 0$ for $v \neq 1$ and $E_1 \not\subset |Z_{min} - D - E_1|$. Hence, summation over the coefficients of $Z_{min} - D - E_1$ gives $(Z_{min} - D - E_1, D) \leq 0$. But, by *(a)*, $\chi(D + E_1) = 1$ and $(D, E_1) = 1$, facts which imply $(Z_{min} - D - E_1, D) \geq 1$. □

The Lemma will be generalized for non-minimal graphs in the next paragraphs.

7.2.27 The Case of Non-minimal Resolutions Although in Theorem 7.2.15 we characterized the minimally elliptic singularities in terms of minimal resolution graphs, we stress that for such germs the difference between the *minimal* and the *minimal good* resolution is rather small. The next facts can be verified directly.

Proposition 7.2.28 ([363, 3.5]) *The list of those minimal resolution graphs of minimally elliptic singularities which are not minimal good is the following:*

(1) E consists of a rational curve with a node singularity;
(2) E consists of a rational curve with a $(2, 3)$-cusp singularity (locally $u^2 = v^3$);
(3) E consists of two smooth rational curves which have first order tangential contact at one point;
(4) E consists of three smooth rational curves all meeting transversally at the same point.

Case (1) provides a special 'cusp singularity', cf. 7.2.19, while the others are 'triangle singularities', cf. 7.2.20.

One can check that in all the above cases in the *minimal good* resolution graph one has $|C| = E$, hence $C = Z_K$ is still valid (cf. 7.2.10). Nevertheless $Z_K < Z_{min}$.

Corollary 7.2.29

(a) If the link of a 'triangle singularity' (cf. 7.2.20) is an integral homology sphere then it is necessarily $\Sigma(2, 3, 7)$.

(b) If the link of a minimally elliptic singularity is an integral homology sphere then it is either $\Sigma(2, 3, 7)$ or $\Sigma(2, 3, 11)$.

Proof

(a) $(2, 3, 7)$ is the only solution of the Diophantine equation $\det(-I) = 1$, that is, of $(1 - 1/\alpha_1 - 1/\alpha_2 - 1/\alpha_3)\alpha_1\alpha_2\alpha_3 = 1$.

(b) Indeed, if the minimal resolution is not good, then by 7.2.28 (X, o) is 'triangle', hence part (a) applies. Assume that the minimal resolution is good. Then by 7.2.15 $Z_{min} = Z_K$. Hence, by the adjunction formula, $(Z_{min}, E_v) = (Z_K, E_v) < 0$ whenever $E_v^2 < -2$. These coefficients appear in $Z_{min} = -\sum_v (Z_K, E_v)E_v^*$; and additionally note that each $E_v^* \in \mathscr{S}$ (since $\det(-I) = 1$). Therefore, by the minimality of Z_{min} in $\mathscr{S}$, necessarily $E_v^2 = -2$ for all vertices except exactly one for which $E_v^2 = -3$. The subgraph consisting of (-2)-curves is an ADE. Then using 7.2.15(2) we conclude. (For the graph of $\Sigma(2, 3, 11)$ see 7.2.33.) □

7.2.30 Non-minimal Resolutions with $|C| = E$ Lemma 7.2.26, valid for minimal resolutions, can be extended for resolutions with $|C| = E$ as follows (the verification is left to the reader). (This will allow us to prove several statements for resolutions with $|C| = E$, hence, in particular, for the minimal good resolution too.)

Since $|C| = E$, one still has $C = Z_K$, and C is the cohomological cycle. Therefore, for any computation sequence $\{z_i\}_i$ of Z_{min}, the unique step with $(z_i, E_{v(i)}) = 2$ happens exactly for the minimal index i for which $z_{i+1} \geq C$. We say that the sequence 'jumps' at $E_{v(i)}$. Furthermore, part (b) of Lemma 7.2.26 has the following version: *For any pair E_0 and E_1 $(E_0 \neq E_1)$ there exists a computation sequence for Z_{min} which starts with E_0 and jumps at E_1. Moreover, if E_0 is a component whose coefficient in C is ≥ 2, then there exists a computation sequence for Z_{min} which starts and jumps at E_0.*

Theorem 7.2.31 *Consider a resolution of a minimally elliptic singularity with $|C| = E$. Then the following facts hold:*

(a) $\mathscr{S}_{an} = \mathscr{S}$, *hence* $Z_{max} = Z_{min}$ *too.*

(b) *If* $H^1(\widetilde{X}, \mathbb{Z}) = 0$ *(in particular,* $\mathscr{S}'_{an}$ *is well-defined) then* $\mathscr{S}'_{an} = \mathscr{S}'$.

(c) *Assume that* $c_1(\mathscr{L}) \in -\mathscr{S}'$ *for a certain* $\mathscr{L} \in \mathrm{Pic}(\widetilde{X})$. *Then* $h^1(\widetilde{X}, \mathscr{L}) = 1$ *if* $\mathscr{L}$ *is trivial, and* $= 0$ *otherwise.*

(d) *Assume that* $c_1(\mathscr{L}) \in -\mathscr{S}' \setminus \{0\}$. *Let* $\{z_i\}_i$ *be a computation sequence for* Z_{min} *which jumps at* E_1 *with* $(E_1, c_1(\mathscr{L})) > 0$. *Then* $h^1(\widetilde{X}, \mathscr{L}(-z_i)) = 0$ *for any* z_i.

(e) *Fix any* $\mathscr{L} \in \mathrm{Pic}(\widetilde{X})$, *set* $c_1 := c_1(\mathscr{L})$, *and use the notations of 6.6.4; in particular,* $s(-c_1) = -c_1 + e \in \mathscr{S}'$. *Then*

$$h^1(\widetilde{X}, \mathscr{L}) = \chi(-c_1) - \chi(s(-c_1)) + \begin{cases} 1 & \text{if } \mathscr{L}(-e) \text{ is trivial,} \\ 0 & \text{otherwise.} \end{cases}$$

Proof *(b)* Fix some $l' \in \mathscr{S}' \setminus \{0\}$, and a component E_0 of E. By Corollaries 6.2.11 and 6.7.23 we need to show that $h^1(\mathscr{O}(-l'-E_0)) < h^0(\mathscr{O}_{E_0}(-l'))$, since in this case $H^0(\mathscr{O}(-l'-E_0)) \to H^0(\mathscr{O}(-l'))$ is not onto.

Assume that there exists a computation sequence $\{z_i\}_{i=1}^t$ with $z_1 = E_0$ and which jumps at some E_1 such that $(E_1, l') < 0$. Then consider the infinite sequence $\{x_i\}_i$: $l'+z_1, \dots, l'+z_i, \dots, l'+Z_{min}, l'+Z_{min}+z_1, \dots, l'+2Z_{min}, l'+2Z_{min}+z_1, \dots$. Then $H^1(\mathscr{O}(-x_{i+1})) \to H^1(\mathscr{O}(-x_i))$ is onto, hence $\alpha : H^1(\mathscr{O}(-l'-nZ_{min}-E_0)) \to H^1(\mathscr{O}(-l'-E_0))$ is onto for any $n \geq 0$. Compose this with $\beta : H^1(\mathscr{O}(-l'-E_0)) \to H^1(nZ_{min}, \mathscr{O}(-l'-E_0))$ to get an exact sequence. But, by the theorem of formal functions, β is an isomorphism for $n \gg 0$, hence $\alpha = 0$ and $H^1(\mathscr{O}(-l'-E_0)) = 0$.

By Lemma 7.2.26 (see also the comment in 7.2.30), if such a computation sequence does not exist, then E_0 is the only component with $(l', E_0) < 0$, and the coefficient of E_0 in Z_{min} is 1. In this case we consider a computation sequence $\{z_i\}_i$ which starts with E_0 and jumps at some other E_1, and a sequence $\{y_i\}_i$ which starts with E_1 and jumps at E_0. Take the infinite sequence $\{x_i\}_i$: $l'+z_1, \dots, l'+z_i, \dots, l'+Z_{min}, l'+Z_{min}+y_1, \dots, l'+Z_{min}+y_i, \dots, l'+2Z_{min}, l'+2Z_{min}+y_1, \dots$. Then $H^1(\mathscr{O}(-x_{i+1})) \to H^1(\mathscr{O}(-x_i))$ is onto, except when we pass from $l'+Z_{min}-E_1$ to $l'+Z_{min}$, in which case the corank is 1. Hence, α has corank at most one, $h^1(\mathscr{O}(-l'-E_0)) \leq 1$. But $h^0(\mathscr{O}_{E_0}(-l')) \geq 2$.

(a) has the identical proof as *(b)* whenever $H^1(\widetilde{X}, \mathbb{Z}) = 0$, which can be adopted to the general case as well (the vanishing $H^1(\widetilde{X}, \mathbb{Z}) = 0$ is not needed in 7.2.26).

(c) Assume that $-l' := c_1(\mathscr{L}) \neq 0$. If all components E_v are rational, take a computation sequence which starts with some E_0 and jumps at certain E_1 with $(E_1, l') < 0$. Then, as in the proof of *(b)*, $h^1(\mathscr{L}(-E_0)) = 0$. This implies $h^1(\mathscr{L}) = 0$ too. If $E = E_0$ is an elliptic curve, then a similar proof works.

Assume next that $c_1(\mathscr{L}) = 0$. Since $h^1(\mathscr{L}) = h^1(\mathscr{L}|_{Z_K}) = h^0(\mathscr{L}|_{Z_K})$, if $h^1(\mathscr{L}) \neq 0$, then $\mathscr{L}$ is trivial by Lemma 7.2.17.

(d) The proof runs as in *(c)*, where one should replace E_0 by z_i.

(e) Combine Proposition 6.6.4 and part *(c)*. □

7.2.32 Arbitrary (Non-minimal) Resolutions If we consider arbitrary resolution graphs (e.g. which are not minimal good), then it can happen that Z_K has even negative coefficients, or the support of C is smaller than E. In such cases Lemma 7.2.26 and Theorem 7.2.31 fail, in general.

Example 7.2.33 Take the minimal resolution graph of $\{x^2+y^3+z^{11}=0\}$. It can be obtained from E_8 by attaching a new vertex E_0 to the end-vertex of the longest leg with $E_0^2 = -3$. Let $\widetilde{X}_P \to \widetilde{X}$ be the blow up at a point P of E_0 (P smooth in E). Let the strict transform of E_0 be E_0, and let the new component be E_1. Then one verifies that in the minimal resolution $(Z_{min}, E_0) = 1$ and $Z_{min}^2 = -1$. Hence, $\mathscr{O}(-Z_{min})$ has a base point Q on E_0, cf. 7.2.36(c), (it is the intersection of E_0 with the strict transform of $z = 0$). If we blow up $P = Q$ then $E_1^* \in \mathscr{S}_{an}$ and $E_0^* \notin \mathscr{S}_{an}$ in $\widetilde{X}_P$; while if we blow up $P \neq Q$ then $E_1^* \notin \mathscr{S}_{an}$ and $E_0^* \in \mathscr{S}_{an}$. (In fact, by similar argument as in 6.7.25(b), one can prove that for any analytic structure supported

on the graph Γ_P the divisors E_0^* and E_1^* cannot be in $\mathscr{S}_{an}$ simultaneously; indeed, otherwise the corresponding two functions would provide a map $(X, o) \to (\mathbb{C}^2, 0)$ of degree one.)

In particular, the statement of 7.2.31*(a)* is not valid in an arbitrary resolution.

In the most general case, the vanishing result from Theorem 7.2.31(c) has the following analogue. [Hence, part (d) also has its analogue via Proposition 6.6.3.]

Theorem 7.2.34 *Let C be the minimally elliptic cycle.*

(a) For any irreducible component E_1 of $|C|$ there exists a computation sequence for Z_{min} whose unique step $(z_i, E_{v(i)}) = 2$ (the 'jump') happens for $E_{v(i)} = E_1$.

(b) Assume that a line bundle $\mathscr{L} \in \mathrm{Pic}(\widetilde{X})$ satisfies $c_1(\mathscr{L}) \in -\mathscr{S}'$ and $(c_1(\mathscr{L}), C) < 0$. Then $h^1(\widetilde{X}, \mathscr{L}) = 0$.

Proof *(a)* We modify the proof of 7.2.26(b). There exists a computation sequence of Z_{min} which has one of its terms C. [Otherwise there would exist a cycle $0 < D < C$ with $(D, E_v) \le 0$ for all $E_v \subset |C - D|$. But this implies $(D, C - D) \le 0$, which contradicts $\chi(D) > 0$.] Then for such a computations sequence we repeat the proof of 7.2.26(b) with Z_{min} replaced by C. Part *(b)* follows from *(a)* as in 7.2.31. □

7.2.35 Topological description of some further analytic invariants is given next. The same statement is valid in a more general context of elliptic germs, see 7.2.G.

Theorem 7.2.36 ([363]) *Assume that (X, o) is a minimally elliptic singularity. Let Z_{min} be the fundamental cycle in the minimal resolution ϕ. Then:*

(a) $Z_{min} = Z_{max}$;

(b) If $Z_{min}^2 \le -2$, then $\phi^*\mathfrak{m}_{X,o} = \mathscr{O}_{\widetilde{X}}(-Z_{min})$, hence $\mathrm{mult}(X, o) = -Z_{min}^2$;

(c) If $Z_{min}^2 = -1$, then $\phi^*\mathfrak{m}_{X,o} = \mathfrak{m}_Q\mathscr{O}_{\widetilde{X}}(-Z_{min})$ for a smooth point Q of E, and $\mathrm{mult}(X, o) = 2$;

(d) $\mathrm{emb}(X, o) = \max(3, -Z_{min}^2)$;

(e) If $Z_{min}^2 \le -3$ then $\dim \mathscr{O}_{X,0}/\mathfrak{m}_{X,o}^k = \chi(\mathscr{O}_{kZ_{min}}) + 1$.

(f) If $Z_{min}^2 = -4$ then (X, o) is a complete intersection, If $Z_{min}^2 < -4$ then (X, o) is not a complete intersection.

7.2.C The Elliptic Sequence, General Case

The analytic and topological properties of an elliptic singularity are guided by its elliptic sequence. The elliptic sequence is defined from the minimal graph. It was introduced by S. S.-T. Yau and Laufer in order to give an intrinsic structure of the cohomology of certain line bundles, of the Picard group, and to provide a topological upper bound for p_g. In the numerically Gorenstein

case the sequence works perfectly, it answers all the posed questions (see Sects. 7.2.D and 7.2.F). However, in the general (non-numerically Gorenstein) case, the sequence of Yau–Laufer, being a sequence of integral cycles, loses certain important connections with the (non-integral) canonical cycle. However, it still preserves several universal properties.

In this subsection we provide the definition of the elliptic sequence (general case) and we list several of its properties. In the next Sect. 7.2.D we discuss the numerically Gorenstein case (with a simplified definition and more properties). In Sect. 7.2.E we propose a new elliptic sequence for the non–numerically Gorenstein case, which recaptures the connection with the canonical cycle (and it has some additional universal properties). ♣

7.2.37 We fix the *minimal* resolution graph of an elliptic singularity.

The intrinsic properties of an elliptic graph are guided by its *elliptic sequence*. It is a monotone sequence of elements of $L_{>0}$, containing the fundamental cycle. In fact, all the members are fundamental cycles of certain connected subgraphs.

For any effective reduced cycle $l \in L$, with connected support $|l|$, let Z_l denote the fundamental cycle of $|l|$. We say that the reduced cycle D is 'connected' if $|D|$ is connected. It is convenient to identify a reduced cycle with its support.

Definition 7.2.38 ([766], [767, Def. 3.3]) Let C be the minimally elliptic cycle.

If $(C, Z_{min}) < 0$ then the elliptic sequence consists of one element, namely $\{Z_{min}\}$.

If $(C, Z_{min}) = 0$, let D_1 be the maximal connected reduced cycle such that $|C| \le D_1 \le E$, and $(E_v, Z_{min}) = 0$ for all $E_v \le D_1$. Since $Z_{min}^2 < 0$, $D_1 \ne E$.

Assume that the support D_{i-1} is already defined. If $(C, Z_{D_{i-1}}) = 0$, let D_i be the maximal connected reduced cycle such that $|C| \le D_i \le D_{i-1}$, and $(E_v, Z_{D_{i-1}}) = 0$ for every $E_v \le D_i$. Again D_i is properly contained in D_{i-1}. This process stops after finitely many steps, say at D_n, which has the property $(C, Z_{D_n}) < 0$.

Write $D_0 := E$, $Z_{D_0} := Z_{min}$. Then the elliptic sequence is $\{Z_{D_0}, \dots, Z_{D_n}\}$. Its length is $\ell_{seq} := n + 1$ $(n \ge 0)$.

Definition 7.2.39 For each vertex v let $d(v, |C|)$ be the combinatorial distance in the graph from v to C. (This is well-defined thanks to Remark 7.2.9 (d).)

For example, if $E_v \subset |C|$ then the distance is zero, if $(E_v, C) > 0$ then it is 1.

Theorem 7.2.40 *With the above notations one has:*

1. $C \le Z_{D_n} < \cdots < Z_{D_0}$.
2. $\chi(Z_{D_j}) = 0$ *for all* $0 \le j \le n$, *and* $(Z_{D_k}, Z_{D_j}) = 0$ *for* $k \ne j$.
3. *For any* $0 \le \ell \le n$ *set* $F_\ell := \sum_{j=0}^{\ell} Z_{D_j}$. *Then* $\chi(F_\ell) = 0$ *and* $F_\ell \in \mathscr{S}$.
4. *For any* $1 \le \ell \le n$, F_ℓ *is the smallest element of* $\mathscr{S}$ *which is greater than a cycle of type* $F_{\ell-1} + E_1$ *with* E_1 *arbitrarily chosen from* $|C|$.

   The set $\{l \in \mathscr{S},\ (l, C) < 0\}$ *has a smallest element. It is equal to* F_n.
5. *If* $Z \in \mathscr{S}$ *and* $\chi(Z) = 0$ *then* $Z \in \{0, F_0, \dots, F_n\}$.

6. *If $E_v \le D_{j-1}$ but $E_v \not\le D_j$ then $(E_v, D_{j+1}) = 0$. (In other words, if one wishes to extend the elliptic graph supported on D_j to a larger elliptic graph via a new vertex E_v, then E_v cannot intersect any curve from D_{j+1}.) In particular, if $\mathcal{V}_Z := \{v \in \mathcal{V} : (E_v, Z_{min}) < 0\}$ then $n \le \min_{v\in\mathcal{V}_Z} d(v, |C|)$.*

Proof (1) Use $|C| \le D_n \lneqq \cdots \lneqq D_0$ and $C = Z_{|C|}$, cf. Example 7.2.22.

(2) $h^0(\mathcal{O}_{Z_{D_j}}) = 1$ and $h^1(\mathcal{O}_{Z_{D_j}}) \ge h^1(\mathcal{O}_C) = 1$. But $\chi(Z_{D_j}) \ge 0$ too, hence $\chi(Z_{D_j}) = 0$. For the second part use the definition of D_j.

(3) $\chi(F_\ell) = 0$ follows from part (2). Next we prove $F_\ell \in \mathcal{S}$.

If $E_v \le D_\ell$ then $(E_v, Z_{D_j}) \le 0$ for all $j \le \ell$, hence $(F_\ell, E_v) \le 0$.

Assume that $E_v \le D_{j-1}$ but $E_v \not\le D_j$ for some $1 \le j \le \ell$. Then $(E_v, Z_{D_k}) \le 0$ for $k \le j-1$ since $D_{j-1} \le D_k$. If $(E_v, Z_{D_j}) = 0$, that is, E_v does not intersect the support D_j, then E_v does not intersect the smaller supports $\{D_k\}_{\ell\ge k\ge j}$ either, hence $(E_v, Z_{D_k}) = 0$ for all $k \ge j$. Hence we are done.

Next assume that $(E_v, Z_{D_j}) > 0$, hence E_v intersects D_j, say along the component E_u. Then we observe two facts. First, (†) $(E_v, Z_{D_{j-1}}) < 0$ since otherwise E_v would be in D_j. Second, Z_{D_j} can be completed by a computation sequence to Z_{D_0}, but both of them have the same $\chi = 0$, hence (‡) $(Z_{D_j}, E_v) = 1$. Then (†) and (‡) imply $(Z_{D_j} + Z_{D_{j-1}}, E_v) \le 0$. If $j = \ell$ then again we are done.

If $j < \ell$ then $(Z_{D_j}, C) = 0$ and $Z_{D_{j+1}}$ exists, and it is a summand of F_ℓ. We show that $(Z_{D_{j+1}}, E_v) = 0$. This means that E_v does not intersect the support D_{j+1} hence neither the smaller supports $\{D_k\}_{\ell\ge k\ge j+1}$, hence $(Z_{D_k}, E_v) = 0$ for all $k \ge j+1$.

Assume the opposite: $(Z_{D_{j+1}}, E_v) > 0$. Then necessarily $(Z_{D_j}, E_v) > 0$ too. Then consider the cycle $l := Z_{D_{j+1}} + Z_{D_j} + E_v$. Then $\chi(l) = \chi(Z_{D_{j+1}} + Z_{D_j}) + 1 - (Z_{D_{j+1}} + Z_{D_j}, E_v) = 1 - (Z_{D_{j+1}} + Z_{D_j}, E_v) < 0$, a fact which contradicts the ellipticity.

(4) If we add $F_{\ell-1}$ to the computation sequence which starts with E_1 and ends with the fundamental cycle Z_{D_ℓ} of D_ℓ, we get exactly that increasing sequence (see Proposition 6.6.3) which starts with $F_{\ell-1} + E_1$ and ends with the smallest element of $\mathcal{S}$ which is larger than $F_{\ell-1} + E_1$. In particular, this element is exactly F_ℓ.

Assume that $l \in \mathcal{S}$ and $(l, C) < 0$. Fix a component $E_v \subset |C|$ such that $(E_v, l) < 0$. Then $l \ge Z_{D_0}$. Since $(E_v, Z_{D_0}) = 0$ we obtain $l \ge Z_{D_0} + E_v$ too. By the first part $l \ge F_1$. Again, since $(E_v, F_1) = 0$ we have $l \ge F_1 + E_v$, and the whole procedure is repeated. Hence $l \ge F_n$. Since F_n has the required property, it is the smallest such element.

(5) If $Z \ne 0$ then $Z \in \mathcal{S}$ implies $Z \ge Z_{min} = F_0$. Assume that $Z \ge F_\ell$ for some $\ell < n$, but $Z \ne F_\ell$. Then using $\chi(Z) = \chi(F_\ell) + \chi(Z - F_\ell) - (F_\ell, Z - F_\ell)$ we get $\chi(Z - F_\ell) = (F_\ell, Z - F_\ell)$. Since $\chi(Z - F_\ell) \ge 0$ and $(F_\ell, Z - F_\ell) \le 0$, necessarily $\chi(Z - F_\ell) = 0$. Then, by 7.2.8, $Z - F_\ell \ge C$. Hence, by (4), $Z \ge F_{\ell+1}$.

Finally, assume that $Z > F_n$. Then, similarly as above, $\chi(Z - F_n) = (F_n, Z - F_n) = 0$. $(F_n, Z - F_n) = 0$ implies that $(F_n, E_v) = 0$ for any $E_v \subset |Z - F_n|$. But $\chi(Z - F_n) = 0$ implies (by 7.2.8) that $Z - F_n \ge C$. These two facts combined show that $(F_n, C) = 0$, which leads to a contradiction.

(6) If $E_v \le D_{j-1}$, $E_v \not\le D_j$ and $(E_v, D_{j+1}) > 0$ then $(E_v, D_j) > 0$ too, hence $\chi(Z_{D_{j+1}} + Z_{D_j} + E_v) < 0$, cf. the proof of (3). □

7.2.41 The elliptic sequence is an invariant of the *minimal* resolution graph, hence it is topological. We invite the reader to review the definition of another topological invariant, Path^i, which is a topological upper bound for p_g, cf. 6.8.36. Since Path^i is well-defined only in a good resolution (and as a bound is efficient only if each E_v is rational), whenever when we deal with Path^i we will assume that the link is a rational homology sphere and in the proofs we will assume that the minimal resolution is good.

Parts (1)–(2) of the next theorem relate Path^i with ℓ_{seq} and formulate *the universal topological upper bound* for p_g (independently of the analytic structure), while part (3) provides the exact value p_g valid for any analytic structure.

Theorem 7.2.42 *Assume that the link is a rational homology sphere. Then one has the following inequalities:*

1. $\mathrm{Path}^i \le \ell_{seq}$.
2. *In particular,* $p_g \le \ell_{seq} = 1 + n \le 1 + \min_{v\in\mathcal{V}_Z} d(v, |C|)$, *where* $\mathcal{V}_Z := \{v \in \mathcal{V} : (E_v, Z_{min}) < 0\}$.
3. $p_g = 1 + \#$, *where* # *is the number of those indices* $\{\ell : 0 \le \ell \le n-1\}$ *with the property that there exists* $f_\ell \in H^0(\mathcal{O}(-F_\ell)) \setminus H^0(\mathcal{O}(-F_\ell - E_1))$ *for some* $E_1 \subset |C|$.

Proof In the proof we assume that each E_v is rational and smooth. We invite the reader to adopt the proof in any other situation.

(1) Since $(C, Z_{D_n}) < 0$ there exists $E_0 \subset |C|$ with $(E_0, Z_{D_n}) < 0$. By 7.2.26 there exists a computation sequence $\{z_i\}_{i=1}^t$ for $C = Z_{|C|}$ such that $(z_i, E_{v(i)}) = 1$ for $i < t-1$ and $(z_{t-1}, E_{v(t-1)}) = 2$, where $E_{v(t-1)}$ is exactly E_0. We mark this step by (†).

We will also denote the first cycle of the sequence by E_1. It is in $|C|$.

This computation sequence can be completed to a computation sequence of Z_{D_j}, $\{z_i^{(j)}\}_{i=1}^{t(j)}$, such that for $t \le i < t(j)$ one has $(z_i^{(j)}, E_{v(i)}) = 1$ (for $\chi(Z_{D_j}) = \chi(C) = 0$).

If we concatenate these sequences, $\{z_i^{(0)}\}_i$, $\{z_i^{(1)}\}_i$, …, $\{z_i^{(n)}\}$, we get $\{z_i^c\}_{i=1}^{\sum_j t(j)}$, which connects E_1 to $F_n := \sum_{j=0}^n Z_{D_j}$, and in it exactly $n+1$ times happens that $\chi(z_{i+1}^c) < \chi(z_i^c)$. When this happens then $\chi(z_{i+1}^c) = \chi(z_i^c) - 1$, and they occur exactly when we add the last component of C, namely during steps marked by (†).

Next, we continue the sequence $\{z^c\}_i$ with $F_n + \{z^c\}_i$. Then by Lemma 7.2.40(3) $(F_n + z_i^c, E_{v(i)}) \le (z_i^c, E_{v(i)}) \le 2$, and $(F_n + z_i^c, E_{v(i)})$ might be 2 only at steps marked by (†). But at these steps $(F_n, E_{v(i)}) = (F_n, E_0) = (Z_{D_n}, E_0) < 0$, hence $(F_n + z_i^c, E_{v(i)}) < 2$ always, and the χ-values along the sequence $F_n + \{z^c\}_i$ are non-decreasing. This remains true for $kF_n + \{z^c\}_i$ for any $k \ge 1$, hence we get an infinite sequence $\{\ell_i\}_i$ whose multiplicities tend to infinity, and which satisfies $\sum_i \max\{0, \chi(\ell_i) - \chi(\ell_{i+1})\} = n+1$. This proves the inequality $\mathrm{Path}^i \le \ell_{seq}$.

(2) Use $p_g \le \mathrm{Path}^i$ from (6.8.14)–(6.8.16) and 7.2.40(6) (or directly part (3): $p_g = 1 + \# \le 1 + n$).

(3) Consider the infinite computation sequence constructed in the proof of (a), but only starting from F_n. Along these $h^1(\mathscr{O}_{E_{v(i)}}(-z_i)) = 0$, hence $H^q(\mathscr{O}_{z_i}) \to H^q(\mathscr{O}_{F_n})$ is onto for $q = 0$ and isomorphism for $q = 1$. Taking limit we have the same for $H^q(\mathscr{O}) \to H^q(\mathscr{O}_{F_n})$. Therefore $H^1(\mathscr{O}(-F_n)) = 0$.

Now consider again that part of the concatenated computation sequence from the proof of (1), which connects Z_{D_0} and F_n. We analyse the behaviour of $H^1(\mathscr{O}(-z_i^c))$ along the sequence via $0 \to \mathscr{O}(-z_i^c) \to \mathscr{O}(-z_{i-1}^c) \to \mathscr{O}_{E_{v(i)}}(-z_{i-1}^c) \to 0$.

Between $F_{\ell-1}$ and F_ℓ there are two non-trivial steps, when $\Delta_i := h^1(\mathscr{O}(-z_i^c)) - h^1(\mathscr{O}(-z_{i-1}^c))$ can be non-zero. The first one is $F_{\ell-1} \mapsto F_{\ell-1} + E_1$, the second one is $F_{\ell-1} + C - E_0 \mapsto F_{\ell-1} + C$. At the second step we always have $\Delta_i = -1$. In the first case $\Delta_i = 0$ or 1: it is zero exactly when the restriction $r_\ell : H^0(\mathscr{O}(-z_{i-1}^c)) \to H^0(\mathscr{O}_{E_{v(i)}}(-z_{i-1}^c)) = \mathbb{C}$ is onto, that is, when f_ℓ exists. Therefore, $h^1(\mathscr{O}(-Z_{D_0})) = n - (n - \#) = \#$. But using the exact sequence $0 \to \mathscr{O}(-Z_{min}) \to \mathscr{O} \to \mathscr{O}_{Z_{min}} \to 0$ we have (cf. 6.6.9), $p_g = h^1(\mathscr{O}(-Z_{min})) + h^1(\mathscr{O}_{Z_{min}}) = \# + 1$. □

Remark 7.2.43 S.S.-T. Yau in [766] considered another sequence, the 'Laufer sequence', and he proved that p_g is not greater than the length of the Laufer sequence. On the other hand, J. Stevens in (the first preprint version of) [660] proved that the elliptic sequence and the Laufer sequence coincide. Hence, these two results imply the inequality $p_g \le n + 1$ as well.

7.2.D *The Elliptic Sequence, Numerically Gorenstein Case*

We provide a new definition of the elliptic sequence, which is adopted to the case when (X, o) is numerically Gorenstein and $\widetilde{X}$ is a minimal resolution. The presentation is based on [766, 767] and [473]. In Proposition 7.2.47 we show that the elliptic sequences defined in 7.2.C and in the present subsection in fact coincide (in the numerically Gorenstein case). We also establish some additional properties, on which the cohomological manipulations of the forthcoming subsections will rely. ♣

7.2.44 Assume that (X, o) is numerically Gorenstein and $\widetilde{X}$ is a minimal resolution. Then, cf. 7.2.8 and 6.3.3,

$$C \le Z_{min} \le Z_K. \tag{7.2.1}$$

7.2.45 The Construction of the Elliptic Sequence [Laufer, S. S.-T. Yau]. The elliptic sequence consists of the sequence $\{Z_{B_j}\}_{j=0}^m$, where Z_{B_j} is the fundamental

cycle of $B_j \subset E$. We define $\{B_j\}_j$ inductively as follows. For $j = 0$ take $B_0 = E$ hence $Z_{B_0} = Z_{min}$. By (7.2.1) $C \leq Z_{B_0} \leq Z_K$.

If $Z_{B_0} < Z_K$, we define $B_1 := |Z_K - Z_{B_0}|$. Then B_1 is connected. Indeed, assume that B_1 has more connected components $\{B_{1,t}\}_t$; then write $Z_K - Z_{B_0} = \sum_t D_t$ with $|D_t| = B_{1,t}$. Then the vanishing $\chi(Z_K - Z_{B_0}) = \chi(Z_{B_0}) = 0$ implies $\chi(D_t) = 0$ for every t. Then each $B_{1,t}$ supports a minimally elliptic cycle. This contradicts the uniqueness of the minimally elliptic cycle from Proposition 7.2.8. Next, using again $\chi(Z_{B_0}) = 0$, one gets $(Z_{B_0}, Z_K - Z_{B_0}) = 0$. Since $(Z_{B_0}, E_v) \leq 0$ for all E_v, it follows that for any $E_v \subset B_1$ the equality $(Z_{B_0}, E_v) = 0$ holds. Hence, by the non-degeneracy of the intersection form, $B_1 \neq B_0$. Moreover, for any $E_v \subset B_1$ one has: $(E_v, Z_K - Z_{B_0}) = (E_v, Z_K)$, hence B_1 supports a numerically Gorenstein singularity with canonical cycle $Z_K - Z_{B_0}$. This singularity is again elliptic. Indeed, $\min_{l \in L(B_1)_{>0}} \chi(l) \geq \min_{l \in L(B_0)_{>0}} \chi(l) = 0$, but $\chi(Z_K - Z_{B_0}) = 0$, hence $\min_{l \in L(B_1)_{>0}} \chi(l) = 0$. In particular, again by (7.2.1), $C \leq Z_{B_1} \leq Z_K - Z_{B_0}$.

Now we repeat the above arguments. If $Z_{B_1} < Z_K - Z_{B_0}$, we define $B_2 := |Z_K - Z_{B_0} - Z_{B_1}|$ and we verify that it supports an elliptic singularity with canonical cycle $Z_K - Z_{B_0} - Z_{B_1}$ and for any $E_v \subset B_2$ the vanishing $(E_v, Z_{B_1}) = 0$ holds. If we repeat this procedure, after a finite step we will obtain $Z_{B_m} = Z_K - Z_{B_0} - \cdots - Z_{B_{m-1}}$, i.e. the fundamental cycle and the canonical cycle of B_m coincides, hence by 7.2.15, B_m supports a minimally elliptic singularity with $C = Z_{B_m}$. In particular $B_m = |C|$.

The length of the elliptic sequence $\{Z_{B_j}\}_{j=0}^m$ is $m + 1$.

If we contract the connected exceptional divisor $B_j \subset \widetilde{X}$, we obtain a unique singular point; this will be denoted by $(\widetilde{X}/B_j, o_j)$ $(0 \leq j \leq m)$. It is also convenient to introduce the notations

$$C_t = \sum_{i=0}^{t} Z_{B_i} \quad \text{and} \quad C'_t = \sum_{i=t}^{m} Z_{B_i} \quad (0 \leq t \leq m).$$

By these notations, $C_0 = Z_{min}$, $C'_m = C$, and $C_m = C'_0 = Z_K$. The next lemma summarizes the immediate properties of the elliptic sequence.

Lemma 7.2.46

(a) $B_0 = E$, $B_1 = |Z_K - Z_{B_0}|$, $B_2 = |Z_K - Z_{B_0} - Z_{B_1}|$, $\ldots$, $B_m = |C|$; *each* B_j *is connected and* $B_j \gneqq B_{j+1}$. *Moreover,* $Z_{min} = Z_{B_0} \gneqq Z_{B_1} \gneqq \cdots \gneqq Z_{B_m} = C$.

(b) If $E_v \leq B_{j+1}$ *then* $(E_v, Z_{B_j}) = 0$ *for every* v *and* j. *In particular,* $(Z_{B_i}, Z_{B_j}) = (C_i, Z_{B_j})$ *for every* $0 \leq i < j \leq m$.

(c) $Z_K = \sum_{i=0}^m Z_{B_i}$.

(d) $(E_v, C'_t) = (E_v, Z_K)$ *for any* $E_v \subset |C'_t|$. *In other words,* C'_t *is the canonical cycle of* $|C'_t| = B_t$ *(i.e. of* $(\widetilde{X}/B_t, p_t)$*).*

(e) $C_t \in \mathscr{S}$.

Proof (a)–(d) follow from the construction. The proof of (e) is as follows. If $E_v \le B_t$ then $(E_v, Z_{B_j}) \le 0$ for any $j \le t$ by the definition of the fundamental cycle, hence $(E_v, C_t) \le 0$. If $E_v \not\le B_t$ then $(E_v, C_t) = (E_v, Z_K - C'_{t+1})$. Now, $(E_v, Z_K) \le 0$ (by the minimality of the resolution) and $(E_v, C'_{t+1}) \ge 0$ (because $|C'_{t+1}| \subset B_t$). □

Proposition 7.2.47 *Let us consider the supports $\{D_j\}_{j=0}^n$ defined in 7.2.38. Then $m = n$ and $B_j = D_j$. Hence all the properties of the 'general elliptic sequence' from the previous subsection 7.2.C will be true for $\{B_j\}_j$ and $\{Z_{B_j}\}_j$ as well.*

Proof Clearly, $D_0 = B_0$. Moreover, the continuation of both sequences is decided by the same criterion: by 7.2.46(b) one has $(C, Z_{B_0}) = (Z_K, Z_{B_0})$, hence $(Z_{B_0}, C) = 0 \Leftrightarrow Z_K > Z_{min}$. Next we show that $D_1 = B_1$. From 7.2.46(b) we get $B_1 \le D_1$. Since D_1 is connected, if $B_1 \ne D_1$, then there exists $E_v \le D_1$, $E_v \not\le B_1$, which intersects the support B_1. Then, $(E_v, Z_{B_0}) = 0$, but $(E_v, Z_K - Z_{B_0}) > 0$ (since B_1 is the support of $Z_K - Z_{B_0}$), hence $(E_v, Z_K) > 0$, a fact which contradicts with the minimality of the resolution. Then we proceed by induction. □

Properties (d) and (e) of 7.2.46 characterize the elliptic sequence.

Lemma 7.2.48 *Assume that (X, o) is a numerically Gorenstein elliptic singularity.*

(a) If a cycle $Z \ge 0$ satisfies $(E_v, Z - Z_K) \ge 0$ for every $E_v \subset |Z|$, then, in fact, $(E_v, Z - Z_K) = 0$ for every $E_v \subset |Z|$, and $Z \in \{C'_0, C'_1, \dots, C'_m, 0\}$.
(b) If $Z \in \mathscr{S}$ and $Z \le Z_K$ then $Z \in \{0, C_0, C_1, \dots, C_m\}$.

Part (b) above can be compared with (and is complemented by) (5) of 7.2.40.

Proof If $(E_v, Z - Z_K) \ge 0$, then $(Z, Z - Z_K) \ge 0$, hence $\chi(Z) \le 0$. This and the ellipticity guarantee that $\chi(Z) = 0$ and $(E_v, Z - Z_K) = 0$ for every $E_v \subset |Z|$. If $|Z| = B_0$ then necessarily $Z = Z_K$. If $|Z| \ne B_0$, then $(E_v, Z_K - Z) \le 0$ for $E_v \not\subset |Z|$ (since $(E_v, Z_K) \le 0$ and $(E_v, Z) \ge 0$). Hence $Z_K - Z \in \mathscr{S}$, thus $Z_K - Z \ge Z_{B_0}$, or $Z \le Z_K - Z_{B_0}$. In particular, $|Z| \subset B_1$. If $|Z| = B_1$, then Z is the canonical cycle C'_1 of B_1, otherwise, by the same argument as above $Z \le Z_K - Z_{B_0} - Z_{B_1}$, hence $|Z| \subset B_2$. Continuation of the process proves (a). For (b) apply (a) for $Z_K - Z$. □

Next we will list some cohomological properties of the computation sequence.

7.2.49 Since $C \le Z_{B_j} \le Z_{min}$,

$$h^1(\mathscr{O}_{Z_{B_j}}) = 1 \text{ hence } \chi(Z_{B_j}) = \chi(C_j) = \chi(C'_j) = 0 \text{ for all } 0 \le j \le m.$$

7.2.50 Fix two integers $0 \le i < j \le m$. Since Z_{B_i} is a fundamental cycle with $Z_{B_j} < Z_{B_i}$, there is a computation sequence which starts with Z_{B_j} and ends with Z_{B_i}. This has the form $z_0, \dots, z_t$ with $z_0 = Z_{B_j}$, $z_t = Z_{B_i}$, $z_{l+1} = z_l + E_{v(l)}$ and $(E_{v(l)}, z_l) > 0$ for $0 \le l \le t - 1$. In fact, since $h^1(\mathscr{O}_{Z_{B_i}}) = h^1(\mathscr{O}_{Z_{B_j}}) = 1$, cf.

7.2.49, for any $0 \le l \le t-1$ the curve $E_{v(l)}$ is smooth rational and $(E_{v(l)}, z_l) = 1$. We will call such a sequence a 'computation sequence which connects Z_{B_j} and Z_{B_i}'.

7.2.51 Let D be a cycle with $(E_v, D) = 0$ for any $E_v \subset B_i$, and consider a computation sequence $\{z_l\}_l$ which connects Z_{B_j} and Z_{B_i} (where $j > i$). In the exact sequence $0 \to \mathscr{O}_{E_{v(l)}}(-D - z_l) \to \mathscr{O}_{D+z_{l+1}} \to \mathscr{O}_{D+z_l} \to 0$ $(0 \le l \le t-1)$ the Chern number is $(E_{v(l)}, -D - z_l) = -1$, hence the natural map

$$H^*(\mathscr{O}_{D+Z_{B_i}}) \to H^*(\mathscr{O}_{D+Z_{B_j}}) \text{ is an isomorphism.}$$

For example, if $D = C_{i-1}$ and $j = m$, then

$$H^*(\mathscr{O}_{C_i}) \to H^*(\mathscr{O}_{C_{i-1}+C}) \ (1 \le i \le m) \text{ is an isomorphism.}$$

7.2.52 Note that $\chi(\mathscr{O}_C(-C_{i-1})) = 0$ $(1 \le i \le m)$. Consider also the exact sequence

$$0 \to \mathscr{O}_C(-C_{i-1}) \to \mathscr{O}_{C_{i-1}+C} \to \mathscr{O}_{C_{i-1}} \to 0.$$

Since C is 2-connected (see 7.2.16 and 7.2.17), by 7.2.17 $\mathscr{O}_C(-C_{i-1})$ is trivial if and only if it has a non-zero section, and this is happening if and only if $h^k(\mathscr{O}_C(-C_{i-1})) = 1$ for $k = 0, 1$. Hence (using 7.2.51 too) one has $h^1(\mathscr{O}_{C_{i-1}}) \le h^1(\mathscr{O}_{C_i}) \le h^1(\mathscr{O}_{C_{i-1}}) + 1$ for $1 \le i \le m$. Since $h^1(\mathscr{O}_{C_0}) = 1$, one obtains (cf. [767, (3.9)], see also 7.2.42):

$$h^1(\mathscr{O}_{C_i}) \le i+1 \text{ for } 0 \le i \le m, \text{ hence } p_g = h^1(\mathscr{O}_{C_m}) \le m+1 = \ell_{seq}. \qquad (7.2.2)$$

7.2.E The 'second' Elliptic Sequence, General Case

We construct a new elliptic sequence following [456]. In the numerically Gorenstein case we recover the sequence from 7.2.D, however, in general, the output is different from the one from 7.2.C. For motivations (and gained advantages and differences) see 7.2.54 and 7.2.63. ♣

7.2.53 In this section we will assume that the link is a rational homology sphere.

7.2.54 Motivation As we already noticed in 7.2.47, the general construction of the elliptic sequence from 7.2.C specializes to the construction from subsection 7.2.D, valid in the numerically Gorenstein case. However, the sequence in the numerically Gorenstein case has several advantages: it has several additional properties, which are not transparent at all in the non–numerically Gorenstein situation.

Let us point out some differences. Note that in both cases, the length of the elliptic sequence is an upper bound for p_g. However, in the numerically Gorenstein case, by [579], any numerically Gorenstein topological type supports a Gorenstein analytic structure. Furthermore, as we will see in the next subsections, the Gorenstein property is equivalent to $p_g = \ell_{seq}$ (that is, the universal upper bound $p_g \le \ell_{seq}$ becomes an identity) whenever the link is a rational homology sphere. In particular, in the numerically Gorenstein case, there exists an analytic structure (X, o) supported on the minimal graph such that $p_g(X, o)$ is exactly ℓ_{seq}. The existence of an analytic structure with similar property is not clear at all in the non–numerically Gorenstein case, at least not via the elliptic sequence constructed in 7.2.C.

The primary reason for this difference is that while in the numerically Gorenstein case the elliptic sequence has a very subtle relationship with the canonical cycle (see its construction in 7.2.45, or 7.2.46*(c)*), in the construction 7.2.C, or in the list of properties in 7.2.40, such a relationship is not visible at all.

The goal of the present section is to construct a new elliptic sequence, which eliminates such deficiencies of the 'old' sequence. The 'old' and 'new' sequences are different (though they coincide for numerically Gorenstein graphs), but their lengths are the same. Moreover, both of them have their own specific universal properties (this is the reason, why we kept both of them, cf. 7.2.63). We follow [456].

7.2.55 In the next lemma (X, o) is an arbitrary (not necessarily elliptic) singularity. If $\widetilde{X}$ is a minimal resolution then $Z_K \in \mathscr{S}'$, cf. 6.3.3. Let $[Z_K] \in H$ be its class, and $s_{[Z_K]} \in L'$ be its unique minimal lift into $\mathscr{S}'$, cf. 6.6.15. Thus $Z_K - s_{[Z_K]} \in L_{\ge 0}$.

Lemma 7.2.56 *If $Z_K = s_{[Z_K]}$ then $p_g = 0$. If $Z_K > s_{[Z_K]}$ then $p_g = h^1(\mathscr{O}_{Z_K - s_{[Z_K]}})$. Moreover, $h^1(\widetilde{X}, \mathscr{L}) = h^1(Z_K - s_{[Z_K]}, \mathscr{L})$ for any $\mathscr{L} \in \mathrm{Pic}(\widetilde{X})$ with $c_1(\mathscr{L}) \in -\mathscr{S}'$.*

Proof By 6.4.7 if $\lfloor Z_K \rfloor = 0$ then $p_g = 0$. Otherwise, again by 6.4.7, $h^1(\mathscr{O}_{\lfloor Z_K \rfloor}) = p_g$. Next, consider the computation sequence from Proposition 6.6.3 applied for $l' = r_{[Z_K]}$, connecting $r_{[Z_K]}$ to $s(r_{[Z_K]}) = s_{[Z_K]}$. By induction we prove that $h^1(\mathscr{O}_{Z_K - z_i}) = p_g$. For $i = 0$ we just verified it, since $Z_K - r_{[Z_K]} = \lfloor Z_K \rfloor$. Then use the cohomological exact sequence associated with $0 \to \mathscr{O}_{E_{v(i)}}(-Z_K + z_{i+1}) \to \mathscr{O}_{Z_K - z_i} \to \mathscr{O}_{Z_K - z_{i+1}} \to 0$ and the vanishing $h^1(\mathscr{O}_{E_{v(i)}}(-Z_K + z_{i+1})) = 0$.

More generally, $h^1(\widetilde{X}, \mathscr{L}) = h^1(Z_K - z_i, \mathscr{L})$ for any i by similar argument. □

7.2.57 Assume that $\widetilde{X}$ is the minimal resolution of an elliptic singularity. Assume that $Z_K \notin L$. Since the resolution is minimal, $Z_K \in \mathscr{S}'$, hence $Z_K \ge s_{[Z_K]}$. By Lemma 7.2.56 $Z_K > s_{[Z_K]}$. We will use the following notations: $B_{-1} := E$, $Z_{B_{-1}} := s_{[Z_K]}$ and $B_0 := |Z_K - s_{[Z_K]}|$. (Note that $Z_{B_{-1}} \in L' \setminus L$.)

Lemma 7.2.58

(a) B_0 *is connected,* $|C| \le B_0 \lneq E$, *and* $(E_v, Z_{B_{-1}}) = 0$ *for any* $E_v \le B_0$.
(b) B_0 *supports a numerically Gorenstein elliptic topological type with canonical cycle* $Z_K - s_{[Z_K]}$.

Proof

(a) Write $l := Z_K - s_{[Z_K]}$. Then $\chi(s_{[Z_K]}) = \chi(Z_K - l) = \chi(l)$. Since (X, o) is elliptic $\chi(s_{[Z_K]}) = \chi(l) \ge 0$ (†). Also, $(s_{[Z_K]}, l) \le 0$ since $s_{[Z_K]} \in \mathscr{S}'$ (‡). On the other hand, $0 = \chi(Z_K) = \chi(l + s_{[Z_K]}) = \chi(l) + \chi(s_{[Z_K]}) - (l, s_{[Z_K]})$. Then by (†) and (‡) the expressions from the right hand side are ≥ 0, hence necessarily $\chi(l) = (l, s_{[Z_K]}) = 0$. If l has more connected components, say $\cup_i l_i$, then $\chi(l_i) = 0$ for all i, hence each l_i contains/dominates a minimally elliptic cycle (cf. [363]), a fact which contradicts the uniqueness of the minimally elliptic cycle. Hence $|l| = B_0$ is connected and $|C| \le B_0$. Furthermore, $(l, s_{[Z_K]}) = 0$ and $|s_{[Z_K]}| = E$ show that $|l| \ne E$.
(b) $|C| \le B_0 \lneq E$ shows that $\min_{|l| \le B_0,\, l>0} \chi(l) = 0$, hence B_0 supports an elliptic topological type. Moreover, from $(l, s_{[Z_K]}) = 0$ we read that for any E_v from the support of l one has $(E_v, s_{[Z_K]}) = 0$, hence $(E_v, Z_K - s_{[Z_K]}) = (E_v, Z_K)$, hence $Z_K - s_{[Z_K]} \in L$ is the canonical cycle on B_0. □

Then, as a continuation of the sequence, starting from B_0 and from its integral canonical class $Z_K - s_{[Z_K]}$, we construct the sequence $\{Z_{B_j}\}_{j=0}^m$ as in the numerically Gorenstein case. We say that the elliptic sequence $\{Z_{B_j}\}_{j=-1}^m$ has length $m+1$ and 'pre–term' $Z_{B_{-1}} = s_{[Z_k]} \in L'$.

In order to have a uniform notation, in the numerically Gorenstein case we set $Z_{B_{-1}} := 0$ (which, in fact, is $s_{[Z_K]}$). In all cases (see also 7.2.46(b))

$$(E_v, Z_{B_j}) \text{ for any } E_v \le B_{j+1} \quad (-1 \le j < m). \tag{7.2.3}$$

The analogue of 7.2.40(6) is the following 'linearity' property of the structure of Γ.

Lemma 7.2.59 *Consider an elliptic graph* Γ *and the supports* $B_{-1}, B_0, \dots, B_m$. *Assume that we can glue to the graph a new vertex* v_{new} *by an edge* (v, v_{new}), $v \in \mathscr{V}$, *such that the new graph* Γ_{new} *is still elliptic. Then the* E_v*–multiplicity of the fundamental cycle* Z_{min} *is 1 and* $v \notin B_1$. *Furthermore, if* Γ *is numerically Gorenstein, then* v *is necessarily an end-vertex, and the* E_v*–multiplicity of* Z_K *is 1 too.*

Proof Suppose that $v \in B_1$. Then we have that the multiplicity of E_v in $Z_{B_0} + Z_{B_1}$ is at least 2, but $\chi(Z_{B_0} + Z_{B_1}) = 0$, cf. (7.2.3). Therefore, $\chi(Z_{B_0} + Z_{B_1} + E_{new}) < 0$, which contradicts the ellipticity of the large graph. By Laufer's algorithm [359] there exists a computation sequence of $Z_{min}(\Gamma_{new})$ such that one of its terms is $Z_{min} = Z_{min}(\Gamma)$ while the next one is $Z_{min} + E_{new}$. Since $\chi(Z_{min}) = \chi(Z_{min} + E_{new}) = 0$, we get that the coefficient $m_{E_v}(Z_{min})$ of E_v in Z_{min} is 1. In the numerically Gorenstein case, since $E_v \not\le B_1$ we get that that $m_{E_v}(Z_K) = 1$ too. By the

adjunction formula then v is either an end-vertex (as in the statement) or it has two neighbours both with multiplicity 1. But this last case would generate an infinite string, all members with multiplicity one, which cannot happen. □

7.2.60 The Cycles C_t and C'_t Set $C_t := \sum_{i=-1}^{t} Z_{B_i}$ and $C'_t := \sum_{i=t}^{m} Z_{B_i}$, $-1 \leq t \leq m$. E.g. $C_m = Z_K$ and, in general, C'_j is the canonical cycle of B_j. Furthermore, $\chi(Z_{B_j}) = \chi(C_j) = \chi(C'_j) = 0$. The next lemma extends 7.2.48.

Lemma 7.2.61 *Assume that $l' \in S'$, $[l'] = [Z_K]$ and $l' \leq Z_K$. Then $l' \in \{C_{-1}, C_0, \cdots, C_m\}$.*

Proof Since $0 \leq l' - Z_{B_{-1}} \leq Z_K - Z_{B_{-1}}$ and from (7.2.3) $l' - Z_{B_{-1}} \in \mathscr{S}(B_0)$, the statement reduces to the numerically Gorenstein case. □

Remark 7.2.62 The supports $\{B_i\}_{i=0}^m$ satisfy another universality property as well. Assume first that the graph is numerically Gorenstein, and let $I \subset \mathscr{V}$, $I \neq \emptyset$, such that I supports a numerically Gorenstein (connected) subgraph. Then I is one of the supports $\{B_i\}_{i=0}^m$.

Indeed, suppose, that $E_I := \sum_{i \in I} E_v \neq B_0$. Then, by induction, it is enough to prove that $E_I \leq B_1$. Let the canonical cycle on I be $Z \in L$. Then $(Z_K - Z, E_v) = 0$ for every $E_v \subset |Z|$. Otherwise, if $E_v \not\subset |Z|$, we have $(Z_K, E_v) \leq 0$ and $(Z, E_v) \geq 0$, so $(Z_K - Z, E_v) \leq 0$. This means, that $Z_K - Z \in \mathscr{S}'$, $Z_K - Z > 0$, $Z_K - Z \in L$. These imply that $Z_K - Z \geq Z_{min}$, hence $Z \leq C'_1$ and $E_I = |Z| \leq B_1$.

Assume next that the graph is not numerically Gorenstein. Then we claim that the support of any numerically Gorenstein (connected) subgraph belongs to $\{B_i\}_{i=0}^m$. First we show that the largest numerically Gorenstein subgraph is supported by B_0. (Then the rest follows from the previous paragraph.) Indeed, if I is its support and Z is the canonical cycle on this support, then similarly as above, $Z_K - Z \in \mathscr{S}'$, hence $Z_K - Z \geq s_{[Z_K]}$. This reads as $Z \leq Z_K - s_{[Z_K]}$, or $|Z| \leq B_0$.

Remark 7.2.63 Both elliptic sequences $\{D_j\}_j$ and $\{B_j\}_j$ have some geometric universal properties. For $\{D_j\}_j$ see 7.2.40(5), for $\{B_j\}_j$ see 7.2.61 and 7.2.62.

Note also that in the proof of $\text{Path}^i \leq \ell_{seq}$ in 7.2.42 the fact that the computation sequence is formed by fundamental cycles was deeply exploited (in the second case $Z_{B_{-1}} = s_{[Z_K]}$ is not even an integral cycle). On the other hand, the choice $Z_{B_{-1}} = s_{[Z_K]}$ solves craftily the structure of numerically Gorenstein subgraphs and the needed relationship with Z_K.

7.2.64 Next we compare the length of $\{D_j\}_{j=0}^n$ and $\{B_j\}_{j=-1}^m$. In the proof we will use Theorems 7.2.81 and 7.2.85, which imply that if we fix an elliptic numerically Gorenstein minimal resolution graph Γ, then (X, o) is Goresntein if and only if $p_g = m + 1$ (and the link is a rational homology sphere). Recall also that any numerically Gorestein topological type carries a Gorenstein structure [579].

Theorem 7.2.65 *Fix any (not necessarily numerically Gorenstein) elliptic graph associated with a minimal resolution. Then*

(a) there exists E_v in the support of B_0 with $(E_v, Z_{min}) < 0$;
(b) $m + 1 = n + 1$;
(c) there exists an analytic type for which $p_g = n + 1$. This happens e.g. when (X_0, o_0) is chosen to be Gorenstein.

Proof

(a) Although the statement is topological, it is convenient to fix a special analytic structure on (X, o), which produces a fast proof. Since $Z_{min} \in \mathcal{S}$, there exists an analytic structure for which this cycle is realized as a divisor of $f \circ \phi$ for a certain function f [567]. Assume that $(E_v, Z_{min}) = 0$ for any $E_v \subset B_0$. Then the strict transforms of $\{f = 0\}$ do not intersect B_0, hence $\mathcal{O}_{\widetilde{X}}(-Z_{min})|_{Z_K(B_0)}$ is trivialized by f. Therefore, using 7.2.56 for the structure sheaf, we get that $h^1(Z_k - s_{[Z_K]}, \mathcal{O}_{\widetilde{X}}(-Z_{min})) = h^1(Z_k - s_{[Z_K]}, \mathcal{O}_{Z_k - s_{[Z_K]}}) = p_g$. On the other hand, by 7.2.56 again, for $\mathcal{O}_{\widetilde{X}}(-Z_{min})$ we get $h^1(Z_k - s_{[Z_K]}, \mathcal{O}_{\widetilde{X}}(-Z_{min})) = h^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-Z_{min}))$. In particular, $h^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-Z_{min})) = p_g$.

Now consider the long cohomology exact sequence of $0 \to \mathcal{O}_{\widetilde{X}}(-Z_{min}) \to \mathcal{O}_{\widetilde{X}} \to \mathcal{O}_{Z_{min}} \to 0$, and using the fact that $H^0(\mathcal{O}_{\widetilde{X}}) \to H^0(\mathcal{O}_{Z_{min}}) = \mathbb{C}$ is onto, we get that $h^1(\mathcal{O}_{\widetilde{X}}(-Z_{min})) = p_g - h^1(\mathcal{O}_{Z_{min}}) = p_g - 1$. This leads to a contradiction.

(b) The direction $m + 1 \leq n + 1$ and part *(c)* follow from the fact that there exists an analytic structure with $p_g = m + 1$. Indeed, fix a Gorenstein structure on B_0 (cf. [579]), and construct $\widetilde{X}$ otherwise arbitrarily (that is, extend $\widetilde{X}_0$ arbitrarily by analytic plumbing). Then, via 7.2.85 $m+1 = p_g$. Hence, by 7.2.42, $m+1 = p_g \leq n + 1$.

We prove $m+1 \geq n+1$ by induction on n. The $n = 0$ case is trivial. Next we assume that $n > 0$ and that the statement holds for singularities with 'D–length' n.

By part *(a)* $B_0 \not\leq D_1$ (†). Next, denote by $B_0(D_1)$ the 'B_0 support' of the elliptic graph associated with D_1. Since $B_0(D_1)$ is included in D_1, and it is a connected numerically Gorenstein support, by 7.2.62 $B_0(D_1) \leq B_0$. But, $B_0(D_1) \leq D_1$ and (†) show that this inclusion should be strict: $B_0(D_1) \lneq B_0$.

Let us denote by $maxp_g(D)$ the maximum p_g which can be realized by different analytic structures supported on a connected support/subgraph D.

Now, $maxp_g(D_1) = maxp_g(B_0(D_1))$ by Lemma 7.2.56. Since $B_0(D_1) \leq B_0$ we have $maxp_g(B_0(D_1)) \leq maxp_g(B_0)$. However, since B_0 is a numerically Gorenstein support, and its maximal p_g is realized by a Gorenstein structure, which has the property that its cohomological cycle is exactly its canonical cycle with support B_0, any smaller support has strictly smaller $maxp_g$. Since $B_0(D_1) \lneq B_0$, we get that $maxp_g(B_0(D_1)) < maxp_g(B_0) = m + 1$.

On the other hand, the D-length of D_1 is n (since the D–elliptic sequence of D_1 is $\{D_1, \ldots, D_n\}$). Hence for D_1 the inductive step works. In particular,

$maxp_g(D_1) = n$. This combined with the statements from the previous paragraph gives $m+1 = maxp_g(B_0) > maxp_g(B_0(D_1)) = maxp_g(D_1) = n$. That is, $m+1 \geq n+1$. □

Corollary 7.2.66 *Assume that the link is a rational homology sphere and the minimal resolution is good. Then* $m+1 = n+1 = \mathrm{Path}^i$. *Therefore, the general topological upper bound* $p_g \leq \mathrm{Path}^i$ *for* p_g *in the case of elliptic singularities is sharp.*

Proof From (6.8.14) and 7.2.42 one has $p_g \leq \mathrm{Path}^i \leq 1+n$ for any analytic structure. On the other hand, there exists an analytic structure with $p_g = n+1$. □

Example 7.2.67 (The Elliptic Sequence of Elliptic Kulikov Singularities) Consider the notations of 7.2.14, and write $c := \min_i\{c_i\}$.

Assume first that (X, o) is Gorenstein, hence $c_i = c$ for all i. Then the elliptic sequence can be determined as follows: $B_0 = E$, and for any $0 < j < c$ we obtain B_j by eliminating those components E_v which have $d(v, F) \leq j$. Each B_j supports a Kulikov singularity with $c(B_j) = c - j$, hence the elliptic sequence of B_{c-1} has length 1. In particular, $\ell_{seq} = c$. From definitions it also follows that $\min_{v\in\mathscr{V}_Z} d(v, |C|)+1 = c$. From 7.2.14 we have $p_g \geq c$, while from 7.2.42 $p_g \leq c$. Hence in this case $p_g = c = \ell_{seq} = 1 + \min_{v\in\mathscr{V}_Z} d(v, |C|)$.

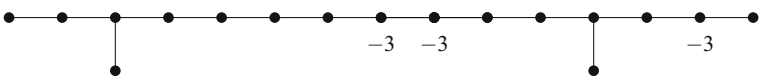
Assume next that the integers c_i are not all the same. Then consider the connected subgraph, which contains those vertices v with $d(v, |C|) \leq c-1$. It determines the support B_0 (the other terms of the sequence are determined as above). Let $\widetilde{X}_{Gor}$ be a small neighbourhood of these components in $\widetilde{X}$. Then $\widetilde{X}_{Gor}$ is the resolution of a Gorenstein Kulikov singularity (X_{Gor}, o) with the same c. Then $c = p_g(X_{Gor}, o) \leq p_g(X, o) \leq \ell_{seq} = c$. Therefore, for any elliptic Kulikov singularity

$$p_g = \mathrm{Path}^i = \ell_{seq} = 1 + \min_{v\in\mathscr{V}_Z} d(v, |C|) = c.$$

It is instructive to determine the supports D_j as well: $D_0 = E$, and for any $0 < j < c$ we obtain D_j by eliminating those components E_v, which have $d(v, F) \leq j$.

Example 7.2.68 In several cases one has $p_g = \mathrm{Path}^i = \ell_{seq} = 1 + \min_{v\in\mathscr{V}_Z} d(v, |C|)$ (see e.g. the Kulikov singularities 7.2.67 above). Nevertheless, there are also examples when the 'distance bound' is weak: $\ell_{seq} < 1 + \min_{v\in\mathscr{V}_Z} d(v, |C|)$.

Take for example the next elliptic numerically Gorenstein graph, whose unmarked vertices are rational (-2) curves. Then $\min_{v\in\mathscr{V}_Z} d(v, |C|) = 6$ and $\ell_{seq} = 4$.



Example 7.2.69 If Γ is minimally elliptic then $\ell_{seq} = 1$. If Γ is numerically Gorenstein and $\ell_{seq} = 1$ then $Z_K = Z_{min}$ and Γ is necessarily minimally elliptic (see 7.2.D). However, in general, $\ell_{seq} = 1$ does not imply that Γ is minimally elliptic. Indeed, non–numerically Gorenstein elliptic singularities with $\ell_{seq} = 1$ exist (replace e.g. the (-7) curve from the graph of 7.2.99 with a (-8) curve).

Remark 7.2.70 Assume that the minimal resolution of an elliptic singularity (with rational homology sphere link) is good. Let Γ be the corresponding graph with elliptic sequence $B_{-1}, \ldots, B_m$, cf. 7.2.E. In [456] the following facts are proved:

(i) Let $\widetilde{X}(B_0)$ be a small convenient neighbourhood of B_0 in $\widetilde{X}$. Then if $\widetilde{X}$ satisfies the WECC then $\widetilde{X}(B_0)$ is the resolution of a Goresntein singularity. (In particular, in the numerically Gorenstein case, WECC $\Rightarrow$ Gorenstein.)

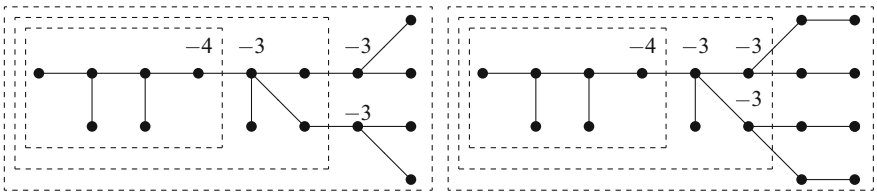
(ii) Γ supports an analytic structure with WECC if and only if it satisfies the 'monomial condition' (which allows one to construct a splice quotient analytic structure supported on Γ, cf. 5.4.8).

(iii) Assume that Γ supports an analytic structure with WECC. Then any such analytic structure satisfies ECC too.

(iv) A new topological characterization (in the terms of the supports $\{B_j\}_{j=-1}^m$) of the elliptic graphs, which might support analytic structures satisfying WECC (or, equivalently, ECC) can be formulated as follows. Indeed, one has the following 'Extension Criterion of the elliptic sequence'. *There exists a singularity with minimal resolution $\widetilde{X}$ with dual graph Γ, and which satisfies WECC, if and only if for every $0 \le i \le m$ and for any vertex $v \in B_i \setminus B_{i+1}$, v has at most one neighbour in B_{i-1}.*

(v) The numerically Gorenstein graph Γ from 5.4.5 is elliptic, but it does not satisfy the 'monomial condition'.

Example 7.2.71 The next pair of graphs exemplify part (iv). The unmarked decorations are (-2)'s. The graphs are numerically Gorenstein with $m = 2$. The dash–boxes show the supports $B_2 \subset B_1 \subset B_0$. The first graph satisfies the 'Extension Criterion', while the second one not.



7.2.F *Maximally Elliptic Numerically Gorenstein Germs*

Singularities with $p_g = 1$ (without Gorenstein assumption), or even Gorenstein singularities with $p_g = 2$ (or $p_g \geq 2$) cannot be characterized topologically. In this second case, one can easily construct pairs of hypersurface singularities with the same resolution graph but different p_g (see e.g. 6.8.3). Nevertheless, the main message of the next subsections is that for elliptic Gorenstein singularities with $H_1(E, \mathbb{Z}) = 0$ the Artin–Laufer program (of rational/minimally elliptic germs) can be continued: the geometric genus and the Hilbert–Samuel function can be read from the graph. We follow [473]. ♣

7.2.72 In this and the next section we assume that (X, o) is a numerically Gorenstein elliptic singularity and $\widetilde{X}$ is a minimal resolution.

We start with several characterizations of the 'extreme property' $p_g = m + 1 = \ell_{seq}$. Following S.S.-T. Yau we call those elliptic singularities which satisfy this property 'maximally elliptic'. For results of Yau regarding such germs consult [767]; see also [689]. Here we follow [473], where all the proofs can be found.

Theorem 7.2.73 (First Characterization of $p_g = m + 1$) *Assume that (X, o) is a numerically Gorenstein elliptic singularity. Then the following facts are equivalent:*

(a) $p_g = m + 1$;
(b) $h^1(\mathcal{O}_{C_j}) = j + 1$ *for all* $0 \leq j \leq m$;
(c) The line bundles $\mathcal{O}_C(-C_j)$ *are trivial in* $Pic(C)$, *and* $H^0(\mathcal{O}_{C_j+C}) \to H^0(\mathcal{O}_{C_j})$ *is surjective for any* $0 \leq j \leq m - 1$;
(d) $h^1(\mathcal{O}(-C_j)) = m - j$ *for all* $0 \leq j \leq m$;
(e) $h^1(\mathcal{O}_{C'_j}) = m - j + 1$ *for all* $0 \leq j \leq m$;
(f) For any $0 \leq j \leq m - 1$, *there exists* $f_j \in H^0(\widetilde{X}, \mathcal{O}(-C_j))$, *such that for any* $E_v \subset B_{j+1}$ *the vanishing order of* f_j *on* E_v *is exactly the multiplicity of* C_j *at* E_v;
(g) The line bundles $\mathcal{O}_{C'_{j+1}}(-C_j)$ *are trivial for* $0 \leq j \leq m - 1$;
(h) The line bundles $\mathcal{O}_{C'_{j+1}}(-Z_{B_j})$ *are trivial for* $0 \leq j \leq m - 1$.

Corollary 7.2.74 *Assume that (X, o) is an elliptic Gorenstein singularity.*

(a) Then $p_g = m + 1$ *if and only if the line bundles* $\mathcal{O}_{C'_{j+1}}(-Z_{B_j})$ *are trivial for all* $1 \leq j \leq m - 1$.
(b) If $p_g = m + 1$, *then* $h^1(\widetilde{X}, \mathcal{O}(-lZ_{min})) = h^1(Z_K, \mathcal{O}_{Z_K}(-lZ_{min})) = p_g - 1$ *for any* $l \geq 1$.

Example 7.2.75 Example 6.8.3 (with $m = 2$) shows that without the assumption $H^1(E, \mathbb{Z}) = 0$ the property $p_g = m + 1$ is not always true, even if we deal with hypersurface singularities.

Definition 7.2.76 In the sequel, we call the line bundles $\mathscr{O}_{C'_{j+1}}(-Z_{B_j})$, $0 \le j \le m-1$, 'obstruction line bundles'.

By Theorem 7.2.73, the triviality of these line bundles is of main interest.

7.2.77 For C, the minimally elliptic cycle, from 7.2.17 one has

$$\begin{cases} \mathscr{L} \in \mathrm{Pic}^0(C) \text{ is trivial} \Leftrightarrow H^0(C, \mathscr{L}) \neq 0; \\ \text{moreover, if } \mathscr{L} \text{ is trivial then } h^0(C, \mathscr{L}) = 1. \end{cases}$$

7.2.78 In the case of an elliptic singularity, for any $0 \le j \le m$, $\mathrm{Pic}^0(Z_{B_j})$ is as simple as $\mathrm{Pic}^0(C)$. Indeed, consider the exponential cohomology sequences of the cycles Z_{B_j} and C. Then the natural maps $H^1(|Z_{B_j}|, \mathbb{Z}) \to H^1(|C|, \mathbb{Z})$ and $H^1(\mathscr{O}_{Z_{B_j}}) \to H^1(\mathscr{O}_C)$ are isomorphisms (cf. 7.2.9(d) and 7.2.51). Hence $\mathrm{Pic}^0(Z_{B_j}) \to \mathrm{Pic}^0(C)$ is an isomorphism as well. Moreover, if we consider a computation sequence $\{Z_l\}_l$ which connects $C = Z_{B_m}$ and Z_{B_j}, cf. 7.2.50, then from the exact sequences $0 \to \mathscr{O}_{E_{v(l)}}(-Z_l) \otimes \mathscr{L} \to \mathscr{L}|Z_{l+1} \to \mathscr{L}|Z_l \to 0$ we obtain that $H^0(Z_{B_j}, \mathscr{L}) = H^0(C, \mathscr{L}|_C)$. Therefore, for any line bundle $\mathscr{L} \in \mathrm{Pic}^0(Z_{B_j})$:

$$\begin{cases} h^0(Z_{B_j}, \mathscr{L}) \le 1, \text{ and} \\ h^0(Z_{B_j}, \mathscr{L}) = 1 \text{ if and only if } \mathscr{L} \text{ is trivial.} \end{cases}$$

Theorem 7.2.79 *Assume that (X, o) is elliptic, numerically Gorenstein and satisfies $p_g = m+1$. Fix an integer $0 \le j \le m$. Then the following facts are equivalent.*

(a) $\mathscr{L} \in \mathrm{Pic}^0(C'_j)$ is trivial;
(b) $h^0(C'_j, \mathscr{L}) \ge m - j + 1$.

Furthermore, if $\mathscr{L}$ is trivial, then $h^0(C'_j, \mathscr{L})$ is exactly $m - j + 1$.

Regarding the line bundles $\mathscr{L} := \mathscr{O}_{C'_j}(-C_{j-1}) \in \mathrm{Pic}(C'_j)$, whenever (X, o) is Gorenstein, the condition 7.2.79(*b*) can be replaced with an equivalent condition formulated in terms of $h^0(\mathscr{O}_{C'_j}(-C_{j-1})) = h^1(\mathscr{O}_{C'_j}) = p_g(\widetilde{X}/B_j, o_j)$.

Corollary 7.2.80 *Assume that (X, o) is an elliptic Gorenstein singularity, and we fix an integer $0 \le j \le m$. Then $h^1(\mathscr{O}_{C'_j}) = m - j + 1$ if and only if $\mathscr{L} = \mathscr{O}_{C'_j}(-C_{j-1})$ is trivial in $\mathrm{Pic}^0(C'_j)$.*

For example, for $j = m$ we obtain that $\mathscr{O}_C(-C_{m-1})$ is trivial in $\mathrm{Pic}^0(C)$ (since $h^1(\mathscr{O}_C) = 1$ is automatically satisfied). For $j = m-1$ one has: $p_g((\widetilde{X}/B_{m-1}, o_{m-1})) = 2$ if and only if $\mathscr{O}_{C'_{m-1}}(-C_{m-2})$ is trivial in $\mathrm{Pic}^0(C'_{m-1})$.

Theorem 7.2.81 (Second Characterization of $p_g = m + 1$) *Assume that (X, o) is a numerically Gorenstein elliptic singularity. Consider its minimal resolution $\widetilde{X} \to X$ and the singularities $(\widetilde{X}/B_j, o_j)$ obtained by the contraction of B_j, $0 \le j \le m$. Then (X, o) satisfies the 'extreme property' $p_g = m + 1$ if and only if the singularities $(\widetilde{X}/B_j, o_j)$ are Gorenstein for all $0 \le j \le m - 1$.*

(Notice that $(\widetilde{X}/B_m, o_m)$ is automatically Gorenstein by Theorem 7.2.15).

Example 7.2.82 Assume that (X, o) is numerically Gorenstein and elliptic.

(a) If $m = 1$ then $p_g \le 2$, and

$$p_g = 2 \Leftrightarrow \mathscr{O}_C(-Z_{min}) \text{ is trivial } \Leftrightarrow (X, o) \text{ is Gorenstein.}$$

Examples with $p_g = 1$ exist (cf. 7.2.H); in fact, in the generic case $p_g = 1$.

(b) Assume that $m = 2$. Then we have two obstruction line bundles. For the generic analytic structure $p_g = 1$ as above. If (X, o) is Gorenstein, then $p_g = h^1(\mathscr{O}_{Z_K}) > h^1(\mathscr{O}_C) = 1$, hence $p_g \ge 2$. From 7.2.74, $p_g = 3$ if and only if $\mathscr{O}_C(-Z_{B_1})$ is trivial. From 7.2.80 $\mathscr{O}_C(-Z_{min} - Z_{B_1})$ is trivial, hence if (X, o) is Gorenstein, then:

$$p_g = 3 \Leftrightarrow \mathscr{O}_C(-Z_{min}) \text{ is trivial} \Leftrightarrow (\widetilde{X}/B_1, o_1) \text{ is Gorenstein.}$$

Now we are ready to state the main result of this subsection, which guarantees that some of the obstruction line bundles have finite order under certain conditions.

Theorem 7.2.83 *Consider the minimal resolution of an elliptic Gorenstein singularity (X, o) with elliptic sequence $\{Z_{B_j}\}_{j=0}^m$ ($m \ge 1$), and fix an integer $1 \le k \le m$.*

Assume that for $k + 1 \le j \le m$ the obstruction line bundles $\mathscr{O}_{C'_j}(-Z_{B_{j-1}}) \in \mathrm{Pic}^0(\mathscr{O}_{C'_j})$ are trivial. (If $k = m$ then this assumption is vacuous.)

Then $\mathscr{O}_{C'_k}(-Z_{B_{k-1}}) \in \mathrm{Pic}^0(\mathscr{O}_{C'_k})$ has finite order. In fact, for any $0 \le r \le k - 1$, there exist integers $1 \le l_r \le k - r$ such that the line bundle $\mathscr{O}_{C'_k}(l_0 \cdots l_r\ Z_{B_r})$ is trivial.

In particular, the order of $\mathscr{O}_{C'_k}(-Z_{B_{k-1}})$ in $\mathrm{Pic}^0(\mathscr{O}_{C'_k})$ is not greater than k!.

Example 7.2.84 (Cf. 7.2.82(b)) Assume that (X, o) is elliptic Gorenstein, and $m = 2$. Recall that $\mathscr{O}_C(-Z_{B_1} - Z_{min}) \approx \mathscr{O}_C$. The above theorem (for $m = k = 2$) says that the order of $\mathscr{O}_C(-Z_{min})$ divides 2, hence:

$$\mathscr{O}_C(-Z_{min}) \text{ is trivial} \Leftrightarrow p_g = 3; \qquad \mathscr{O}_C(-Z_{min}) \text{ has order } 2 \Leftrightarrow p_g = 2.$$

The main consequence of the above theorem.

Theorem 7.2.85 *Assume that (X, o) is an elliptic Gorenstein singularity with $H^1(E, \mathbb{Z}) = 0$. Then $p_g = m + 1 =$ the length of the elliptic sequence in the minimal resolution of (X, o). In particular, p_g is a topological invariant.*

Proof If $H^1(E,\mathbb{Z}) = 0$, then by the exponential exact sequence $\mathrm{Pic}^0(D) \approx H^1(\mathscr{O}_D)$ for any positive cycle D; in particular, it is torsion free. Therefore, Theorem 7.2.83, as an inductive step, proves that all the obstruction line bundles are trivial. Hence $p_g = m + 1$ by Theorem 7.2.73. □

Corollary 7.2.86 *Consider the minimal resolution graph of a numerically Gorenstein elliptic singularity with rational homology sphere link. Then* $\mathrm{Path}^i = \ell_{seq}$.

Proof For any analytic realization one has $p_g \le \mathrm{Path}^i \le \ell_{seq}$, cf. 6.8.36 and Theorem 7.2.42. Moreover, from [579], this topological type supports a Gorenstein structure. For this analytic type, by Theorem 7.2.85, $\ell_{seq} = p_g$. □

Remark 7.2.87 If $m \ge 1$, then in Theorem 7.2.85 both assumptions are necessary. Without Gorenstein condition p_g generically is 1 ([363, page 1279] or 7.2.98 here). Moreover, Example 6.8.3 shows that $p_g < m + 1$ can occur if $H^1(E,\mathbb{Z}) \neq 0$.

Remark 7.2.88 For a Gorenstein singularity (X, o) with $H^1(E,\mathbb{Z}) = 0$ the following facts are equivalent. (Notice that the right hand side is completely topological.)

$$p_g = 2 \;\Leftrightarrow\; \chi(Z_{min}) = 0 \text{ and } Z_K = Z_{min} + C.$$

Indeed, if (X, o) is Gorenstein with $p_g = 2$, then $h^1(Z_{min})$ cannot be zero (this would imply $p_g = 0$); and cannot be ≥ 2 (this would imply via 6.4.19 that $Z_{min} = Z_K$, which characterizes the minimally elliptic singularities). Hence $h^1(Z_{min}) = 1$. Therefore (X, o) is elliptic. The rest follows from the above results.

For particular cases see [763, 767]. In fact, Yau in [763] classified all possible resolution graphs of hypersurface singularities with $p_g = 2$ (250 cases).

Example 7.2.89 Set $(X_i, 0) \subset (\mathbf{C}^3, 0)$ given by $(X_1, 0) = \{z^2 + y^3 + x^{9+6m} = 0\}$, $(X_2, 0) = \{z^2 + y^3 + x^{11+6m} = 0\}$, $(X_3, 0) = \{z^3 + y^3 + x^{4+3m} = 0\}$, $(X_4, 0) = \{z^3 + y^3 + x^{5+3m} = 0\}$ (where $m \ge 0$). Then, in all these cases, $H^1(E,\mathbb{Z}) = 0$, $(X_i, 0)$ is elliptic, and $p_g = m + 1 = \ell_{seq}$.

7.2.G The Multiplicity and Hilbert–Samuel Function

Assume that (X, o) is an elliptic Gorenstein singularity with $p_g = m + 1$. For minimally elliptic singularities all the statements below were proved by Laufer [363]. For the proofs in this general case see [469, 473].

In this section $\widetilde{X}$ is the minimal resolution.

First we indicate that the multiplicity $\mathrm{mult}(X, o)$ depends only on Z_{min}^2.

Theorem 7.2.90 *Assume that (X, o) is an elliptic Gorenstein singularity with $p_g = m + 1 = \ell_{seq}$. (If $m = 0$ or $H^1(E, \mathbb{Z}) = 0$ then $p_g = m + 1$ is automatically satisfied.) Then the following facts hold.*

If $Z_{min}^2 \leq -2$, then $\phi^* \mathfrak{m}_{X,o} = \mathfrak{m}_{X,o} \mathscr{O}_{\widetilde{X}} = \mathscr{O}_{\widetilde{X}}(-Z_{min})$, hence $\operatorname{mult}(X, o) = -Z_{min}^2$.

If $Z_{min}^2 = -1$, then $\phi^* \mathfrak{m}_{X,o} = m_Q \mathscr{O}_{\widetilde{X}}(-Z_{min})$ for some smooth point Q of E, and $\operatorname{mult}(X, o) = 2$.

Example 7.2.91 (Comparison with Tomari's Bound) If (X, o) is Gorenstein elliptic and the link is rational homology sphere, then Tomari's T-invariant can also be identified with ℓ_{seq}. Indeed, if (X, o) is elliptic, then $T = p_g$ follows from (6.8.19). Hence $T = \ell_{seq}$ follows. [If we drop the $H_1(L_X, \mathbb{Q}) = 0$ assumption, then for Gorenstein elliptic singularities $T = p_g$ still holds, see [531].]

7.2.92 Next we present several formulae regarding the Hilbert–Samuel function.

Theorem 7.2.93 *Assume that (X, o) is an elliptic Gorenstein singularity with $p_g = m + 1$. Then the following facts hold:*

(a) $\mathfrak{m}_{X,o}^k = \phi_*(\mathscr{O}(-kZ_{min}))_o$ *for any $k \geq 1$ provided that $Z_{min}^2 \leq -3$;*

(b) $\dim \mathscr{O}_{X,o}/m_{X,o}^k = \chi(kZ_{min}) + 1$, *and* $\dim \mathfrak{m}_{X,o}^k/\mathfrak{m}_{X,o}^{k+1} = -kZ_{min}^2$ *for any $k \geq 1$ provided that $Z_{min}^2 \leq -3$;*

(c) $\operatorname{emb}(X, o) = \dim \mathfrak{m}_{X,o}/\mathfrak{m}_{X,o}^2 = \max\{3, -Z_{min}^2\}$.

7.2.94 A possible generator set of the $\mathscr{O}_{X,o}$-algebra $R := \oplus_{k \geq 0} \phi_*(\widetilde{X}, \mathscr{O}(-kZ_{min}))_o$ can be identified as follows (see also [587]).

If $Z_{min}^2 \leq -3$ then R is generated by elements of degree $k = 1$.

If $Z_{min}^2 = -2$, then R is generated by $\mathfrak{m}_{X,o}$ in degree 1 and an element $y \in \mathfrak{m}_{X,o} \setminus \mathfrak{m}_{X,o}^2$ in degree two.

If $Z_{min}^2 = -1$, then R is generated by $\mathfrak{m}_{X,o}$ in degree 1, by an element $y \in \mathfrak{m}_{X,o} \setminus \mathfrak{m}_{X,o}^2$ in degree 2, and $z \in \mathfrak{m}_{X,o} \setminus (\mathfrak{m}_{X,o}^2, y)$ in degree 3.

Remark 7.2.95 (Cf. with [363]) Let (X, o) be an elliptic Gorenstein singularity with $p_g = \ell_{seq}$. By 7.2.93, if $-Z_{min}^2 = 1, 2$ or 3 then (X, o) is a hypersurface singularity. If $-Z_{min}^2 = 4$ then $\operatorname{emb}(X, o) = 4$. But, by a result of Serre [628, Prop. 3], codimension 2 Gorenstein singularities are complete intersections. For $-Z_{min}^2 > 4$ the singularity (X, o) is not a complete intersection. E.g., if $-Z_{min}^2 = 5$, then (X, o) embeds into $(\mathbb{C}^5, 0)$. Moreover, $\dim \mathfrak{m}_{X,o}^2/\mathfrak{m}_{X,o}^3 = 10$ and $\dim \mathfrak{m}_{\mathbb{C}^5,0}^2/\mathfrak{m}_{\mathbb{C}^5,0}^3 = 15$, hence there are 5 defining equations with independent quadratic parts.

7.2.H Non-maximally Elliptic Singularities

7.2.96 The Non-maximally Elliptic (or Non-Gorenstein) Case We assume that (X, o) is numerically Gorenstein, and we use the notation of the Sect. 7.2.D. We also write X_j for $(\widetilde{X}/B_j, o_j)$. We consider the set of integers

$$\mathscr{A}_g := \{j \,:\, 0 \le j \le m,\ X_j \text{ is Gorenstein}\},$$

$$\mathscr{A}_f := \{j \,:\, 0 \le j \le m-1,\ H^0(\mathscr{O}_{\widetilde{X}}(-C_{j+1}) \subsetneq H^0(\mathscr{O}_{\widetilde{X}}(-C_j)\} \cup \{m\}.$$

Set also $\alpha := \min \mathscr{A}_g$ and $\beta := \min \mathscr{A}_f$. If the link is a rational homology sphere then some of the previous results can be summarized as follows:

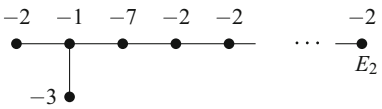
- By 7.2.85 and 7.2.81 $\mathscr{A}_g = \{j \,:\, \alpha \le j \le m\}$. Hence $\#\mathscr{A}_g = m + 1 - \alpha = p_g(X_\alpha)$. In particular, $\#\mathscr{A}_g = p_g(X_\alpha) \le p_g$.
- By 7.2.73 $\mathscr{A}_g \subset \mathscr{A}_f$, and by 7.2.42 $\#\mathscr{A}_f = p_g$. Hence, $m + 1 - \alpha - p_g(X_\alpha) = \#\mathscr{A}_g \le \#\mathscr{A}_f = p_g$.

In fact these statements can be improved, and even some of the assumptions can be dropped. For the next, and even more general statements, the interested reader may consult [531]. The proofs rely on (or are similar to) those presented above.

(1) $p_g(X, o) = p_g(X_\alpha) = \#\mathscr{A}_f = \#\mathscr{A}_g$. In particular, $p_g(X, o) = 1 \Leftrightarrow \alpha = m \Leftrightarrow \beta = m$. Furthermore, $p_g(X, o) = m + 1 \Leftrightarrow$ each X_j is Gorenstein.
(2) $p_g(X_{\beta+1}) = p_g(X, o) - 1$ (where, by convention, $p_g(X_{m+1}) = 0$).
(3) $\alpha \le \beta$. If $H^1(\widetilde{X}, \mathbb{Z}) = 0$ then $\alpha = \beta$, and $\mathscr{A}_f = \mathscr{A}_g = \{j \,:\, \alpha \le j \le m\}$.
(4) If $H^1(\widetilde{X}, \mathbb{Z}) = 0$ then any integer between 1 and $m + 1$ can be realized as p_g for a certain analytic structure.

The next example illustrates the case $H^1(\widetilde{X}, \mathbb{Z}) = 0, \alpha = \beta = 1$.

Example 7.2.97 Consider the hypersurface singularity $(X_m, o) = \{x^2 + y^3 + z^{6m+7} = 0\}$, $m \ge 1$. One verifies that $p_g = m + 1 = \ell_{seq}$ and the graph is:



(In the graph the number of (-2) vertices right to (-7) is m.)

Let $\widetilde{Y}$ be a neighbourhood of B_1 (E without E_2). Let E_1 be the neighbour of E_2. Since $Z_{B_1}^2 = -1$, $\mathscr{O}_{\widetilde{Y}}(-Z_{B_1})$ has a base point P, cf. 7.2.36, and this point is exactly $E_1 \cap E_2$. Let us modify the analytic structure on $\widetilde{X}$ such that we keep the structure on $\widetilde{Y}$, but the gluing of E_2 to $\widetilde{Y}$ is performed in such a way that in the new situation the intersection point $E_1 \cap E_2$ is not the base point P. Then the new singularity $\widetilde{X}/E$ will be not Gorenstein (compare with the discussion from 7.2.99), nevertheless

$\widetilde{Y}/B_1$ still supports a Gorenstein singularity (equisingular to (X_{m-1}, o)). Therefore, in this case, $\alpha = \beta = 1$ and $p_g = m$.

A possible set of equations can be done as follows [531].

We define the singularity $(X, o) \subset (\mathbb{C}^4, 0)$ by the 2-minors of the matrix

$$\begin{pmatrix} x & y & z \\ y - 3w^2 & z + w^{3m} & x^2 + 6wy - 2w^3 \end{pmatrix}.$$

One shows that its minimal good resolution graph is the same as the graph above. Indeed, first take the partial resolution X' in $\mathbb{C}^4 \times \mathbb{P}^1$ of X defined by the equations

$$sx = t(y - 3w^2), \quad sy = t(z + w^{3m}), \quad sz = t(x^2 + 6wy - 2w^3),$$

where $[s : t]$ are the coordinates of $\mathbb{P}^1$. The exceptional set E' is rational and X' has only one singular point on E', which is equisingular with (X_{m-1}, o). (Hence, in the good resolution, the strict transform of E' is E_2.). In this case $Z_{max} = C_1$ (with a base point), $\text{mult}(X, o) = 3$ and $\text{emb}(X, o) = 4$.

7.2.I *The 'generic' Analytic Structure of Elliptic Singularities*

The main result of this section is the following.

Theorem 7.2.98 ([363]) *Fix the topological type of an elliptic singularity (X, o). Then for generic analytic structure one has $p_g(X, o) = 1$.*

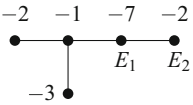
Proof If $m = 0$ then $p_g = 1$ for any analytic structure, hence we will assume that $m > 1$. According to Theorem 7.2.42 (3) we have to prove that for any $0 \le \ell \le m - 1$ and for the generic analytic structure the restriction $r : H^0(\mathscr{O}_{\widetilde{X}}(-C_\ell)) \to H^0(\mathscr{O}_{E_1}(-C_\ell))$ is trivial, where E_1 is a component of $|C|$.

First assume that C consists of an elliptic curve, say E_1. Choose $E_2 \neq E_1$ such that $P := E_1 \cap E_2 \neq \emptyset$. Let B be a ball around P with local coordinated (u, v), where $\{u = 0\} = B \cap E_1$ and $\{v = 0\} = B \cap E_2$. Assume that C_ℓ has local equation $u^n v^m$ in B. If we modify the gluing at P of E_1 and E_2 such that the new E_2 will have equation $v = \epsilon$ ($|\epsilon| \ll 1$), then the difference between the old and new bundles $\mathscr{O}_{E_1}(-C_\ell)$ is the bundle $\mathscr{O}_{E_1}(\Delta_\epsilon)$, where Δ_ϵ is the divisor supported in B with equation $(v - \epsilon)^m / v^m$. For different ϵ's these bundles are different, hence for generic ϵ in the new glued situation $\mathscr{O}_{E_1}(-C_\ell)$ is not trivial. Hence its cohomology groups are zero.

In the sequel we assume that all the irreducible components E_v are rational and smooth. In particular, $H^0(\mathscr{O}_{E_1}(-C_\ell)) \simeq \mathbb{C}$. Fix again $E_1 \subset |C|$ and $E_2 \not\subset |C|$ such that $P = E_1 \cap E_2 \neq \emptyset$. We show that it is possible to modify the gluing of E_1 and E_2 near P such that $\mathscr{O}_C(-C_\ell)$ becomes a non-trivial line bundle. Then its H^0 is zero by 7.2.17, hence the restriction r becomes trivial too.

Let $\widetilde{Y}$ be a small neighbourhood of $|C|$, then $|C|$ contracts in $\widetilde{Y}$ into a minimally elliptic singularity. It is automatically Gorenstein with $p_g(\widetilde{Y}/|C|) = 1$, cf. 7.2.15; let $\omega \in H^0(\widetilde{Y} \setminus |C|, \Omega^2)$ be a Gorenstein form. Its pole is $Z_K(|C|)$, it is supported everywhere along $|C|$ (since it is the fundamental cycle of $|C|$). Let its order along E_1 be o. Let B be a ball around P with similar notations as above. Then if we modify the gluing such that $\{v = 0\} = E_2 \cap B$ will be replaced by $v + \epsilon u^{o-1}$, then by the discussion of 6.8.22, for generic small ϵ, $\mathscr{O}_C(-C_\ell)$ will become non-trivial. □

Example 7.2.99 In some cases the non-existence of functions f_ℓ from 7.2.42 can be verified in the following way as well. Take e.g. the following elliptic graph.



Then $m = 1$ and $C_0 = Z_{min} = C + E_2$. Assume that there exists a function $f \in \phi_*(\mathscr{O}_{\widetilde{X}}(-Z_{min}))_o \setminus \phi_*(\mathscr{O}_{\widetilde{X}}(-Z_{min} - E_1))_o$. Let $\widetilde{Y}$ be a neighbourhood of $|C|$. Then f restricted to $\widetilde{Y}$ is a section of $\mathscr{O}_{\widetilde{Y}}(-C)$ (C being the fundamental cycle of its support). Since $C^2 = -1$, $\mathscr{O}_{\widetilde{Y}}(-C)$ has a base point, cf. 7.2.36, and this base point should be $P = E_1 \cap E_2$. Now, if we do not modify the analytic structure of $\widetilde{Y}$ this base point will stay unmodified. However, if we glue E_2 to E_1 such that their intersection point Q will be different, not the base point P of $\mathscr{O}_{\widetilde{Y}}(-C)$, then by the above discussion, in the new situation, such an f cannot exist. Hence p_g will become 1 by 7.2.42(3).

For an analytic structure of this second singularity $(X, o)_{P \neq Q}$ see 7.2.97.

One verifies (using e.g. the equations of 7.2.97) that $Z_{max} = E_1^*$ for $(X, o)_{P \neq Q}$. Hence E_1^* is the compact part of the divisor of a function, while E_2^* is not. In fact, for $(X, o)_{P \neq Q}$, $nE_2^* \notin \mathscr{S}_{an}$ for any $n \geq 1$. [Indeed, $nE_2^* \in \mathscr{S}_{an}$ implies that the restricted line bundle $\mathscr{O}_C(-nZ_{min})$ is trivial, but since $\mathrm{Pic}(C)$ has no torsion, then $\mathscr{O}_C(-Z_{min})$ is also trivial, a fact which contradicts 7.2.82(a).

Remark 7.2.100 Consider the graph from 7.2.99, and *any* analytic structure supported on it. Let us reprove topologically that E_1^* and E_2^* cannot be realized simultaneously as divisors of functions. Indeed, since the multiplicity of E_1^* along E_2 is one, the linking number of the two links (or the intersection of the two topological Milnor fibers) is one. Hence, if both functions (f_1, f_2) existed then $(f_1, f_2) : (X, o) \to (\mathbb{C}^2, 0)$ would be an analytic covering of degree 1, hence an isomorphism.

Remark 7.2.101 (Existence of a Non-projective Modification) Consider the situation $(X, o) = (X, o)_{P \neq Q}$ of Example 7.2.99. Let X' be the space obtained from $\widetilde{X}$ by analytic contraction of the support $|C|$ of the elliptic cycle. It has a normal singular point o', the image of $|C|$. The resolution $\phi : \widetilde{X} \to X$ and contraction

$\widetilde{X} \to X'$ induce an analytic modification $\phi' : X' \to X$. Let E' be the exceptional curve of ϕ' (the image by the contraction of E_2).

Since E_2^* is not the compact (exceptional) part of the divisor of a function, $E' \subset X'$ is not a Cartier divisor (at o'). Furthermore, since the link of X' at o' is integral homology sphere, the local Picard group of X' at o' is torsion free, hence $nE' \subset X'$ is not Cartier (at o') for any $n \in \mathbb{Z}_{>0}$.

This also shows that the modification $\phi' : X' \to X$ is non-projective: it cannot be obtained (as in the case of projective modifications) by blowing up of an ideal $\mathscr{I}$ of $\mathscr{O}_{X,o}$. Indeed, for a blowing up of an ideal $\mathscr{I}$, $\phi'^{-1}\mathscr{I} \cdot \mathscr{O}_{X'}$ is an invertible sheaf, globally generated by $\{a_i \circ \phi'\}_i$, where $\{a_i\}_i$ is a finite set of generators of the ideal $\mathscr{I}$. In such a case, the divisor of the invertible sheaf $\phi'^{-1}\mathscr{I} \cdot \mathscr{O}_{X'}$ is locally Cartier at every point of the exceptional set. That is, the projectivity of ϕ' is equivalent to the local Cartier property of nE' for some $n \geq 1$, which is not the case.

Remark 7.2.102

(a) Recall that any numerically Gorenstein topological type admits Gorenstein structure [579]. However, if $\min \chi \geq 0$, the *generic* analytic structure is Gorenstein only if it is Klein or minimally elliptic [363, Th. 4.3]. Moreover, for any Γ with $M(\Gamma)$ a $\mathbb{Q}HS^3$, the *generic* analytic structure is $\mathbb{Q}$-Gorenstein only if it is rational or minimally elliptic [453, Prop. 5.9.1].

(b) For singularities with generic analytic structure (and rational homology sphere link) several analytic invariants are computed topologically in [453, 455].

7.3 Weighted Cubes and Generalized Computation Sequences

To any good resolution graph Γ and characteristic element $k \in \mathrm{Char}$, we consider the weight function $\chi_k : L \to \mathbb{Z}$, and a natural cubical decomposition of $\mathbb{R}^s$ associated with the embedding $L \simeq \mathbb{Z}^s \hookrightarrow \mathbb{Z}^s \otimes \mathbb{R} = \mathbb{R}^s$, where $s = |\mathscr{V}|$ and the identification $L \simeq \mathbb{Z}^s$ is given by the base vectors $\{E_v\}_{v \in \mathscr{V}}$. Then, for each $n \geq \min_{l \in L}\{\chi_k(l)\}$, we define the topological space $S_{k,n}$, as the union of all cubes, which have all vertices of weight $\leq n$. We show that the tower $\{S_{k,n}\}_n$ depends only on the 3-manifold $M(\Gamma)$ and the spin^c structure associated with k. The tower $\{S_{k,n}\}_n$ carries an extremely deep information about $M(\Gamma)$; the final goal is to determine their homotopy types. ♣

Via the spaces $\{S_{k,n}\}_n$ this section prepares the theory of graded roots and lattice cohomology, which will be discussed in Chap. 11. On the other hand, by placing its content in this chapter, we regard it as a natural continuation of the Artin–Laufer program: indeed, the main technical machinery consists of the generalizations of Laufer's type computation sequences and certain 'combinatorial contractions' associated with them (cf. 7.3.A—7.3.D).

In this procedure the computation sequences play the role of special (weight non-increasing) 'combinatorial flows', and they will induce homotopical deformation retractions. In this way, all the machinery of computation sequences receives a much deeper homotopical reinterpretation.

7.3.1 Cubes in $L \otimes \mathbb{R}$ and the Spaces $\{S_{k,n}\}_n$ [480] Fix a connected plumbing graph Γ with negative definite intersection form, and *we assume that the plumbed 3-manifold $M(\Gamma)$ is a rational homology sphere*. (However, for possible generalisations see [480] or 7.3.6 here.) In our applications for resolution graphs, we will assume that the corresponding resolutions are good.

We use the standard notations for the lattice L, which has the distinguished base elements $\{E_v\}_{v\in\mathcal{V}}$. Using this basis, one identifies L with $\mathbb{Z}^s$ with its fixed standard basis, still denoted by $\{E_v\}_{v\in\mathcal{V}}$.

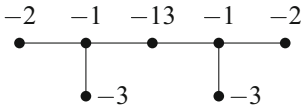
$\mathbb{Z}^s \otimes \mathbb{R} \simeq \mathbb{R}^s$ has a natural decomposition into cubes given by the inclusion $\mathbb{Z}^s \hookrightarrow \mathbb{R}^s$. The zero-dimensional cubes are exactly the lattice points $\mathbb{Z}^s$. Any $l \in \mathbb{Z}^s$ and subset $I \subset \mathcal{V}$ of cardinality q defines a q-dimensional cube $\square_q = (l, I)$, which has its vertices in the lattice points $(l + \sum_{v\in I'} E_v)_{I'}$, where I' runs over all subsets of I.

Next, we fix a characteristic element $k \in \mathrm{Char}$ and we consider the Riemann–Roch function $\chi_k : L \to \mathbb{Z}$, $\chi_k(l) = -(l, l + k)/2$. Here we regard χ_k as a weight function on the set of cubes: the weights of zero-dimensional cubes are defined by $w_0(l) = \chi_k(l)$, while, in general, $w_q((l, I)) := \max\{\chi_k(v) : v \text{ is a vertex of } (l, I)\}$.

Definition 7.3.2 For every $n \in \mathbb{Z}$, define $S_n \subset \mathbb{R}^s$ as the union of all the cubes $\square_q$, of any dimension, with $w(\square_q) \le n$ (with induced topology of $\mathbb{R}^s$). Clearly, $S_n \neq \emptyset$ exactly when $n \ge m_k$, where $m_k := \min_{l\in\mathbb{Z}^s} \chi_k(l)$. (If we wish to emphasize the k-dependence we write $S_{k,n}$.)

One has the natural inclusions $S_{m_k} \subset \ldots \subset S_n \subset S_{n+1} \subset \ldots$. It turns out that the topology of the spaces $\{S_n\}_{n\ge m_k}$ might be rather interesting. The tower has a finiteness property: only finitely many S_n have nontrivial topology (are non-contractible), but an S_n with n 'small' might have rather complicated homological groups. In the body of this book we will list several examples. The simplest one (with $H^1(S_n) \neq 0$) is the following.

Example 7.3.3 ([480]) Consider the following graph:



E_1 E_2 E_0 E_2' E_1'

E_3 E_3'

On the right hand side we give names to the base elements. Set $\chi := \chi_K$. We prefer to write any $l \in L$ in the form $l = l_x + zE_0 + l_y$, where $l_x = \sum x_i E_i$, $l_y = \sum y_i E_i'$, $x_i,\ y_i,\ z \in \mathbb{Z}$ $(i = 1, 2, 3)$; or $(x_1, x_2, x_3; z; y_1, y_2, y_3)$. Then $Z_K =$

$(7, 14, 5; 3; 7, 14, 5)$ and $Z_{min} = (3, 6, 2; 1; 3, 6, 2)$, with $\chi(Z_{min}) = \chi(2Z_{min}) = -1$. In fact, $m_K = -1$ too. Then it turns out that S_{-1} consists of two connected components, both contractible, S_0 has three components, two contractible ones and one with the homotopy type of S^1, and S_n is contractible for $n \geq 1$.

In order to see (at least part of) these facts, we will describe explicitly S_{-1} and S_0. Since $\chi(l) = \chi(Z_K - l)$, we can use the central χ-symmetry of the lattice points with respect to $Z_K/2$. If $z = 0$ then $\chi(l) = \chi(l_x) + \chi(l_y)$, and since l_x and l_y are supported by rational subgraphs, $\chi(l_x) \geq 0$, $\chi(l_y) \geq 0$. Thus $\chi(l) \geq 0$. Similar argument shows $\chi(l) \geq 0$ for $z \leq 0$ too. Hence, the lattice points in S_{-1} have $z = 1$ or $z = 2$, and they correspond via the above symmetry.

If $z = 1$ then $\chi(l) = \chi(l_x) + \chi(l_y) - x_2 - y_2 + 1$. Therefore, with the notation $f(x) := \chi(l_x) - x_2 = (\, 2x_1^2 + x_2^2 + 3x_3^2 - 2x_1x_2 - 2x_2x_3 - x_2 - x_3\,)/2$ we have $\chi(l) = f(x) + f(y) + 1$. Since the minimum of f over $\mathbb{R}^3$ is > -2, its minimum over $\mathbb{Z}^3$ is ≥ -1. Therefore, $\chi(l) = -1$ if and only if $f(x) = f(y) = -1$. By a computation, the integral solutions of $f(x) = -1$ are the triplets

$$A := \{(1, 2, 1), (1, 3, 1), (2, 3, 1), (2, 4, 1), (2, 4, 2), (2, 5, 2), (3, 5, 2), (3, 6, 2)\}.$$

Therefore, points $(x, 1, y)$ with $x \in A$ and $y \in A$ (denoted simply by $(A, 1, A)$) are in S_{-1}. Let $B = (7, 14, 5) - A$. Then, by symmetry, we get that the set of lattice points of S_{-1} is $(A, 1, A) \cup (B, 2, B)$. They determine two contractible connected components of S_{-1} in which Z_{min} and $2Z_{min}$ are 'representatives'.

Next, we solve $\chi(l) = 0$ with $z = 1$. Then, $f(x) + f(y) = -1$. $f(x) = 0$ has 24 integral solutions, namely the union of the triplets $A' := \{(0, 0, 0), (0, 1, 0), (1, 1, 0), (1, 2, 0), (0, 1, 1), (1, 1, 1), (0, 2, 1), (2, 2, 1), (1, 4, 1), (3, 4, 1), (2, 5, 1), (3, 5, 1)\}$ and the triplets of type $A'' = (4, 8, 3) - A'$. Set $\tilde{A} := A \cup A' \cup A''$, and $\tilde{B} = (7, 14, 5) - \tilde{A}$. Then the points of type $X := (A, 1, \tilde{A}) \cup (\tilde{A}, 1, A) \cup (B, 2, \tilde{B}) \cup (\tilde{B}, 2, B)$ are in S_0. Since $\tilde{A} \cap B$ is not empty, all the points from X can be connected by segments. In fact, S_0 has three connected components, one of them contains the zero cycle, the other one contains Z_K, and the third one, $\mathscr{C}_0$, contains all the points from X.

Finally, notice that the two intersection points $P = \tilde{A} \cap B = (4, 8, 3)$ and $Q = A \cap \tilde{B} = (3, 6, 2)$ create a loop in $\mathscr{C}_0$. Indeed, half of it is the connecting path of $(P; 1; Q)$ and $(Q; 1; P)$ through points in X with $z = 1$, the other half connects $(P; 2; Q)$ with $(Q; 2; P)$ through points in X with $z = 2$. This loop can be contracted only in S_1.

This example suggests that in general it is rather hard *to solve the corresponding Diophantine equations* and *to analyse the adjacent properties/positions* of the solutions (in order to get the cubes which build up the topological space S_n). Furthermore, the example also suggests that this combinatorial/artihmetical structure might be extremely rich, it might cover indeed a big amount of deep information.

7.3.4 Assume that k and k' determine the same Spin^c–structure of $M(\Gamma)$, cf. 6.10.22, hence $k' = k + 2l$ for some $l \in L$. Then $\chi_{k'}(x - l) = \chi_k(x) - \chi_k(l)$ for any $x \in L$. This means that the transformation $x \mapsto x' := x - l$ realizes an

identification of the 'S_n-spaces' associated with k and k': $S_{k,n} = S_{k',n-\chi_k(l)}$. Hence, fixing a representative k from each class $[k] \in \mathrm{Spin}^c(M(\Gamma))$ we can speak about the tower of spaces $\{S_{k,n}\}_n$, indexed by $[k] \in \mathrm{Spin}^c(M(\Gamma))$.

Proposition 7.3.5 ([480]) *Similarly as above, assume that $M(\Gamma)$ is a $\mathbb{Q}HS^3$. Then the tower of spaces $\{S_{k,n}\}_n$, indexed by $[k] \in \mathrm{Spin}^c(M(\Gamma))$, up to homotopy equivalence, depends only on $M = M(\Gamma)$, it is independent of the choice of the negative definite plumbing graph Γ, which provides M.*

Proof First we assume that Γ' is obtained from Γ by **'blowing up the vertex** v_0', cf. 2.2.5.2. We denote the new (-1)-vertex of Γ' by E_{new}. We will use the notations $L(\Gamma)$, $L(\Gamma')$, $L'(\Gamma)$, $L'(\Gamma')$. Similarly, write I, I' for the corresponding intersection forms. The following facts can be verified:

(i) Consider the maps $\pi_* : L(\Gamma') \to L(\Gamma)$ defined by $\pi_*(\sum x_v E_v + x_{new}E_{new}) = \sum x_v E_v$, and $\pi^* : L(\Gamma) \to L(\Gamma')$ defined by $\pi^*(\sum x_v E_v) = \sum x_v E_v + x_{v_0}E_{new}$. Then the projection formula $I'(\pi^* x, x') = I(x, \pi_* x')$ holds. This shows that $I'(\pi^* x, \pi^* y) = I(x, y)$ and $I'(\pi^* x, E_{new}) = 0$ for any $x, y \in L(\Gamma)$.

(ii) Set the (nonlinear) map: $c : L'(\Gamma) \to L'(\Gamma')$, $c(l') := \pi^*_{\mathbb{Q}}(l') + E_{new}$. Then $c(\mathrm{Char}(\Gamma)) \subset \mathrm{Char}(\Gamma')$ and c induces a bijection between the two orbit spaces $\mathrm{Char}(\Gamma)/2L(\Gamma)$ and $\mathrm{Char}(\Gamma')/2L(\Gamma')$.

(iii) Consider $k \in \mathrm{Char}(\Gamma)$ and write $k' := c(k) \in \mathrm{Char}(\Gamma')$. Then for any $x \in L(\Gamma)$ one has: $\chi_k(x) = \chi_{k'}(\pi^* x)$. Moreover, for any $z \in L(\Gamma')$, write z in the form $\pi^*\pi_* z + aE_{new}$ for some $a \in \mathbb{Z}$. Then

$$\chi_{k'}(z) = \chi_{k'}(\pi^*\pi_* z) + \chi_{k'}(aE_{new}) = \chi_k(\pi_*(z)) + a(a+1)/2. \tag{7.3.1}$$

This shows that if $\chi_{k'}(\pi^* x + aE_{new}) \le n$, then $\chi_k(x) \le n$ too. In particular, the projection $\pi_{\mathbb{R}}$ in the direction of E_{new} induces a well-defined map $\pi_{\mathbb{R},n} : S_n(\Gamma') \to S_n(\Gamma)$. We claim that (whenever $S_n(\Gamma)$ is non-empty) $\pi_{\mathbb{R},n}$ is a homotopy equivalence (with all fibers non-empty and contractible).

We proceed in two steps. First we prove that $\pi_{\mathbb{R},n} : S_n(\Gamma') \to S_n(\Gamma)$ is onto.

Consider a zero dimensional cube (i.e. lattice point) $x \in S_n(\Gamma)$. Then $\chi(x) \le n$. But then $\chi_{k'}(\pi^* x) = \chi(x) \le n$ too, hence $\pi^*(x) \in S_n(\Gamma')$ and $x = \pi_{\mathbb{R}}(\pi^* x) \in \mathrm{im}(\pi_{\mathbb{R}})$.

Next, take a cube $(x, I) \subset S_n(\Gamma)$ ($I \subset \mathcal{V}$). This means that $\chi(x + E_{I'}) \le n$ for any $I' \subset I$. But then

$$\pi^*(x + E_{I'}) = \pi^* x + E_{I'} + \epsilon \cdot E_{new}, \tag{7.3.2}$$

where $\epsilon = 0$ if $v_0 \notin I'$ and $\epsilon = 1$ otherwise. Hence

$$\chi_{k'}(\pi^* x + E_{I'}) = \chi_{k'}(\pi^*(x + E_{I'}) - \epsilon E_{new}) \overset{(7.3.1)}{=} \chi(x + E_{I'}) \le n. \tag{7.3.3}$$

Therefore $(\pi^* x, I) \in S_n(\Gamma')$ and $\pi_{\mathbb{R},n}$ projects $(\pi^* x, I)$ isomorphically onto (x, I).

In particular, $\pi_{\mathbb{R},n}$ is onto. Next, we show that $\pi_{\mathbb{R},n}$ is in fact a homotopy equivalence. In order to prove this fact it is enough to verify that if $\square \in S_n(\Gamma)$ and $\square^\circ$ denotes its relative interior, then $\pi_{\mathbb{R},n}^{-1}(\square^\circ) = \pi_{\mathbb{R}}^{-1}(\square^\circ) \cap S_n(\Gamma')$ is contractible.

Let us start again with a lattice point $x \in S_n(\Gamma)$. Then $\pi_{\mathbb{R}}^{-1}(x) \cap S_n(\Gamma')$ is a real interval (whose endpoints are lattice points, considered in the real line of the E_{new} coordinate). Let us denote it by $\mathscr{I}(x)$. Now, if $\square = (x, I)$, then we have to show that all the intervals $\mathscr{I}(x + E_{I'})$ associated with all the subsets $I' \subset I$ have a common lattice point. But this is exactly what we verified above: the E_{new} coordinate of $\pi^* x$ is such a common point. Therefore, $\pi_{\mathbb{R}}^{-1}(\square^\circ) \cap S_n(\Gamma')$ has a strong deformation retraction (in the E_{new} direction) to the contractible space $(\pi^* x, I)^\circ$.

For any $l \in L$ let $N(l) \subset \mathbb{R}^s$ denote the union of all cubes which have l as one of their vertices. Let $U(l)$ be its interior. Set $U_n(l) := U(l) \cap S_n(\Gamma)$. If $l \in S_n(\Gamma)$ then $U_n(l)$ is a contractible neighbourhood of l in $S_n(\Gamma)$. Also, $S_n(\Gamma)$ is covered by such $\{U_n(l)\}_l$. Moreover, $\pi_{\mathbb{R},n}^{-1}(U_n(l))$ has the homotopy type of $\pi_{\mathbb{R},n}^{-1}(l)$, hence it is contractible. More generally, for any cube $\square$,

$$\pi_{\mathbb{R},n}^{-1}(\cap_v U_n(v) \ : \ v \text{ vertex of } \square) \sim \pi_{\mathbb{R},n}^{-1}(\square^\circ),$$

which is contractible by the above discussion. Since all the intersections of $U_n(l)$'s are of these type, we get that the inverse image of any such intersection is contractible. Hence by Čech covering (or Leray spectral sequence) argument, $\pi_{\mathbb{R},n}$ induces an isomorphism $H^*(S_n(\Gamma'), \mathbb{Z}) = H^*(S_n(\Gamma), \mathbb{Z})$. In order to prove the homotopy equivalence, one can use quasifibration, defined in [151] (see also [131], e.g. the relevant Theorem 6.1.5). Since $\pi_{\mathbb{R},n} : S_n(\Gamma') \to S_n(\Gamma)$ is a quasifibration, and all the fibers are contractible, the homotopy equivalence follows.

Assume next that we **blow up an intersection point** $E_{v_0} \cap E_{v_1}$. We start very similarly, however at some point there is a major difference.

With very similar notations, in this case we define $\pi^*(\sum_v x_v E_v) = \sum_v x_v E_v + (x_{v_0} + x_{v_1}) E_{new}$. Then all the statements till (7.3.2)) remain valid (including the key (7.3.1)). The first difference is in (7.3.2). Indeed, in this case

$$\pi^*(x + E_{I'}) = \pi^* x + E_{I'} + \epsilon \cdot E_{new}, \tag{7.3.4}$$

where ϵ is the cardinality of $I' \cap \{v_0, v_1\}$. This can be 0, 1 or 2. Therefore, if $\{v_0, v_1\} \not\subset I$, then $\epsilon \in \{0, 1\}$ for any I', hence for such cubes (x, I) all the arguments from the first part work without any modification.

Assume in the sequel that $\{v_0, v_1\} \subset I$. Write $J = I \setminus \{v_0, v_1\}$.

There are two cube candidates of $L(\Gamma') \otimes \mathbb{R}$ which might cover the cube $(x, I) \subset S_n(\Gamma)$. One of them is $(\pi^* x, I)$ (as in the above case). By (7.3.1) for any $V \subsetneq \{v_0, v_1\}$ and any $J' \subset J$ we have $\pi^*(x + E_{J' \cup V}) = \pi^* x + E_{J' \cup V} + \epsilon E_{new}$ with $\epsilon \in \{0, 1\}$, hence by (7.3.1) $\pi^* x + E_{J' \cup V} \in S_n(\Gamma')$. However, vertices of type $\pi^* x + E_{J' \cup \{v_0, v_1\}}$ might not be in $S_n(\Gamma')$.

Another candidate is $(\pi^*x + E_{new}, I)$. In this case, the points of type $\pi^*x + E_{J'\cup V} + E_{new}$ are in $S_n(\Gamma')$ (where $J' \subset J$ and $\emptyset \neq V \subset \{v_0, v_1\}$), however vertices of type $\pi^*x + E_{J'} + E_{new}$ of $(\pi^*x + E_{new}, I)$ might be not.

So both cubes a priori are obstructed if we apply (7.3.1) only. Next we analyze these obstructions and we show that at least one of the two candidate cubes works.

Assume that $\pi^*x + E_{J'\cup\{v_0,v_1\}} \in S_n(\Gamma')$ for any $J' \subset J$. Then (π^*x, I) is a good lift of (x, I) in $S_n(\Gamma')$. In the second case, assume that $\pi^*x + E_{J''} + E_{new} \in S_n(\Gamma')$ for any $J'' \subset J$. Then $(\pi^*x + E_{new}, I)$ is a good lift of (x, I).

Finally assume that non of these two assumptions hold. The negation of the first one is that there exists $J' \subset J$ such that $\chi_k(x + E_{J'\cup\{v_0,v_1\}}) = n$. [Indeed, otherwise, if $\chi_k(x + E_{J'\cup\{v_0,v_1\}}) = m < n$ then by (7.3.1) $\chi_{k'}(\pi^*x + E_{J'\cup\{v_0,v_1\}}) = \chi_{k'}(\pi^*(x + E_{J'\cup\{v_0,v_1\}}) - 2E_{new}) = m + 1 \leq n$, a contradiction.] The negation of the second one is that there exists $J'' \subset J$ such that $\chi_k(x + E_{J''}) = n$. [Again, otherwise, if $\chi_k(x + E_{J''}) = m < n$ then $\chi_{k'}(\pi^*x + E_{J''} + E_{new}) = \chi_{k'}(\pi^*(x + E_{J''}) + E_{new}) = m + 1 \leq n$.]

On the other hand, we claim that for a cube (x, I), $J = I \setminus \{v_0, v_1\}$, (v_0, v_1) connected by an edge e, the simultaneous realizability of

$$\begin{cases} \chi_k(x + E_{J'\cup\{v_0,v_1\}}) = n & \text{for some } J' \subset J \\ \chi_k(x + E_{J''}) = n & \text{for some } J'' \subset J \\ \chi_k(x + E_{I'}) \leq n & \text{for all } I' \subset I \end{cases}$$

cannot happen.

Assume that we have such a situation. Then, by replacing x by $x + E_{J'\cap J''}$ and deleting $J' \cap J''$ from J', J'' and I respectively, we can assume that $J' \cap J'' = \emptyset$.

Next, let us write x in the dual basis, namely $x = \sum_v x_v E_v^*$. Then $\chi(x + E_{I'}) = \chi(x) + \chi(E_{I'}) + \sum_{v\in I'} x_v$, where $\chi(E_{I'})$ equals the number of connected components of the support $|I'|$. Let Γ_l and Γ_r be the two connected components of the graph $\Gamma \setminus \{e\}$, and accordingly we decompose $J' \cup \{v_0, v_1\} = A_l \sqcup A_r$, $J'' = B_l \sqcup B_r$, where $A_l \cup B_l \subset \mathscr{V}(\Gamma_l)$ and $A_r \cup B_r \subset \mathscr{V}(\Gamma_r)$. Then we define $a_l := \chi(A_l) + \sum_{v\in A_l} x_v$, and similarly a_r, b_l and b_r. Then, using $\chi(E_{J'\cup\{v_0,v_1\}}) = \chi(E_{A_l}) + \chi(E_{A_r}) - 1$, we obtain

$$\begin{cases} n = \chi_k(x + E_{J'\cup\{v_0,v_1\}}) = \chi(x) + a_l + b_l - 1, \\ n = \chi_k(x + E_{J''}) = \chi(x) + b_l + b_r. \end{cases}$$

Next, we modify in Γ_r the subsets: what was in A_r we put in B_r and viceversa. In this way we get two subsets of I, namely $I' = A_l \cup B_r$ and $I'' = B_l \cup A_r$. Note that in this case the two ends of the edge e do not belong to the same set I' or I''. Therefore,

$$\begin{cases} \chi_k(x + E_{I'}) = \chi(x) + a_l + b_r, \\ \chi_k(x + E_{I''}) = \chi(x) + b_l + a_r. \end{cases}$$

In particular, $2n+1 = 2\chi(x) + a_l + a_r + b_l + b_r = \chi_k(x + E_{I'}) + \chi_k(x + E_{I''}) \leq 2n$, which is a contradiction. Hence, in all cases the cube (x, I) has a lift, and the argument of the first case (blowing up a vertex) can be repeated. □

Remark 7.3.6 A possible generalization of the set of weighted cubes and spaces S_n is provided via a set of compatible *weight functions*. Let $\mathscr{Q}_q$ denote the set of $q-$cubes. A set of functions $w_q : \mathscr{Q}_q \to \mathbb{Z}$ $(0 \leq q \leq |\mathscr{V}|)$ is called a *set of compatible weight functions* if the following hold:

(a) for any integer $n \in \mathbb{Z}$, the set $w_0^{-1}(\,(-\infty, n]\,)$ is finite;
(b) for any $\square_q \in \mathscr{Q}_q$ and for any of its faces $\square_{q-1} \in \mathscr{Q}_{q-1}$ one has $w_q(\square_q) \geq w_{q-1}(\square_{q-1})$.

Then one can define S_n as $\cup_q\{\square \in \mathscr{Q}_q \,:\, w_q(\square_q) \leq n\}$.

This gives the possibility to construct the spaces S_n (with 'relevant' properties) even in the case of graphs with $g_v > 0$. E.g. if E_v is a smooth elliptic curve, then the segment $[0, E_v] \in \mathscr{Q}_1$ is 'special': though for the vertices $\chi(0) = \chi(E_v) = 0$, hence $\max\{\chi(v) : v \text{ is a vertex of } [0, E_v]\} = 0$, for $w_1([0, E_v])$ is more appropriate to take $w_1([0, E_v]) = 1$ (for motivation see e.g. 6.8.B). For more see [480].

7.3.A *The Topology of the Spaces $\{S_{k,n}\}_n$. Deformation Retractions*

We show how χ-monotone computation sequences provide deformation retractions for the spaces $S_{k,n}$. We give a new topological characterization of the rational singularities via the contractibility of all (non-empty) spaces $S_{k,n}$. Elliptic singularities can also be characterized via the spaces $S_{K,n}$. In the elliptic case $S_{K,0}$ has $\ell_{seq} + 1$ connected components, all contractible. ♣

7.3.7 In order to analyse the topology of the space $S_n = S_{n,k}$ it is convenient to introduce the set of finite rectangles indexed by pairs $l_1, l_2 \in L$ with $l_1 \leq l_2$.

Definition 7.3.8 For any such pair $l_1 \leq l_2$ set $R(l_1, l_2) := \{x \in \mathbb{R}^s \,:\, l_1 \leq x \leq l_2\}$. Define also $R(l_1, \infty) := \{x \in \mathbb{R}^s \,:\, l_1 \leq x\}$.

Lemma 7.3.9 *Fix $k \in$ Char and write $S_n = S_{k,n}$.*

(I) There exist $l_+ \in L$ and an increasing infinite sequence of cycles $\{l_i\}_{i\geq 0}$ $(l_i \in L)$ with $l_0 = l_+$, such that

(a) for any $i \geq 0$ one has $l_{i+1} = l_i + E_{v(i)}$ for some $v(i) \in \mathscr{V}$,
(b) if $l_i = \sum_v m_{i,v} E_v$, then $\lim_{i\to\infty} m_{i,v} = \infty$ for all v,
(c) $\chi_k(l_{i+1}) \geq \chi_k(l_i)$.

Similarly, there exists $l_- \in L$ and an increasing infinite sequence of cycles $\{y_i\}_{i\geq 0}$, satisfying $y_0 = l_-$, the analogs of (a)–(b), and (c) $\chi_k(-y_{i+1}) \geq \chi_k(-y_i)$.

(II) Take l_- and l_+ as in (I). Without loss of generality we can assume that $-l_- \leq l_+$. Then the inclusions $R(-l_-, \infty) \cap S_n \subset S_n$ and $R(-l_-, l_+) \cap S_n \subset S_n$ are homotopy equivalences for any $n \in \mathbb{Z}$.

Proof

(I) Take a cycle $Z \in L$ such that $(Z, E_v) < 0$ for any v. Let $\{z_i\}_{i=0}^t$ be an increasing sequence with $z_0 = 0$, $z_t = Z$, $z_{i+1} = z_i + E_{v(i)}$ $(0 \leq i < t)$. Then for m sufficiently large, one can take $l^+ = mZ$, and the sequence $\{m'Z + z_i\}$, where $m' \geq m$ and $0 \leq i < t$. (That is, $mZ, \ldots, mZ + z_i, \ldots, (m+1)Z, \ldots, (m+1)Z + z_i, \ldots$.) Similarly, $l_- = mZ$ (for $m \gg 0$) and a similar type of sequence works for $\{y_i\}_i$ too.

(II) Clearly for fixed n, there exist $i' \gg 0$ and $j' \gg 0$ such that $S_n \subset R(-y_{j'}, x_{i'})$. We will show that if we fix this $y_{j'}$ then the inclusion $R(-y_{j'}, x_i) \cap S_n \subset R(-y_{j'}, x_{i+1}) \cap S_n$ admits a strong deformation retraction for any i. At the level of rectangles there is a natural deformation retraction $r_s : R(-y_{j'}, x_{i+1}) \to R(-y_{j'}, x_i)$ $(s \in [0,1])$. Indeed, take r_s to be the identity on $R(-y_{j'}, x_i)$ for any s. Furthermore, if $t \in [0,1]$ and $z \in R(-y_{j'}, x_i)$, such that the $E_{v(i)}$-coefficients of z and x_i equal, then one defines $r_s(z + tE_{v(i)}) = z + stE_{v(i)}$. The point is that if z is as above, then $(x_i, E_{v(i)}) \geq (z, E_{v(i)})$, hence $\chi_k(z + E_{v(i)}) - \chi_k(z) \geq \chi_k(x_{i+1}) - \chi_k(x_i) \geq 0$ too. This means that the restriction $r_s : R(-y_{j'}, x_{i+1}) \cap S_n \to R(-y_{j'}, x_i) \cap S_n$ is well-defined and it is a strong deformation retraction such that r_1 is the identity and r_0 has its image in $R(-y_{j'}, x_i) \cap S_n$.

We proceed with the y-sequence similarly. □

Corollary 7.3.10 *For any $k \in \mathrm{Char}$ the space S_n is contractible for any $n \gg 0$.*

Proof Fix $l_- \leq l_+$ as in Lemma 7.3.9*(I)*. Let n be so large that $R(-l_-, l_+) \subset S_n$. Then, by Lemma 7.3.9*(II)* S_n has the same homotopy type as $R(-l_-, l_+)$. □

7.3.11 Distinguished Representatives As we already said in 7.3.4, if $k' = k + 2l$ for some $l \in L$ then $S_{k,n} = S_{k',n-\chi_k(l)}$. Hence, it is natural to choose one representative from each spin^c structure. For several results the choice is irrelevant, however, certain choices have certain advantages. Our preferred choice is the *distinguished representative*, or *distinguished characteristic element* $k_r := -Z_K + 2s_h$, cf. 6.10.22, where $s_h \in L'$ is the smallest representative of h in $\mathscr{S}'$, cf. 6.6.A.

The first motivation for the choice of k_r is the following. Recall that the rationality criterion for graphs is $\chi(l) > 0$ for any $l \in L_{>0}$, hence it is decided in the 'first quadrant' $L_{\geq 0}$ of L. More generally, for arbitrary graphs, the essential properties of $\chi : L \to \mathbb{Z}$ are already coded in the restriction $\chi|L_{\geq 0}$. The choice $k_r = -Z_K + 2s_h$ guarantees that the essential properties of $\chi_{k_r} : L \to \mathbb{Z}$ are coded again in $L_{\geq 0}$ (or, equivalently, for a fixed h, the essential information of

$\chi_{\mathbb{Q}}|\{l' \in L' : [l'] = h\}$ is coded in $\chi_{\mathbb{Q}}|s_h + L_{\geq 0}$. Another advantage in the definition of k_r is that $s_h \in \mathscr{S}'$ (exploited deeply, see several proofs below).

Lemma 7.3.12 *Fix $h \in H$ and $k_r = -Z_K + 2s_h$. Then the following facts hold.*

(a) In Lemma 7.3.9 one may take $l_- = 0$. This means that $R(0, \infty) \cap S_{k_r,n} \subset S_{k_r,n}$ is a homotopy equivalence for any n. In particular, by Lemma 7.3.9, there exists $l_+ \geq 0$ such that $R(0, l_+) \cap S_{k_r,n} \subset S_{k_r,n}$ is a homotopy equivalence for any n.

(b) Assume that $Z_K \geq 0$ (e.g., as in the minimal good resolution, cf. 6.3.4). Then one can take $l_+ = \lfloor Z_K \rfloor$. Hence, $S_{k_r,n}$ has the homotopy type of $R(0, \lfloor Z_K \rfloor) \cap S_{k_r,n}$.

(c) For any $x \geq 0$ one has $\chi_{k_r}(x) \geq \chi_K(x)$. Therefore, $\min \chi_{k_r} \geq \min \chi_K$.

(Compare the statements with Lemma 6.8.40.)

Proof

(a) Fix a large $l_- > 0$ given by 7.3.9. Then one can connect $-l_-$ to 0 by an increasing sequence along which χ_{k_r} is non-increasing. Indeed, for any $y < 0$ there exists v such that $E_v \subset |y|$ and $\chi_{k_r}(y + E_v) \leq \chi_{k_r}(y)$. If not, then $(E_v, y + s_h) \leq 0$ for all $E_v \subset |y|$. But the same inequality automatically works for all other components. Hence $y + s_h \in \mathscr{S}'$ with $y < 0$, which contradicts 6.6.15.

(b) Adopt the proof of Lemma 6.8.40(b).

(c) $\chi_{k_r}(x) = \chi_K(x) - (s_h, x) \geq \chi_K(x)$ for any $x \geq 0$. Then use *(a)*. □

Corollary 7.3.13 *$S_{K,n}$ (i.e. when $h = 0$ and $s_h = 0$) is connected for all $n \geq 1$.*

The analogue statement is not true for $h \neq 0$, see 11.5.20.

Proof Via 7.3.12 it is enough to show that any $x_0 \geq 0$ with $\chi(x_0) \leq n$, can be connected to the zero cycle by a path inside $S_{K,n}$. For this, it is enough to verify the following two facts:

(i) For any $x > 0$ with $\chi(x) > 0$, there exists $E_v \subset |x|$ such that $\chi(x - E_v) \leq \chi(x)$.

Indeed, otherwise $(E_v, x - E_v) > 1$ for any $E_v \subset |x|$. This is equivalent to $(E_v, x - Z_K) \geq 0$, which implies $(x, x - Z_K) \geq 0$, or $\chi(x) \leq 0$, a contradiction.

(ii) For any $x > 0$ with $\chi(x) \leq 0$, there exists $E_v \subset |x|$ such that $\chi(x - E_v) \leq 1$.

In order to prove this, denote $\chi(x)$ by $c \leq 0$; write $x = \sum_v m_v E_v$ and set $m := \sum_v m_v$. Necessarily $m \geq 2$ since $\chi(E_v) = 1$ for any v. Next, assume that $\chi(x - E_v) > 1$ for all $E_v \subset |x|$. Then, in the identity $-(E_v, x - E_v) = \chi(x) - \chi(x - E_v) - 1$ the right hand side is $< c - 2$. This means that $-(E_v, x) < c - 2 - E_v^2 = c - (E_v, Z_K)$ for any $E_v \subset |x|$. This implies $(x, x - Z_K) + cm > 0$, or $c(m - 2) > 0$, which is a contradiction.

Then the construction of the wanted path connecting x_0 with 0 is clear: in the path any x is succeeded by some $x - E_v$ according to the cases (i) or (ii). □

Theorem 7.3.14 (Characterization of Rational Graphs via the Spaces S_n [478]) *Let Γ be a connected, negative definite plumbing graph whose plumbed 3-manifold is a rational homology sphere. Then the following facts are equivalent:*

(a) Γ *is rational;*
(b) $S_{K,n}$ *is contractible for every* $n \geq \min \chi$*;*
(b′) $S_{K,n}$ *is connected for every* $n \geq \min \chi$*;*
(c) $S_{k,n}$ *is contractible for all* $k \in \mathrm{Char}$ *and* $n \geq \min \chi_k$.

Additionally, if Γ is rational and $k_r = K + 2s_h$, then $\min \chi_{k_r} = 0$.

Proof By Lemma 7.3.12 we need to consider only effective lattice points.

$(b') \Rightarrow (a)$ $S_{K,0}$ contains $0 \in L$ and it is connected. Therefore, there exists no $x > 0$ with $\chi(x) \leq 0$. Indeed, since $\chi(E_v) = 1$ for all v, there is no path in $R(0, l_+) \cap S_{K,0}$ connecting 0 and such an x. Hence Γ is rational by Theorem 7.1.2.

$(a) \Rightarrow (c)$ By 7.3.11 we can assume that $k = k_r$. We prove that in $R(0, l_+) \cap S_n$ we can take $l_+ = 0$. For this we need to show that for any $x > 0$ (which automatically satisfies $\chi(x) > 0$ by rationality) there exists $E_v \subset |x|$ such that $\chi_{k_r}(x - E_v) \leq \chi_{k_r}(x)$. The proof is similar to the verification of (i) in the proof of 7.3.13, using additionally $(s_h, E_v) \leq 0$. [The implications $(c) \Rightarrow (b) \Rightarrow (b')$ are clear.]

For the last statement use 7.3.12*(c)*. □

Theorem 7.3.15 (Characterization of Elliptic Graphs via the Spaces $S_{K,n}$ [478]) *Under the same assumption as in 7.3.14 the following facts are equivalent:*

(a) Γ *is elliptic;*
(b) $S_{K,n}$ *is empty for* $n < 0$ *and* $S_{K,0}$ *is not connected.*

Furthermore, if Γ is elliptic, then let $\ell + 1$ be the number of connected components of $S_{K,0}$. Then ℓ equals the length of the elliptic sequence ℓ_{seq} (cf. 7.2.C).

In fact, each connected component contains a unique maximal lattice point, and their collection is $\{0, F_0, \ldots, F_n\}$ *(see 7.2.C for notations).*

Proof $(a) \Rightarrow (b)$ Use $\min \chi = 0$ and 7.3.14. For $(b) \Rightarrow (a)$, since $\min \chi = 0$, Γ should be either rational or elliptic. But it cannot be rational because of 7.3.14.

Next, we prove that $\ell = \ell_{seq}$. We will use the notations of 7.2.C.

If $\mathscr{C}_0^i$ is a connected component of $S_{K,0}$, and $Z \in \mathscr{C}_0^i$ is a maximal lattice point in it, then $Z \in \mathscr{S}$. Hence, by 7.2.40 (5) $Z \in \{F_{-1}, F_0, \ldots, F_n\}$, where $F_{-1} = 0$. Therefore, $\mathscr{C}_0^i$ contains at least one of these elements. It is also clear that each F_j belongs to some $\mathscr{C}_0^i$. Hence, we need to show that there is no pair $F_k < F_j$ $(-1 \leq k < j \leq n)$ in the same connected component of $S_{K,0}$. This also shows that $\mathscr{C}_0^i$ has a unique maximal lattice point, which (after a possible reordering) is exactly F_i.

Assume that such a pair exists. This means that there exists a path $\{l_i\}_i$, $l_{i+1} = l_i + \epsilon_i E_{v(i)}$, $\epsilon_i = \pm 1$, which starts with F_k and ends with F_j, and $\chi(l_i) = 0$ for all i. We show that if such a path exists then it can be replaced by another increasing path with the very same properties. Indeed, if $\epsilon_i = -1$ for some i, then there should

exist a subsequence $l_s, l_{s+1}, \dots, l_t, l_{t+1}$ of the path, such that $\epsilon_s = -1$, $\epsilon_i = 1$ for all $s+1 \le i \le t$, and $v(s) = v(t)$, but $v(i) \ne v(s)$ for all $s < i < t$. Write $l_t - l_{s+1}$ as $Z = \sum_{s<i<t} E_{v(i)}$. Then $l_{t+1} - l_s = Z$ too. Since $0 = \chi(l_{t+1}) = \chi(l_s + Z) = \chi(Z) - (l_s, Z)$, and $0 = \chi(l_t) = \chi(l_{s+1} + Z) = \chi(Z) - (l_{s+1}, Z)$, we get $(Z, E_{v(s)}) = 0$, or equivalently, $(E_{v(i)}, E_{v(s)}) = 0$ for all $s < i < t$.

Set $\bar{l}_i := l_i + E_{v(s)}$ for $s+1 \le i \le t$; hence $\bar{l}_{s+1} = l_s$ and $\bar{l}_t = l_{t+1}$.

Then, $\chi(\bar{l}_{i+1}) - \chi(\bar{l}_i) = \chi(l_{i+1} + E_{v(s)}) - \chi(l_i + E_{v(s)}) = -(E_{v(s)}, l_{i+1} - l_i) = -(E_{v(s)}, E_{v(i)}) = 0$. Hence, the path sequence $l_s, l_{s+1}, \dots, l_t, l_{t+1}$ can be replaced with $\bar{l}_{s+1}, \dots, \bar{l}_t$, for which $\epsilon_i = 1$ always.

Hence, using induction, we can assume that the path is increasing. But any increasing path starts as $F_k, F_k + E_v, \dots$ (for a certain v), and $\chi(F_k + E_v) - \chi(F_k) = 1 - (F_k, E_v) \ge 1$ (since $F_k \in \mathscr{S}$), which leads to a contradiction. □

Remark 7.3.16

(a) Later in 7.3.D we will show that in the elliptic case all connected components of $S_{k,n}$ ($n \ge 0$, $k \in$ Char) are in fact contractible.
(b) Lemma 7.3.12*(c)* shows that in the elliptic case $\min \chi_{k_r} = 0$ for all k_r.

7.3.17 The proof of Theorem 7.3.15 suggests that the rational and elliptic graphs can and should be discussed uniformly, in both cases $\min \chi = 0$ and in both cases $S_{K,n}$ are contractible for $n \ge 1$. If $\ell + 1$ denotes the number of components of $S_{K,0}$ then $\ell = 0$ corresponds to the rational case and $\ell > 0$ to the elliptic case.

This uniform discussion can be continued: In both of the rational and elliptic cases, for any class $[k]$, one has $\min \chi_{k_r} = 0$. (This follows from 7.3.12*(c)*.)

Let us add the following statement in this spirit.

Proposition 7.3.18 *Assume that Γ is minimally elliptic and it is associated with the minimal resolution. Assume also that the resolution is good and $M(\Gamma)$ is a rational homology sphere. Then $S_{k_r,n}$ is contractible for any $n \ge 0$ and $[k_r] \ne [K]$.*

Proof Let us give first a different proof for $(a) \Rightarrow (c)$ of 7.3.14: if Γ is rational and $k = k_r$ then we can take $l_+ = 0$, that is, there exists a sequence $\{l_i\}_{i \ge 0}$ as in 7.3.9*(a)-(b)-(c)* with $l_0 = 0$. For this take a sequence $\{z_i\}_{i=0}^t$ which connects 0 and Z_{min} provided by the Laufer algorithm 6.6.7. Then $z_0 = 0$, $z_1 = E_1$ and $(E_{v(i)}, z_i) = 1$ for all $1 \le i < t$. Hence $\chi_{k_r}(z_1) = 1 - (s_h, E_1) \ge 1$ and $\chi_{k_r}(z_{i+1}) - \chi_{k_r}(z_i) = -(s_h, E_{v(i)}) \ge 0$. Therefore, the infinite sequence $\{mZ_{min} + z_i\}$ with $m \ge 0$ and $0 \le i < t$ works.

Next, assume that Γ is as in the statement of the proposition. Then the above proof (valid for rational Γ) can be adopted. Indeed, since $[k_r] \ne [K]$, $s_h \ne 0$, there exists $w \in \mathscr{V}$ with $(s_h, E_w) < 0$. By Lemma 7.2.26 there exists a computation sequence $\{z_i\}_{i=0}^t$ for Z_{min} so that the last $E_{v(t-1)}$ is E_w, $(z_i, E_{v(i)}) = 1$ for $1 \le i \le t-2$, and $(z_{t-1}, E_{v(t-1)}) = 2$. Then the above construction works in this case too. □

Remark 7.3.19 Having in mind the $\mathbb{Z}_2$ symmetry of spin^c structures, the restriction $[k_r] \neq [K]$ in 7.3.18 looks strange. Indeed, since $\chi_{-k}(-x) = \chi_k(x)$, one has $S_{-k,n} = S_{k,n}$, hence their (non)contractibility should happen simultaneously. However, K is integral for a minimally elliptic graph, hence $[-K] = [K]$. Therefore, for $k_r = [K] = [-K]$ the space $S_{k_r,0}$ is not connected, and for every other k_r it is contractible.

7.3.B *Measure of Non-rationality. 'Bad' Vertices*

We measure how far an arbitrary tree Γ from a rational graph is. This can be done in different ways: we search for a subset of vertices $\overline{\mathcal{V}} \subset \mathcal{V}$ such that either by eliminating these vertices, or by modification of their decorations, we obtain a rational graph. Different versions of such subset $\overline{\mathcal{V}}$ are called WR, R or SR–sets, depending on the type of the construction. The cardinality and the type WR, R, SR of $\overline{\mathcal{V}}$ indicate the 'badness' of Γ. Graphs with an SR–set of cardinality ≤ 1 are called 'almost rational' (AR). ♣

7.3.20 Recall that decreasing all the Euler numbers of a tree, with all the vertices decorated by $g_v = 0$, we obtain a rational graph, cf. 7.1.5(a). The next definition aims to identify those vertices where such a decrease is really necessary. [Similar subsets of $\mathcal{V}$ were already considered in different articles like [478, 480, 483, 547], mostly under the name 'bad vertices'. Since the definition of the 'badness' was not uniform, and we also wish to distinguish different levels of 'badness', we prefer to give new names for such families.]

Definition 7.3.21 Let Γ be a negative definite connected tree, with arbitrary genus decorations $\{g_v\}_v$ ($g_v \geq 0$).

(a) A subset of vertices $\overline{\mathcal{V}} = \{v_1, \dots, v_\nu\} \subset \mathcal{V}$ is called *WR–set*, if deleting these vertices and their adjacent edges we get a graph with all connected components rational.

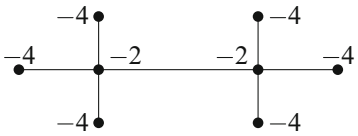
(b) Fix a disjoint decomposition $\overline{\mathcal{V}} \sqcup \mathcal{V}^*$ of the vertices. Let Γ' be a connected maximal subgraph of Γ with vertices in $\mathcal{V}^*$, and let $v \in \overline{\mathcal{V}}$ be an adjacent vertex to Γ'. They determine a maximal subgraph of Γ, which has the vertices of $\Gamma' \cup v$ and all connecting edges. Let $\widetilde{\Gamma}$ be the graph obtained from this subgraph by modification of the decorations of v: the genus $[g_v]$ is changed into $[0]$, and the Euler number E_v^2 is changed into a 'sufficiently negative' integer $e'_v \leq E_v^2$.

The subset of vertices $\overline{\mathcal{V}} = \{v_1, \dots, v_\nu\} \subset \mathcal{V}$ is called *R–set*, if for *any* choice of Γ' and v as above, the graph $\widetilde{\Gamma}$ (with $e'_v \ll 0$) is rational.

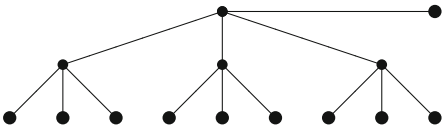
(c) A subset of vertices $\overline{\mathcal{V}} = \{v_1, \dots, v_\nu\} \subset \mathcal{V}$ is called *SR–set*, if by replacing the Euler numbers $e_v = E_v^2$ indexed by $v \in \overline{\mathcal{V}}$ by some more negative integers $e'_v \leq e_v$ we get a rational graph. A graph is called AR-graph ('almost rational graph') if it admits a SR–set of cardinality ≤ 1.

Example 7.3.22

(a) A possible $*$–set ($*$ = WR, R, SR) can be chosen in many different ways, it is not unique even if it is minimal with the corresponding property.
(b) Usually we allow non-minimal $*$–sets as well.
(c) Any R–set is a WR–set. The converse usually is false: in the next graph a single node can serve as a WR–set, but the minimal R–set has two elements.



(d) Any SR–set is an R–set. The converse usually is false: in the next graph the three nodes of the middle row constitute an R–set but any SR–set must contain the 4 nodes.



• $= (-2)$-curve
• $= (-100)$-curve

(e) Any rational graph is AR; for rational graphs the empty set is an SR–set. The class of AR graphs is closed while taking subgraphs or/and decreasing the Euler numbers. If $H_1(L_X, \mathbb{Q}) = 0$ and Γ admits an R–set of cardinality ≤ 1 then it is AR. Indeed, if the Euler decoration of v_0 is $\ll 0$, then the multiplicity of E_{v_0} in Z_{min} is 1, and the Laufer criterion 7.1.2*(iv)* gives the statement.
(f) If $H_1(L_X, \mathbb{Q}) = 0$ then the set of nodes is an SR–set. Hence any star-shaped graph (with $g = 0$) is AR. For any star-shaped graph (with arbitrary g) the central vertex is an R–set.
(g) Any elliptic graph with $H_1(L_X, \mathbb{Q}) = 0$ is AR. Indeed, take a computation sequence $\{z_i\}_i$ for Z_{min} starting from $z_0 = E$. Then $(z_i, E_{v(i)}) = 1$ all the time except for exactly one step $i = i_0$ when this intersection is 2. Then $\{v(i_0)\}$ is an SR–set.
(h) Consider the graph Γ of $S^3_{-p/q}(K)$ (for $p/q > 0$ and $K \subset S^3$ algebraic knot). For notations see 3.5.13. We claim that Γ is AR: if we modify the -1 decoration of v_1 into -2, we get a sandwiched graph Γ_{-2} (for definition see 7.1.5(i)). Indeed, consider the 'smooth' graph Γ_1. Blow up the -1 vertex v_1. The new decoration of v_1 will be -2, while a new -1 vertex is created. Then blow up this new vertex $(m_1 + k_1 - 1)$ times. Then its new decoration will be $-m_1 - k_1$, while it has $(m_1 + k_1 - 1)$ (-1)-vertex neighbours. Fix one of them, and blow up $k_2 - 1$ times. If one continues this procedure, one gets a graph which contains Γ_{-2} as a subgraph.

(i) Let $\{K_i\}_{i=1}^{\nu}$ be algebraic knots and set $K = \#_i K_i$. For $d > 0$ the negative definite graph Γ of $S^3_{-d}(K)$ is given in 3.5.12. We claim that the smallest SR–set consists of the set of (-1)-vertices (their number is ν).

The fact that any (-1)-vertex belongs to any SR–set follows from Laufer's Ratonality Criterion 7.1.2*(iv)*. The claim follows from the fact that if we replace all the (-1) Euler numbers of Γ by -2 then the new graph Γ_{-2} is rational.

Let $\Gamma_{i,-2}$ be the minimal connected subgraph of Γ_{-2} which contains the modified graph Γ_i and the vertex v_+. By (h) we know that $\Gamma_{i,-2}$ is sandwiched. Let $Z \in L(\Gamma_{-2})$ be the cycle whose restriction to the modified Γ_i is Z_i (the compact divisor of the equation, cf. 3.5.12) and has multiplicity one at v_+. Since $Z \in \mathscr{S}(\Gamma_{-2})$ we get that $Z_{min}(\Gamma_{-2}) \leq Z$, hence the multiplicity of v_+ in $Z_{min}(\Gamma_{-2})$ is 1. This means that if we run Laufer's Rationality Criterion starting from $E(\Gamma_{-2})$ (cf. 7.1.2*(iv)*), the subgraphs $\Gamma_{i,-2}$ have no interactions. Since the subgraphs $\Gamma_{i,-2}$ do not obstruct the rationality criterion, the graph Γ_{-2} will not obstruct it either.

Note also that $\{v_+\}$ is a WR–set of Γ. Hence for this graph the cardinality of the minimal SR–set is much larger than the cardinality of a minimal WR–set.

7.3.C *Lattice Fibrations. Universal Cycles in the Fibers*

Recall (see e.g. Sect. 7.3.A, especially the first paragraph of the proof of 7.3.18) that if Γ is rational, then 0 is a χ_{k_r}–minimal lattice point, and $0 \hookrightarrow S_{k_r,n}$ $(n \geq 0)$ admits a strong deformation retraction: there is a χ_{k_r}–non-increasing (combinatorial) flow contracting any $S_{k_r,n}$ (and $L \otimes \mathbb{R}$) to 0.

Here is the motivation and the intuitive picture of the next construction.

Let us start with the lattice L and a representative $k = -Z_K + 2l'_h$. Then (dictated by some 'badness properties' of some of the vertices, indexed by $\overline{\mathscr{V}}$) we will write the set of vertices $\mathscr{V}$ as a disjoint union $\overline{\mathscr{V}} \sqcup \mathscr{V}^*$, such that any sublattice of type $\bar{l} + L(\mathscr{V}^*)$ (where $\bar{l} = \sum_{v \in \overline{\mathscr{V}}} \ell_v E_v \in L(\overline{\mathscr{V}})$) behaves as a rational lattice, that is, it can be contracted to one of its lattice points via a χ_k–non-increasing flow. (In other words, 'L, or the spaces S_n, project to $L(\overline{\mathscr{V}})$ with contractible fibers'.) On the other hand, the χ_k–minimal point of $\bar{l} + L(\mathscr{V}^*)$, where $\bar{l} + L(\mathscr{V}^*)$ contracts, depends essentially on $\bar{l}$; it is a crucial universal point $x_{l'_h}(\bar{l})$ of $\bar{l} + L(\mathscr{V}^*)$. The aim of different reduction theorems is to recover different invariants of the weighted lattice (L, χ_k) from $\{\chi_k(x_{l'_h}(\bar{l}))\}_{\bar{l} \in L(\overline{\mathscr{V}})}$.

In this subsection we define and analyse the points $x_{l'_h}(\bar{l})$. If $l'_h = s_h$ then some additional 'positivity' properties hold for them. ♣

7.3.23 The Definition of the Lattice Points $x(\bar{l})$ Let us fix a resolution of a germ (whose link is not necessarily a rational homology sphere). Suppose we have a family of *distinguished* vertices $\overline{\mathscr{V}} := \{v_k\}_{k=1}^{\nu} \subseteq \mathscr{V}$ (usually chosen by a certain geometric property). Then we split the set of vertices $\mathscr{V}$ into the disjoint union

$\overline{\mathscr{V}} \sqcup \mathscr{V}^*$. Let $\{m_v(x)\}_v$ denote the coefficients of a cycle $x \in L \otimes \mathbb{Q}$, that is $x = \sum_{v \in \mathscr{V}} m_v(x) E_v$.

We use the notation $\bar{l} := \sum_{v \in \overline{\mathscr{V}}} \ell_v E_v \in L(\overline{\mathscr{V}})$, and we fix $h \in H$ and a representative $l'_h \in L'$ with $[l'_h] = h$. Then the cycles $x(\bar{l})$ are defined as follows.

Proposition 7.3.24 ([478, 7.6], [350]) *For any $\bar{l} \in L(\overline{\mathscr{V}})$ there exists a unique cycle $x_{l'_h}(\bar{l}) = x(\bar{l}) \in L$ (depending on the choice of l'_h) satisfying the next properties:*

(a) $m_v(x(\bar{l})) = \ell_v$ for any distinguished vertex $v \in \overline{\mathscr{V}}$;
(b) $(x(\bar{l}) + l'_h, E_v) \le 0$ for every 'non-distinguished vertex' $v \in \mathscr{V}^*$;
(c) $x(\bar{l})$ is minimal with the two previous properties (with respect to $\le$).

Furthermore, the cycle $x(\bar{l})$ automatically satisfies

$$x(\bar{l}) + \bar{l}_1 \le x(\bar{l} + \bar{l}_1) \quad \text{for any } \bar{l}_1 \ge 0, \bar{l}_1 \in L(\overline{\mathscr{V}}). \tag{7.3.5}$$

Proof The proof of *(a)–(c)* is similar to the proof of Proposition 6.6.3. For the last statement note that $x(\bar{l} + \bar{l}_1) - \bar{l}_1$ satisfies *(a)–(b)* for $\bar{l}$, hence *(c)* implies (7.3.5). □

If we wish to emphasize the dependence on l'_h we write $x_{l'_h}(\bar{l})$.

Lemma 7.3.25 *Fix $z \in L$ with $\overline{\mathscr{V}} \sqcup \mathscr{V}^*$-components $(\bar{z}, z^*)$ (hence $z = \bar{z} + z^*$).*

(a) $x_{l'_h + z}(\bar{l}) = x_{l'_h}(\bar{l} + \bar{z}) - z$.
(b) If $(z, E_v) = 0$ for every $v \in \mathscr{V}^*$, then $x_{l'_h}(\bar{l} + \bar{z}) = x_{l'_h}(\bar{l}) + z$.

Proof Use the universal properties 7.3.24 of the cycles $x(\bar{l})$. E.g., for (b), $x_{l'_h}(\bar{l} + \bar{z}) \le x_{l'_h}(\bar{l}) + z$ follows from the universal property of $x_{l'_h}(\bar{l} + \bar{z})$, while the opposite inequality follows from the universal property of $x_{l'_h}(\bar{l})$. □

The cycles $x(\bar{l})$ satisfy the following universal property as well.

Lemma 7.3.26 *Choose l'_h arbitrarily, set $k = -Z_K + 2l'_h$, and fix some $\bar{l} \in L(\overline{\mathscr{V}})$. Assume that a certain $x \in L$ satisfies $m_v(x) = m_v(x(\bar{l}))$ for all $v \in \overline{\mathscr{V}}$, and $x \le x(\bar{l})$.*

Then there is a 'generalized Laufer's computation sequence' connecting x with $x(\bar{l})$. The sequence $\{z_i\}_{i=0}^t$ is constructed as follows. Set $z_0 = x$. Assume that z_i is already constructed. If for some $v \in \mathscr{V}^*$ one has $(z_i + l'_h, E_v) > 0$ then take $z_{i+1} = z_i + E_{v(i)}$, where $v(i)$ is such an index. If z_i satisfies 7.3.24(b), then stop and set $t = i$. Then this procedure stops after finite steps and z_t is exactly $x(\bar{l})$.

Moreover, along the computation sequence $\chi_k(z_{i+1}) \le \chi_k(z_i)$ for any $0 \le i < t$. (Equality holds if $g(E_{v(i)}) = 0$ and $(z_i + l'_h, E_{v(i)}) = 1$.)

Proof Adopt 6.6.3, 6.6.4, 6.6.7 and 7.1.2. □

The generalized computation sequence usually is not unique, one can make several choices for $v(i)$ at each step i.

If the choice of the *distinguished vertices* $\overline{\mathscr{V}}$ is guided by some specific geometric features, then the cycles $x(\bar{l})$ inherit further properties.

7.3.27 Properties of $*$–Sets We start with an addendum of Lemma 7.3.26.

Proposition 7.3.28 *Let $\overline{\mathscr{V}}$ be an WR–set. Choose l'_h and set $k = -Z_K + 2l'_h$. Then $\bar{l} + L(\mathscr{V}^*) = \{x \in L : m_v(x) = m_v(x(\bar{l}))$ for all $v \in \overline{\mathscr{V}}\}$ contracts to $x(\bar{l})$ such that along the contraction χ_k is non-increasing. In particular, $\chi_k(x) \geq \chi_k(x(\bar{l}))$ for any $x \in \bar{l} + L(\mathscr{V}^*)$.*

Proof We follow Lemma 7.3.9 and (proof of) Theorem 7.3.14 *(a)⇒(c)* as models, applied for $\bar{l} + L(\mathscr{V}^*)$. Lemma 7.3.26 shows that l_- can be taken as $x(\bar{l})$. Here we verify that $x(\bar{l})$ might serve for l_+ too. For this we need to show that for any $y > x(\bar{l})$ there exists $E_v \subset |y - x(\bar{l})|$ so that $\chi_k(y - E_v) \leq \chi_k(y)$. If not, then for any such E_v one has $(l'_h + y, E_v) \geq 2 + E_v^2 = (Z_K, E_v)$. But $(l'_h + x(\bar{l}), E_v) \leq 0$, hence $(y - x(\bar{l}) - Z_K, E_v) \geq 0$ for every $E_v \subset |y - x(\bar{l})|$. By summation, $(y - x(\bar{l}) - Z_K, y - x(\bar{l})) \geq 0$, or $\chi(y - x(\bar{l})) \leq 0$, a fact which contradicts the WR (rationality) property. □

In some of the applications we do not really need the cycles $x(\bar{l})$ themselves, but rather the values $\chi_k(x(\bar{l}))$ only. They can be determined inductively.

Proposition 7.3.29 *Assume that $\overline{\mathscr{V}}$ is an R–set. Choose l'_h and set $k = K + 2l'_h$ as above. Then for any $\bar{l} \in L(\overline{\mathscr{V}})_{\geq 0}$ and $v \in \overline{\mathscr{V}}$*

$$\chi_k(x_{l'_h}(\bar{l} + E_v)) = \chi_k(x_{l'_h}(\bar{l})) + \chi(E_v) - (x_{l'_h}(\bar{l}) + l'_h, E_v).$$

Proof We show that $\chi_k(x(\bar{l} + E_v)) = \chi_k(x(\bar{l}) + E_v)$. Consider a computation sequence $\{z_i\}_i$ connecting $z_0 = x(\bar{l}) + E_v$ with $x(\bar{l} + E_v)$, cf. (7.3.5) and 7.3.26. We claim that $(z_i + l'_h, E_{v(i)}) = 1$ for any i. Let Γ' be a connected maximal subgraph with vertices in $\mathscr{V}^*$ and adjacent to v. Let w be its vertex adjacent to v. Clearly we can assume that the computation sequence 'fills in' first the component Γ', this first part will be denoted by $\{z_i\}_{i=0}^t$ (containing all $E_{v(i)}$ supported in Γ'). If $(x(\bar{l}) + E_v + l'_h, E_w) \leq 0$ then this sequence is empty (it will not propagate into Γ'). Otherwise, necessarily, $(x(\bar{l}) + E_v + l'_h, E_w) = 1$ (since $(x(\bar{l}) + l'_h, E_w) \leq 0$ and $(E_v, E_w) = 1$; hence $(x(\bar{l}) + l'_h, E_w) = 0$ follows a posteriori). Hence $z_1 = x(\bar{l}) + E_v + E_w$. Next, one considers the connected graph $\widetilde{\Gamma}$ containing the vertices $\Gamma' \cup \{v\}$ as in Definition 7.3.21*(b)*. Then $\tilde{z}_i := z_i - x(\bar{l})$ $(0 \leq i \leq t)$ is supported on $\widetilde{\Gamma}$, and $\{\tilde{z}_i\}_{i=0}^t$ is the beginning of a computation sequence connecting E_v with $Z_{min}(\widetilde{\Gamma})$. Indeed, $(z_i - x(\bar{l}), E_{v(i)}) \geq (z_i + l'_h, E_{v(i)})$ (†), hence if $(z_i + l'_h, E_{v(i)}) \geq 1$ then $(\tilde{z}_i, E_{v(i)}) \geq 1$ too. However, $\widetilde{\Gamma}$ being rational, by the criterion 7.1.2*(iv)*, necessarily $(\tilde{z}_i, E_{v(i)}) = 1$ in $\widetilde{\Gamma}$, hence in Γ too. Thus, by (†), $(z_i + l'_h, E_{v(i)}) = 1$. □

This has the following application regarding the assumptions of the 'Unbalanced Vanishing Theorem' 6.4.12.

Corollary 7.3.30 *Assume that $\overline{\mathscr{V}}$ is an R–set. Fix any $s \in \mathscr{S}$. Then there exists a computation sequence $\{x_i\}_{i=0}^{t}$ ($x_i \in L_{\geq 0}$, $x_0 = 0$, $x_{i+1} = x_i + E_{v(i)}$) such that $x_t = s$ and $(E_{v(i)}, x_i) \leq 1$ whenever $v(i) \notin \overline{\mathscr{V}}$.*

Hence, a sequence required by Theorem 6.4.12 with $\mathscr{U} = \overline{\mathscr{V}}$ always exists.

Proof Combine the proofs of Lemma 6.4.15 and Proposition 7.3.29. □

Theorem 7.3.31 (Unbalanced Vanishing Theorem—Conceptual Version) *Assume that $\overline{\mathscr{V}}$ is an R–set. Then there exists a constant M (computable from Γ) such that $h^1(\mathscr{L}) = 0$ whenever $c_1(\mathscr{L}) \in -\mathscr{S}'$ and $(c_1(\mathscr{L}), E_v) \geq M$ for all $v \in \overline{\mathscr{V}}$.*

7.3.32 For special choices of l'_h one has the following 'positivity' statements.

Lemma 7.3.33

(a) If $l'_h = s_h$ or $l'_h = r_h$ then $x_{l'_h}(\bar{l}) \geq \bar{l}$ whenever $\bar{l} \geq 0$.

(b) In particular, $x_{s_h}(\bar{0}) = 0$ and $\chi_{k_r}(x_{s_h}(\bar{0})) = 0$.

Proof Set $l'_h = s_h$ and for a certain $\bar{l} \geq 0$ assume that $x(\bar{l}) - \bar{l} = y_1 - y_2$ with y_i supported on $\mathscr{V}^*$, effective and without common components. Then $x(\bar{l}) + s_h - \bar{l} - y_1 = s_h - y_2$; $(s_h - y_2, E_v) \leq 0$ if $E_v \not\subset |y_2|$ and $(x(\bar{l}) + s_h - \bar{l} - y_1, E_v) \leq 0$ otherwise. Hence $s_h - y_2 \in \mathscr{S}'$, and $y_2 = 0$ by the minimality of s_h. Hence $x(\bar{l}) \geq \bar{l}$. In particular, $x_{s_h}(\bar{0}) \geq 0$. But $s_h \in \mathscr{S}'$, hence $x_{s_h}(\bar{0}) \leq 0$ by the minimality property of $x_{s_h}(\bar{0})$.

If $l'_h = r_h$, write $x(\bar{l}) = \bar{l} + x^*$ and $r_h = \bar{r_h} + r_h^*$. Then $(x^* + r_h^*, E_v) \leq (-\bar{l} - \bar{r_h}, E_v) \leq 0$ for every $v \in \mathscr{V}^*$, hence $x^* + r_h^* \geq 0$ by 2.1.19. But then $x^* \geq 0$ too. □

7.3.D Concatenated Computation Sequences of AR Graphs

We analyse AR graphs. For such graphs we prove that any connected component of a non-empty $S_{k,n}$ is contractible. ♣

7.3.34 Assume that Γ is an AR resolution graph. In particular $M(\Gamma)$ is a rational homology sphere. AR property means that there exists $v_0 \in \mathscr{V}$ such that if we replace $e_{v_0} = E_{v_0}^2$ by $e'_{v_0} \ll e_{v_0}$ then we get a rational graph. If $H_1(L_X, \mathbb{Q}) = 0$ then the fact that $\{v_0\}$ is an AR–set is equivalent to the fact that $\{v_0\}$ is an R–set, cf. 7.3.22(e). In the sequel we fix such a vertex v_0.

Our goal is to prove the following property.

Theorem 7.3.35 *If Γ is AR, then for any $k \in \mathrm{Char}$ and $n \geq m_k = \min \chi_k$ any connected component of $S_{k,n}$ is contractible.*

Note that the statement is independent of the choice of k in its class, cf. 7.3.11. In the sequel we will choose the distinguished representative k_r, and we write S_n for $S_{k_r,n}$. We also write $\mathscr{V} = \overline{\mathscr{V}} \sqcup \mathscr{V}^*$, where $\overline{\mathscr{V}} = \{v_0\}$. For each $\ell \in \mathbb{Z}$ we consider the cycles $\bar{l} := \ell E_{v_0} \in L(\overline{\mathscr{V}})$ and $x(\bar{l}) \in L$ from 7.3.C. We abridge $x_{s_h}(\ell E_{v_0})$ as $x(\ell)$.

In order to prove the theorem we construct an increasing path $\gamma = \{l_i\}_{i\geq 0}$ in L (cf. 6.8.B), which determines the 1-chain $C_\gamma := \cup_{i\geq 0}[l_i, l_{i+1}]$ of 1-cubes in $L \otimes \mathbb{R}$, such that $C_\gamma \cap S_n \hookrightarrow S_n$ is a homotopy equivalence. The construction and the statement of the theorem constitute the prototype of the more general Reduction Theorem 11.4.3. We also mention that this construction was the original intuitive motivation and starting point in the definition of the graded roots, cf. 11.2 and 11.2.A.

The construction starts as follows. By Lemma 7.3.12*(a)* the inclusion $R(0,\infty) \cap S_n \subset S_n$ admits a strong deformation retraction. Hence we can restrict ourself to the intersection with the first quadrant. The path $\gamma = \{l_i\}_{i\geq 0}$ is defined as a series of concatenated computation sequences. It contains, as intermediate terms, all the universal cycles $\{x(\ell)\}_{\ell\geq 0}$ in an increasing order. The first term is $l_0 = x(0) = 0$ (cf. Lemma 7.3.33). The part of the sequence connecting $x(\ell)$ and $x(\ell+1)$ starts with $x(\ell)$ and the next term is $x(\ell) + E_{v_0}$. Then, the continuation is a generalized Laufer-type computation sequence connecting $x(\ell) + E_{v_0}$ with $x(\ell+1)$. Indeed, the multiplicity of E_0 in both $x(\ell) + E_{v_0}$ and $x(\ell+1)$ is $\ell+1$, and by (7.3.5) $x(\ell+1) \geq x(\ell) + E_{v_0}$. Hence Lemma 7.3.26 gives a computation sequence $\gamma^{(\ell+1)} = \{l_i^{(\ell+1)}\}_i$, which connects them. Then we proceed inductively.

Set $\tau(\ell) := \chi_{k_r}(x_{s_h}(\ell))$. Let o be the order of $[E^*_{v_0}]$ in L'/L and $p := m_{v_0}(oE^*_{v_0})$.

Lemma 7.3.36

(a) The path $\{l_i\}_i$ is increasing: $l_{i+1} = l_i + E_{v(i)}$.
(b) For any E_v-coefficient one has $\lim_{\ell\to\infty} m_v(x_{s_h}(\ell)) = \infty$ (where $v \in \mathscr{V}$).
(c) (Quasiperiodicity) $x_{s_h}(\ell + tp) = x_{s_h}(\ell) + toE^*_{v_0}$.
(d) χ_{k_r} along each part (subsequence) $\gamma^{(\ell)}$ is constant.
(e) $\tau(\ell+1) = \tau(\ell) + 1 - (x_{s_h}(\ell) + s_h, E_{v_0})$.
(f) There exists ℓ_0 such that $\tau(\ell+1) \geq \tau(\ell)$ for $\ell \geq \ell_0$.

Proof For *(b)* use part *(c)*, which follows from the universal properties of $x(\ell)$ and $x(\ell + tp)$ (cf. 7.3.24*(c)*) as in (7.3.5). For *(d)-(e)* use 7.3.29. For *(f)* set M as $\max_{0\leq\ell<p}\{(x(\ell), E_{v_0})\}$. Then by *(e)* $(x(\ell + tp), E_{v_0}) \leq M - to$ for any $0 \leq \ell < p$ and $t \geq 0$. □

Theorem 7.3.37 *Consider the 1-chain $C_\gamma := \cup_{i\geq 0}[l_i, l_{i+1}]$ in $L \otimes \mathbb{R}$ as above. Then for any n the inclusion $C_\gamma \cap S_n \subset S_n$ is a homotopy equivalence. In particular, since each connected component of $C_\gamma \cap S_n$ is contractible, Theorem 7.3.35 follows.*

Proof There is a natural projection $L(\mathscr{V}) = \mathbb{Z}^s \to L(\overline{\mathscr{V}}) = \mathbb{Z}$, which induces a projection $p : L(\mathscr{V}) \otimes \mathbb{R} = \mathbb{R}^s \to L(\overline{\mathscr{V}}) \otimes \mathbb{R} = \mathbb{R}$. It restricts to the quadrants $R(0,\infty)_L \to R(0,\infty)_{\mathbb{Z}}$ (the real rectangles in $L(\mathscr{V})$ and $L(\overline{\mathscr{V}})$ respectively)

and it extends to maps at the level of cubes as well. We wish to show that the inclusion $C_\gamma \cap S_n \cap R(0,\infty)_L \subset S_n \cap R(0,\infty)_L$ is a homotopy equivalence. In fact, we can replace this pair by a compact pair. Indeed, let us fix ℓ_0 as in Lemma 7.3.36*(f)*, and consider the part of the path Γ starting from $x(\ell_0)$, namely $x(\ell_0), x(\ell_0)+E_{v_0}, \gamma^{(\ell_0+1)}, x(\ell_0+1)+E_{v_0}, \gamma^{(\ell_0+2)}, \cdots$. By Lemma 7.3.36 this is χ_{k_r}–non-decreasing. Hence, similarly as in 7.3.9 with $l_+ = x(\ell_0)$, or in 7.3.28, we get that the above pair has the homotopy type of $C_\gamma \cap S_n \cap R(0, x(\ell_0))_L \subset S_n \cap R(0, x(\ell_0))_L$.

Next, we slice $S_n \cap R(0, x(\ell_0))_L$ with the fibers of $p : S_n \cap R(0, x(\ell_0))_L \to R(0,\ell_0)_{\mathbb{Z}}$. The base space $R(0,\ell_0)_{\mathbb{Z}}$ is a segment, and it has 0–cubes $\{\ell\}_{0\le\ell\le\ell_0}$ and 1-cubes $\{[\ell,\ell+1]\}_{0\le\ell<\ell_0}$. For each ℓ we show that the inclusions $C_\gamma \cap S_n \cap R(0,x(\ell_0)) \cap p^{-1}(\ell) \subset S_n \cap R(0,x(\ell_0)) \cap p^{-1}(\ell)$ and $C_\gamma \cap S_n \cap R(0,x(\ell_0)) \cap p^{-1}([\ell,\ell+1]) \subset S_n \cap R(0,x(\ell_0)) \cap p^{-1}([\ell,\ell+1])$ admit strong deformation retractions and all these homotopy equivalences are compatible with the corresponding boundary maps and the retractions glue together to the path C_γ.

The retractions will be defined via certain paths as in several previous proofs. For each $0 \le \ell \le \ell_0$ we construct two increasing sequences: $\alpha^{(\ell)}$ connects ℓE_{v_0} to $x(\ell)$, while $\beta^{(\ell)}$ connects $x(\ell)$ to $x(\ell_0) - (\ell_0-\ell)E_{v_0}$. Both $\alpha^{(\ell)}$ and $\beta^{(\ell)}$ sit in $p^{-1}(\ell)$.

$\alpha^{(\ell)}$ consists of the concatenated sequence: $\gamma^{(1)} + (\ell-1)E_{v_0}, \ldots, \gamma^{(\ell-1)} + E_{v_0}, \gamma^{(\ell)}$.

$\beta^{(\ell)}$ consists of the concatenated sequence: $\gamma^{(\ell+1)} - E_{v_0}, \ldots, \gamma^{(\ell_0)} - (\ell_0-\ell)E_{v_0}$.

Shifting $\gamma^{(k)}$-sequences is monotone in the following sense: for any $v \ne v_0$

$$\begin{cases} \text{If } \chi_{k_r}(l) \ge \chi_{k_r}(l+E_v) \text{ then } \chi_{k_r}(l+E_{v_0}) \ge \chi_{k_r}(l+E_v+E_{v_0}) \text{ too,} \\ \text{If } \chi_{k_r}(l+E_{v_0}) \le \chi_{k_r}(l+E_v+E_{v_0}) \text{ then } \chi_{k_r}(l) \le \chi_{k_r}(l+E_v) \text{ too.} \end{cases}$$

This shows that χ_{k_r} along $\alpha^{(\ell)}$ is non-increasing, while along $\beta^{(\ell)}$ is non-decreasing.

Let us consider first the homotopy type of the pair $(A^{(\ell)} \subset B^{(\ell)}) = (C_\gamma \cap S_n \cap R(0,x(\ell_0)) \cap p^{-1}(\ell) \subset S_n \cap R(0,x(\ell_0)) \cap p^{-1}(\ell))$. Using again retractions along the sequences $\alpha^{(\ell)}$ and $\beta^{(\ell)}$, the pair retracts to the pair $(\{x(\ell)\} \subset \{x(\ell)\})$.

Let us note the following phenomenon. In $p^{-1}(\ell+1)$ by the above argument we can contract $S_n \cap R(0,x(\ell_0)) \cap p^{-1}(\ell+1))$ to $x(\ell+1)$. However, since χ_{k_r} is constant along $\gamma^{(\ell+1)}$, we can contract it to the point $x(\ell)+E_{v_0}$ too. This can be done if we replace the two paths $\alpha^{(\ell+1)}$ and $\beta^{(\ell+1)}$ in the fiber $p^{-1}(\ell+1)$ with $\alpha^{(\ell)}+E_{v_0}$, respectively with the concatenated $\gamma^{(\ell+1)}$ and $\beta^{(\ell+1)}$.

In fact, we see that the pair $(A^{(\ell+1)} \subset B^{(\ell+1)})$ can be retracted to several pairs, either to the pair $(\{x(\ell)+E_{v_0}\} \subset \{x(\ell)+E_{v_0}\})$, or to $(\{x(\ell+1)\} \subset \{x(\ell+1)\})$, or even to $(C_{\gamma^{(\ell+1)}}, C_{\gamma^{(\ell+1)}})$ as well. Here $C_{\gamma^{(\ell+1)}}$ is the path connecting $x(\ell)+E_{v_0}$ to $x(\ell+1)$ in $p^{-1}(\ell+1)$ given by the sequence $\gamma^{\ell+1}$.

Moreover, the pair $C_\gamma \cap S_n \cap R(0,x(\ell_0)) \cap p^{-1}([\ell,\ell+1]) \subset S_n \cap R(0,x(\ell_0)) \cap p^{-1}([\ell,\ell+1])$ retracts along $\alpha^{(\ell)}+E_{v_0}$ and the concatenated $(\gamma^{(\ell+1)}, \beta^{(\ell+1)})$ to the pair of identical segments $([x(\ell), x(\ell)+E_{v_0}] \subset [x(\ell), x(\ell)+E_{v_0}])$.

Let us summarize. The subspace $C_\gamma \cap S_n \cap R(0,x(\ell_0)) \cap p^{-1}([\ell,\ell+1])$ admits a strong deformation retraction to the segment $[x(\ell), x(\ell)+E_{v_0}])$, $C_\gamma \cap S_n \cap$

$R(0, x(\ell_0)) \cap p^{-1}(\ell+1)$ to the 1–chain $C_{\gamma^{(\ell+1)}}$ connecting $x(\ell) + E_{v_0}$ to $x(\ell+1)$, $C_\gamma \cap S_n \cap R(0, x(\ell_0)) \cap p^{-1}([\ell+1, \ell+2])$ to the segment $[x(\ell+1), x(\ell+1)+E_{v_0}]$), etc.

These steps iterated show that $C_\gamma \cap S_n \subset S_n$ admits a strong deformation retraction. □

Example 7.3.38

(a) If Γ is rational, then Theorems 7.3.14 and 7.3.37 combined show that χ_{k_r} along the sequence γ constructed above is non-decreasing, hence along γ the space $S_n \cap R(0, \infty)$ can be contracted to the origin ($n \geq 0$).

(b) Assume that Γ is elliptic and $k_r = -Z_K$. We wish to compare the set $\{x(\ell)\}_{\ell \geq 0}$ with the set $\{F_{-1} = 0, F_1, \ldots, F_n\}$ associated in 7.2.40 (3) with the elliptic sequence (see also 7.3.15 for another characterization of the set $\{F_{-1} = 0, F_1, \ldots, F_n\}$).

First note that $\chi(x(\ell+1)) > \chi(x(\ell))$ if and only if $x(\ell) \in \mathscr{S}$ (use 7.3.36*(e)* and the definition of $x(\ell)$). Furthermore, we claim that

$$\{x(\ell) : \chi(x(\ell)) = 0,\ x(\ell) \in \mathscr{S}\} = \{F_{-1}, \ldots, F_n\}. \tag{7.3.6}$$

Hence, the sequence $\{x(\ell)\}_{\ell \geq 0}$ contains (at certain easily identifiable places) the guiding cycles $\{F_i\}_i$ of the geometry of the elliptic germs.

Indeed, the inclusion $\subset$ in (7.3.6) follows from 7.2.40(5). Conversely, fix one of the cycles F_i. Let ℓ_i be its E_{v_0}–multiplicity. Then both $x(\ell_i)$ and F_i have the same E_{v_0} multiplicity. Hence by 7.3.28 we have $0 = \chi(F_i) \geq \chi(x(\ell_i)) \geq 0$. Thus $\chi(x(\ell_i)) = 0$.

Furthermore, from 7.2.40(3) we have $F_i \in \mathscr{S}$, hence by the minimality of $x(\ell_i)$ we obtain $F_i \geq x(\ell_i)$. But then, since $E_{v_0} \not\subset |F_i - x(\ell_i)|$, we get $(F_i - x(\ell_i), E_{v_0}) \geq 0$. This together with $F_i \in \mathscr{S}$ shows that $x(\ell_i) \in \mathscr{S}$ too.

Hence, under this correspondence, $m_{v_0}(F_i) = m_{v_0}(x(\ell_i))$.

Recall also that by 7.3.15 the cycles $\{F_{-1}, \ldots, F_n\}$ are the (unique) maximal elements of the connected component of $S_{K,0}$ (hence $n+1 = \ell_{seq}$).

Remark 7.3.39 In general, it is not easy to find the cycles $x(\ell)$ (say, associated with k_r). Fortunately, in several applications (see e.g. 11.3) one does not need all the coefficients of these cycles, only the values $\tau(\ell) = \chi_{k_r}(x_{s_h}(\ell))$. In most of the cases they are computed inductively using 7.3.36*(e)*, hence basically one needs only to know $(x_{s_h}(\ell) + s_h, E_{v_0})$ for any ℓ.

Example 7.3.40 For the determination of the universal cycles $\{x_{s_h}(\ell)\}_\ell$ and the corresponding τ-function in the case of star-shaped graphs and surgery manifolds see 11.3.A, 11.3.B and 11.5.

Chapter 8
Multivariable Divisorial Filtration

8.1 Multi-Variable Series

8.1.A Multigradings

We consider rings graded by an abelian group A, and (under some finiteness assumption) we define their Poincaré series. Furthermore, in the presence of a finite abelian group action, we define their equivariant versions as well. ♣

8.1.1 Let A be an abelian group. We say that the polynomial ring $S := \mathbb{C}[\mathbf{x}] = \mathbb{C}[x_1, \dots, x_n]$ is *A–multigraded* if a group homomorphism $\deg : \mathbb{Z}^n \to A$ is given, which associates to each monomial $\mathbf{x}^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ an element $\mathbf{a} = \deg(\alpha) \in A$ (denoted also by $\deg(\mathbf{x}^\alpha)$). For any $\mathbf{a} \in A$ let $S_\mathbf{a}$ be the vector space $\mathbb{C}\langle \mathbf{x}^\alpha \rangle_{\deg(\alpha)=\mathbf{a}}$; its elements are called homogeneous polynomials of degree $\mathbf{a}$. Then $S = \oplus_{\mathbf{a}\in A} S_\mathbf{a}$, and $S_\mathbf{a} \cdot S_\mathbf{b} \subset S_{\mathbf{a}+\mathbf{b}}$. We assume that $\ker(\deg) \cap \mathbb{N}^n = 0$; hence $S_\mathbf{0} = \mathbb{C}$, and each $S_\mathbf{a}$ is finite dimensional, cf. [425, 8.1].

More generally, *a ring R is A–multigraded* if there is a $\mathbb{Z}$-module isomorphism $R = \oplus_{\mathbf{a}\in A} R_\mathbf{a}$ and $R_\mathbf{a} \cdot R_\mathbf{b} \subset R_{\mathbf{a}+\mathbf{b}}$. Elements of $R_\mathbf{a}$ are called homogeneous of degree $\mathbf{a}$. In most of our cases R is an A–multigraded $\mathbb{C}$-algebra such that $R_\mathbf{0} = \mathbb{C}$ and $R_\mathbf{a}$ is finite dimensional for every $\mathbf{a}$.

$\mathbb{C}$ itself is A–multigraded (for any A) if we take $R_\mathbf{0} = \mathbb{C}$ and $R_\mathbf{a} = 0$ for $\mathbf{a} \neq \mathbf{0}$.

Fix a ring R with an A–multigrading. An R-module M is *A–multigraded* if M admits a direct sum decomposition $M = \oplus_{\mathbf{a}\in A} M_\mathbf{a}$ such that $R_\mathbf{a} \cdot M_\mathbf{b} \subset M_{\mathbf{a}+\mathbf{b}}$ for any $\mathbf{a}$ and $\mathbf{b}$. There is a well-defined notion of graded morphisms of graded modules, hence of graded kernels, images and cokernels as well.

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If M is an A–multigraded $\mathbb{C}$-module (i.e. $M = \oplus_{\mathbf{a}\in A} M_{\mathbf{a}}$ as $\mathbb{C}$–vector spaces) such that each $M_{\mathbf{a}}$ is finite dimensional, then one defines the Poincaré series of M by

$$P(M;\mathbf{t}) := \sum_{\mathbf{a}\in A} \dim(M_{\mathbf{a}})\cdot \mathbf{t}^{\mathbf{a}} \in \mathbb{Z}[[A]]. \tag{8.1.1}$$

Usually, A is naturally embedded in some lattice $\mathbb{Z}^s$. If $A = \mathbb{Z}^s$, and in $\mathbb{Z}^s$ we fix a base $\{e_k\}_k$, then $\mathbb{Z}[[A]]$ is also denoted by $\mathbb{Z}[[\mathbf{t}^{\pm 1}]] := \mathbb{Z}[[t_1^{\pm 1},\dots,t_s^{\pm 1}]]$, and $\mathbf{t}^{\mathbf{a}} = \prod_k t_k^{a_k}$ for $\sum_k a_k e_k = \mathbf{a} \in A$. Set also $\mathbb{Z}[[\mathbb{N}^s]] = \mathbb{Z}[[\mathbf{t}]] := \mathbb{Z}[[t_1,\dots,t_s]]$, the ring of formal power series, $\mathbb{Z}[\mathbf{t}]$, the ring of polynomials, and $\mathbb{Z}[\mathbf{t}][\mathbf{t}^{-1}]$ the ring of Laurent polynomials. Then $\mathbb{Z}[[\mathbf{t}^{\pm 1}]]$ is a module over both rings $\mathbb{Z}[\mathbf{t}]$ and $\mathbb{Z}[\mathbf{t}][\mathbf{t}^{-1}]$.

If $\mathbf{a} \in \mathbb{N}^s \setminus \{\mathbf{0}\}$, then we rewrite the series $\sum_{k\in\mathbb{N}} \mathbf{t}^{k\mathbf{a}}$ in the form of the 'rational' expression $1/(1-\mathbf{t}^{\mathbf{a}})$.

Example 8.1.2

(a) Assume that each $\deg(x_i) \in \mathbb{N}^s \setminus \{\mathbf{0}\}$. Then

$$P(\mathbb{C}[\mathbf{x}];\mathbf{t}) = \frac{1}{\prod_{i=1}^n (1-\mathbf{t}^{\deg(x_i)})} \in \mathbb{Z}[[A]]. \tag{8.1.2}$$

(b) Assume that the polynomials $f_1,\dots,f_r$ are homogeneous with respect to deg. Then the affine coordinate ring $R = \mathbb{C}[\mathbf{x}]/(f_1,\dots,f_r)$ is a graded module. Moreover, if $\{f_j\}_j$ is a regular sequence, then for the complete intersection R one has

$$P(R;\mathbf{t}) = \frac{\prod_{j=1}^r (1-\mathbf{t}^{\deg(f_j)})}{\prod_{i=1}^n (1-\mathbf{t}^{\deg(x_i)})} \in \mathbb{Z}[[A]].$$

For the proof use the Koszul complex associated with $f,\dots,f_r$, cf. [89, 1.6].

Example 8.1.3 (The Equivariant Case)

(a) Let us fix a multigrading as in 8.1.1. Assume that H is a finite abelian group, which acts on $\mathbb{C}^n$ via diagonal matrices. Then H acts on the polynomial ring $\mathbb{C}[\mathbf{x}]$ as well via the characters $\chi_1,\dots,\chi_n \in \widehat{H}$; that is, $h * x_i = \chi_i(h)x_i$ for all i. Then the action preserves the multigrading. For a character $\chi \in \widehat{H}$ we denote by $\mathbb{C}[\mathbf{x}]^{\chi}$ the χ-eigenspace, which has a natural induced A–multigrading. Then the Poincaré series of $\mathbb{C}[\mathbf{x}]^{\chi}$ is given by the *Molien's formula*: [72, 439, 651]

$$P(\mathbb{C}[\mathbf{x}]^{\chi};\mathbf{t}) = \frac{1}{|H|}\cdot \sum_{h\in H} \frac{\chi(h)^{-1}}{\prod_{i=1}^n (1-\chi_i(h)\mathbf{t}^{\deg(x_i)})} \in \mathbb{Z}[[A]].$$

This is equivalent (via Fourier transform) by

$$\sum_{\chi\in\widehat{H}} P(\mathbb{C}[\mathbf{x}]^{\chi},\mathbf{t})\cdot\chi=\frac{1}{\prod_{i=1}^{n}(1-\chi_i\cdot\mathbf{t}^{\deg(x_i)})}\in\mathbb{Z}[[A]][\widehat{H}].$$

Here, again, $1/(1-\chi\cdot\mathbf{t}^{\mathbf{a}})$ means $\sum_{k\in\mathbb{N}}\mathbf{t}^{k\mathbf{a}}\cdot\chi^k\in\mathbb{Z}[[A]][\widehat{H}]$.

(b) Assume that R is a complete intersection $\mathbb{C}[\mathbf{x}]/(f_1,\dots,f_r)$ as in 8.1.2. We consider a group action of H on $\mathbb{C}[\mathbf{x}]$ as above, which has the property that $h*f_j=\rho_j(h)f_j$ for any j, for certain characters $\rho_j\in\widehat{H}$. Then R inherits an action of H compatibly with the multigrading. The Poincaré series of its χ-eigenspace R^{χ} is:

$$P(R^{\chi};\mathbf{t})=\frac{1}{|H|}\cdot\sum_{h\in H}\chi(h)^{-1}\cdot\frac{\prod_{j=1}^{r}(1-\rho_j(h)\mathbf{t}^{\deg(f_j)})}{\prod_{i=1}^{n}(1-\chi_i(h)\mathbf{t}^{\deg(x_i)})}.$$

The proof is a combination of the Molien's formula with the Koszul complex, see e.g. [533, Appendix]. Again, by Fourier transform, this also reads as

$$\sum_{\chi\in\widehat{H}} P(R^{\chi},\mathbf{t})\cdot\chi=\frac{\prod_{j=1}^{r}(1-\rho_j\cdot\mathbf{t}^{\deg(f_j)})}{\prod_{i=1}^{n}(1-\chi_i\cdot\mathbf{t}^{\deg(x_i)})}.$$

Example 8.1.4 (Cyclic Quotient Singularities) Consider the singularity $(X_{n,q},o)$ defined in Example 2.3. Here we focus on the realization *(e)* of Theorem 2.3.1. Accordingly, we consider its affine coordinate ring $R:=\mathbb{C}[\sigma^{\vee}\cap\mathbb{Z}^2]$, where $\sigma^{\vee}$ is the real cone in $\mathbb{R}^2$ generated by $(1,0)$ and (q,n). Let Q be the open–closed 'unit cube' $\{t(1,0)+s(q,n)\,:\,0\le t,s<1\}$, and set $B:=Q\cap\mathbb{Z}^2$. Let us define the 'tautological multidegree', which assigns to the monomial associated with the lattice point (n,m) the degree $(n,m)\in\mathbb{Z}^2$. In this case

$$P(R;\mathbf{t})=\sum_{\mathbf{a}\in\sigma^{\vee}\cap\mathbb{Z}^2}\mathbf{t}^{\mathbf{a}}=\frac{\sum_{\mathbf{b}\in B}\mathbf{t}^{\mathbf{b}}}{(1-\mathbf{t}^{(1,0)})(1-\mathbf{t}^{(q,n)})}.$$

The equivariant version runs as follows (for notation see again 2.3). Consider $\mathbb{C}[z_1,z_2]$ with the action $\xi*(z_1,z_2)=(\xi z_1,\xi^q z_2)$, where $H=\{\xi\in\mathbb{C}\,:\,\xi^n=1\}$. Then the invariant part $\mathbb{C}[z_1,z_2]^H$ is exactly R. Extend the multigrading deg by $z_1\mapsto\frac{1}{n}(1,0)$, $z_2\mapsto\frac{1}{n}(q,n)$, with $A:=(\frac{1}{n}\mathbb{Z})\times\mathbb{Z}$. Set $\chi_0\in\widehat{H}$, $\chi_0(e^{2\pi i/n})=e^{2\pi i/n}$. Then

$$\sum_{\chi\in\widehat{H}} P(\mathbb{C}[z_1,z_2]^{\chi};\mathbf{t})\cdot\chi=\frac{1}{(1-\chi_0\cdot\mathbf{t}^{\frac{1}{n}(1,0)})(1-\chi_0^{q}\cdot\mathbf{t}^{\frac{1}{n}(q,n)})}\in\mathbb{Z}[[A]][\widehat{H}].\tag{8.1.3}$$

Example 8.1.5 Assume that $A = \mathbb{Z}$ and $R = \oplus_{\ell=0}^{\infty} R_\ell$ with $R_0 = \mathbb{C}$. If R is Noetherian then R is finitely generated as a $\mathbb{C}$-algebra and each R_ℓ is a finite dimensional $\mathbb{C}$–vector space. Let $y_1, \dots, y_s$ be a set of homogeneous generators of R. Then, the *Hilbert–Serre theorem* asserts (see e.g. [33, 89, 650]) that there exists a polynomial $K(R,t) \in \mathbb{Z}[t]$ such that

$$P(R;\mathbf{t}) = \frac{K(R,t)}{\prod_{i=1}^{s}(1-t^{\deg(y_i)})} \in \mathbb{Z}[[t]].$$

If d is the (Krull) dimension of R then there exist $a_1, \dots, a_d \in \mathbb{Z}_{>0}$ and $K(R,t) \in \mathbb{Z}[t]$ with $K(R,1) > 0$ such that

$$P(R;\mathbf{t}) = \frac{K(R,t)}{\prod_{i=1}^{d}(1-t^{a_i})} \in \mathbb{Z}[[t]].$$

In particular, $\lim_{t\to 1}(1-t)^d \cdot P(R,t)$ is a non-zero finite number, that is, d is the pole order at $t=1$ of $P(R,t)$, cf. [33, 650].

See [650] for additional properties of $P(R,t)$ when R is Cohen–Macaulay or Gorenstein.

8.1.B *Poincaré series of Weighted Homogeneous Singularities*

Recall that all the finitely generated $\mathbb{Z}_{\geq 0}$-graded $\mathbb{C}$-algebras correspond to affine varieties with good $\mathbb{C}^*$-actions, see 5.1.1. Assume additionally that the singularity is normal and 2-dimensional with $g = 0$. We provide the equivariant Poincaré series $\{P_h\}_{h\in H}$ associated with the graded algebra of the universal abelian covering in terms of the topological Seifert invariants of the link (formula obtained by Dolgachev, Neumann and Pinkham).

In order to understand the asymptotic behaviour of the coefficients of $P_h(R,t) = \sum_{\ell\geq 0} c_\ell t^\ell$, in this book we will consider several 'regularization procedures':

(I) $\ell \mapsto c_\ell$ for $\ell \gg 0$ is a quasipolynomial $1 + N(\ell)$ of degree 1 explicitly computable from the Seifert invariants. In particular, there exists a period $p \in \mathbb{Z}_{>0}$ and a polynomial $n \mapsto \mathfrak{P}_p(n)$ of degree 2 such that $n \mapsto \sum_{\ell<pn} c_\ell$ equals $\mathfrak{P}_p(n)$ for every n. The coefficients of $\mathfrak{P}_p(n)$ (especially its free term, called the 'periodic constant' of P_h, cf. 8.7.1) have a distinguished role in this book, cf. 8.7.9. The formula is based on the computation of the degree 2 polynomial $n \mapsto \sum_{\ell<\alpha n}(1+N(\ell))$ given in 8.1.11.

(II) We will express (in a unique way) the (twisted) rational function $P_h = P_h(R,t)$ as $P_h = P_h^+ + P_h^-$, where P_h^+ is a (twisted) polynomial and P^- is a (twisted) rational function of negative degree. The decomposition (with such properties) is unique, and, in fact, the coefficients of P_h^- are given exactly by $1 + N(\ell)$ for every ℓ. It turns out that P_h and P_h^- determine

each other. The 'polynomial part P_h^+' plays a crucial role as well. The regularizations (I) and (II) are connected by $\mathfrak{P}_p(0) = P_h^+(1)$.

(III) In part (I) the numbers $\{c_\ell\}_\ell$ are (tautologically) the coefficients of the Taylor expansion at $t = 0$ of the rational function determined by the series. Additionally, we consider the Laurent expansion of this function at $t = 1$, $P_h(R, t) = \sum_{i \geq -2} C_i(1-t)^i$. Its coefficients $\{C_i\}_i$ are again important invariants (C_{-2} and C_{-1} are 'classical', see 8.1.12, while C_0 is related with the torsion invariant of the link, see 9.7.A).

The formulae (and their regularization properties) serve as prototype for generalizations for other singularities, when the local algebra is not graded, however it has several filtrations. ♣

8.1.6 Let (X, o) be a weighted homogeneous normal surface singularity, see 3.5.A and 5.1 for notations. We assume $\nu \geq 3$ and $g = 0$. Recall that e denotes the virtual (orbifold) Euler number and χ the virtual Euler characteristic of the link.

By Theorem 5.1.33, the universal abelian covering (X_a, o) is a complete intersection in $\mathbb{C}[z_1, \dots, z_\nu]$. We introduce the rational degrees $\deg(z_i) = (|e|\alpha_i)^{-1} \in \frac{1}{\mathfrak{o}}\mathbb{Z}$. Then all the equations have degree $\deg(f_j) = |e|^{-1}$. In particular, the affine ring $R(X_a)$ is $\frac{1}{\mathfrak{o}}\mathbb{Z}$-graded.

Since for each $h \in H$ one has $h * z_i = \theta([E^*_{is_i}])(h)z_i$ and $h * f_j = \theta([E_0^*])(h)f_j$, by 8.1.3 the equivariant Poincaré series of $R(X_a)$ is

$$\sum_{\chi \in \widehat{H}} P(R(X_a)^\chi, t) \cdot \chi = \theta\Big(\sum_{h \in H} P_h(R(X_a), t) \cdot [h]\Big) \in \mathbb{Z}[[t^{\frac{1}{\mathfrak{o}}}]][\widehat{H}], \tag{8.1.4}$$

where $\sum_{h \in H} P_h(R(X_a), t) \cdot [h] \in \mathbb{Z}[[t^{\frac{1}{\mathfrak{o}}}]][H]$ is

$$\frac{(1 - [E_0^*]\, t^{\frac{1}{|e|}})^{\nu-2}}{\prod_{j=1}^{\nu} (1 - [E^*_{js_j}]\, t^{\frac{1}{|e|\alpha_j}})} = \frac{(1 - [E_0^*]\, t^{-(E_0^*, E_0^*)})^{\nu-2}}{\prod_{j=1}^{\nu} (1 - [E^*_{js_j}]\, t^{-(E^*_{js_j}, E_0^*)})}. \tag{8.1.5}$$

8.1.7 Recall that $\alpha_j[E^*_{js_j}] = [E_0^*]$. Therefore, $\sum_h P_h(R(X_a), t)[h]$ has the following expansion (cf. [506]):

$$(1-[E_0^*]\, t^{\frac{1}{|e|}})^{-2} \cdot \prod_j \frac{1 - \big([E^*_{js_j}]\, t^{\frac{1}{|e|\alpha_j}}\big)^{\alpha_j}}{1 - [E^*_{js_j}]\, t^{\frac{1}{|e|\alpha_j}}} = \sum_{k_0 \geq 0} (1+k_0)[k_0 E_0^*]\, t^{\frac{k_0}{|e|}} \cdot \prod_j \sum_{k_j=0}^{\alpha_j - 1} [k_j E^*_{js_j}]\, t^{\frac{k_j}{|e|\alpha_j}}$$

$$= \sum (1 + k_0)[k_0 E_0^* + \sum_j k_j E^*_{js_j}]\, t^{\frac{1}{|e|}(k_0 + \sum_j k_j/\alpha_j)},$$

where the first sum is over $k_0 \geq 0$ and $0 \leq k_j < \alpha_j$ for each j.

Let us fix $h \in H$ and write it as $h = [l'_{\rm red}]$, $l'_{\rm red} = a_0E_0^* + \sum_j a_jE_{js_j}^* \in L'$ for certain integers $\{a_j\}_{j\geq 0}$. Then using the relations in H, cf. 3.5.4, $[k_0E_0^* + \sum_j k_jE_{js_j}^*] = h$ if and only if there exist integers ℓ and ℓ_j such that

$$k_0E_0^*+\sum_j k_jE_{js_j}^* = a_0E_0^*+\sum_j a_jE_{js_j}^*+\sum_j \ell_j(E_0^*-\alpha_jE_{js_j}^*)+\ell(b_0E_0^*-\sum_j \omega_jE_{js_j}^*),$$

hence $k_0 = a_0 + \sum_j \ell_j + \ell b_0$ and $k_j = a_j - \ell_j\alpha_j - \ell\omega_j$. Then $\ell_j = \lfloor(-\ell\omega_j + a_j)/\alpha_j\rfloor$ and

$$P_h(R(X_a),t) = \sum_{\ell\in\mathbb{Z}} \max\left\{0\,,\,1+N_{l'_{\rm red}}(\ell)\right\}\cdot t^{\ell+\mathfrak{a}}, \tag{8.1.6}$$

where

$$N_{l'_{\rm red}}(\ell) := a_0+\ell b_0-\sum_j\left\lceil\frac{\ell\omega_j-a_j}{\alpha_j}\right\rceil \text{ and } \mathfrak{a} := -(E_0^*,l'_{\rm red}) = \frac{1}{|e|}\Big(a_0+\sum_j\frac{a_j}{\alpha_j}\Big).$$

Since $N_{l'_{\rm red}}(\ell) \leq |e|(\ell+\mathfrak{a})$, the sum in (8.1.6) is in fact over integers $\ell \geq -\mathfrak{a}$.

Above $\mathfrak{a}$ depends on the choice of the integers $\{a_j\}_{j\geq 0}$ in the representative $l'_{\rm red}$. Adding $E_0^* - \alpha_jE_{js_j}^*$ (hence modifying a_0 into a_0+1 and a_j into $a_j-\alpha_j$) has no effect on $\mathfrak{a}$, but adding $b_0E_0^*-\sum_j\omega_jE_{js_j}^*$ increases $\mathfrak{a}$ by 1. However, the expression (8.1.6) is independent of the choice. By a convenient choice of $l'_{\rm red}$ we can assume $\mathfrak{a}\in[0,1)$. With this choice, the summation in (8.1.6) is over integers $\ell\geq 0$.

Example 8.1.8 Let us fix again $h \in H$, and let $l' = a_0E_0^* + \sum_{ik} a_{ik}E_{ik}^* \in L'$ be one of its representatives. l' can be r_h, or s_h, but also *any other representative.* Then $l'_{\rm red} := a_0E_0^* + \sum_{ik} a_{ik}n^i_{k+1,s_i}E^*_{is_i}$ is still a representative of h, and it satisfies $(E_0^*,l') = (E_0^*,l'_{\rm red})$. Usually we write $N_{l'} = N_{l'_{\rm red}}$.

In particular, if we start with the representative $l' = r_h$, then $\mathfrak{a} = -(E_0^*,r_h)$ is in $[0,1)$, and the summation in (8.1.6) is over $\ell\geq 0$. Similarly, if we take $l' = s_h$, then $\mathfrak{a}\geq 0$ and (merely from the above discussion) it might not be necessarily in $[0,1)$. But again, due to the (SI_{red}) relations from 6.6.20 (see also (6.6.28) and the paragraphs after it), the summation is over $\ell\geq 0$.

8.1.9 *Let us assume that the summation in (8.1.6) is over* $\ell\geq 0$ (independently whether $\mathfrak{a}\in[0,1)$ is true or not). Clearly $P_h(R(X_a),t)$ can be written as the sum $P_h^+(R(X_a),t) + P_h^-(R(X_a),t) = P_h^+(t) + P_h^-(t)$, where

$$P_h^+(t) = \sum_{\ell\geq 0}\max\left\{0\,,\,-1-N_{l'}(\ell)\right\}\cdot t^{\ell+\mathfrak{a}}, \quad P_h^-(t) = \sum_{\ell\geq 0}(1+N_{l'}(\ell))\cdot t^{\ell+\mathfrak{a}}.$$

Since $e = -b_0 + \sum_j \omega_j/\alpha_j < 0$, $P_h^+(t)/t^{\mathfrak{a}}$ is a polynomial. Moreover, the 'regularized' series $P_h^-(t)$ also has a distinguished property, namely $P_h^-(t)/t^{\mathfrak{a}}$ can be written as a rational function of negative degree. More precisely,

$$P_h^-(t) = \Big(-\frac{e\cdot t}{(1-t)^2} + \frac{1-e\mathfrak{a}}{1-t} - \sum_{j=1}^{\nu} \sum_{k_j=0}^{\alpha_j-1} \Big\{ \frac{a_j - \omega_j k_j}{\alpha_j} \Big\} t^{k_j} \cdot \frac{1}{1-t^{\alpha_j}} \Big) \cdot t^{\mathfrak{a}}.$$

Corollary 8.1.10 *$P_h(t) = P_h(R(X_a), t)$ can be decomposed in a unique way as $P_h^+(t) + P_h^-(t)$, where $P_h^+(t)/t^{\mathfrak{a}}$ is a polynomial, and $P_h^-(t)/t^{\mathfrak{a}}$ is a rational function of negative degree. In particular, by this property (and the form of their coefficients) $P_h(t)$ and $P_h^-(t)$ determine each other.*

(With the choice $\mathfrak{a} \in [0, 1)$, $P_h^-(t)$ itself has negative rational degree.)

8.1.11 In general, it is rather hard to compute the counting function $n \mapsto \sum_{\ell<n}(1+N_{l'}(\ell))$ of the coefficients of $P_h^-(t)$ explicitly for every n because of the quasi-periodical nature of $N_{l'}$. However, taking multiples of the natural period, the output is simple and conceptual. Indeed,

$$\begin{aligned} \sum_{\ell<\alpha n} (1 + N_{l'}(\ell)) &= \sum_{\ell<\alpha n} \Big(1 + a_0 + \ell b_0 + \sum_j \frac{a_j - \ell\omega_j}{\alpha_j} - \sum_j \Big\{ \frac{a_j - \ell\omega_j}{\alpha_j} \Big\} \Big) \\ &= \frac{|e|}{2}(\alpha n)^2 + \Big(|e|\mathfrak{a} - \frac{|e|}{2} + \frac{\chi}{2} \Big) \alpha n. \end{aligned} \tag{8.1.7}$$

8.1.12 Next, we determine the first three terms of the Laurent expansion of $P_h^-(t)$ at $t = 1$. By a computation, using the identities of 2.2.A or the identity

$$-n\{c/n\} + \frac{n-1}{2} = \sum_{\xi^n=1\neq\xi} \frac{\xi^{-c}}{1-\xi},$$

we obtain (see also [506])

$$\begin{aligned} P_h^-(t)/t^{\mathfrak{a}} &= -\frac{et}{(1-t)^2} + \frac{\chi/2 - e\mathfrak{a}}{1-t} + \sum_{j=1}^{\nu} \sum_{\ell\geq 0} \Big(-\Big\{ \frac{a_j - \omega_j \ell}{\alpha_j} \Big\} + \frac{\alpha_j - 1}{2\alpha_j} \Big) \cdot t^{\ell} \\ &= -\frac{et}{(1-t)^2} + \frac{\chi/2 - e\mathfrak{a}}{1-t} + \sum_{j=1}^{\nu} \frac{1}{\alpha_j} \cdot \sum_{\xi^{\alpha_j}=1\neq\xi} \frac{\xi^{-a_j}}{(1-\xi)(1-\xi^{\omega_j}\cdot t)}. \end{aligned}$$

Using (2.2.10) and (2.2.11) we obtain the expansion

$$P_h^-(t)/t^{\mathfrak{a}} = -\frac{et}{(1-t)^2} + \frac{\chi/2 - e\mathfrak{a}}{1-t} + \frac{2-\chi}{4} - \sum_{j=1}^{v}\left(\mathbf{s}(\omega_j,\alpha_j) + \frac{1}{2}\left\{\frac{a_j}{\alpha_j}\right\} + \sum_{k_j=1}^{a_j}\left(\!\!\left(\frac{k_j\omega_j'}{\alpha_j}\right)\!\!\right)\right) + R_1(t), \tag{8.1.8}$$

for some function $R_1(t)$ with $\lim_{t\to 1} R_1(t) = 0$. Hence (with $\lim_{t\to 1} R_2(t) = 0$):

$$P_h^-(t) = \frac{-e}{(1-t)^2} + \frac{\chi/2 + e}{1-t} + \frac{2-\chi}{4} - \mathfrak{a}\cdot\frac{\chi - e(\mathfrak{a}-1)}{2} - \sum_{j=1}^{v}\left(\mathbf{s}(\omega_j,\alpha_j) + \frac{1}{2}\left\{\frac{a_j}{\alpha_j}\right\} + \sum_{k_j=1}^{a_j}\left(\!\!\left(\frac{k_j\omega_j'}{\alpha_j}\right)\!\!\right)\right) + R_2(t). \tag{8.1.9}$$

Note that the non-regular part $-e/(1-t)^2 + (\chi/2+e)/(1-t)$ of $P_h^-(t)$ is independent of h.

8.1.13 The Non-equivariant Poincaré Series of the Universal Abelian Covering Let $P_{\mathrm{ne}}(R(X_a),t) = \sum_\chi P(R(X_a)^\chi,t)$ be the projection of $\sum_\chi P(R(X_a)^\chi,t)\cdot\chi \in \mathbb{Z}[[t^{\frac{1}{\mathfrak{o}}}]][\widehat{H}]$ into $\mathbb{Z}[[t^{\frac{1}{\mathfrak{o}}}]]$, the Poincaré series of the universal abelian covering associated with its natural $\mathbb{C}^*$ action. Then, using the non-equivariant version of (8.1.5), with the notation $S(k) := \sum_j \frac{k_j}{\alpha_j}$ for $k = (k_1,\dots,k_v)$, one has

$$P_{\mathrm{ne}}(t) = \frac{(1-t^{\frac{1}{|e|}})^{v-2}}{\prod_{j=1}^{v}(1-t^{\frac{1}{|e|\alpha_j}})} = \frac{1}{(1-t^{\frac{1}{|e|}})^2}\cdot\sum_{\substack{0\le k_j<\alpha_j\\ 1\le j\le v}} t^{\frac{1}{|e|}S(k)}. \tag{8.1.10}$$

8.1.C Filtrations

We introduce $\mathbb{Z}^s$–filtrations of $\mathbb{C}$–vector spaces and (under certain finiteness property) their multivariable Hilbert series. Once we have a Hilbert series $H(\mathbf{t})$, the corresponding multivariable Poincaré series $P(\mathbf{t})$ is associated canonically with $H(\mathbf{t})$ by the relation $P(\mathbf{t}) = -H(\mathbf{t})\cdot\prod_{i=1}^{s}(1-t_i^{-1})$ in $\mathbb{Z}[[\mathbf{t}^{\pm 1}]]$ (and *not* as the Poincaré series of some associated $\mathbb{Z}^s$–multigraded vector space). Usually P contains less information than H, however we develop certain criteria, which allow us to regain H from P (the procedure is called 'inversion').

We list several geometrical/algebraic possibilities to define $\mathbb{Z}^s$–filtrations: using order functions, valuations, projections via factorization, collection of weights (e.g. associated with a Newton diagram), or powers of the (maximal) ideal. The case of (plane) curves is exemplified with more details. ♣

8.1.14 Fix the standard basis $\{e_i\}_i$ of $\mathbb{Z}^s$, $e_i = (0, \dots, 0, 1, 0, \dots, 0)$. Write also $e_I = \sum_{i\in I} e_i$ for any $I \subset \{1, \dots, s\}$. Assume that the $\mathbb{C}$-vector space M is $\mathbb{Z}^s$–multigraded, $M = \oplus_{\mathbf{a}\in\mathbb{Z}^s} M_{\mathbf{a}}$, with $\dim M_{\mathbf{a}} < \infty$. Write the Poincaré series $P(\mathbf{t}) = \sum_{\mathbf{a}} \dim(M_{\mathbf{a}})\mathbf{t}^{\mathbf{a}}$ as $\sum_{\mathbf{a}} \mathfrak{p}(\mathbf{a})\mathbf{t}^{\mathbf{a}}$. For any $\mathbf{a} \in \mathbb{Z}^s$ set $F(\mathbf{a}) := \oplus_{\mathbf{a}'\geq\mathbf{a}} M_{\mathbf{a}'} \subset M$. Then $\{F(\mathbf{a})\}_{\mathbf{a}}$ defines a $\mathbb{Z}^s$–filtration of M with

$$F(\mathbf{a}) \subset F(\mathbf{a}') \text{ for any } \mathbf{a} \geq \mathbf{a}'. \tag{8.1.11}$$

Assume that for each fixed $\mathbf{a}$ the set $\{\mathbf{a}' \ : \ \mathbf{a}' \not\geq \mathbf{a},\ M_{\mathbf{a}'} \neq 0\}$ is finite. Then define the *Hilbert series of the filtration* by

$$H(\mathbf{t}) = \sum_{\mathbf{a}\in\mathbb{Z}^s} \mathfrak{h}(\mathbf{a})\mathbf{t}^{\mathbf{a}}, \text{ where } \mathfrak{h}(\mathbf{a}) := \dim(M/F(\mathbf{a})), \tag{8.1.12}$$

or

$$\mathfrak{h}(\mathbf{a}) = \sum_{\mathbf{a}'\not\geq\mathbf{a}} \mathfrak{p}(\mathbf{a}'). \tag{8.1.13}$$

The multigrading and the Poincaré polynomial can be deduced from the filtration by $M_{\mathbf{a}} = F(\mathbf{a})/\sum_i F(\mathbf{a} + e_i)$, hence by the inclusion–exclusion principle

$$P(\mathbf{t}) = -H(\mathbf{t}) \cdot \prod_{i=1}^{s} (1 - t_i^{-1}), \text{ or } \mathfrak{p}(\mathbf{a}) = \sum_{I\subset\{1,\dots,s\}} (-1)^{|I|+1} \mathfrak{h}(\mathbf{a} + e_I). \tag{8.1.14}$$

Remark 8.1.15 Assume that the multigrading of M is supported in the first quadrant of $\mathbb{Z}^s$, that is, $M_{\mathbf{a}} = 0$ whenever $\mathbf{a} \notin \mathbb{N}^s$. Then $P(\mathbf{t}) \in \mathbb{Z}[[\mathbf{t}]]$, and we are tempted to consider the filtration and Hilbert series supported merely on ('the simpler') $\mathbb{N}^s$. In other words, we define $F'(\mathbf{a}) = \oplus_{\mathbf{a}'\geq\mathbf{a}} M_{\mathbf{a}'}$ and $\mathfrak{h}'(\mathbf{a}) = \dim(M/F'(\mathbf{a}))$ for any $\mathbf{a} \in \mathbb{N}^s$ and set $H'(\mathbf{t}) := \sum_{\mathbf{a}\in\mathbb{N}^s} \mathfrak{h}'(\mathbf{a})\mathbf{t}^{\mathbf{a}} \in \mathbb{Z}[[\mathbf{t}]]$. Clearly $H(\mathbf{t})$ and $H'(\mathbf{t})$ determine each other: H' is the part of H supported on $\mathbb{N}^s$, while H can be recovered from H' by the formula $\mathfrak{h}(\mathbf{a}) = \mathfrak{h}'(\max\{\mathbf{a}, \mathbf{0}\})$. Nevertheless, the identity (8.1.14) is valid for the pair P and H and it is not valid for P and H'. This fact—together with the presence of several geometrical filtrations naturally indexed by $\mathbb{Z}^s$ — motivate the choice of $\mathbb{Z}^s$ versus $\mathbb{N}^s$ (even if $P(\mathbf{t}) \in \mathbb{Z}[[\mathbb{N}^s]]$ holds).

8.1.16 In ideal situations we search for the presence of a multigraded structure, however in many geometrical/algebraic situations we do not have a multigrading, but only a *filtration*, that is, a family of subspaces $\{F(\mathbf{a})\}_{\mathbf{a}\in\mathbb{Z}^s}$ of M with the property

(8.1.11). In general, $M/F(\mathbf{a})$ can be infinite dimensional (see e.g. $M = \mathbb{C}[t_1, t_2]$ with the natural monomial $\mathbb{Z}^2$–multigrading). If $\dim(M/F(\mathbf{a})) < \infty$ for any $\mathbf{a} \in \mathbb{Z}^s$, then we define the Hilbert series $H(\mathbf{t})$ as in (8.1.12) and the Poincaré series $P(\mathbf{t})$ by (8.1.14). However, even if H and P are well-defined in this way, in general, we cannot expect the validity of the identities (8.1.13). Furthermore, in general, $\mathfrak{p}(\mathbf{a})$ is not the dimension of certain spaces (it can be even negative), see 8.1.18 or 8.1.22.

Though the multiplication by $\prod_i(1 - t_i^{-1})$ in $\mathbb{Z}[[\mathbf{t}]][\mathbf{t}^{-1}]$ (where $\mathbb{Z}[[\mathbf{t}]][\mathbf{t}^{-1}]$ is considered a $\mathbb{Z}[\mathbf{t}^{\pm 1}]$-module) is injective, in general, the multiplication by $\prod_i(1 - t_i^{-1})$ in $\mathbb{Z}[[\mathbf{t}^{\pm 1}]]$ has a non-trivial kernel. Therefore, in general, taking P instead of H, one loses some information of H. In some cases one can recover H easily from P (e.g., when $s = 1$ and $H(\mathbf{a}) = 0$ for $\mathbf{a} \leq 0$). Nevertheless, the general 'inversion' (and its concrete form) is not automatical at all. When additionally the identities (8.1.13) hold for every $\mathbf{a} \in \mathbb{Z}^s$, we call them the 'inversions' of (8.1.14).

(In the cases when we recover H from P we use special additional properties of H and we prove that with such properties there is a unique H which satisfies (8.1.14) with fixed P. This unique solution is not always the inversion (8.1.13), see e.g. 8.1.26. However, in this book we will strongly distinguish the inversion of type (8.1.13) from any other type of relationship connecting P and H.)

Example 8.1.17 Take $M = \mathbb{C}^2$ with a $\mathbb{Z}^2$–filtration defined as follows: $F(\mathbf{a}) = F(\max\{\mathbf{a}, \mathbf{0}\})$, $F(\mathbf{0}) = M$, $F((i, 0)) = \mathbb{C} \times 0$, $F((0, j)) = 0 \times \mathbb{C}$ and $F((i, j)) = 0$ for $i \geq 1$ and $j \geq 1$. Then $P(\mathbf{t}) = 0$, hence the 'inversion' does not hold.

Example 8.1.18 There are filtrations, which are not induced by multigradings. Assume e.g. that F is a $\mathbb{Z}^3$–filtration with $F(\mathbf{a}) = F(\max\{\mathbf{a}, \mathbf{0}\})$, $F(\mathbf{0}) = M = \mathbb{C}^2$, for the positive generators $\{e_i \in \mathbb{Z}^3\}_{i=1,2,3}$ the subspaces $\{F(e_i)\}_i$ are 1-dimensional and distinct, and $F(e_i + e_j) = 0$ for any i, j. In this case $P(\mathbf{t}) = -1 + t_1 + t_2 + t_3$, hence $\mathfrak{p}(\mathbf{0}) < 0$. On the other hand, in this case, the 'inversion' holds.

Proposition 8.1.19 *Assume that there exist*

(a) a cone $\mathscr{S} = v_0 + \sum_k \mathbb{R}_{\geq 0} \cdot v_k \subset \mathbb{R}^s$ with vertex $v_0 \in \mathbb{R}^s$ and generated by vectors $\{v_k\}_{k=1}^N$, $v_k \in (0, \infty)^s$, such that for any $\mathbf{a} \in \mathbb{Z}^s$ there exists a unique $\min\{\mathbf{b} : \mathbf{b} \geq \mathbf{a},\ \mathbf{b} \in \mathscr{S} \cap \mathbb{Z}^s\}$, called $m(\mathbf{a})$;

(b) a filtration $\{F(\mathbf{a})\}_{\mathbf{a}}$ such that for all $\mathbf{a}$ we have $\dim M/F(\mathbf{a}) < \infty$ and $F(\mathbf{a}) = F(m(\mathbf{a}))$, and there exists $\mathbf{a}_0$ so that $F(\mathbf{a}_0) = M$.

Then $P(\mathbf{t})$ is supported on $\mathscr{S}$, and the 'inversion' (8.1.13) is valid for any $\mathbf{a}$.

Proof From the uniqueness of $m(\mathbf{a})$ it follows that for $\mathbf{a} \leq \mathbf{b}$ one also has $m(\mathbf{a}) \leq m(\mathbf{b})$ (†). From the definition, if $\mathbf{a} \notin \mathscr{S}$ then there exists i with $\mathbf{a}_i < m(\mathbf{a})_i$. Then for any $I \not\ni i$ the property (†) implies $m(\mathbf{a}+e_I) \geq m(\mathbf{a})$, hence $m(\mathbf{a}+e_I) \geq \mathbf{a}+e_I+e_i$ and $F(\mathbf{a} + e_I) = F(\mathbf{a} + e_I + e_i)$. This shows the first statement.

Next we prove (8.1.13) for any $\mathbf{a}$. Choose $\mathbf{k}_0$ with sufficiently small entries such that $\mathbf{k}_0 \leq \min\{\mathbf{a}, \mathbf{a}_0\} - (1, \dots, 1)$, hence in particular $F(\mathbf{k}_0) = M$ and $\mathfrak{h}(\mathbf{k}_0) = 0$. Moreover, choose $\mathbf{k} \geq \mathbf{k_0}$ with sufficiently large entries such that $\{\mathbf{a}' \in \mathscr{S} : \mathbf{a}' \not\geq$

$\mathbf{a}\} \subset \{\mathbf{a}' : \mathbf{k}_0 \le \mathbf{a}' \le \mathbf{k}\}$. Then the sum $\sum_{\mathbf{a}' \not\ge \mathbf{a}} \mathfrak{p}(\mathbf{a}')$ equals $S = \sum \mathfrak{p}(\mathbf{a}')$, (sum over $\mathbf{a}' \not\ge \mathbf{a},\ \mathbf{k}_0 \le \mathbf{a}' \le \mathbf{k}$) since the support of P is in $\mathscr{S}$. But,

$$S = \sum_{\mathbf{a}' \not\ge \mathbf{a},\ \mathbf{k}_0 \le \mathbf{a}' \le \mathbf{k}} \ \sum_{I} (-1)^{|I|+1} \mathfrak{h}(\mathbf{a}' + e_I) = \mathfrak{h}(\mathbf{a}) - \mathfrak{h}(\mathbf{k}_0)$$

by combinatorial cancellation. □

Remark 8.1.20 In the above statement the cone $\mathscr{S}$ cannot be replaced by $\mathbb{R}^s_{\ge 0}$, see 8.1.26. On the other hand, in some cases $\mathbb{R}^s_{\ge 0}$ also works. E.g., if (a) there exists $\mathbf{a}_0$ such that $\mathfrak{h}(\mathbf{a})$ is constant (finite) for any $\mathbf{a} \not\ge \mathbf{a}_0$, (b) $F(\mathbf{a}) = F(\max\{\mathbf{a}, \mathbf{0}\})$, and (c) $F(\mathbf{0}) = M$, then P is finitely supported in $\mathbb{N}^s$ and the inversion holds again (with the above proof). For a concrete example see 8.1.18.

Example 8.1.21 (Filtration Associated with an Order Function) Let (X, o) be an analytic space germ, and $\mathscr{O}_{X,o}$ be its local ring of analytic functions. A function $\mathfrak{v} : \mathscr{O}_{X,o} \to \mathbb{Z} \cup \{\infty\}$ is called *order function* if $\mathfrak{v}(f_1 + f_2) \ge \min\{\mathfrak{v}(f_1), \mathfrak{v}(f_2)\}$ and $\mathfrak{v}(f_1 f_2) \ge \mathfrak{v}(f_1)$ for any $f_1,\ f_2 \in \mathscr{O}_{X,o}$. If additionally $\mathfrak{v}(f_1 f_2) = \mathfrak{v}(f_1) + \mathfrak{v}(f_2)$ holds for any $f_1,\ f_2$, then it is called a *valuation*.

If $\{\mathfrak{v}_1, \ldots, \mathfrak{v}_s\}$ is a collection of order functions and $\mathbf{a} = (a_1, \ldots, a_s)$, then $\mathscr{F}(\mathbf{a}) = \{f \in \mathscr{O}_{X,o} : \mathfrak{v}_i(f) \ge a_i \text{ for all } i\}$ is an ideal of $\mathscr{O}_{X,o}$, and $\{\mathscr{F}(\mathbf{a})\}_{\mathbf{a}}$ is a $\mathbb{Z}^s$–filtration. It satisfies $\mathscr{F}(\mathbf{a}) \cap \mathscr{F}(\mathbf{b}) = \mathscr{F}(\max\{\mathbf{a}, \mathbf{b}\})$.

Example 8.1.22 (Filtration Associated with a Collection of Weights) A system of weights for $\mathscr{O}_{\mathbb{C}^n,0}$ is an n-tuple $\mathbf{w} = (w_1, \cdots, w_n)$ of *positive* integers. For a monomial $\mathbf{x}^\alpha$ one defines its weight by $\mathbf{w}(\mathbf{x}^\alpha) := \sum_i w_i \alpha_i$, and for any $f = \sum_\alpha c_\alpha \mathbf{x}^\alpha \in \mathscr{O}_{\mathbb{C}^n,0}$ one defines the *valuation associated with* $\mathbf{w}$ by

$$\mathfrak{v}(f) = \mathfrak{v}_{\mathbf{w}}(f) = \min_{\alpha\,:\,c_\alpha \ne 0} \mathbf{w}(\mathbf{x}^\alpha) \quad (f \in \mathscr{O}_{\mathbb{C}^n,0}).$$

If we start with s n-tuples $\mathbf{w}_1, \ldots, \mathbf{w}_s$, each of them determines a valuation $\mathfrak{v}_{\mathbf{w}_i}$. Their collection defines a $\mathbb{Z}^s$–filtration $\mathscr{F}_{\mathbb{C}^n,0}$ of $\mathscr{O}_{\mathbb{C}^n,0}$ (cf. 8.1.21). The filtration is associated with a $\mathbb{Z}^s$–multigrading (with $\deg(\mathbf{x}^\alpha) = (\mathbf{w}_1(\alpha), \ldots, \mathbf{w}_s(\alpha)) \in \mathbb{Z}^s_{\ge 0}$). Hence the Poincaré polynomial is the same as the right hand side of (8.1.2).

Let $(X, o) \subset (\mathbb{C}^n, 0)$ be an analytic germ and let $\Phi : \mathscr{O}_{\mathbb{C}^n,0} \to \mathscr{O}_{X,o}$ be the natural projection. Fix also a collection $\mathbf{w}_1, \ldots, \mathbf{w}_s$ of weights for $\mathscr{O}_{\mathbb{C}^n,0}$. The valuations $\mathfrak{v}_{\mathbf{w}_i}$ on $\mathscr{O}_{\mathbb{C}^n,0}$ induce *order functions* on $\mathscr{O}_{X,o}$ by

$$\mathfrak{v}_{X,\mathbf{w}}(\Phi(f)) = \max_{f' : \Phi(f') = \Phi(f)} \mathfrak{v}_{\mathbf{w}}(f'). \tag{8.1.15}$$

Thus, the weights $\{\mathbf{w}_i\}_i$ induce *two* $\mathbb{Z}^s$–filtrations on $\mathscr{O}_{X,o}$. The first one is given by projection: $\mathscr{F}^{pr}(\mathbf{a}) := \Phi(\mathscr{F}_{\mathbb{C}^n,0}(\mathbf{a}))$. The other one, is the filtration $\mathscr{F}^o$ associated with the collection of order functions $\{\mathfrak{v}_{X,\mathbf{w}_i}\}_{i=1}^s$. From definitions it follows that

$\mathscr{F}^{pr}(\mathbf{a}) \subset \mathscr{F}^{o}(\mathbf{a})$. If $s = 1$ then they are equal; however, for $s > 1$ they might differ.

The next example shows a few pathological/typical behaviours.

Take $X = \{x^4+x^2y+xy^2+y^4 = 0\} \subset (\mathbb{C}^2, 0)$ and the weights $\mathbf{w} = \mathbf{w_1} = (2, 1)$ and $\mathbf{w_2} = (1, 2)$. Then $\mathfrak{v}_{X,\mathbf{w}}(\Phi(y^2(x + y^2))) = \mathfrak{v}_{X,\mathbf{w}}(\Phi(-x^4 - x^2y)) = 5$, but $\mathfrak{v}_{X,\mathbf{w}}(\Phi(y^2)) = 2$ and $\mathfrak{v}_{X,\mathbf{w}}(\Phi(x + y^2)) = 2$. Hence usually $\mathfrak{v}_{X,\mathbf{w}}$ is not a valuation. Moreover, the class of $y^2(x + y^2)$ belongs to $\mathscr{F}^{o}((5, 5)) \setminus \mathscr{F}^{pr}((5, 5))$, hence $\mathscr{F}^{o}$ and $\mathscr{F}^{pr}$ are different. By taking the two representatives of $y^2(x + y^2)$ one sees that it is in the intersection $\mathscr{F}^{pr}((5, 4)) \cap \mathscr{F}^{pr}((4, 5))$, hence $\mathscr{F}^{pr}(\mathbf{a}) \cap \mathscr{F}^{pr}(\mathbf{b}) \neq \mathscr{F}^{pr}(\max\{\mathbf{a}, \mathbf{b}\})$. That is, $\mathscr{F}^{pr}$ is *not the filtration of any set of order functions*. [In general, $\mathscr{F}^{pr}$ coincides with a filtration provided by a set of order functions if and only if $\mathscr{F}^{pr}(\mathbf{a}) \cap \mathscr{F}^{pr}(\mathbf{b}) = \mathscr{F}^{pr}(\max\{\mathbf{a}, \mathbf{b}\})$ for all $\mathbf{a}$ and $\mathbf{b}$ [384].]

In the case of previous example, one can also check (use e.g. 8.1.23) that the Poincaré series associated with $\mathscr{F}^{pr}$ is $(1 - \mathbf{t}^{(4,4)})/((1 - \mathbf{t}^{(2,1)})(1 - \mathbf{t}^{(1,2)}))$, hence the coefficient of $\mathbf{t}^{(4,4)}$ is -1 (in particular, it cannot be the dimension of any space).

Usually we assume that our order functions have values in $\mathbb{Z}_{\geq 0} \cup \{\infty\}$, and they are centered at the maximal ideal, that is, $\{f \in \mathscr{O}_{X,o} \mid \mathfrak{v}(f) > 0\} = \mathfrak{m}_{X,o}$.

Example 8.1.23 ([384]) Consider s valuations $\mathfrak{v} := (\mathfrak{v}_1, \dots, \mathfrak{v}_s) : \mathscr{O}_{\mathbb{C}^n,0} \to (\mathbb{Z}_{\geq 0} \cup \{\infty\})^s$, all of them centered at the maximal ideal. Assume that $X = \{f = 0\} \subset (\mathbb{C}^n, 0)$ is a hypersurface. Then the Poincaré series of $\mathscr{O}_{X,o}$ associated with $\mathscr{F}^{pr}$ is

$$P^{pr}(\mathbf{t}) = \frac{1 - \mathbf{t}^{\mathfrak{v}(f)}}{\prod_{j=1}^{n}(1 - \mathbf{t}^{\mathfrak{v}(x_j)})} \in \mathbb{Z}[[\mathbf{t}]]. \tag{8.1.16}$$

Example 8.1.24 (Newton Filtration) Let us fix a Newton diagram in $\mathbb{R}^n_{\geq 0}$. Let $\mathbf{w}$ be the primitive normal vector of one of the facets (with positive integral entries). Then it defines a valuation and filtration of $\mathscr{O}_{\mathbb{C}^n,0}$. If we consider the collection of valuations associated with all facets of the diagram we get the *Newton filtration* of $\mathscr{O}_{\mathbb{C}^n,0}$ associated with the corresponding Newton diagram. This induces two filtrations on the local ring of *any* $(X, o) \subset (\mathbb{C}^n, 0)$ as in 8.1.22. The Poincaré series of $\mathscr{F}^{pr}$ is given in (8.1.16) if (X, o) is a hypersurface (for more general cases see [383]). The Poincaré series $P_{\mathscr{F}^0}$ of $\mathscr{F}^{o}$ for some particular diagrams ('stellar', 'bi–stellar') are computed in [172, 383, 384] as $P_{\mathscr{F}^{pr}}$. In fact, for a hypersurface singularity with non-degenerate principal part the filtrations $\mathscr{F}^{pr}$ and $\mathscr{F}^{o}$ coincide if and only if the diagram is bi–stellar (every two facets have a non-empty intersection).

Example 8.1.25 (Maximal Ideal Filtration) Let $\mathfrak{m}_{X,o} \subset \mathscr{O}_{X,o}$ be the maximal ideal of the local ring of a germ (X, o). Then $\mathscr{F}(k) = \mathfrak{m}^k_{X,o}$ $(k \in \mathbb{Z}_{\geq 0})$ is called the *maximal ideal filtration*.

Although the maximal ideal filtration is the most natural filtration of a local ring, its 'compatibility' with other filtrations is usually rather complicated.

For example, if (X, o) is homogeneous, then the ideals $\mathfrak{m}_{X,o}^k$ are homogeneous. However, in the general weighted homogeneous case, $\mathfrak{m}_{X,o}^k$ cannot be characterized merely in terms of the grading of the ring $R(X)$. For example, if $(X, o) = \{x^2 + y^4 + z^5 = 0\} \subset (\mathbb{C}^3, 0)$, then there is no d such that $\mathfrak{m}^2 = \oplus_{n \geq d} R(X)_n$. Indeed, $\deg(z^2) < \deg(x)$.

Even more, the maximal ideal filtration usually is not divisorial either (for more about divisorial filtrations see the next subsection). This means that for a fixed resolution, usually $\mathfrak{m}_{X,o}^k$ is not of type $H^0(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(-D))$, for a certain $D \in L$. For example, if (a) $Z_{max} = Z_{min}$ (hence $\mathfrak{m}_{X,o}^2 \subset H^0(\mathscr{O}_{\widetilde{X}}(-2Z_{min}))$, and the cycle $2Z_{min}$ is maximal with this property), and (b) there exists $g \in \mathfrak{m}_X \setminus \mathfrak{m}_X^2$ such that $\mathrm{div}_E(g) \geq 2Z_{min}$, then $\mathfrak{m}_{X,o}^2$ is not divisorial. This happens e.g. for the above (X, o) and its minimal good resolution.

Example 8.1.26 (Filtration Induced by Normalization of Curve Singularities) Let us fix a reduced curve singularity $(C, o) \subset (\mathbb{C}^n, 0)$ with s irreducible components $\{C_i\}_{i=1}^s$ and normalizations $n_i : (\mathbb{C}, 0) \to (C_i, o)$. Then $\mathfrak{v}_i : \mathscr{O} = \mathscr{O}_{\mathbb{C}^n,0} \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$, $\mathfrak{v}_i(f) := \mathrm{order}_t(f(n_i(t)))$ is a valuation. The functions $\mathfrak{v} := (\mathfrak{v}_1, \ldots, \mathfrak{v}_s)$ define a $\mathbb{Z}^s$–filtration, and the corresponding Hilbert and Poincaré series are well-defined by (8.1.12) and (8.1.14) respectively. From the definition $\mathfrak{h}(\mathbf{0}) = 0$ and $\mathfrak{h}(\mathbf{a}) = \mathfrak{h}(\max\{\mathbf{a}, \mathbf{0}\})$. Hence H is determined completely by $H(t)|_{\mathbf{a} \geq \mathbf{0}} := \sum_{\mathbf{a} \geq \mathbf{0}} \mathfrak{h}(\mathbf{a})\mathbf{t}^{\mathbf{a}}$. (Nevertheless, by the discussion below, the 'inversion' property (8.1.13) fails.)

For any nonempty subset $I = \{i_1, \ldots, i_{|I|}\} \subset \{1, \ldots, s\}$ consider the germ $(C_I, o) = (\cup_{i \in I} C_i, o)$. As above, each (C_I, o) determines an $|I|$-index filtration of $\mathscr{O}$, hence the Hilbert series H_{C_I} in variables $t_{i_1}, \ldots, t_{i_{|I|}}$ associated with (C_I, o) as well. Then

$$H_{C_I}|_{\mathbf{a} \geq 0}(t_{i_1}, \ldots, t_{i_{|I|}}) = H_C|_{\mathbf{a} \geq 0}(t_1, \ldots, t_s)|_{t_i = 0,\ i \notin I}. \tag{8.1.17}$$

Analogously, we also consider the Poincaré series P_{C_I} of each (C_I, o), defined via H_{C_I} as in (8.1.14). For $I = \emptyset$ we set $P_\emptyset = 0$. The next combinatorial identity (see [217]) shows that one can recover $H(\mathbf{t})$ from the collection of Poincaré series $P_{C_I}(\mathbf{t})$:

$$H(t_1, \ldots, t_r)|_{\mathbf{a} \geq \mathbf{0}} = \frac{1}{\prod_{i=1}^s (1 - t_i)} \cdot \sum_{I \subset \{1, \ldots, s\}} (-1)^{|I|-1} \Big(\prod_{i \in I} t_i \Big) \cdot P_{C_I}(t_{i_1}, \ldots, t_{i_{|I|}}).$$

For example, in the case of the plane curve singularity A_{2n-1}, defined by the equation $x^2 - y^{2n} = 0$, $P(t_1, t_2) = 1 + t_1 t_2 + \cdots + (t_1 t_2)^{n-1}$, $P_{\{x \pm y^n = 0\}}(t_i) = 1/(1 - t_i)$, hence

$$H(t_1, t_2)|_{\mathbf{a} \geq \mathbf{0}} = \frac{1}{(1 - t_1)(1 - t_2)} \left(\frac{t_1}{1 - t_1} + \frac{t_2}{1 - t_2} - t_1 t_2 (1 + \ldots + (t_1 t_2)^{n-1}) \right).$$

The image of $\mathfrak{v}$ (excluding ∞) is $\mathscr{S}_{C,o} := \{\mathbf{a} \in \mathbb{Z}^s : \text{there exists } f \in \mathscr{O} \text{ with } \mathfrak{v}(f) = \mathbf{a}\}$; it is called the *semigroup of* (C, o). It can be equivalently defined by (see [217])

$$\mathscr{S}_{C,o} = \{\mathbf{a} \in \mathbb{Z}^s_{\geq 0} : \mathfrak{h}(\mathbf{a} + e_i) > \mathfrak{h}(\mathbf{a}) \text{ for every } i = 1, \dots, s\}.$$

Fix some $\mathbf{a} \geq 0$ and an index i. Then $\mathfrak{h}(\mathbf{a} + e_i) = \mathfrak{h}(\mathbf{a}) + 1$ if there is an element $\mathbf{b} \in \mathscr{S}_{C,o}$ such that $b_i = a_i$ and $b_j \geq a_j$ for $j \neq i$. Otherwise $\mathfrak{h}(\mathbf{a} + e_i) = \mathfrak{h}(\mathbf{a})$. In particular, H and $\mathscr{S}_{C,o}$ determine each other.

Since each $\mathscr{F}(\mathbf{a})$ contains the ideal of (C, o), we get a filtration on $\mathscr{O}_{C,o}$ too with the same H and P.

Assume next that (C, o) is a *plane curve singularity*. In this case the Poincaré series P has the following topological reincarnation as the multivariable Alexander polynomial $\Delta(t_1, \dots, t_s)$ of the embedded link of (C, o) in S^3 [95]: If $s = 1$, then $P(t)(1 - t) = \Delta(t)$, while if $s > 1$, then $P(\mathbf{t}) = \Delta(\mathbf{t})$. If $s = 1$ this also reads as

$$\frac{\Delta(t)}{1 - t} = P(t) = \sum_i (\mathfrak{h}(i + 1) - \mathfrak{h}(i))t^i = \sum_{s \in \mathscr{S}_{C,o}} t^s. \tag{8.1.18}$$

Furthermore, for plane curve singularities, by [760] the multi-variable Alexander polynomial Δ of (C, o) determines the embedded topological type of (C, o), hence the multi-variable Alexander polynomials for all C_I as well. In particular, by the above correspondence between the Alexander polynomials and Poincaré series, P_C determines the collection $\{P_{C_I}\}_I$ as well. Nevertheless, this reduction procedure from P_C to P_{C_I} is more complicated than the above reduction (8.1.17) valid for the Hilbert series. Indeed, by Torres formula [699]:

$$P_{C_2 \cup \dots \cup C_s}(t_2, \dots, t_s) = P(t_1, \dots, t_s)|_{t_1 = 1} \cdot \left(1 - t_2^{(C_1, C_2)} \cdots t_s^{(C_1, C_s)}\right)^{-1},$$

where (C_i, C_j) is the intersection multiplicity at the origin of C_i and C_j, $i \neq j$.

The above identity of H in terms of P_{C_I} and the above discussion show that P determine H completely; nevertheless this procedure is *not* of type (8.1.13). (In particular, (8.1.13) in general does not hold for an arbitrary filtration.)

On the other hand, regarding the theory of plane curve singularities, the above discussion shows that P, H, Δ and $\mathscr{S}$ determine each other and are complete embedded topological invariants of (C, o).

Finally we give the formula of $P(\mathbf{t})$ in terms of the embedded resolution graph [178] (compare also with ζ^{-1} from 4.1.22). Assume that $(C, o) = (V(f), 0)$ for some reduced $f = \prod_{i=1}^s f_i : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$. Let Γ_f be an embedded resolution graph, and for each $v \in \mathscr{V}$ let m_v^i be the multiplicity of f_i along E_v. Then

$$P(\mathbf{t}) = \prod_{v \in \mathscr{V}} (1 - t_1^{m_v^1} \cdots t_s^{m_v^s})^{-2 + \kappa_v(\Gamma_f)}. \tag{8.1.19}$$

Later we will need the next invariant as well. The *conductor* $c = (c_1, \dots, c_s) \in \mathbb{Z}^s_{\geq 0}$ is the smallest lattice point such that $c + \mathbb{Z}^s_{\geq 0} \subset \mathscr{S}_{C,o}$. For a ring theoretical interpretation (via the conductor ideal) see e.g. [101, 138]. We have

$$c_i = \mu(C_i) + \sum_{j \neq i} (C_i, C_j)_o \quad (1 \leq i \leq s). \tag{8.1.20}$$

8.2 Divisorial Filtration and its Multivariable Series

We fix a resolution of a normal surface singularity with irreducible exceptional divisors $\{E_v\}_v$. Each E_v defines a valuation of $\mathscr{O}_{X,o}$. By considering all of them we obtain an L–filtration of $\mathscr{O}_{X,o}$, called the 'divisorial filtration'. If the link is a rational homology sphere, and (X_a, o) is the universal abelian covering, then we obtain a compatible L'–filtration of $\mathscr{O}_{X_a,o}$ together with an H-action. It provides an equivariant Poincaré and Hilbert series, P and H. The inversion property holds for them, and they determine the analytic semigroup as well.
Next, we analyse the polynomial behaviour of the counting function of the coefficients of P: in the Grauert–Riemenschneider vanishing zone it is the sum of the (twisted) Riemann–Roch expression and the equivariant geometric genus. In general it involves an additional summand of type $h^1(\mathscr{O}_{\widetilde{X}}(-l'))$ as well. (This property will be reinterpreted later as follows: the periodic constant of P is the equivariant geometric genus.) ♣

8.2.1 Let (X, o) be a normal surface singularity, $\phi : (\widetilde{X}, E) \to (X, o)$ be an arbitrary fixed resolution of (X, o). We will define an L–filtration of the local ring of (X, o) and a compatible H-equivariant L'–filtration of the local ring of (X_a, o) (where $H = L'/L$). In the whole discussion regarding the universal abelian covering (X_a, o) and the L'–filtration of its local ring we will assume that the link of (X, o) is a rational homology sphere. At the level of the L–filtration of the $\mathscr{O}_{X,o}$ this assumption is not needed. We will adopt the notations from 2.1, 6.2.5 and (6.2.1).

8.2.2 The Module $\mathbb{Z}[[L']]$ Once a resolution is fixed, hence the natural basis $\{E_v\}_v$ of L is fixed too, $\mathbb{Z}[[L]]$ is identified with $\mathbb{Z}[\mathbf{t}^{\pm 1}] = \mathbb{Z}[[t_1^{\pm 1}, \dots, t_s^{\pm 1}]]$. It is contained in the larger module $\mathbb{Z}[[\mathbf{t}^{\pm 1/d}]] = \mathbb{Z}[[t_1^{\pm 1/d}, \dots, t_s^{\pm 1/d}]]$, the module of formal power series in variables $t_v^{\pm 1/d}$, where $d := |H|$. $\mathbb{Z}[[L']]$ embeds into $\mathbb{Z}[[\mathbf{t}^{\pm 1/d}]]$ as a submodule: it consists of the $\mathbb{Z}$-linear combinations of monomials of type

$$\mathbf{t}^{l'} = t_1^{l'_1} \cdots t_s^{l'_s}, \quad \text{where } l' = \textstyle\sum_v l'_v E_v \in L'.$$

$\mathbb{Z}[[L']]$ also admits several $\mathbb{Z}$-submodules corresponding to different cones of L'; e.g. $\mathbb{Z}[[L'_{\geq 0}]]$ and $\mathbb{Z}[[\mathscr{S}']]$, generated by monomials $\mathbf{t}^{l'}$ with $l' \in L'_{\geq 0}$, or $l' \in \mathscr{S}'$ respectively. Both $\mathbb{Z}[[L'_{\geq 0}]]$ and $\mathbb{Z}[[\mathscr{S}']]$ have natural ring structure.

$\mathbb{Z}[[\mathscr{S}']]$ is a usual formal power series ring in variables $\{\mathbf{t}^{E_v^*}\}_v$: its elements are

$$\Phi(f)(\mathbf{t}) := f(\mathbf{t}^{E_1^*}, \dots, \mathbf{t}^{E_s^*}), \quad \text{where } f(x_1, \dots, x_s) \in \mathbb{Z}[[\mathbf{x}]] = \mathbb{Z}[[x_1, \dots, x_s]]. \tag{8.2.1}$$

Definition 8.2.3 Any series $S(\mathbf{t}) = \sum_{l'} a_{l'} \mathbf{t}^{l'} \in \mathbb{Z}[[L']]$ decomposes in a unique way as

$$S = \sum_{h \in H} S_h, \quad \text{where } S_h = \sum_{[l']=h} a_{l'} \mathbf{t}^{l'}. \tag{8.2.2}$$

S_h is called the h-component of S.

Lemma 8.2.4 *Consider $F(\mathbf{t}) := \Phi(f)(\mathbf{t})$ for some $f \in \mathbb{Z}[[\mathbf{x}]]$ as in (8.2.1). Then*

$$F_h(\mathbf{t}) = \frac{1}{|H|} \cdot \sum_{\rho \in \widehat{H}} \rho(h)^{-1} \cdot f(\rho([E_1^*])\mathbf{t}^{E_1^*}, \dots, \rho([E_s^*])\mathbf{t}^{E_s^*}).$$

Proof We can assume that f is a monomial: if $l' := \sum n_v E_v^*$ and $f = \prod x_v^{n_v}$ then $\Phi(f)(\mathbf{t}) = \mathbf{t}^{l'}$ and $(1/d)\sum_\rho \rho(h)^{-1}\rho([l'])\mathbf{t}^{l'}$ is $\mathbf{t}^{l'}$ if $[l'] = h$ and zero otherwise. □

8.2.A The Series $H(\mathbf{t})$ and $P(\mathbf{t})$

Recall that we have the following notations regarding the universal abelian covering

$$\begin{array}{ccccc} \widetilde{Z} & \xrightarrow{r} & Z & \xrightarrow{\psi_a} & (X_a, o) \\ & & \downarrow{\scriptstyle \widetilde{c}} & & \downarrow{\scriptstyle c} \\ & & (\widetilde{X}, E) & \xrightarrow{\phi} & (X, o) \end{array} \tag{8.2.3}$$

Set $\phi_a = \psi_a \circ r$ and $p = \widetilde{c} \circ r$. By (6.2.3), $p^*(l')$ is an *integral* cycle for any $l' \in L'$.

Definition 8.2.5 The L'–filtration on the local ring of holomorphic functions $\mathscr{O}_{X_a,o}$ is defined as follows. For any $l' \in L'$, we set

$$\mathscr{F}(l') := \{f \in \mathscr{O}_{X_a,o} \mid \operatorname{div}(f \circ \phi_a) \geq p^*(l')\}. \tag{8.2.4}$$

Notice that the natural action of H on (X_a, o) induces an action on $\mathscr{O}_{X_a,o}$, which keeps $\mathscr{F}(l')$ invariant. Therefore, H acts on $\mathscr{O}_{X_a,o}/\mathscr{F}(l')$ as well. For any $l' \in L'$,

let $\mathfrak{h}(l')$ be the dimension of the $\theta([l'])$-eigenspace $(\mathscr{O}_{X_a,o}/\mathscr{F}(l'))_{\theta([l'])}$. Then one defines the Hilbert series $H(\mathbf{t})$ by

$$H(\mathbf{t}) := \sum_{l' \in L'} \mathfrak{h}(l') \cdot \mathbf{t}^{l'} \in \mathbb{Z}[[L']]. \tag{8.2.5}$$

For a fixed l' we write $[l'] = h$. For any $l' > 0$ one has the exact sequence $0 \to \mathscr{O}_{\widetilde{Z}}(-p^*(l')) \to \mathscr{O}_{\widetilde{Z}} \to \mathscr{O}_{p^*(l')} \to 0$. Its $\theta(h)$-eigenspaces are (cf. (6.2.5))

$$0 \to \mathscr{O}_{\widetilde{X}}(-l') \to \mathscr{O}_{\widetilde{X}}(-r_h) \to \mathscr{O}_{l'-r_h}(-r_h) \to 0. \tag{8.2.6}$$

In particular, for $l' > 0$,

$$\begin{aligned} \mathfrak{h}(l') = \dim \Big(\frac{H^0(\widetilde{Z}, \mathscr{O}_{\widetilde{Z}})}{H^0(\widetilde{Z}, \mathscr{O}_{\widetilde{Z}}(-p^*(l')))} \Big)_{\theta(h)} &= \dim \frac{H^0(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(-r_h))}{H^0(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(-l'))} \\ &= \dim \frac{\phi_*(\mathscr{O}_{\widetilde{X}}(-r_h))_o}{\phi_*(\mathscr{O}_{\widetilde{X}}(-l'))_o}. \end{aligned} \tag{8.2.7}$$

By 6.6.3, for any $l' \in L'$ there exists a unique minimal $s(l') \in \mathscr{S}'$ such that $l' \le s(l')$ and $[l'] = [s(l')]$. Moreover, there exists a computation sequence $\{z_i\}_i$ connecting l' with $s(l')$. Along this sequence $H^0(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(-z_i))$ is constant, hence the inclusion $H^0(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(-s(l'))) \subset H^0(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(-l'))$ is an isomorphism. Therefore,

$$\mathfrak{h}(l') = \mathfrak{h}(s(l')) \text{ and } \mathscr{F}(l')_{\theta(h)} = \mathscr{F}(s(l'))_{\theta(h)}. \tag{8.2.8}$$

[We note that if $s_{\widetilde{Z}}$ denotes the Laufer operator from 6.6.3 associated with the lattice from $\widetilde{Z}$ then $s_{\widetilde{Z}}(p^*(l')) \le p^*(s(l'))$, however, usually $s_{\widetilde{Z}}(p^*(l')) \ne p^*(s(l'))$. Moreover, usually $\mathscr{F}(l') \ne \mathscr{F}(s(l'))$ as well.]

Example 8.2.6 In (8.2.7) if $l' \in L$ then $r_h = 0$. Thus the 0-component of $H(\mathbf{t})$ is

$$H_0(\mathbf{t}) = \sum_{l \in L} \dim \Big(\frac{\mathscr{O}_{X,o}}{\{f \in \mathscr{O}_{X,o} : \mathrm{div}_E(f \circ \phi) \ge l\}} \Big) \cdot \mathbf{t}^l.$$

This is the Hilbert series of $\mathscr{O}_{X,o}$ associated with the divisorial filtration $L \ni l \mapsto \mathscr{F}_0(l) = \{f \in \mathscr{O}_{X,o} : \mathrm{div}_E(f \circ \phi) \ge l\}$ of all irreducible exceptional divisors of ϕ.

8.2.7 Next, we define the Poincaré series $P(\mathbf{t}) = \sum_{l' \in \mathscr{S}'} \mathfrak{p}(l') \mathbf{t}^{l'}$ associated with the filtration $\{\mathscr{F}(l')\}_{l'}$, as in (8.1.14).

$$P(\mathbf{t}) = -H(\mathbf{t}) \cdot \prod_v (1 - t_v^{-1}), \text{ or } \mathfrak{p}(l') = \sum_{I \subset \{1,\dots,s\}} (-1)^{|I|+1} \mathfrak{h}(l' + E_I), \tag{8.2.9}$$

where $E_I = \sum_{v\in I} E_v$. By (8.2.8) and 8.1.19, the series $P(\mathbf{t})$ is supported in $\mathscr{S}'$, and the inversion identities (8.1.13) are valid. Namely,

$$\mathfrak{h}(l') = \sum_{l\in L,\, l\not\geq 0} \mathfrak{p}(l'+l). \tag{8.2.10}$$

Multi-index filtrations were considered by Campillo, Delgado and Gusein-Zade in several contexts, this definition of $P(\mathbf{t})$ is compatible with their definition, see e.g. [96–100].

For a more general version see (8.2.16). One has the following facts too.

Lemma 8.2.8

(a) P, in fact, is supported in $\mathscr{S}'_{an}$, that is, $P(\mathbf{t}) \in \mathbb{Z}[[\mathscr{S}'_{an}]]$.
(b) For any $l' \in \mathscr{S}'$ one has

$$\mathfrak{p}(l') = \sum_{I\subset\mathscr{V}} (-1)^{|I|+1} \dim \frac{H^0(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(-l'))}{H^0(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(-l'-E_I))}$$

$$= \sum_{I\subset\mathscr{V}} (-1)^{|I|} \dim \frac{H^0(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(-l'-E_I))}{H^0(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(-l'-E))} = \sum_{I\subset\mathscr{V}} (-1)^{|I|} \dim \frac{\phi_*(\mathscr{O}_{\widetilde{X}}(-l'-E_I))_o}{\phi_*(\mathscr{O}_{\widetilde{X}}(-l'-E))_o}.$$

Proof (a) If $l' \notin \mathscr{S}'_{an}$ then by 6.7.23 there exists $v \in \mathscr{V}$ such that $\{\tilde{l}' \in L' : \tilde{l}' \geq l',\ \tilde{l}'_v = l'_v\} \cap \mathscr{S}'_{an,[l']} = \emptyset$. Therefore, for any $I \not\ni v$ one has $H^0(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(-l'-E_I)) = H^0(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(-l'-E_I-E_v))$. For (b) use (8.2.7). □

Remark 8.2.9

(a) The support of P can be smaller than $\mathscr{S}'_{an}$. E.g., it can happen that Z_{max} is not in the support of P, see Example 8.3.6.
(b) The support of P is not closed under the operator min, see e.g. 8.3.6.
(c) $\mathfrak{h}(l') < \mathfrak{h}(l'+E_v)$ iff there exists $\tilde{l}' \in \mathscr{S}'_{an}$ such that $\tilde{l}' \geq l'$ and $\tilde{l}'_v = l'_v$. This happens exactly when $\sum_{l\in L_{\geq 0},\ l_v=0} \mathfrak{p}(l'+l) > 0$ (use (8.2.10)).
(d) $\mathfrak{h}(l') < \mathfrak{h}(l'+E_v)$ for every $v \in \mathscr{V}$ iff $l' \in \mathscr{S}'_{an}$ (cf. 6.7.23). Therefore,

$$\mathscr{S}'_{an} = \{l' \in L' : \mathscr{O}_{\widetilde{X}}(-l') \text{ has no fixed components}\} \tag{8.2.11}$$

From the above properties it follows that H and/or P determine $\mathscr{S}'_{an}$.

8.2.10 We will present the inversion property (8.2.10) slightly more generally, targeting the cohomology groups of an arbitrary line bundle. Our main goal is to describe for any $\mathscr{L} \in \mathrm{Pic}(\widetilde{X})$ and effective cycle $l \in L_{>0}$ the dimensions

$$\dim \frac{H^0(\widetilde{X}, \mathscr{L})}{H^0(\widetilde{X}, \mathscr{L}(-l))} \quad \text{and} \quad \dim\ H^1(\widetilde{X}, \mathscr{L}). \tag{8.2.12}$$

It turns out that in their determination the crucial ingredient is:

$$\mathfrak{p}_{\mathscr{L}} = \sum_{I \subseteq \mathscr{V}} (-1)^{|I|+1} \dim \frac{H^0(\widetilde{X}, \mathscr{L})}{H^0(\widetilde{X}, \mathscr{L}(-E_I))}. \tag{8.2.13}$$

If $(c_1(\mathscr{L}), E_v) < 0$ for some $v \in \mathscr{V}$, then for any $I \not\ni v$ one has an isomorphism $H^0(\mathscr{L}(-E_{I\cup v})) \to H^0(\mathscr{L}(-E_I))$, hence

$$\mathfrak{p}_{\mathscr{L}} = 0 \text{ whenever } -c_1(\mathscr{L}) \notin \mathscr{S}'. \tag{8.2.14}$$

Regarding 'natural line bundles', $\mathfrak{p}_{\mathscr{O}_{\widetilde{X}}(-l')} = \mathfrak{p}(l')$ by Lemma 8.2.8, that is

$$P(\mathbf{t}) = \sum_{l' \in L'} \mathfrak{p}_{\mathscr{O}_{\widetilde{X}}(-l')} \mathbf{t}^{l'}. \tag{8.2.15}$$

The next proposition targets the structure of the invariants from (8.2.12). In the sequel sometimes we write K instead of $c_1(\mathscr{O}_{\widetilde{X}}(K_{\widetilde{X}})) \in L'$.

Theorem 8.2.11 ([484]) *Let (X, o) be a normal surface singularity (whose link is not necessarily a rational homology sphere). Fix one of its good resolutions $\widetilde{X} \to X$. Then for any line bundle $\mathscr{L} \in \mathrm{Pic}(\widetilde{X})$ the following facts hold:*

1. *For every effective cycle $l \in L_{\geq 0}$*

$$\dim \frac{H^0(\mathscr{L})}{H^0(\mathscr{L}(-l))} = \sum_{a \in L_{\geq 0}, a \not\geq l} \mathfrak{p}_{\mathscr{L}(-a)}. \tag{8.2.16}$$

2. *There exists a constant $\mathrm{const}_{[\mathscr{L}]}$, depending only on the class of*

$$[\mathscr{L}] \in Pic(\widetilde{X})/L = \mathrm{Cl}(X, o),$$

such that

$$-h^1(\mathscr{L}) = \sum_{a \in L,\, a \not\geq 0} \mathfrak{p}_{\mathscr{L}(a)} + \mathrm{const}_{[\mathscr{L}]} + \frac{(K - 2c_1(\mathscr{L}))^2 + |\mathscr{V}|}{8}. \tag{8.2.17}$$

3. *Then numbers $\mathfrak{p}_{\mathscr{L}}$ determine the constants $\mathrm{const}_{[\mathscr{L}]}$ (up to the combinatorics of Γ) as well. Indeed, if $\mathscr{L}$ is chosen such that $c_1(\mathscr{L}) \in K - \mathscr{S}'$, then*

$$\sum_{a \in L,\, a \not\geq 0} \mathfrak{p}_{\mathscr{L}(a)} = -\mathrm{const}_{[\mathscr{L}]} - \frac{(K - 2c_1(\mathscr{L}))^2 + |\mathscr{V}|}{8}. \tag{8.2.18}$$

In other words, if $c_1(\mathscr{L})$ is in the 'Grauert–Riemenschneider vanishing zone' of L', then the counting function from the left hand side of the equation (8.2.18)

is a quadratic multivariable polynomial in $c_1(\mathscr{L})$, whose constant term is $-\mathrm{const}_{[\mathscr{L}]} - (K^2 + |\mathscr{V}|)/8$.

(All the sums above are finite thanks to (8.2.14) and 2.1.19.)

Proof Fix $v \in \mathscr{V}$. By comparing (8.2.16) for l and $l + E_v$ and using induction, it is enough to prove (8.2.16) only for $l = E_v$ and arbitrary $\mathscr{L}$.

We rewrite the right-hand side of (8.2.16) in several steps. Via (8.2.14), it is $\sum_a \mathfrak{p}_{\mathscr{L}(-a)}$, where a runs over $a = \sum_{u \neq v} a_u E_u$, with $0 \leq a_u < k$ for some fixed $k \gg 0$. Then, using (8.2.13) and grouping the subsets of $\mathscr{V}$ in pairs $(I, I \cup v)_{I \not\ni v}$, we obtain that the right hand side of (8.2.16) for $l = E_v$ is

$$\sum_a \sum_{I \not\ni v} (-1)^{|I|} \dim \frac{H^0(\mathscr{L}(-a - E_I))}{H^0(\mathscr{L}(-a - E_I - E_v))}. \tag{8.2.19}$$

Since all the involved spaces of global sections are finite codimensional subspaces in $H^0(\mathscr{L})$, by combinatorial cancelation, (8.2.19) equals

$$\sum_{I \not\ni v} (-1)^{|I|} \dim \frac{H^0(\mathscr{L}(-kE_I))}{H^0(\mathscr{L}(-kE_I - E_v))}. \tag{8.2.20}$$

We claim, that the inclusion $H^0(\mathscr{L}(-kE_I - E_v)) \hookrightarrow H^0(\mathscr{L}(-kE_I))$ is an isomorphism for $I \neq \emptyset$, $I \not\ni v$, and $k \gg 0$. In order to prove this, embed again both groups in $H^0(\mathscr{L})$. Then $H^0(\mathscr{L}(-l)) \setminus \{0\} = \{s \in H^0(\mathscr{L}) \setminus \{0\} : \mathrm{div}_{\mathscr{L}}(s) \geq l\}$ for any $l \in L_{\geq 0}$. On the other hand, since $(\mathrm{div}_{\mathscr{L}}(s) - c_1(\mathscr{L}), E_u) = 0$ for all $u \in \mathscr{V}$ and $s \neq 0$, the compactly supported part $D(s)$ of $\mathrm{div}_{\mathscr{L}}(s)$ satisfies $D(s) \in c_1(\mathscr{L}) + \mathscr{S}'$. On the other hand, since $\mathscr{S}'$ is generated by $\{E_u^*\}_u$ whose coefficients are all strictly positive, $\mathscr{S}'$ has the following property: for any fixed $c_1 \in L'$ there is a $k \gg 0$ such that for all $D \in c_1 + \mathscr{S}'$ with $D \geq kE_I$ one has $D \geq kE_I + E_v$ too, cf. 2.1.19. Hence, the only contribution in (8.2.20) comes from $I = \emptyset$, and (8.2.16) for $l = E_v$ follows.

For 8.2.11(2), one compares the sheaves $\mathscr{L}$ and $\mathscr{L}(-l)$ for $l \in L_{\geq 0}$. The cohomology long exact sequence of $0 \to \mathscr{L}(-l) \hookrightarrow \mathscr{L} \twoheadrightarrow \mathscr{L} \otimes \mathscr{O}_l \to 0$ provides

$$\dim \frac{H^0(\mathscr{L})}{H^0(\mathscr{L}(-l))} = \chi(\mathscr{L} \otimes \mathscr{O}_l) - h^1(\mathscr{L}(-l)) + h^1(\mathscr{L}). \tag{8.2.21}$$

Then (8.2.16) and the Riemann-Roch formula shows that the expression

$$h^1(\mathscr{L}) + \sum_{a \not\leq 0} \mathfrak{p}_{\mathscr{L}(a)} + \frac{(K - 2c_1(\mathscr{L}))^2 + |\mathscr{V}|}{8}$$

is the same for $\mathscr{L}$ and $\mathscr{L}(-l)$.

For the last part use the identity (8.2.17) and the vanishing Theorem 6.4.3. □

Example 8.2.12 ([484]) Let $P_0(\mathbf{t}) = \sum_{l\in\mathscr{S}} \mathfrak{p}(l)\mathbf{t}^l$ be the 0-component of $P(\mathbf{t})$. Theorem 8.2.11 for the trivial class of line bundles $[\mathscr{L}] = [0] \in \mathrm{Cl}(X, o)$ reads as follows:

1. For $\mathscr{L} = \mathscr{O}_{\widetilde{X}}$ we obtain

$$-p_g = \mathrm{const}_{[0]} + \frac{K^2 + |\mathscr{V}|}{8}. \tag{8.2.22}$$

2. For $\mathscr{L} = \mathscr{O}_{\widetilde{X}}(-l)$ with $l \in (-K + \mathscr{S}') \cap L$

$$\sum_{\tilde{l}\in L,\ \tilde{l}\not\geq l} \mathfrak{p}(\tilde{l}) = \chi(l) + p_g. \tag{8.2.23}$$

   That is, the counting function of the coefficients of $P_0(\mathbf{t})$, associated with the special truncation $\{\tilde{l} \in \mathscr{S},\ \tilde{l} \not\geq l\}$, evaluated in the chamber $-K + \mathscr{S}'$, equals the quadratic polynomial $\chi(l) + p_g$.
3. More generally, for $\mathscr{L} = \mathscr{O}_{\widetilde{X}}(-l)$ with arbitrary $l \in L$,

$$h^1(\mathscr{O}_{\widetilde{X}}(-l)) = -\sum_{\tilde{l}\in L,\ \tilde{l}\not\geq l} \mathfrak{p}(\tilde{l}) + \chi(l) + p_g. \tag{8.2.24}$$

   If $l \leq 0$, then the sum on the right hand side is empty.

In particular, $P_0(\mathbf{t})$ determines completely p_g and the functions $l \mapsto \chi(l)$, $l \mapsto h^1(\mathscr{O}_{\widetilde{X}}(l))$ $(l \in L)$.

Example 8.2.13 (Equivariant Version. The Case of Natural Line Bundles [484]) Next, we assume that the link of (X, o) is a rational homology sphere. In particular, the universal abelian covering is well defined with its H-action. Recall that the geometric genus of (X_a, o) is the sum $\sum_h h^1(\mathscr{O}_{\widetilde{X}}(-r_h))$ (of the equivariant genera of (X, o)) corresponding to the eigenspace decomposition of $H^1(\mathscr{O}_Z)$.

We also simplify the notation $\mathrm{const}_{[\mathscr{O}(l')]}$ into $\mathrm{const}_{[l']}$.

The equivariant analogues of the formulae from Example 8.2.12 are the following. Let l'_h be either r_h or s_h. Then for any fixed h the following facts hold.

1. For $\mathscr{L} = \mathscr{O}_{\widetilde{X}}(-l'_h)$ we obtain

$$\begin{aligned} -h^1(\mathscr{O}_{\widetilde{X}}(-l'_h)) &= \mathrm{const}_{-h} + \frac{(K + 2l'_h)^2 + |\mathscr{V}|}{8} \\ &= \mathrm{const}_{-h} + \frac{K^2 + |\mathscr{V}|}{8} - \chi(l'_h). \end{aligned} \tag{8.2.25}$$

2. For $\mathscr{L} = \mathscr{O}_{\widetilde{X}}(-l')$ with $l' \in -K + \mathscr{S}'$, $l' = l + l'_h$ with $l \in L$,

$$\begin{aligned}\sum_{[\tilde{l}']=[l'],\ \tilde{l}' \not\geq l'} \mathfrak{p}(\tilde{l}') =& \chi_{K+2l'_h}(l) + h^1(\mathscr{O}_{\widetilde{X}}(-l'_h)) \\ =& \chi(l') + h^1(\mathscr{O}_{\widetilde{X}}(-l'_h)) - \chi(l'_h).\end{aligned} \tag{8.2.26}$$

3. More generally, for $\mathscr{L} = \mathscr{O}_{\widetilde{X}}(-l')$ with arbitrary $l' \in L'$, $l' = l + l'_h$ with $l \in L$,

$$\begin{aligned}h^1(\mathscr{O}_{\widetilde{X}}(-l')) =& - \sum_{[\tilde{l}']=[l'],\ \tilde{l}' \not\geq l'} \mathfrak{p}(\tilde{l}') + \chi_{K+2l'_h}(l) + h^1(\mathscr{O}_{\widetilde{X}}(-l'_h)) \\ =& - \sum_{[\tilde{l}']=[l'],\ \tilde{l}' \not\geq l'} \mathfrak{p}(\tilde{l}') + \chi(l') + h^1(\mathscr{O}_{\widetilde{X}}(-l'_h)) - \chi(l'_h).\end{aligned} \tag{8.2.27}$$

Therefore, $P(\mathbf{t})$ determines completely each $h^1(\mathscr{O}_{\widetilde{X}}(l'))$ $(l' \in L')$.

Remark 8.2.14 The following comment is appropriate. In the above formulae (e.g. in 8.2.12 and 8.2.13) the term consisting of the sum of the coefficients of P can be replaced (via (8.2.10)) by the corresponding coefficient of the Hilbert series $H(\mathbf{t})$. E.g., (8.2.23) reads as $\mathfrak{h}(l) = \chi(l) + p_g$. The corresponding versions in terms of the Hilbert series are simpler (and from the analytic point of view even more conceptual). The reason why we prefer the summation expressions is the following. Later in this book we will develop the topological analogues of the above identities. The point is that $P(\mathbf{t})$ will have a topological analogue, namely $Z(\mathbf{t})$ (see 8.4), however, the analogue of $H(\mathbf{t})$ will be defined ('merely') as the inversion of $Z(\mathbf{t})$, that is, by the summation of its coefficients. Hence, later we will hunt in the topological side for sum–expressions as above, where the coefficients of P will be replaced by those of Z (see e.g. (9.6.14) or section 10.3).

8.2.15 A Reformulation of $P(\mathbf{t})$ For $l' - r_h \in L_{>0}$ from (8.2.6) it follows that

$$\mathfrak{h}(l') = \chi(\mathscr{O}_{l'-r_h}(-r_h)) - h^1(\mathscr{O}_{\widetilde{X}}(-l')) + h^1(\mathscr{O}_{\widetilde{X}}(-r_h)).$$

Since $\chi(\mathscr{O}_{l'-r_h}(-r_h)) = \chi(l') - \chi(r_h)$, this reads as

$$\mathfrak{h}(l') = \chi(l') - h^1(\mathscr{O}_{\widetilde{X}}(-l')) - \chi(r_h) + h^1(\mathscr{O}_{\widetilde{X}}(-r_h)).$$

Hence, using the definition of P, we get

$$P(\mathbf{t}) = \sum_{l' \in \mathscr{S}'} \sum_{I \subset \mathscr{V}} (-1)^{|I|+1} \Big(\chi(l' + E_I) - h^1(\mathscr{O}_{\widetilde{X}}(-l' - E_I)) \Big) \mathbf{t}^{l'}. \tag{8.2.28}$$

Consider now the computation sequence $\{x_i\}_i$, depending on l', connecting $l' + E_I$ with $s(l' + E_I)$, cf. 6.6.3 ($x_0 = l' + E_I$, $x_t = s(l' + E_I)$, $x_{i+1} = x_i + E_{v(i)}$), $(E_{v(i)}, x_i) > 0$). Then, at each step one has $\chi(x_i) - h^1(\mathscr{O}_{\widetilde{X}}(-x_i)) = \chi(x_{i+1}) - h^1(\mathscr{O}_{\widetilde{X}}(-x_{i+1}))$.

Therefore, for every cycle x_i along the computation sequence one has

$$P(\mathbf{t}) = \sum_{l' \in \mathscr{S}'} \sum_{I} (-1)^{|I|+1} \Big(\chi(x_i) - h^1(\mathscr{O}_{\widetilde{X}}(-x_i)) \Big) \mathbf{t}^{l'}. \tag{8.2.29}$$

In particular, if in (8.2.29) (or in (8.2.28)) one can obtain the vanishing of the h^1–contributions (in some other way, using the particularities of a germ), then $P(\mathbf{t})$ becomes topologically computable from the graph (for that germ).

In the next paragraphs and subsections we provide several families of singularities, where $P(\mathbf{t})$ is computed explicitly, even topologically. Although among these cases there are some overlaps, the different proofs and arguments emphasize different aspects of the corresponding identities.

Example 8.2.16 Consider the **cyclic quotient singularity** whose minimal resolution ϕ has only one irreducible component E with self-intersection $-n$ ($n \geq 2$). Then $H = \mathbb{Z}_n$, $\mathscr{O}_{X_a,o} = \mathbb{C}\{z_1, z_2\}$. Moreover, $E^* = E/n$. The action of H is given by $h * z_i = \theta([E^*])(h) z_i$; hence, $z_1^i z_2^j$ is in the $\theta([E^*])^{i+j}$-eigenspace. Hence, with $\deg(z_i) = \frac{1}{n} \in A = \frac{1}{n}\mathbb{Z}$, the Poincaré series is $P(\mathbb{C}\{\mathbf{z}\}, t) = \sum_{\ell \geq 0} (1 + \ell) t^{\ell/n}$, and the Poincaré series of the H-eigenspaces are the following (see also 8.1.4 with $q = 1$):

$$P((\mathbb{C}\{\mathbf{z}\}^{\theta([aE^*])}, t) = \sum_{k \geq 0} (1 + a + nk) t^{k + \frac{a}{n}} \quad (0 \leq a < n).$$

Therefore

$$\sum_{\chi \in \widehat{H}} P(\mathbb{C}\{\mathbf{z}\}^{\chi}, t) \cdot \chi = \frac{1}{(1 - \theta([E^*]) \cdot t^{E^*})^2} \in \mathbb{Z}[[t]][\widehat{H}]. \tag{8.2.30}$$

Now we analyse the filtration $\mathscr{F}$ (and compare with the *monomial filtration* from above). The space $\widetilde{Z} = Z$ is just the blow up ϕ_a of X_a at 0 with exceptional divisor E_a a (-1)-curve. Since $p^*(E) = nE_a$ (see also 4.1.39), we get for $nk' \in \mathbb{Z}$ that $\mathscr{F}(k'E)$ contains all the monomials of degree $\geq nk'$. We claim that this inclusion is, in fact, an isomorphism. Indeed, if $f = \sum_{i+j=nk'} c_i z_1^i z_2^j \in \mathbb{C}\{\mathbf{z}\}$, such that f is not identically zero, then $\phi_a^*(f)$ will have vanishing order exactly nk' (and never higher) along E_a, independently of the choice of the coefficients $c_i \in \mathbb{C}$. Therefore,

for $k' = k + a/n$ ($k \in \mathbb{Z}$), $(\mathscr{F}(k'E)/\mathscr{F}(k'E + E))_{\theta([aE^*])}$ is exactly the vector space of monomials of degree $nk + a$ ($0 \le a < n$), and $P(\mathbf{t}) = 1/(1 - t^{E^*})^2$. Its h-components are

$$\sum_{h \in H} P_h(t) \cdot h = \frac{1}{(1 - [E^*] \cdot t^{E^*})^2} \in \mathbb{Z}[[t]][H]. \tag{8.2.31}$$

Example 8.2.17 (Cyclic Quotient Singularities with $q > 1$) We follow the notations (and statements) of Example 8.1.4; furthermore, we fix the minimal resolution ϕ of (X, o). We construct the monomorphism $\iota : A \to L'$ given by $\iota(\frac{1}{n}(1, 0)) = E_1^*$ and $\iota(\frac{1}{n}(q, n)) = E_s^*$ (where E_1 and E_s correspond to the two end-vertices of the graph). This provides a *monomial L'–filtration* of $\mathbb{C}[\sigma^\vee \cap \mathbb{Z}^2]$ by $\mathscr{G}(l') := \oplus_{\iota(\mathbf{a}) \ge l'} \mathbb{C}\langle \mathbf{z}^{\mathbf{a}} \rangle$. Let E_a be the (non-irreducible) exceptional curve of $\widetilde{Z}$. Then $\mathrm{div}_{E_a} \phi_a^*(\mathbf{z}^{\mathbf{a}}) = p^*(\iota(\mathbf{a}))$. Hence $\mathscr{G}(l') \subset \mathscr{F}(l')$ for any l'. We claim that, in fact, $\mathscr{G}(l') = \mathscr{F}(l')$. This follows from the next lemma.

Lemma 8.2.18 *Let $\phi_a : (\widetilde{Z}, E_a) \to (\mathbb{C}^2, 0)$ be a modification whose graph is a string and the strict transforms of the coordinates of $\mathbb{C}^2$ are at the two ends. Then, whenever $\sum c_{\mathbf{a}} \mathbf{z}^{\mathbf{a}} \not\equiv 0$, one has*

$$\mathrm{div}_{E_a}(\phi_a^*(\textstyle\sum c_{\mathbf{a}} \mathbf{z}^{\mathbf{a}})) = \min_{c_{\mathbf{a}} \neq 0} \mathrm{div}_{E_a}(\phi_a^*(\mathbf{z}^{\mathbf{a}})).$$

Indeed, if the above equality did not hold, then along a component, $\phi_a^*(\sum c_{\mathbf{a}} \mathbf{z}^{\mathbf{a}})$ would have a 'higher vanishing' (compared with the one imposed by 'min'). This can happen only if there exist at least two different monomials of type $\mathbf{z}^{\mathbf{a}}$ such that their multiplicities along two neighboring irreducible exceptional divisors are the same. But this is not possible (otherwise, all the multiplicities are the same by (2.2.1)). That is, the 'jump'–phenomenon from 6.7.26 does not occur in this case.

The identity of the two filtrations imply that $\mathfrak{h}(l')$ is the cardinality of $\{\mathbf{a} : \iota(\mathbf{a}) \not\ge l',\ [\iota(\mathbf{a})] = [l']\}$. Since the identity (8.2.10) determines $P(\mathbf{t})$ uniquely, we get that

$$\mathfrak{p}(l') = \begin{cases} 1 \text{ if } l' = \iota(\mathbf{a}) \text{ for some } \mathbf{a}, \\ 0 \text{ otherwise.} \end{cases}$$

Therefore, (8.1.3) transforms into $P(\mathbf{t}) = (1 - \mathbf{t}^{E_1^*})^{-1}(1 - \mathbf{t}^{E_s^*})^{-1}$. (This can be compared with the 'topological/combinatorial' series $Z(\mathbf{t})$ from 8.4.)

The coincidence of the monomial and divisoarial filtrations is more general, see the case of singularities satifying the 'end curve condition' in 8.5.21.

Remark 8.2.19 The filtration $\{\mathscr{F}(l')\}_{l' \in L'}$ is associated with a fixed resolution ϕ. Taking different resolutions we obtain different filtrations. However, they can be compared: if ϕ_2 dominates ϕ_1 then $\mathscr{F}_2$ is finer than $\mathscr{F}_1$.

For example, any modification of $(\mathbb{C}^2, 0)$ (by blowing up infinitely near points of the origin) provides a multivariable filtration of $\mathscr{O}_{\mathbb{C}^2,0}$, which geometrically can

be associated with simultaneous intersection properties of the germs with a fixed collection of certain plane curve singularities.

8.3 Linear Subspace Arrangements Associated with the Filtration

Consider the Poincaré series $P(\mathbf{t})$ of (X, o) associated with the divisorial filtration. (In particular, in the L' equivariant case we assume that the link is a rational homology sphere, in the discussion of invariants indexed by L this assumption is not needed, cf. 8.2.1).
For each $l' \in L'$ we construct a (finite dimensional) linear subspace arrangement $\{A_v(l')\}_{v\in\mathcal{V}} \subset A(l')$ such that the Euler characteristic of the projectivized complement $\mathbb{P}(A(l') \setminus \cup_v A_v(l'))$ is $\mathfrak{p}(l')$. $A(l')$ sits naturally in a finite dimensional vector space $T(l')$, which admits a linear subspace arrangement $\{T_v(l')\}_{v\in\mathcal{V}}$ too, such that $A_v(l') = A(l') \cap T_v(l')$ for any v. Furthermore, the arrangement $\{T_v(l')\}_{v\in\mathcal{V}} \subset T(l')$ is purely topological. Both arrangements are rich sources of analytic/topological invariants, which 'stay in pairs'. We follow [487]. ♣

8.3.1 Fix a normal surface singularity as in 8.2.1, one of its resolutions and the filtration $\{\mathcal{F}(l')\}_{l'\in L'}$ from 8.2.5. For any $l' \in L'$, the linear space (cf. (8.2.7))

$$(\mathcal{F}(l')/\mathcal{F}(l'+E))_{\theta([l'])} = H^0(\mathcal{O}_{\widetilde{X}}(-l'))/H^0(\mathcal{O}_{\widetilde{X}}(-l'-E))$$

naturally embeds into

$$T(l') := H^0(\mathcal{O}_E(-l')).$$

Let its image be denoted by $A(l')$. Furthermore, for every $v \in \mathcal{V}$, consider the linear subspace $T_v(l')$ of $T(l')$ given by

$$T_v(l') := H^0(\mathcal{O}_{E-E_v}(-l'-E_v)) = \ker\big(\, H^0(\mathcal{O}_E(-l')) \to H^0(\mathcal{O}_{E_v}(-l'))\,\big) \subset T(l').$$

Then the image $A_v(l')$ of $H^0(\mathcal{O}_{\widetilde{X}}(-l'-E_v)/H^0(\mathcal{O}_{\widetilde{X}}(-l'-E))$ in $T(l')$ satisfies $A_v(l') = A(l') \cap T_v(l')$.

Definition 8.3.2 The (finite dimensional) arrangement of linear subspaces $\mathcal{A}_{\text{top}}(l') = \{T_v(l')\}_v$ in $T(l')$ is called the 'topological arrangement' at $l' \in L'$. The arrangement of linear subspaces $\mathcal{A}_{\text{an}}(l') = \{A_v(l') = T_v(l') \cap A(l')\}_v$ in $A(l')$ is called the 'analytic arrangement' at $l' \in L'$. The corresponding projectivized arrangement complements will be denoted by $\mathbb{P}(T(l') \setminus \cup_v T_v(l'))$ and $\mathbb{P}(A(l') \setminus \cup_v A_v(l'))$ respectively.

If $l' \notin \mathscr{S}'$ then there exists v such that $(E_v, l') > 0$, that is $h^0(\mathscr{O}_{E_v}(-l')) = 0$, proving that $T_v(l') = T(l')$. Hence $A_v(l') = A(l')$ too. In particular, both arrangement complements are empty. In fact, if $l' \notin \mathscr{S}'_{an}$, then by similar argument, the analytic arrangement complement is empty too (use 8.2.9(d)).

The connection with the series P is provided by the following topological Euler characteristic formula.

Lemma 8.3.3 *Assume that $\{V_\alpha\}_{\alpha\in\Lambda}$ is a finite family of linear subspaces of a finite dimensional linear space V. For $I \subset \Lambda$ set $V_I := \cap_{\alpha\in I} V_\alpha$ (where $V_\emptyset = V$). Then*

$$\chi_{\rm top}(\mathbb{P}(V \setminus \cup_\alpha V_\alpha)) = \sum_{I\subset\Lambda} (-1)^{|I|} \dim V_I.$$

If $\Lambda \neq \emptyset$, then this also equals $\sum_I (-1)^{|I|+1}\mathrm{codim}(V_I \subset V)$.

Proof Use the inclusion–exclusion principle and $\dim V_I = \chi_{\rm top}(\mathbb{P}V_I)$. □

Corollary 8.3.4 *For any $l' \in \mathscr{S}'_{an}$ one has*

$$\mathfrak{p}(l') = \chi_{\rm top}(\,\mathbb{P}(A(l') \setminus \cup_v A_v(l'))\,).$$

Proof Use Lemmas 8.2.8(b) and 8.3.3. □

Similar identities appear in several article of Campillo, Delgado and Gusein-Zade, see e.g. [96]. (However, in their articles the linear subspace arrangements $\mathscr{A}_{\rm top}(l')$ and $\mathscr{A}_{\rm an}(l')$ do not appear, they were introduced in [486, 487]. A connection between the two approaches can be realized using the language and technique presented after Theorem 8.4.11.)

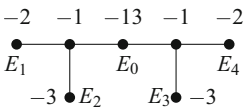
The corresponding dimensions of the linear subspaces in $\mathscr{A}_{\rm an}(l')$ are as follows.

Lemma 8.3.5 *For any $l' \in L'$ and $I \subset \mathscr{V}$ one has:*

$$\dim A(l') = \mathfrak{h}(l'+E) - \mathfrak{h}(l'), \quad \dim \cap_{v\in I} A_v(l') = \mathfrak{h}(l'+E) - \mathfrak{h}(l'+E_I).$$

Thus, we can expect that the analytic arrangement is rather sensitive to the modification of the analytic structure, and, in general, does not coincide with the topological arrangement.

Example 8.3.6 Consider the (splice type) complete intersection singularity in $(\mathbb{C}^4, 0)$ given by $z_1^2 + z_2^3 - z_3^2 z_4 = z_4^2 + z_3^3 - z_2^2 z_1 = 0$. Its graph is



One verifies that $\mathrm{div}(z_i) = E_i^*$ for $1 \le i \le 4$; $Z_{min} = E_0^* \notin \mathscr{S}_{an}$; $Z_{max} = 2E_0^* = \min\{E_2^*, E_3^*\}$. We wish to find $\mathfrak{p}(Z_{max})$. Note that $T(Z_{max}) = H^0(\mathscr{O}_E(-Z_{max})) \simeq H^0(\mathscr{O}_{E_0}(-Z_{max})) \simeq \mathbb{C}^3$. On the other hand, $A(Z_{max})$ is the image of $H^0(\mathscr{O}(-Z_{max})) = \mathfrak{m}_{X,o}$. Since z_1, z_4 and $\mathfrak{m}_{X,o}^2$ are contained in $H^0(\mathscr{O}(-Z_{max} - E)$, $A(Z_{max})$ is 2-dimensional, generated by the classes of z_2 and z_3. (To see the linear independence of their classes, check their divisors.) Hence $A(Z_{max}) \neq T(Z_{max})$.

Moreover, $\cup_v A_v(Z_{max})$ is the union of the two coordinate axes. Hence $\mathbb{P}(\mathbb{C}^2 \setminus (\mathbb{C} \cup \mathbb{C})) = \mathbb{C}^*$ and $\mathfrak{p}(Z_{max}) = 0$.

This shows that the support of P can be smaller than $\mathscr{S}'_{an}$. Note also that both E_2^* and E_3^* are in the support of P (with coefficients 1), but $Z_{max} = \min\{E_2^*, E_3^*\}$ is not.

Although $A(Z_{max}) \neq T(Z_{max})$, they still can be compared. Indeed, $T(Z_{max}) = H^0(\mathscr{O}_{E_0}(2))$ and $\cup_v T_v(Z_{max})$ is a union of two 2-planes (corresponding to global sections of $\mathscr{O}_{E_0}(2)$ vanishing at the two intersection points of E_0 with the other components). Hence $T \setminus \cup_v T_v = (A \setminus \cup_v A_v) \times \mathbb{C}$, and $\chi_{\mathrm{top}}(\mathbb{P}(T \setminus \cup_v T_v)) = 0$ too.

Here $T_0 = 0$, which is contained in A, and A intersects all the other strata of $\{T_v\}_v$ generically. This guarantees that $\chi_{\mathrm{top}}(\mathbb{P}(A \setminus \cup_v A_v)) = \chi_{\mathrm{top}}(\mathbb{P}(T \setminus \cup_v T_v))$ holds.

Our next goal is to show that whenever the link of the singularity is a rational homology sphere the *topological arrangement* $\mathscr{A}_{\mathrm{top}}$ is indeed topological: it depends only on the combinatorics of the resolution graph.

We will need the following technical definition.

Lemma 8.3.7

(1) For any $l' \in L'$ and subset $I \subset \mathscr{V}$ there exists a unique minimal subset $J(l', I) \subset \mathscr{V}$ which contains I, and has the following property:

$$\text{there is no } v \in \mathscr{V} \setminus J(l', I) \text{ with } (E_v, l' + E_{J(l',I)}) > 0. \tag{8.3.1}$$

(2) $J(l', I)$ can be found by the next algorithm: one constructs a sequence $\{I_m\}_{m=0}^k$ of subsets of $\mathscr{V}$, with $I_0 = I$, $I_{m+1} = I_m \cup \{v(m)\}$, where the index $v(m)$ is determined as follows. Assume that I_m is already constructed. If I_m satisfies (8.3.1) then we stop and $m = k$. Else, there exists at least one v with $(E_v, l' + E_{I_m}) > 0$. Take $v(m)$ one of them and continue the algorithm with I_{m+1}. Then $I_k = J(l', I)$.

Proof For *(1)* notice that if J_1 and J_2 satisfies the wished requirement (8.3.1) of $J(l', I)$ then $J_1 \cap J_2$ satisfies it too. Part *(2)* is a version of the Algorithm 6.6.7. □

Remark 8.3.8 With the notation of Lemma 8.3.7

$$\chi(l' + E_{J(l',I)}) = \min_{\bar{I} \subset \mathscr{V} \setminus I} \chi(l' + E_{I \cup \bar{I}}). \tag{8.3.2}$$

Indeed, let $K \subset \mathscr{V} \setminus I$ be such that $\chi(l' + E_{I\cup K}) = \min_{\bar{I}\subset\mathscr{V}\setminus I} \chi(l' + E_{I\cup\bar{I}})$. Note also that $\chi(l' + E_{I_m})$ along the computation sequence from 8.3.7(2) is non-increasing. Hence, it is enough to show that the sequence $\{I_m\}_m$ can be constructed in such a way that for some $k_0 \le k$ one has $I_{k_0} = I \cup K$. Indeed, assume that I_m is already constructed and it satisfies $I_m < I \cup K$. Set $\bar{I}_m := I \cup K \setminus I_m$. By assumption $\chi(l' + E_{I\cup K}) \le \chi(l' + E_{I_m})$, which implies $(E_{\bar{I}_m}, l' + E_{I_m}) \ge \chi(E_{\bar{I}_m}) > 0$. Hence, there exists $v \in \bar{I}_m$ so that $I_{m+1} = I_m \cup \{v\}$ satisfies $I_{m+1} \le I \cup K$.

Proposition 8.3.9 *Assume that the resolution graph is a tree of rational curves. For any $l' \in L'$ and $I \subset \mathscr{V}$ write $J(I) := J(l', I)$. Then the following facts hold.*

(a) One has the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \to & H^0(\mathscr{O}_{E-E_{J(I)}}(-l' - E_{J(I)})) & \to & H^0(\mathscr{O}_E(-l')) & \overset{k}{\twoheadrightarrow} & H^0(\mathscr{O}_{E_{J(I)}}(-l')) & \to & 0 \\
 & & \simeq\downarrow j & & \| & & \downarrow i & & \\
0 & \to & H^0(\mathscr{O}_{E-E_I}(-l' - E_I)) & \to & H^0(\mathscr{O}_E(-l')) & \to & H^0(\mathscr{O}_{E_I}(-l')) & & \\
 & & \| & & \| & & & & \\
 & & \cap_{v\in I} T_v(l') & \hookrightarrow & T(l') & & & &
\end{array}$$

where j is an isomorphism (hence $\cap_{v\in I} T_v(l') = \cap_{v\in J(I)} T_v(l')$), i is injective and k is surjective.

(b) $\dim \cap_{v\in J(I)} T_v(l') = \chi(\mathscr{O}_{E-E_{J(I)}}(-l' - E_{J(I)})) = \chi(l' + E) - \chi(l' + E_{J(I)})$.

(c) In particular, if $J(I_1) = J(I_2)$ then $\cap_{v\in I_1} T_v(l') = \cap_{v\in I_2} T_v(l')$, and if $J(I_1) \subsetneq J(I_2)$ then $\cap_{v\in I_1} T_v(l') \supsetneq \cap_{v\in I_2} T_v(l')$. Therefore, $J(I)$ is the unique maximal subset $I_{max} \subset \mathscr{V}$, such that $I \subset I_{max}$, and $\cap_{v\in I} T_v(l') = \cap_{v\in I_{max}} T_v(l')$.

(d) Part (b) for $I = \emptyset$ reads as follows: $\dim T(l') = \dim \cap_{v\in J(\emptyset)} T_v(l') = \chi(l' + E) - \chi(l' + E_{J(\emptyset)})$. *Hence, if $l' \in \mathscr{S}'$ then* $\dim T(l') = -(l', E) + 1$.

(e) $\operatorname{codim}(\cap_{v\in I} T_v(l') \hookrightarrow T(l')) = \chi(l' + E_{J(I)}) - \chi(l' + E_{J(\emptyset)})$.

(f) In particular, the arrangement complement is non-empty if and only if $J(\emptyset) = \emptyset$ (if and only if $l' \in \mathscr{S}'$).

Proof First we prove the following fact: let $F \le E$ be an effective non-zero cycle and we assume that for any $E_w \le F$ one has $(E_w, l') \le 0$. Then $h^1(\mathscr{O}_F(-l')) = 0$. The proof runs over induction: choose E_w from the support of F so that $(F - E_w, E_w) \le 1$, then use the cohomological long exact sequence of $\mathscr{O}_F(-l') \to \mathscr{O}_{F-E_w}(-l')$.

From the definition, $\cap_{v\in I} T_v(l') = H^0(\mathscr{O}_{E-E_I}(-l' - E_I))$. To prove *(a)* note that the group $I \mapsto \cap_{v\in I} T_v(l')$ is stable along the steps of the algorithm 8.3.7(2). Hence j is an isomorphism. Similarly, along these steps i is injective. Since $h^1(\mathscr{O}_{E-E_J}(-l' - E_J)) = 0$ by the above fact, k is onto and *(b)* follows too. For *(c)* use *(b)* and the fact that $\chi(l' + J(I_2)) > \chi(l' + J(I_1))$ whenever $J(I_1) \subsetneq J(I_2)$. The proof of *(f)* runs as follows. If there exists $v \in J(\emptyset)$ then $J(\emptyset) = J(v)$ hence $T_v(l') = T(l')$. Otherwise, if $l' \in \mathscr{S}'$ and $J \ne \emptyset$, then $\chi(l' + E_J) - \chi(l') = -(l', E_J) + \chi(E_J) > 0$. □

Corollary 8.3.10 *If the graph is a tree of rational curves then the isotopy type of the arrangement $\mathscr{A}_{\text{top}}$ depends only on the combinatorial data of the graph.*

Remark 8.3.11 Note that $J(l', I)$—associated with $l' \in L'$ and $I \subset \mathscr{V}$—, and also the topological linear subspace arrangement $\mathscr{A}_{\text{top}}$, depend only on the E^*-coefficients of l' and on the *shape of the graph* Γ, that is, on the valencies $\{\kappa_v\}_v$ but not on the Euler numbers $\{e_v\}_v$.

At topological Euler characteristic level one has:

Corollary 8.3.12 *If the graph is a tree of rational curves and $l' \in \mathscr{S}'$ then*

$$\chi_{\text{top}}(\,\mathbb{P}(T(l') \setminus \cup_v T_v(l'))\,) = \sum_{I \subset \mathscr{V}} (-1)^{|I|+1} \chi(l' + E_{J(l',I)}).$$

Proof Use Lemma 8.3.3 and Proposition 8.3.9*(b)*. □

Example 8.3.13 Consider the situation of the example 8.3.6, and set $l' = Z_{min}$. Then $T(Z_{min}) = \mathbb{C}^2$ and $\cup_v T_v(Z_{min})$ consists of the union of two different lines. Therefore $\chi_{\text{top}}(\,\mathbb{P}(T(Z_{min} \setminus \cup_v T_v(Z_{min}))\,) = 0$.

At Z_{min} the complement of the analytic arrangement is empty.

Theorem 8.3.14 *Assume the following cases:*

(a) either (X, o) is rational, ϕ is an arbitrary resolution, and $l' \in \mathscr{S}'$ is arbitrary,
(b) or (X, o) is a minimally elliptic singularity with $H^1(\widetilde{X}, \mathbb{Z}) = 0$, ϕ is a resolution with $|C| = E$, and we also assume that either $l' = 0$ or for the fixed $l' \in \mathscr{S}'$ there exists a computation sequence for Z_{min}, which contains E as one of its terms, and it jumps at some E_1 with $(E_1, l') < 0$ (cf. 7.2.B).

Then the topological and analytic arrangements at l' agree, $\mathscr{A}_{\text{top}}(l') = \mathscr{A}_{\text{an}}(l')$.

Proof We need to show that $H^0(\mathscr{O}_{\widetilde{X}}(-l')) \to H^0(\mathscr{O}_E(-l'))$ is surjective. This follows from the vanishing of $H^1(\mathscr{O}_{\widetilde{X}}(-l' - E))$ from Theorem 7.1.11*(b)* applied for $z_i = E$ (rational case), respectively from Theorem 7.2.31*(d)* (minimally elliptic case, $l' \neq 0$). For $l' = 0$ the surjectivity is clear. □

Example 8.3.15 For minimally elliptic singularities it can happen that $\mathscr{A}_{\text{top}}(l') \neq \mathscr{A}_{\text{an}}(l')$, even for the minimal resolution. E.g., in the case of the minimal good resolution of $\{x^2 + y^3 + z^7 = 0\}$, or in the case of minimal resolution of $\{x^2 + y^3 + z^{11} = 0\}$ (which is good), for $l = Z_{min}$ one has $\dim T(Z_{min}) = 2$ and $\dim A(Z_{min}) = 1$.

8.4 The Topological Series $Z(\mathbf{t})$

We provide several equivalent definitions of the series $Z(\mathbf{t})$, the topological 'pair' of the analytic Poincaré series $P(\mathbf{t})$. ♣

8.4.1 Examples 8.2.16 and 8.2.17, and also the case of singularities with $\mathbb{C}^*$-actions, cf. e.g. (8.1.5), motivate the definition of the next series, as a combinatorial/topological 'candidate' for $P(\mathbf{t})$.

We fix a good resolution and we assume that each $E_v \simeq \mathbb{P}^1$.

Definition 8.4.2 We define the **rational function** $Z(\mathbf{t})$ in variables $x_v = \mathbf{t}^{E_v^*}$ by

$$Z(\mathbf{t}) := \Phi(z)(\mathbf{t}), \quad \text{where } z(\mathbf{x}) := \prod_{v \in \mathscr{V}} (1 - x_v)^{\kappa_v - 2}. \tag{8.4.1}$$

Hence $Z(\mathbf{t}) = \prod_v (1 - \mathbf{t}^{E_v^*})^{\kappa_v - 2}$. By 8.2.4, for any $h \in H$ the h-component of $Z(\mathbf{t})$ is

$$Z_h(\mathbf{t}) := \frac{1}{|H|} \cdot \sum_{\rho \in \widehat{H}} \rho(h)^{-1} \cdot \prod_{v \in \mathscr{V}} (1 - \rho([E_v^*])\mathbf{t}^{E_v^*})^{\kappa_v - 2}. \tag{8.4.2}$$

Here, as usual, H is the discriminant group L'/L, and $\widehat{H}$ is its dual.

In the sequel we identify the rational function $Z(\mathbf{t})$ with its Taylor expansion at the origin, as an element of $\mathbb{Z}[[\mathscr{S}']]$ (cf. 8.2.1).

This definition provides a well-defined series even if the link is not a rational homology sphere. However, *in subsection 8.4 and 8.4.A we will assume that the link is a rational homology sphere* (in 8.4.B and 8.4.C we will remove this assumption).

8.4.3 We start the list of properties of $Z(\mathbf{t})$ by the next observation, cf. [97, 100]. If Σ is a topological space, let $S^a\Sigma$ ($a \geq 0$) denote its symmetric product $\Sigma^a/\mathfrak{S}_a$. For $a = 0$, by convention, $S^0\Sigma$ is a point. Then, by Macdonald formula [401],

$$\sum_{a \geq 0} \chi_{\mathrm{top}}(S^a\Sigma)\, x^a = (1 - x)^{-\chi_{top}(\Sigma)}. \tag{8.4.3}$$

Note that if E_v° denotes $E_v \setminus \cup_{u \neq v} E_u$ then $\chi_{\mathrm{top}}(E_v^\circ) = 2 - \kappa_v$.

Corollary 8.4.4 (The First Appearance of $Z(\mathbf{t})$ [97, 100, 481, 482, 484, 486, 489]) *With the notation* $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \cdots x_s^{a_s}$,

$$\begin{aligned} z(\mathbf{x}) &= \prod_v \sum_{a_v \geq 0} \chi_{\mathrm{top}}(S^{a_v} E_v^\circ)\, x_v^{a_v} = \sum_{\mathbf{a} \geq 0} \prod_v \chi_{\mathrm{top}}(S^{a_v} E_v^\circ)\mathbf{x}^{\mathbf{a}} \\ &= \sum_{\mathbf{a} \geq 0} \Big(\prod_v (-1)^{a_v} \binom{\kappa_v - 2}{a_v} \Big)\, \mathbf{x}^{\mathbf{a}}, \end{aligned} \tag{8.4.4}$$

where, for $b \in \mathbb{Z}$, $\binom{b}{0} = 1$ *and* $\binom{b}{a} = b(b-1)\cdots(b-a+1)/a!$ *for* $a \in \mathbb{Z}_{>0}$.

In the next paragraphs we provide another interpretation of $Z(\mathbf{t})$. The combinatorial inductive argument of the next proof will also serve as a model for the proof of several other identities (e.g. for the proofs of 8.4.13 and 8.5.16).

For the definition of the cycle $J(l', I)$ associated with $l' \in L'$ and $I \subset \mathscr{V}$ see 8.3.7.

Theorem 8.4.5 (The Second Appearance of $Z(\mathbf{t})$ [486, 487])

$$Z(\mathbf{t}) = \sum_{l' \in \mathscr{S}'} \sum_{I \subset \mathscr{V}} (-1)^{|I|+1} \chi(l' + E_{J(l',I)}) \mathbf{t}^{l'}. \tag{8.4.5}$$

This combined with 8.3.8 gives

$$Z(\mathbf{t}) = \sum_{l' \in \mathscr{S}'} \sum_{I \subset \mathscr{V}} (-1)^{|I|+1} \min_{\bar{I} \supset I} \chi(l' + E_{\bar{I}}) \cdot \mathbf{t}^{l'}. \tag{8.4.6}$$

Proof With the notation $l' = \sum_v a_v E_v^*$ set

$$y(\mathbf{x}) := \sum_{\mathbf{a} \geq 0} \sum_{I} (-1)^{|I|+1} \chi(l' + E_{J(l',I)}) \mathbf{x}^{\mathbf{a}}. \tag{8.4.7}$$

We wish to show that $y(\mathbf{x}) = z(\mathbf{x})$. In the proof we use induction over $|\mathscr{V}|$. The verification of the $|\mathscr{V}| = 1$ case is left to the reader. Hence, we assume $|\mathscr{V}| > 1$. Fix a vertex $w \in \mathscr{V}$ so that $\kappa_w = 1$. Let $\Gamma_0 := \Gamma \setminus \{w\}$, and let u be that vertex of Γ_0 which is adjacent to w in Γ. Let $\mathbf{x}_0$ be the $\mathbf{x}$–vector associated with $\mathscr{V}(\Gamma_0)$. Clearly, one has

$$z_\Gamma(\mathbf{x}) = z_{\Gamma_0}(\mathbf{x}_0) \cdot (1 - x_u)/(1 - x_w).$$

We will establish similar identity for y_Γ. For this we write $l'_0 := R(l') = \sum_{v \neq w} a_v E_v^{*,\Gamma_0}$ for any $l' = \sum_v a_v E_v^*$ (here E_v^{*,Γ_0} is the anti-dual of E_v in Γ_0). This is the restriction, the dual operator $L(\Gamma)' \to L(\Gamma_0)'$ of the inclusion $L(\Gamma_0) \to L(\Gamma)$. Hence, for $Z \in L(\Gamma_0)$

$$(l', Z) = (l'_0, Z) \text{ and } (-E_u^{*,\Gamma_0}, Z) = (E_w, Z). \tag{8.4.8}$$

First, we fix some $l' \in \mathscr{S}'$ and a subset $I \subset \mathscr{V}$ with $w \notin I$. If $w \in J(l', I)$, we may assume (cf. the notations of 8.3.7) that $I_{k-1} = J(l', I) \setminus w$. Since $(l', E_w) \leq 0$, $\kappa_w = 1$ and $(l' + E_{I_{k-1}}, E_w) > 0$, we get that, in fact, $(l' + E_{I_{k-1}}, E_w) = 1$. Hence

$$\chi(l' + E_{J(l',I)}) = \chi(l' + E_{J(l',I)\setminus w}).$$

Comparing the two algorithms on Γ and Γ_0 we get that $J(l', I) \setminus w = J^{\Gamma_0}(l'_0, I)$, and

$$\chi(l' + E_{J(l',I)\setminus w}) - \chi(l' + E_I) = \chi(l'_0 + E_{J^{\Gamma_0}(l'_0,I)}) - \chi(l'_0 + E_I).$$

By (8.4.8) one also has $\chi(l'+E_I)-\chi(l')=\chi(l'_0+E_I)-\chi(l'_0)$. All these implies the next identity, where in the right hand side all invariants are considered in Γ_0:

$$\chi(l'+E_{J(l',I)})-\chi(l')=\chi(l'_0+E_{J^{\Gamma_0}(l'_0,I)})-\chi(l'_0). \tag{8.4.9}$$

The same is true if $w\notin J(l',I)$. Next, fix again $l'\in\mathscr{S}'$ and take $I\subset\mathscr{V}$ with $w\in I$.

We need to distinguish two cases. In the first case we assume that $(l',E_u)=0$. This happens exactly when $l'_0-E_u^{*,\Gamma_0}\notin\mathscr{S}'(\Gamma_0)$. In this case, for any $K\subset\mathscr{V}\setminus\{u,w\}$ one has $J(l',K\cup w\})=J(l',K\cup\{w,u\})$. Indeed, $(l'+E_{K\cup w},E_u)\geq(E_w,E_u)=1$, hence in the very first step of the algorithm of $J(l',K\cup w)$ we can add E_u. Thus,

$$\sum_{M\not\ni w}(-1)^{|M|}\chi(l'+E_{J(l',M\cup w)})=0. \tag{8.4.10}$$

Next, assume that $l'_0-E_u^{*,\Gamma_0}\in\mathscr{S}'(\Gamma_0)$. Then, comparing the two algorithms we get

$$J(l',I)=J^{\Gamma_0}(l'_0-E_u^{*,\Gamma_0},I\setminus w)\cup w,$$

$$\chi(l'+E_{J(l',I)})-\chi(l'+E_I)=\chi(l'_0-E_u^{*,\Gamma_0}+E_{J^{\Gamma_0}(l'_0-E_u^{*,\Gamma_0},I\setminus w)})-\chi(l'_0-E_u^{*,\Gamma_0}+E_{I\setminus w}).$$

Finally, (8.4.8) implies

$$\chi(l'+E_w+E_{I\setminus w})-\chi(l'+E_w)=\chi(l'_0-E_u^{*,\Gamma_0}+E_{I\setminus w})-\chi(l'_0-E_u^{*,\Gamma_0}).$$

Hence

$$\chi(l'+E_{J(l',I)})-\chi(l'+E_w)=\chi(l'_0-E_u^{*,\Gamma_0}+E_{J^{\Gamma_0}(l'_0-E_u^{*,\Gamma_0},I\setminus w)})-\chi(l'_0-E_u^{*,\Gamma_0}). \tag{8.4.11}$$

Since for any constant c, one has $\sum_{I:I\not\ni w}(-1)^{|I|+1}c=\sum_{I:I\ni w}(-1)^{|I|+1}c=0$, the identities (8.4.9), (8.4.10) and (8.4.11) read as

$$\sum_{\mathbf{a}\geq 0}\sum_{I\not\ni w}(-1)^{|I|+1}\chi(l'+E_{J(l',I)})\mathbf{x}^{\mathbf{a}}=y_{\Gamma_0}(\mathbf{x}_0)\cdot\sum_{a_w\geq 0}x_w^{a_w};$$

$$\sum_{\mathbf{a}\geq 0}\sum_{I\ni w}(-1)^{|I|+1}\chi(l'+E_{J(l',I)})\mathbf{x}^{\mathbf{a}}=-y_{\Gamma_0}(\mathbf{x}_0)x_u\cdot\sum_{a_w\geq 0}x_w^{a_w}.$$

Hence $y_\Gamma(\mathbf{x})=y_{\Gamma_0}(\mathbf{x}_0)(1-x_u)/(1-x_w)$. □

Remark 8.4.6 If we compare the computation sequences from 8.3.7 and 8.2.15 we realize that there is a possible choice of the computation sequence $\{x_i\}_i$ connecting $l' + E_I$ with $s(l' + E_I)$ such that $l' + E_{J(l',I)}$ is one of its terms. Therefore, (8.2.29) applied for $x_i = l' + E_{J(l',I)}$ reads as

$$P(\mathbf{t}) = \sum_{l' \in \mathscr{S}'} \sum_{I \subset \mathscr{V}} (-1)^{|I|+1} \Big(\chi(l' + E_{J(l',I)}) - h^1(\mathscr{O}_{\widetilde{X}}(-l' - E_{J(l',I)})) \Big) \mathbf{t}^{l'}.$$

This combined with (8.4.5) gives

$$Z(\mathbf{t}) - P(\mathbf{t}) = \sum_{l' \in \mathscr{S}'} \sum_{I \subset \mathscr{V}} (-1)^{|I|+1}\, h^1(\mathscr{O}_{\widetilde{X}}(-l' - E_{J(l',I)})\, \mathbf{t}^{l'}.$$

Corollary 8.4.7 (The Third Appearance of $Z(\mathbf{t})$ [486, 487])

$$Z(\mathbf{t}) = \sum_{l' \in \mathscr{S}'} \chi_{\mathrm{top}}(\, \mathbb{P}(T(l') \setminus \cup_v T_v(l'))\,) \cdot \mathbf{t}^{l'}.$$

Proof Combine Corollary 8.3.12 and Proposition 8.4.5. □

Corollary 8.4.8 *$P(\mathbf{t}) = Z(\mathbf{t})$ in the following cases:*

(a) (X, o) is rational, and ϕ is an arbitrary resolution,
(b) or (X, o) is a minimally elliptic singularity, and it satisfies the assumptions of Theorem 8.3.14.

Proof Use 8.3.14 (i.e. $\mathscr{A}_{\mathrm{an}}(l') = \mathscr{A}_{\mathrm{top}}(l')$), 8.3.4 and 8.4.7. □

Remark 8.4.9

(i) The first proof of the identity $P(\mathbf{t}) = Z(\mathbf{t})$ for rational singularities was given in [100]. A different proof appears in [481], which holds for rational and minimally elliptic germs as well.
(ii) Part *(b)* can be improved for some additional cases even if $\mathscr{A}_{\mathrm{an}}(l') \neq \mathscr{A}_{\mathrm{top}}(l')$. E.g., if (X, o) is a minimally elliptic singularity whose minimal resolution is good, and if ϕ is this resolution, then $P(\mathbf{t}) = Z(\mathbf{t})$, cf. 8.5.A. In 8.5.A we prove the identity $P(\mathbf{t}) = Z(\mathbf{t})$ for the more general family of splice quotient singularities. In fact, $P(\mathbf{t}) = Z(\mathbf{t})$ implies that (X, o) is a splice quotient, cf. 8.5.19 and 8.5.35.
(iii) See 8.6.B for several examples when $Z(\mathbf{t}) \neq P(\mathbf{t})$.

8.4.10 $\mathbb{P}(T(l') \setminus \cup_v T_v(l'))$ as a Space of Effective Cartier Divisors For any cycle $Z \in L$, $Z \geq E$, let $\mathrm{ECa}(Z)$ be the set of effective Cartier divisors on Z. Their supports are zero-dimensional in E. Taking the class of a Cartier divisor provides the *Abel map* $\mathrm{ECa}(Z) \to \mathrm{Pic}(Z)$. Let $\mathrm{ECa}^{l'}(Z)$ be the subset of $\mathrm{ECa}(Z)$, which consists of divisors whose associated line bundles have Chern class l'. Set

$c^{l'} : \mathrm{ECa}^{l'}(Z) \to \mathrm{Pic}^{l'}(Z)$ for the restricted Abel map. Regarding the existence of $\mathrm{ECa}^{l'}(Z)$ as an algebraic variety we make the following comment. First, by a theorem of Artin [29, 3.8], there exists an affine algebraic variety Y and a point $y \in Y$ such that (Y, y) and (X, o) have isomorphic formal completions. Then, according to Hironaka [266], (Y, y) and (X, o) are analytically isomorphic. In particular, we can regard Z as a projective algebraic scheme, in which case $\mathrm{ECa}^{l'}(Z)$ together with the algebraic Abel map, as part of the general theory, was constructed by Grothendieck [245], see also the article of Kleiman [319]. In particular, $c^{l'} : \mathrm{ECa}^{l'}(Z) \to \mathrm{Pic}^{l'}(Z)$ is algebraic. For several properties see [452, 453, 456].

From the definition, $\mathrm{ECa}^{l'}(Z) \neq \emptyset$ if and only if there exists $\mathscr{L} \in \mathrm{Pic}^{l'}(Z)$, which has no fixed components (the generic global section does not vanish along any of E_v's). If this case $l' \neq 0$ and $H^0(\mathscr{L}|_{E_v}) \neq 0$ for any v, hence $-l' \in \mathscr{S}' \setminus \{0\}$. Conversely, if $-l' \in \mathscr{S}' \setminus \{0\}$ then one can construct an element of $\mathrm{ECa}^{l'}(Z)$ by (l', E_v) generic cuts along each E_v. Hence, it is natural to modify the definition (for $l' = 0$), and redefine formally $\mathrm{ECa}^0(Z)$ as a point, the space of the 'empty divisor' $\{\emptyset\}$. It is sent by the Abel map to $\mathscr{O}_Z$. In this new version, $\mathrm{ECa}^{l'}(Z) \neq \emptyset \Leftrightarrow -l' \in \mathscr{S}'$.

In [452] it is proved that for any $-l' \in \mathscr{S}'$, $\mathrm{ECa}^{l'}(Z)$ is irreducible, quasiprojective, smooth and of dimension $-(l', Z)$. (Here we will need only the case $Z = E$.)

For any $\mathscr{L} \in \mathrm{Pic}^{l'}(Z)$ define $H^0(Z, \mathscr{L})_{\mathrm{reg}}$, the set of 'regular' sections (or, of sections without fixed components) by $H^0(Z, \mathscr{L}) \setminus \cup_v H^0(Z - E_v, \mathscr{L}(-E_v))$. Then the preimage of $\mathscr{L}$ by the Abel map is $H^0(Z, \mathscr{L})_{\mathrm{reg}}/H^0(\mathscr{O}_Z^*)$ [318, §3].

Next, assume that $Z = E$ and $l' \in \mathscr{S}'$ (note the sign change). Since $h^1(\mathscr{O}_E) = 0$, $\mathrm{Pic}^{-l'}(E)$ is a point, say $\{\mathscr{O}_E(-l')\}$, and $H^0(\mathscr{O}_E^*) = \mathbb{C}^*$. Hence,

$$\mathrm{ECa}^{-l'}(E) = H^0(\mathscr{O}_E(-l'))_{\mathrm{reg}}/\mathbb{C}^* = \mathbb{P}(T(l') \setminus \cup_v T_v(l')). \tag{8.4.12}$$

Theorem 8.4.11 (The Forth Appearance of $Z(\mathbf{t})$ [457]) *If the link is a rational homology sphere then*

$$Z(\mathbf{t}) = \sum_{l' \in L'} \chi_{\mathrm{top}}(\mathrm{ECa}^{-l'}(E)) \cdot \mathbf{t}^{l'}.$$

This identity, or the identity from 8.4.7 via (8.4.12) can be reproved by combining the above 'effective Cartier divisor' interpretation with the configuration space interpretation from 8.4.4.

To see this, fix $l' \in \mathscr{S}'$, and write $a_v = -(l', E_v)$ and $\mathbb{P} := \mathbb{P}(T(l') \setminus \cup_v T_v(l'))$.

A section s of $T(l') = H^0(\mathscr{O}_E(-l'))$ is in the arrangement complement if and only if it does not vanish along the curves E_v, hence its divisor $\mathrm{div}(s)$ on each E_v consists of a_v points (counted with multiplicities). $\mathbb{P}$ has a natural stratification according to the support of $\mathrm{div}(s)$. Let $\mathbb{P}^\circ$ be the set of sections (up to a nonzero multiplicative constant) such that $\mathrm{supp}(\mathrm{div}(s)) \cap \mathrm{Sing}(E) = \emptyset$. $\mathbb{P} \setminus \mathbb{P}^\circ$ has a stratification as follows. For a collection $\{e_i; p_i, q_i\}_i$ of triples, where $e_i = (v_i, u_i)$ is an edge and (p_i, q_i) is a pair of positive integers, we associate a stratum consisting

of those sections (up to local invertible germs in $\mathbb{C}\{x, y\}/(xy)$) which at $E_{v_i} \cap E_{u_i}$ have local form $(\sum_{m \geq p_i} \alpha_m x^m + \sum_{n \geq q_i} \beta_n y^n)$, with $\alpha_{p_i} \neq 0$, $\beta_{q_i} \neq 0$. Hence this stratum is a fibration over $\prod_i \mathbb{C}^*$, in particular, its Euler characteristic is zero. Since $\mathbb{P} \setminus \mathbb{P}^\circ$ is a union of such strata, $\chi_{top}(\mathbb{P}) = \chi_{top}(\mathbb{P}^\circ)$. On the other hand, $\mathbb{P}^\circ$ can be identified with their divisors in $E \setminus \mathrm{Sing}(E)$, hence $\chi_{top}(\mathbb{P}^\circ) = \prod_v \chi_{top}(S^{a_v} E_v^\circ)$. Then use 8.4.4.

8.4.12 The next incarnation of $Z(\mathbf{t})$ uses **weighted cubes**. They will play a key role in the definition of the lattice complex and the lattice cohomology in 11.1.A–11.1.B. Here we will discuss their connection with the series $Z(\mathbf{t})$.

The set of 'combinatorial' q–cubes (where $q \in \mathbb{Z}_{\geq 0}$) consists of pairs $(l', I) \in L' \times \mathscr{P}(\mathscr{V})$, $|I| = q$, where $\mathscr{P}(\mathscr{V})$ denotes the power set of $\mathscr{V}$. $\square_q = (l', I)$ can be identified with the 'vertices' $\{l' + \sum_{v \in I'} E_v\}_{I' \subset I}$ of a 'Euclidean' q–cube in $L' \otimes \mathbb{R}$. One defines the weight function

$$w : L' \to \mathbb{Q}, \qquad w(k) := \chi(l') = -(l', l' + K)/2. \tag{8.4.13}$$

This extends to a weight-function defined on the set of all q–cubes

$$w(\square_q) = w((l', I)) = \max_{I' \subset I} \Big\{ w(l' + \sum_{v \in I'} E_v) \Big\}.$$

Theorem 8.4.13 (The Fifth Appearance of $Z(\mathbf{t})$ [482]) *Let Γ be a connected negative definite graph such that $M(\Gamma)$ is a rational homology sphere. Then*

$$\begin{aligned} Z(\mathbf{t}) &= \sum_{l' \in L'} \sum_{I \subset \mathscr{V}} (-1)^{|I|+1} \max_{I' \subset I} \chi(l' + E_{I'})\, \mathbf{t}^{l'} \\ &= \sum_{l' \in L'} \sum_{I \subset \mathscr{V}} (-1)^{|I|+1}\, w((l', I))\, \mathbf{t}^{l'}. \end{aligned} \tag{8.4.14}$$

(The first expression should be compared with (8.4.6).)

Since $\sum_I (-1)^{|I|} = 0$, the statement of the theorem remains unchanged even if we replace the definition of $w(l')$ by $w(l') + c$ for any constant c.

Proof We follow the strategy of the proof of 8.4.5. For each $l' = \sum_v a_v E_v^* \in L'$, $a_v \in \mathbb{Z}$, write $\mathbf{x}^a := \prod_v x_v^{a_v}$ and set the weighted cube counting series

$$y_\Gamma(\mathbf{x}) := \sum_{l' \in L'} \Big(\sum_{I \subset \mathscr{V}} (-1)^{|I|+1}\, w((l', I)) \Big) \mathbf{x}^a.$$

We determine y_Γ by induction on $|\mathscr{V}|$. If $|\mathscr{V}| = 1$ and $E^2 = e < 0$, then

$$y_\Gamma(x) = \sum_{a\in\mathbb{Z}} \Big[-\frac{a(a+e+2)}{-2e} + \max\Big\{ \frac{a(a+e+2)}{-2e}, \frac{(a-e)(a+2)}{-2e} \Big\} \Big] \cdot x^a.$$

This is $\sum_{a\geq 0}(a+1)x^a = 1/(1-x)^2$.

Next, we assume $|\mathscr{V}| \geq 2$. Let w be a fixed end-vertex of Γ. Set $\Gamma_0 := \Gamma \setminus w$ the graph obtained by deleting w and its supporting edge. If $l' = \sum_v a_v E_v^* \in L'(\Gamma)$, we write $l'_0 = \sum_{v\neq w} a_v E_v^{*,\Gamma_0} \in L'(\Gamma_0)$ for its restriction $R(l')$. We write y_Γ as a sum $y_\Gamma^{(1)} + y_\Gamma^{(2)}$, where the first series is the sum over those subsets I which does not contain w, while the second is the sum over the other subsets. In the first case

$$y_\Gamma^{(1)}(\mathbf{x}) = y_{\Gamma_0}(\mathbf{x}_0) \cdot \sum_{a_w\in\mathbb{Z}} x_w^{a_w}, \tag{8.4.15}$$

where $\mathbf{x}_0$ are the variables $\{x_v\}_{v\neq w}$ corresponding to Γ_0. Indeed, one has $w((l', I)) - w(l') = w((l'_0, I)) - w(l'_0)$ and $\sum_{I\not\ni w}(-1)^{|I|}w(l') = \sum_{I\not\ni w}(-1)^{|I|}w(l'_0) = 0$, hence

$$\begin{aligned}
\sum_{l'\in L'(\Gamma)} \sum_{I\not\ni w} (-1)^{|I|+1}\, w((l', I))\mathbf{x}^a &= \sum_{l'\in L'(\Gamma)} \sum_{I\not\ni w} (-1)^{|I|+1} \big(w((l', I)) - w(l')\big)\mathbf{x}^a \\
&= \sum_{l'\in L'(\Gamma)} \sum_{I\not\ni w} (-1)^{|I|+1} \big(w((l'_0, I)) - w(l'_0)\big)\mathbf{x}^a \\
&= \sum_{l'_0\in L'(\Gamma_0)} \sum_{a_w\in\mathbb{Z}} x_w^{a_w} \sum_{I\not\ni w} (-1)^{|I|+1}\, w((l'_0, I)) \prod_{v\neq w} x_v^{a_v}.
\end{aligned}$$

In the second sum $w \in I$; set $I = I' \cup w$ with $w \notin I'$. Since

$$\chi(l' + E_{I'} + E_w) - \chi(l' + E_{I'}) = 1 + a_w - (E_{I'}, E_w),$$

where $(E_{I'}, E_w) \in \{0, 1\}$, one gets that

$$\begin{cases} \text{if } a_w \geq 0, \text{ then } w((l', I)) = w((l' + E_w, I')), \\ \text{if } a_w < 0, \text{ then } w((l', I)) = w((l', I')). \end{cases}$$

Hence $y_\Gamma^{(2)}(\mathbf{x})$ splits into two sums:

$$\sum_{l'_0\in L'(\Gamma_0)} \sum_{I=I'\cup w} (-1)^{|I|+1} \Big(\sum_{a_w<0} w((l', I'))\, \mathbf{x}^a \; + \sum_{a_w\geq 0} w((l' + E_w, I'))\, \mathbf{x}^a \Big).$$

For the second one we use $E_w = -e_w E_w^* - E_u^*$, where u is the adjacent vertex of w in Γ. Then, computing both sums by similar argument as in the case of $y_\Gamma^{(1)}$, we get

$$y_\Gamma^{(2)}(\mathbf{x}) = -y_{\Gamma_0}(\mathbf{x}_0) \cdot \sum_{a_w<0} x_w^{a_w} - y_{\Gamma_0}(\mathbf{x}_0) \cdot x_u \cdot \sum_{a_w\geq 0} x_w^{a_w}. \tag{8.4.16}$$

The contributions (8.4.15) and (8.4.16) combined provide $y_\Gamma(\mathbf{x}) = y_{\Gamma_0}(\mathbf{x}_0)(1 - x_u)/(1-x_w)$. This as an inductive step, together with the identity valid for $|\mathcal{V}| = 1$, give $y_\Gamma(\mathbf{x}) = z_\Gamma(\mathbf{x}) = \prod_v (1 - x_v)^{\kappa_v - 2}$. □

8.4.A The Extension of $Z(\mathbf{t})$ to the Grothendieck Ring

Assume that the fixed graph is a tree of rational vertices. The information contained in $Z(\mathbf{t})$ can be improved if we replace in the 'third appearance'

$$Z(\mathbf{t}) = \sum_{l'\in\mathscr{S}'} \chi_{\mathrm{top}}(\mathbb{P}(T(l') \setminus \cup_v T_v(l'))\mathbf{t}^{l'}$$

the topological Euler characteristic of $\mathbb{P}(T(l') \setminus \cup_v T_v(l'))$ with the class of this space in the Grothendieck ring of complex quasi-projective varieties.
In section 8.4.A, 8.4.B and 8.4.C we follow [457, 486, 487]. The extension (8.4.17) below can be compared with similar extensions of certain analytic series (like $P(\mathbf{t})$) to the level of Grothendieck rings as in [99]. ♣

8.4.14 Let $\mathbb{L}$ be the class of the 1-dimensional affine space. Then, by inclusion–exclusion principle (as the analogue of 8.3.3) one has the following. If $\{V_\alpha\}_{\alpha\in\Lambda}$ is a finite family of linear subspaces of a finite dimensional linear space V, and for $I \subset \Lambda$ one writes $V_I := \cap_{\alpha\in I} V_\alpha$, then

$$[V \setminus \cup_\alpha V_\alpha] = \sum_I (-1)^{|I|}\mathbb{L}^{\dim(V_I)}, \quad [\mathbb{P}(V \setminus \cup_\alpha V_\alpha)] = (\sum_I (-1)^{|I|}\mathbb{L}^{\dim(V_I)})/(\mathbb{L}-1).$$

According to this, one defines

$$Z(\mathbb{L}, \mathbf{t}) = \sum_{l'\in\mathscr{S}'} [\mathbb{P}(T(l') \setminus \cup_v T_v(l'))]\, \mathbf{t}^{l'}, \tag{8.4.17}$$

which, using 8.3.9 reads as

$$\begin{aligned} Z(\mathbb{L}, \mathbf{t}) &= \frac{1}{\mathbb{L}-1} \cdot \sum_{l'\in\mathscr{S}'} \sum_{I\subset\mathcal{V}} (-1)^{|I|}\, \mathbb{L}^{\chi(l'+E)-\chi(l'+E_{J(l',I)})}\, \mathbf{t}^{l'} \\ &= \sum_{l'\in\mathscr{S}'} \sum_{I\subset\mathcal{V}} (-1)^{|I|} \cdot \frac{\mathbb{L}^{\chi(l'+E)-\chi(l'+E_{J(l',I)})} - 1}{\mathbb{L}-1}\, \mathbf{t}^{l'}. \end{aligned} \tag{8.4.18}$$

Note that $\lim_{\mathbb{L}\to 1} Z(\mathbb{L},\mathbf{t}) = Z(\mathbf{t})$. The analogue of the topological/combinatorial identity 8.4.5 is the following.

Theorem 8.4.15 ([457, 486, 487]) *If the link is a rational homology sphere then*

$$Z(\mathbb{L},\mathbf{t}) = \frac{\prod_{(u,v)\in\mathscr{E}} (1-\mathbf{t}^{E_u^*}-\mathbf{t}^{E_v^*}+\mathbb{L}\mathbf{t}^{E_u^*+E_v^*})}{\prod_{v\in\mathscr{V}} (1-\mathbf{t}^{E_v^*})(1-\mathbb{L}\mathbf{t}^{E_v^*})}. \tag{8.4.19}$$

Proof Follow the steps and all the identities of the proof of 8.4.5. For an independent proof see the proof of Theorem 8.4.20. □

Definition 8.4.16 One defines similarly the analytic version as well: $P(\mathbb{L},\mathbf{t}) = \sum_{l'\in\mathscr{S}'_{an}} [\mathbb{P}(A(l')\setminus\cup_v A_v(l'))]\,\mathbf{t}^{l'}$ (cf. [99]).

Example 8.4.17 Assume that the topological and analytic arrangements at l' agree: $\mathscr{A}_{\rm top}(l') = \mathscr{A}_{\rm an}(l')$. (For concrete cases when this happens see 8.3.14.) Then $P(\mathbb{L},\mathbf{t}) = Z(\mathbb{L},\mathbf{t})$, hence $P(\mathbb{L},\mathbf{t})$ is given by (8.4.19).

Example 8.4.18 (Continuation of Example 8.3.6) Note that in Example 8.3.6 one has $P(\mathbf{t}) = Z(\mathbf{t})$, see also 8.5.A. However, for $l' = Z_{max}$ the entries of the motivic series are different. Namely, $[\mathbb{P}(A(l')\setminus\cup_v A_v(l'))] = [\mathbb{C}^*] = \mathbb{L}-1$, while $[\mathbb{P}(T(l')\setminus\cup_v T_v(l'))] = [\mathbb{C}\times\mathbb{C}^*] = \mathbb{L}^2-\mathbb{L}$.

8.4.B Motivic Extension of $Z(\mathbf{t})$ via the Space of Divisors

We provide the 'right' geometric interpretation of the right hand side of the identity (8.4.19): it is the generating function of the classes $[\mathrm{ECa}^{-l'}(E)]$ in the Grothendieck group. We follow [457]. ♣

8.4.19 The Motivic Series Associated with $\{\mathrm{ECa}^{-l'}(E)\}_{l'}$ (General (X,o)) Let us assume that $\widetilde{X}$ is a good resolution of a normal surface singularity (X,o) such that *each E_v is rational, however, we allow cycles in the graph* (hence in this subsection the link is not necessarily a rational homology sphere). For any $l'\in\mathscr{S}'$ let $\mathrm{ECa}^{-l'}(E)$ be the space of effective Cartier divisors of E, cf. 8.4.10 or [452]. It is a non–empty quasiprojective variety, cf. [245, 318, 319, 452]. Let us define the generating function of their classes in the Grothendieck group

$$Z^{\rm ECa}(\mathbb{L},\mathbf{t}) := \sum_{l'\in\mathscr{S}'} [\mathrm{ECa}^{-l'}(E)]\cdot\mathbf{t}^{l'}. \tag{8.4.20}$$

Theorem 8.4.20 ([457]) *With the above notations*

$$Z^{\mathrm{ECa}}(\mathbb{L},\mathbf{t})=\frac{\prod_{(u,v)\in\mathscr{E}}(1-\mathbf{t}^{E_u^*}-\mathbf{t}^{E_v^*}+\mathbb{L}\mathbf{t}^{E_u^*+E_v^*})}{\prod_{v\in\mathscr{V}}(1-\mathbf{t}^{E_v^*})(1-\mathbb{L}\mathbf{t}^{E_v^*})}.$$

Proof We improve the discussion from the proof of 8.4.11, see also [451, p. 31–32]. Write $l'=\sum_v a_v E_v^*$ with $a_v\in\mathbb{Z}_{\geq 0}$. Recall that effective Cartier divisors are local nonzero sections of the sheaf $\mathscr{O}_E$ up to local invertible elements of $\mathscr{O}_E^*$. At any intersection point $p\in E_v\cap E_u$, $(v,u)=e\in\mathscr{E}$, with local coordinates $(x,y)\in U$, $\{x=0\}=E_v\cap U$, $\{y=0\}=E_u\cap U$, an effective Cartier divisor supported at p has the form $D_{k_v,k_u}=c_v^e x^{k_v^e}+c_u^e y^{k_u^e}$ (in $\mathbb{C}\{x,y\}/(xy)$, up to an invertible element), where $c_v^e,\ c_u^e\in\mathbb{C}^*$. It is nonempty if and only if $k_v^e\geq 1$ and $k_u^e\geq 1$. $\mathrm{ECa}^{-l'}(E)$ has a natural stratification according to the support of the divisors. The sum of the degrees of the divisors with support on E_v should be a_v, out of this, say a_v°, is provided by those supported on E_v°, and the others by divisors supported on intersection points of type $E_v\cap E_u$. Such a divisor has a contribution k_u^e in the degree. Hence $a_v=a_v^\circ+\sum_{(v,u)=e\in\mathscr{E}}k_u^e$ $(\dagger_v)$. Hence, with fixed $\{a_v\}_v$, we consider the stratification of $\mathrm{ECa}^{-l'}(E)$ according to the system $S(l'):=\{\{a_v^\circ\}_v,(k_v^e,k_u^e)_{(v,u)\in\mathscr{E}}\}$ satisfying $(\dagger_v)$ for any v and $k_u^e,k_v^e\geq 1$ for any such edge e which has a contribution. For $s=\{\{a_v^\circ\}_v,(k_v^e,k_u^e)_{(v,u)\in\mathscr{E}_s}\}\in S(l')$ (where $\mathscr{E}_s$ is the set of edges, which contribute in this stratum) the corresponding stratum, $\mathrm{ECa}_s^{-l'}(E)$, satisfies

$$[\mathrm{ECa}_s^{-l'}(E)]=(\mathbb{L}-1)^{|\mathscr{E}_s|}\cdot\prod_v[S^{a_v^\circ}E_v^\circ].$$

First we compute the class $[S^{a_v^\circ}E_v^\circ]$ for any v. Let E_v^c be a set consisting of κ_v-1 distinct point, and x a formal variable. Then $\sum_{i\geq 0}x^i[S^iE_v^\circ]\cdot\sum_{i\geq 0}x^i[S^iE_v^c]=\sum_{i\geq 0}x^i[S^i\mathbb{C}]=\sum_{i\geq 0}x^i\mathbb{L}^i$. Hence

$$\sum_{i\geq 0}x^i[S^iE_v^\circ]=(1-x)^{\kappa_v-1}/(1-\mathbb{L}x).$$

Next, let S be the set of all the systems when we vary l', that is, $S=\cup_{l'\in\mathscr{S}'}S(l')$. Then, in S, $\{a_v\}_v$ is not fixed anymore, and the integers $a_v^\circ\geq 0$, $k_v^e,k_u^e\geq 1$ run independently. Therefore, with the usual substitution $x_v=\mathbf{t}^{E_v^*}$,

$$\begin{aligned}Z^{\mathrm{ECa}}(\mathbb{L},\mathbf{x})&=\sum_{l'}\sum_{s\in S(l')}\Big((\mathbb{L}-1)^{|\mathscr{E}_s|}\cdot\prod_v[S^{a_v^\circ}E_v^\circ]\cdot\prod_v x_v^{a_v^\circ+\sum_{(v,u)=e\in\mathscr{E}_s}k_u^e}\Big)\\&=\sum_{s\in S}\Big(\prod_v x_v^{a_v^\circ}[S^{a_v^\circ}E_v^\circ]\cdot(\mathbb{L}-1)^{|\mathscr{E}_s|}\cdot\prod_v x_v^{\sum_{(v,u)=e\in\mathscr{E}_s}k_u^e}\Big)\end{aligned}$$

$$= \prod_v \frac{(1-x_v)^{\kappa_v-1}}{1-\mathbb{L}x_v} \cdot \sum_{\mathscr{E}'\subset\mathscr{E}} \prod_{(v,u)=e\in\mathscr{E}'} (\mathbb{L}-1)\cdot\frac{x_v}{1-x_v}\cdot\frac{x_u}{1-x_u}$$

$$= \prod_v \frac{(1-x_v)^{\kappa_v-1}}{1-\mathbb{L}x_v} \cdot \prod_{(v,u)=e\in\mathscr{E}} \Big(1+(\mathbb{L}-1)\cdot\frac{x_v}{1-x_v}\cdot\frac{x_u}{1-x_u}\Big)$$

$$= \prod_v \frac{(1-x_v)^{\kappa_v-1}}{1-\mathbb{L}x_v} \cdot \prod_{(v,u)=e\in\mathscr{E}} \frac{1-x_v-x_u+\mathbb{L}x_vx_u}{(1-x_v)(1-x_u)},$$

which is equivalent to the needed expression. □

Example 8.4.21 If the graph is a tree then $\mathrm{ECa}^{-l'}(E) = \mathbb{P}(T(l')\setminus\cup_v T_v(l'))$ (cf. (8.4.12)) and $Z^{\mathrm{ECa}}(\mathbb{L},\mathbf{t}) = Z(\mathbb{L},\mathbf{t})$, hence we recover Theorem 8.4.15.

Remark 8.4.22 Since $Z_K - E = \sum_v(\kappa_v-2)E_v^*$ we have $\mathbf{t}^{Z_K-E}\cdot Z(\frac{1}{\mathbf{t}}) = Z(\mathbf{t})$. Similarly, the motivic expression satisfies the functional equation

$$\mathbb{L}^{c(\Gamma)-1}\cdot\mathbf{t}^{Z_K-E}\cdot Z^{\mathrm{ECa}}(\mathbb{L},\mathbf{t})\big|_{\mathbf{t}^{E_v^*}\mapsto(\mathbb{L}\mathbf{t}^{E_v^*})^{-1}} = Z^{\mathrm{ECa}}(\mathbb{L},\mathbf{t}).$$

8.4.C $Z(\mathbb{L},\mathbf{t})$ *and* $P(\mathbb{L},\mathbf{t})$ *for Cusp Singularities*

In the definition of $P(\mathbf{t})$ and $Z(\mathbf{t})$, both indexed by $\mathscr{S}'$, we used the assumption that the link is a rational homology sphere. In the case of $P(\mathbf{t})$ it was really necessary, since the definition was based on the existence of the natural line bundles (defined via the universal abelian covering). (Indeed, the cohomology of a natural line bundle and of a line bundle with the same Chern class might be different, hence the identification of the natural bundles is crucial.)
However, we suggest that using the linear subspace arrangements an extension can be done even if the link is not a rational homology sphere, at least when all the exceptional curves are rational. We analyse the case of cusps in details. In this case several analytic vanishing statements are present, facts which make the definition and results work. We follow [457]. ♣

We assume that (X,o) is a cusp singularity, and we fix its minimal resolution $\widetilde{X}$.

8.4.23 The Analytic Series $P(\mathbf{t})$ The definition of $P(\mathbf{t}) = \sum_{l'}\mathfrak{p}(l')\mathbf{t}^{l'}$ can be done in (at least) two ways. **The first definition** identifies the coefficients of $P(\mathbf{t})$ by any of the identities from Lemma 8.2.8,

$$\mathfrak{p}(l') = \sum_{I\subset\mathscr{V}} (-1)^{|I|}\ \dim\frac{H^0(\widetilde{X},\mathscr{O}_{\widetilde{X}}(-l'-E_I))}{H^0(\widetilde{X},\mathscr{O}_{\widetilde{X}}(-l'-E))} \tag{8.4.21}$$

once we clarify the meaning of $\mathscr{O}_{\widetilde{X}}(-l') \in \mathrm{Pic}(\widetilde{X})$ for any $l' \in \mathscr{S}'$. The bundle $\mathscr{O}_{\widetilde{X}}(-l')$ can be identified in a 'naive', or in a 'precise' way.

By Theorem 7.2.31 $h^1(\widetilde{X}, \mathscr{L})$ depends only on the Chern class of $\mathscr{L}$, hence in the 'naive' identification we can put for $\mathscr{O}_{\widetilde{X}}(-l')$ *any* bundle with Chern class $-l'$, and the expression from (8.4.21) is well-defined. (Cf. with identities of type (8.2.21).)

The second way identifies precisely the bundles $\mathscr{O}_{\widetilde{X}}(-l') \in \mathrm{Pic}(\widetilde{X})$, and even a L'/L-covering, which replaces the universal abelian covering. Once the covering is fixed, the analogues of the natural line bundles are defined via an eigensheaf decomposition as in 6.2.9, and all the analytic filtrations $\{\mathscr{F}(l')\}_{l'}$ can be defined as in 8.2.A, and all the statements of that subsection can be reproved. The corresponding covering is the 'discriminant covering' of the cusp, discussed in 4.2.8.

As we said earlier, the cohomology of line bundles are topological, hence $\mathfrak{p}(l')$ (defined in (8.4.21)) a posteriori, should also be topological. Let us rewrite it in terms of χ. We fix some $l' \in \mathscr{S}'$ and write $\overline{I} := \mathscr{V} \setminus I$. Then from the exact sequence

$$0 \to \mathscr{O}_{\widetilde{X}}(-l' - E) \to \mathscr{O}_{\widetilde{X}}(-l' - E_I) \to \mathscr{O}_{E_{\overline{I}}}(-l' - E_I) \to 0, \tag{8.4.22}$$

and from the vanishing $h^1(\mathscr{O}_{\widetilde{X}}(-l' - E)) = 0$ (use 7.2.31 for $l' + E \in \mathscr{S}' \setminus \{0\}$), we get that $\mathfrak{p}(l') = \sum_I (-1)^{|I|} h^0(\mathscr{O}_{E_{\overline{I}}}(-l' - E_I))$. Let $J := J(l', I)$ be as in Lemma 8.3.7. Then $h^0(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(-l' - E_I)) = h^0(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(-l' - E_J))$, hence again by (8.4.22) we get

$$\mathfrak{p}(l') = \sum_I (-1)^{|I|} h^0(\mathscr{O}_{E_{\overline{J}}}(-l' - E_J)).$$

We will separate the case $l' = 0$. In this case for $I = \emptyset$ one has $\overline{J} = \mathscr{V}$ hence $h^0(\mathscr{O}_{E_{\overline{J}}}(-E_J)) = 1$. But for $I \neq \emptyset$, $h^0(\mathscr{O}_{E_{\overline{J}}}(-E_J)) = 0$ (since $J = \mathscr{V}$). Hence $\mathfrak{p}(0) = 1$.

If $l' \neq 0$ then $h^1(\mathscr{O}_{E_{\overline{J}}}(-l' - E_J)) = 0$. This for $I = \emptyset$ follows from 7.2.31, for $I \neq \emptyset$ follows from the fact that each component of $\overline{J}$ is a string and all the Chern degrees are ≥ 0. In particular, $h^0(\mathscr{O}_{E_{\overline{J}}}(-l' - E_J)) = \chi(\mathscr{O}_{E_{\overline{J}}}(-l' - E_J)) = \chi(l' + E) - \chi(l' + E_J)$. Hence

$$P(\mathbf{t}) = \mathbf{t}^0 + \sum_{l' \in \mathscr{S}' \setminus \{0\}} \sum_I (-1)^{|I|+1} \chi(l' + E_{J(l',I)}) \cdot \mathbf{t}^{l'}. \tag{8.4.23}$$

(This can be compared with the 'second appearance' of $Z(\mathbf{t})$ from 8.4.5, valid in the rational homology sphere link case.)

8.4.24 The Topological Series $Z(\mathbf{t})$ The topological linear subspace arrangement can be defined analogously as in the rational homology sphere case, once the bundles $\mathscr{O}_E(-l')$ are identified as above. Namely, one sets $T(l') := H^0(\mathscr{O}_E(-l'))$

and $T_v(l') := H^0(\mathscr{O}_{E-E_v}(-l'-E_v))$ for each $v \in \mathscr{V}$, and finally (as in 8.4.7)

$$Z(\mathbf{t}) = \sum_{l' \in \mathscr{S}'} \chi_{\mathrm{top}}(\,\mathbb{P}(T(l') \setminus \cup_v T_v(l'))\,) \cdot \mathbf{t}^{l'}.$$

8.4.25 The Second Definition of the Analytic Series $P(\mathbf{t})$ Define the image of $H^0(\mathscr{O}_{\widetilde{X}}(-l')) \to H^0(\mathscr{O}_E(-l'))$ as $A(l')$ and set $A_v(l') := A(l') \cap T_v(l')$ (as in 8.3) and set

$$P'(\mathbf{t}) = \sum_{l' \in \mathscr{S}'} \chi_{\mathrm{top}}(\,\mathbb{P}(A(l') \setminus \cup_v A_v(l'))\,) \cdot \mathbf{t}^{l'}.$$

Lemma 8.4.26 *$P(\mathbf{t}) = P'(\mathbf{t}) = Z(\mathbf{t})$, or,*

$$Z(\mathbf{t}) = \mathbf{t}^0 + \sum_{l' \in \mathscr{S}' \setminus \{0\}} \sum_{I} (-1)^{|I|+1} \chi(l' + E_{J(l',I)}) \cdot \mathbf{t}^{l'}. \tag{8.4.24}$$

Proof Notice that $H^0(\mathscr{O}_{\widetilde{X}}(-l')) \to H^0(\mathscr{O}_E(-l'))$ is onto (since $h^1(\mathscr{O}_{\widetilde{X}}(-l'-E)) = 0$). Hence the analytic and the topological linear subspace arrangements are the same and $P' = Z$. But $P' = P$ follows similarly as in Corollary 8.3.4, using 8.3.3. □

8.4.27 The Computation of $P(\mathbf{t}) = Z(\mathbf{t})$ via $\{E_v^*\}_v$'s Assume first that $|\mathscr{V}| \geq 3$.

(a) Assume that $l' = E_v^*$ for some v. Then $J = \emptyset$ for $I = \emptyset$ and $J = \mathscr{V}$ otherwise. Hence, by (8.4.23) $\mathfrak{p}(l') = 1$. Similarly, if $l' = kE_v^*$ for some $v \in \mathscr{V}$ and $k \geq 2$ then $J = \emptyset$ for $I = \emptyset$, and $J = \mathscr{V}$ if $v \in I$, and $J = \mathscr{V} \setminus v$ if $I \neq \emptyset$ and $v \notin I$. Hence, again by (8.4.23) $\mathfrak{p}(l') = 1$.

(b) Assume that v and v' are adjacent vertices, and $l' = kE_v^* + k'E_{v'}^*$ with $k, k' > 0$. Let $w \neq v'$ adjacent vertex to v. Let $\mathscr{P}_v := \{I \subset \mathscr{V}, v \in I\}$. Then its elements can be put in pairs $(I, I \cup w)$ with $w \notin I$, and with $J(l', I) = J(l', I \cup w)$. Hence the contribution in (8.4.23) corresponding to the sum over $\mathscr{P}_v$ is zero. The same is true, by similar argument, for the subset $\mathscr{P}_{v,v'} := \{I \subset \mathscr{V}, v \notin I, v' \in I\}$. Hence we remain with subsets I with $I \cap \{v, v'\} = \emptyset$. For them $J = \emptyset$ if $I = \emptyset$ (with contribution $\chi(l')$), and $J = \mathscr{V} \setminus \{v, v'\}$ (with contribution $\chi(l') + 1$) else. Hence $\mathfrak{p}(l') = 1$ again.

(c) We claim that $\mathfrak{p}(l') = 0$ in all other cases. Write l' as $\sum_{v \in S} a_v E_v^*$ with all $a_v > 0$. Assume that the E^* support S of l' is not $\mathscr{V}$. Fix a maximal connected string Γ_1 in $\mathscr{V} \setminus S$. Let $v, v' \in S$ be the two adjacent vertices of Γ_1. Then, by excluding the cases already discussed, we know that $v \neq v'$ and $\Gamma_1 \cup \{v, v'\} \neq \mathscr{V}$. Then we compute $\mathfrak{p}(l')$ via (8.4.23). Similarly as in case (b), the contribution in the sum corresponding to the subset I with $I \cap \{v, v'\} \neq \emptyset$ is zero (choose the adjacent w in Γ_1). Hence, in the sequel we consider sets

I with $I \cap \{v, v'\} = \emptyset$. For each such I, note that $\tilde{J} := J \setminus \{v, v'\}$ has the property that $\chi(l' + E_J) = \chi(l' + E_{\tilde{J}})$ (†). Indeed, if at some step of the algorithm from Lemma 8.3.7(2) we have $I_m = \tilde{J}$, and by the algorithm we have to add v (or v'), then $(E_v, l' + E_{I_m}) = 1$, hence (†) holds. Write I as $I_1 \cup I_2$ with $I_1 \subset \Gamma_1$, while $I_2 \cap \Gamma_1 = \emptyset$. Then $\tilde{J}$ also decomposes as $J_1 \cup J_2$ with similar properties. Note also that $(l' + E_{J_2}, E_{J_1}) = 0$ Hence $\sum_{I_1 \cup I_2}(-1)^{|I_1|+|I_2|}\chi(l' + E_{\tilde{J}}) = \sum_{I_1 \cup I_2}(-1)^{|I_1|+|I_2|}(\chi(l' + E_{J_2}) + \chi(E_{J_1})) = \sum_{I_2}(-1)^{|I_2|}\chi(l' + E_{J_2})\sum_{I_1}(-1)^{|I_1|} + \sum_{I_1}(-1)^{|I_1|}\chi(E_{J_1})\sum_{I_2}(-1)^{|I_2|} = 0$.

Finally assume that $S = \mathscr{V}$. Similarly as in (†) above, we have $\chi(l' + E_J) = \chi(l' + E_I)$, which equals $\chi(l' + E) - \chi(\mathscr{O}_{E_{\tilde{I}}}(-l' - E_I))$. (This identity can be proved via $h^1(\mathscr{O}_{E_{\tilde{I}}}(-l' - E_I)) = 0$ as well.) Now the vanishing $\sum_I (-1)^{|I|}\chi(\mathscr{O}_{E_{\tilde{I}}}(-l' - E_I)) = \sum_I (-1)^{|I|}(\chi(E_{\tilde{I}}) - (E_{\tilde{I}}, l' + E_I)) = 0$ follows from combinatorial arguments.

8.4.28 Hence we proved that whenever $|\mathscr{V}| \geq 3$ then

$$P(\mathbf{t}) = Z(\mathbf{t}) = 1 + \sum_{v \in \mathscr{V}} \frac{\mathbf{t}^{E_v^*}}{1 - \mathbf{t}^{E_v^*}} + \sum_{(u,v) \in \mathscr{E}} \frac{\mathbf{t}^{E_v^*}}{1 - \mathbf{t}^{E_v^*}} \cdot \frac{\mathbf{t}^{E_u^*}}{1 - \mathbf{t}^{E_u^*}}.$$

By direct verification, the same formula holds for $\mathscr{V} = \{v, u\}$ as well. Namely

$$P(\mathbf{t}) = Z(\mathbf{t}) = 1 + \frac{\mathbf{t}^{E_v^*}}{1 - \mathbf{t}^{E_v^*}} + \frac{\mathbf{t}^{E_u^*}}{1 - \mathbf{t}^{E_u^*}} + 2 \cdot \frac{\mathbf{t}^{E_v^*}}{1 - \mathbf{t}^{E_v^*}} \cdot \frac{\mathbf{t}^{E_u^*}}{1 - \mathbf{t}^{E_u^*}}.$$

If $\mathscr{V} = \{v\}$ then $P(t) = Z(t) = 1 + \mathbf{t}^{E^*}/(1 - \mathbf{t}^{E^*})^2$.

Remark 8.4.29 The above formula is totally different from the 'standard zeta function expression' $\prod_v (1 - \mathbf{t}^{E_v^*})^{\kappa_v - 2}$. In fact, for cusps, when $\kappa_v = 2$ for all v, this zeta function expression is $\equiv 1$. Note also that $Z^{\mathrm{ECa}}(\mathbb{L}, \mathbf{t})|_{\mathbb{L}=1} = \prod_v (1 - \mathbf{t}^{E_v^*})^{\kappa_v - 2} \equiv 1$ too, hence $Z^{\mathrm{ECa}}(\mathbb{L}, \mathbf{t})|_{\mathbb{L}=1}$ is not the right extension for cusps either.

However, as we will see in 8.4.32, $P(\mathbf{t})$ can be related with $Z^{\mathrm{ECa}}(\mathbb{L}, \mathbf{t})$.

8.4.30 The Motivic $P(\mathbb{L}, \mathbf{t})$ and $Z(\mathbb{L}, \mathbf{t})$ for a Cusp Singularity We set

$$Z(\mathbb{L}, \mathbf{t}) = \sum_{l' \in \mathscr{S}'} [\mathbb{P}(T(l') \setminus \cup_v T_v(l'))] \cdot \mathbf{t}^{l'}, \quad P(\mathbb{L}, \mathbf{t}) = \sum_{l' \in \mathscr{S}'} [\mathbb{P}(A(l') \setminus \cup_v A_v(l'))] \cdot \mathbf{t}^{l'}.$$

Since the vanishing $h^1(\mathscr{O}_{\tilde{X}}(-l' - E)) = 0$ guarantees that the analytic and topological arrangements are the same, immediately we obtain $P(\mathbb{L}, \mathbf{t}) = Z(\mathbb{L}, \mathbf{t})$.

Next, for any $l' \in \mathscr{S}'$ we consider the Abel map $c^{-l'} : \mathrm{ECa}^{-l'}(E) \to \mathrm{Pic}^{-l'}(E)$, and for any $\mathscr{L} \in \mathrm{Pic}^{-l'}$ the class $[\mathbb{P}(H^0(E, \mathscr{L})_{\mathrm{reg}})] = [(c^{-l'})^{-1}(\mathscr{L})]$. Hence,

$$Z(\mathbb{L}, \mathbf{t}) = \sum_{l' \in \mathscr{S}'} [\mathbb{P}(H^0(E, \mathscr{O}_E(-l'))_{\mathrm{reg}})] \cdot \mathbf{t}^{l'}.$$

Lemma 8.4.31 $Z(\mathbb{L},0)=1$. *Furthermore, for* $l'\in\mathscr{S}'\setminus\{0\}$ *one has*

$$[\mathbb{P}(H^0(E,\mathscr{O}_E(-l'))_{\mathrm{reg}})]=[\mathrm{ECa}^{-l'}(E)]/(\mathbb{L}-1).$$

Proof First we claim that $\mathrm{Pic}^0(E)$, regarded as the kernel of $c_1: H^1(\mathscr{O}_E^*)\to L'$ in $H^1(\mathscr{O}_E^*)$, is the multiplicative group $\mathbb{C}^*$. Indeed, the exponential map $exp(2\pi i\cdot): \mathscr{O}_E\mapsto\mathscr{O}_E^*$ induces an isomorphism $H^1(\mathscr{O}_E)/H^1(E,\mathbb{Z})=\mathbb{C}/\mathbb{Z}\overset{exp(2\pi i\cdot)}{\longrightarrow}\mathbb{C}^*$.

Note that if $f: X\to Y$ is an algebraic locally trivial fibration with fiber F, then $[X]=[Y][F]$. In the proof, instead of the local triviality of the Abel map, we will prove the local triviality of its restrictions on the strata of $\mathrm{ECa}^{-l'}(E)$ considered in the proof of Theorem 8.4.20.

First, consider the Abel map $c^{-E_v^*}: \mathrm{ECa}^{-E_v^*}(E)\to\mathrm{Pic}^{-E_v^*}(E)$ for a fixed $v\in\mathscr{V}$ (for more details see also [452]). The source consists of divisors of degree one supported on E_v°, hence it can be identified with E_v°. The Abel map can be described by Laufer integration. Recall that $p_g=1$ and (X,o) is Gorenstein. Then we fix the Gorenstein differential form in $\widetilde{X}\setminus E$ with pole E, and we apply the discussion from Example 6.8.22. We obtain that $c^{-E_v^*}: E_v^\circ\to\mathbb{C}^*$ is a tautological bijection.

Next, consider for any $k\geq 1$ the Abel map $c^{-kE_v^*}: \mathrm{ECa}^{-kE_v^*}(E)\to\mathrm{Pic}^{-kE_v^*}(E)$. One has a multiplicative structure $\prod_{i=1}^k\mathrm{ECa}^{-E_v^*}(E)\to\mathrm{ECa}^{-kE_v^*}(E)$ given by union (sum) of divisors, while $\prod_{i=1}^k\mathrm{Pic}^{-E_v^*}(E)\to\mathrm{Pic}^{-kE_v^*}(E)$ given by the tensor product of line bundles, that is, multiplication in $\mathbb{C}^*$. These operators commute with the Abel maps. Hence $c^{-kE_v^*}: S^kE_v^\circ=S^k\mathbb{C}^*\to\mathbb{C}^*$ is given by $c^{-kE_v^*}(\sum_{i=1}^k p_i)=\prod_i c^{-E_v^*}(p_i)$ (product in $\mathbb{C}^*$). This is surjective and it is an algebraic fibration over $\mathbb{C}^*$.

Finally, consider a stratum corresponding to $e=(v,u)\in\mathscr{E}$ consisting of divisors of type D_{k_v,k_u}. As a space, it is isomorphic to $\mathbb{C}^*$. We claim that the Abel map restricted to it is an isomorphism. Indeed, if we blow up $p=E_u\cap E_v$ several times conveniently at its infinitesimal close points, this strata can be identified with the divisors on some newly created exceptional divisor $E_{new}^\circ\simeq\mathbb{C}^*$ given by $x^{d_v}+a=0$, where $a\in\mathbb{C}^*$ is the parameter of the stratum, and $d_v=\gcd\{k_v,k_u\}$. By the discussion from the previous case, applied for $S^{d_v}E_{new}^\circ\to\mathbb{C}^*$, we get that the subset given by $\{x^{d_v}+a=0\}_{a\in\mathbb{C}^*}$ maps isomorphically to $\mathbb{C}^*$.

Since the stratification and the Abel map are algebraic, the proof is finished. □

Corollary 8.4.32 *For a cusp singularity* $Z(\mathbb{L},\mathbf{t})=P(\mathbb{L},\mathbf{t})$ *is given by*

$$Z(\mathbb{L},\mathbf{t})=1+\frac{Z^{\mathrm{ECa}}(\mathbb{L},\mathbf{t})-1}{\mathbb{L}-1},$$

where $Z^{\mathrm{ECa}}(\mathbb{L},\mathbf{t})$ *is given in Theorem 8.4.20.*

We invite the reader to verify that

$$\Big(1+\frac{Z^{\mathrm{ECa}}(\mathbb{L},\mathbf{t})-1}{\mathbb{L}-1}\Big)\Big|_{\mathbb{L}=1}$$

provides exactly the expressions from 8.4.28 for $Z(\mathbf{t})$.

Example 8.4.33 For a cusp with one vertex one has $Z(\mathbb{L},\mathbf{t})=1+\sum_{k\geq 1}[\mathbb{P}^{k-1}]\cdot t^{kE^*}$.

Remark 8.4.34 The expression in Theorem 8.4.28 is surprising: the series

$$z(\mathbf{x})=1+\sum_{v\in\mathscr{V}}\frac{x_v}{1-x_v}+\sum_{(u,v)\in\mathscr{E}}\frac{x_v}{1-x_v}\cdot\frac{x_u}{1-x_u}\in\mathbb{Z}[[\mathbf{x}]]$$

appears in a natural way in a rather different context in the literature. Indeed, identify the minimal good resolution graph Γ with a simplicial complex $|\Gamma|$ with vertex set $\mathscr{V}$ having only 1-simplices corresponding to the edges. Then the Hilbert series of the graded Stanley–Reisner ring associated with $|\Gamma|$ is exactly $z(\mathbf{x})$ [89, 425]. (This might provide some additional information about the local equations of the cusp (X,o), cf. [663]).

8.5 Singularities Satisfying the 'end curve condition'

We fix a singularity (X,o) with a rational homology sphere link and a good resolution ϕ with dual graph Γ. The final goal of this section is to prove that the following facts are equivalent: (a) the analytic type of (X,o) and the resolution ϕ satisfies the 'end curve condition', (b) (X,o) is splice quotient (associated with Γ), (c) $P(\mathbf{t})=Z(\mathbf{t})$ (both associated with ϕ).
For such a singularity we prove that the next analytic invariants can be read from the combinatorics of Γ: $\mathscr{S}'_{an}$, $\mathscr{S}_{an}$, Z_{max}, base points of $\mathscr{O}_{\widetilde{X}}(-Z_{max})$, $\mathrm{mult}(X,o)$, $h^1(\mathscr{O}_{\widetilde{X}}(-l'))$ (e.g. p_g). We also show that the 'divisorial filtration' coincides with the 'monomial filtration' (projected weight filtration). ♣

8.5.1 We fix a singularity (X,o) and one of its good resolutions $\phi:\widetilde{X}\to X$ with graph Γ. We assume that $H^1(\widetilde{X},\mathbb{Z})=0$. As above, $\mathscr{V}$ and $\mathscr{V}^e=\{v\in\mathscr{V}\,:\,\kappa_v=1\}$ denote the set of vertices and ends of Γ, and $H=L'/L=H_1(L_X,\mathbb{Z})$.

Definition 8.5.2 Consider an irreducible curve-germ (H,p) on $\widetilde{X}$, transversal to E, intersecting exactly one E_v ($v\in\mathscr{V}$) at the point p. A *cut function* for this curve is a function $f\in\mathscr{O}_{X,o}$ such that the divisor of $f\circ\phi$ is supported on $E\cup H$. (H,p)

is an *analytic principal cut of* E_v if it has a cut function. Usually we refer to it as 'p–cut'.

Definition 8.5.3 ([515]) The 'end curve condition' for the resolution ϕ requires the existence of an analytic p–cut H_e for *each end component* E_e, $e \in \mathscr{V}^e$. We call the corresponding cut function 'end curve function'.

Example 8.5.4 In the following cases the end curve condition is satisfied:

(a) rational singularities, where ϕ is an arbitrary resolution (use 7.1.13*(a)*);
(b) minimally elliptic singularities, and ϕ satisfies $|C| = E$ (use 7.2.31*(b)*);
(c) weighted homogeneous singularities, where ϕ is the minimal good resolution (see 5.1.39); and more generally,
(d) splice quotient singularities (by their construction, the coordinate functions of $\mathbb{C}^{|\mathscr{V}^e|}$ induce end curve functions, see [515, (7.2)(6)], or 5.4.11, 5.4.12 and the comments in 5.4.13 above).

8.5.5 End Curve Sections [515] For a fixed $v \in \mathscr{V}$, let o_v be the order of $[E_v^*]$ in H. Since $\mathrm{Pic}(\widetilde{X})$ has no torsion, if H_v is a p–cut of E_v, then one can take a cut function f_v such that $\mathrm{div}(f_v \circ \phi) = o_v(H_v + E_v^*)$. We will call it 'reduced cut function'. Note also that $\mathscr{O}_{\widetilde{X}}(-E_v^*) = \mathscr{O}_{\widetilde{X}}(H_v)$ in $\mathrm{Pic}(\widetilde{X})$.

Consider now the universal abelian covering (X_a, o) of (X, o), maps $c, \widetilde{c}$ and p as in (6.2.1) or (8.2.3), and a reduced cut function f_v for some $v \in \mathscr{V}$ as above. Then $f_v \circ \phi \circ p$ is an o_v-power of some $z_v \in H^0(\widetilde{Z}, \mathscr{O}_{\widetilde{Z}})$. Indeed, the universal abelian covering factors through the $\mathbb{Z}_{o_v}$ cyclic covering $\{f_v = z^{o_v}\} = X_{f_v,o_v}$ of (X, o), and the algorithm from subsection 4.1.C provides the holomorphic germ $z = z_v$ defined on this covering, which lifts to $\widetilde{Z}$. Since z_v is in the $\theta([E_v^*])$-eigenspace, by 6.2.9 one also has that $z_v \in H^0(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(-E_v^*))$, and the divisor of z_v, as global section of $\mathscr{O}_{\widetilde{X}}(-E_v^*)$, is H_v. (In this construction of z_v from f_v there is a multiplicative ambiguity by an o_v-root of unity that we will disregard.)

If $\phi : \widetilde{X} \to X$ satisfies the *end curve condition*, the sections z_e for $e \in \mathscr{V}^e$ constructed from (a fixed set of) reduced cut functions will be called *end curve sections*.

Usually, for each $e \in \mathscr{V}^e$, one fixes *one* p–cut. Nevertheless, in order to run the theory (and inductive steps) properly, for a graph which has *only one vertex*, one fixes *two* disjoint p–cuts, hence one gets two end curve sections. (For this special case, the reader is invited to replace some of the indices used below accordingly.)

8.5.6 Monomials in $H^0(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(-l'))$ Fix a subset $\mathscr{W} \subset \mathscr{V}$, and assume that ϕ admits cut functions and p–cuts H_w for any $w \in \mathscr{W}$. Fix $l' \in L'$. For any collection $\alpha_w \in \mathbb{Z}_{\geq 0}$ ($w \in \mathscr{W}$), with $[\sum_{w\in\mathscr{W}} \alpha_w E_w^*] = [l']$ in H, take the monomial $z^\alpha = \prod_{w\in\mathscr{W}} z_w^{\alpha_w}$ in $\{z_w\}_w$, considered as a function on $\widetilde{Z}$. It is in the $\theta([l'])$-eigenspace, and it is a meromorphic section of $\mathscr{O}_{\widetilde{X}}(-l')$. As a section of $\mathscr{O}_{\widetilde{X}}(-l')$, its divisor is $\sum_w \alpha_w(H_w + E_w^*) - l'$. Therefore, z^α is a global holomorphic section of $\mathscr{O}_{\widetilde{X}}(-l')$ if and only if $\sum_w \alpha_w E_w^* - l' \in L_{\geq 0}$. We write $D(\alpha) := \sum_{w\in\mathscr{W}} \alpha_w E_w^*$.

Definition 8.5.7 Assume that ϕ satisfies the end curve condition. Consider the end curve sections $\{z_e\}_{e\in\mathscr{V}^e}$ associated with a fixed set of end curve functions. An element of $H^0(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(-l'))$ of the form z^α with $\mathscr{W} = \mathscr{V}^e$ is called *monomial section*.

A global section z^α of $\mathscr{O}_{\widetilde{X}}(-l')$, where $\{\alpha_v\}_{v\in\mathscr{W}}$ is indexed by a subset $\mathscr{W} \supsetneq \mathscr{V}^e$ (provided that each $\{z_w\}_{w\in\mathscr{W}}$ exists) is called an *extended monomial section*.

Example 8.5.8 For any $l' \in L'$ and $l \in L_{\geq 0}$, the classes of non-zero monomial sections in $H^0(\widetilde{X}, \mathscr{O}(-l'))/H^0(\widetilde{X}, \mathscr{O}(-l'-l))$ are indexed by

$$\{\alpha = \{\alpha_e\}_{e\in\mathscr{V}^e} \mid \alpha_e \in \mathbb{Z}_{\geq 0} \text{ for all } e \in \mathscr{V}^e;\ a \geq 0,\ a \not\geq l, \text{ where } a = D(\alpha) - l' \in L\}.$$

E.g., if $l = E_v$ for some $v \in \mathscr{V}$ with $(l', E_v) > 0$, then this index set is empty: the existence of such an α would imply $0 > (D(\alpha) - l', E_v) \geq 0$. This chimes in with the fact that $H^0(\mathscr{O}_{\widetilde{X}}(-l'))/H^0(\mathscr{O}_{\widetilde{X}}(-l'-E_v))$ embeds into $H^0(E_v, \mathscr{O}_{E_v}(-l')) = 0$.

8.5.9 Restrictions We introduce the needed notations of an inductive procedure. Let ϕ and Γ be as above, and we assume $|\mathscr{V}| > 1$. We fix a vertex $v \in \mathscr{V}^e$; let E_w be the unique irreducible exceptional curve which intersects E_v. Set $\bar{E} = \cup_{u\in\mathscr{V}\setminus v} E_u$. Let $\widetilde{X}(\bar{E})$ be a sufficiently small neighbourhood of $\bar{E}$ in $\widetilde{X}$, and let $\bar{\Gamma}$ be the dual graph of $\bar{E}$. We denote by $\bar{E}_u$ $(= E_u)$, $u \in \mathscr{V} \setminus v$, the base elements of the new lattice $\bar{L}$, and by $\bar{E}_u^*$ their $\bar{\Gamma}$-duals in $\bar{L}'$. Notice that $\bar{E}$ analytically can be contracted (denote this by $\bar{\phi}$) giving rise to a singularity $(\bar{X}, o)$. Set $\bar{\mathscr{V}}^e$ for the ends of $\bar{\Gamma}$ and $\mathscr{V}^{ev} = \mathscr{V}^e \setminus v$.

If (X, o) admits the p–cuts $\{H_e\}_{e\in\mathscr{V}^e}$ in $\widetilde{X}$ determined by the cut functions $\{f_e\}_{e\in\mathscr{V}^e}$, then $\widetilde{X}(\bar{E})$ inherits some compatible p–cuts and cut functions. Indeed:

Lemma 8.5.10 ([533, (2.15)]) *The curves $\{H_e\}_{e\in\mathscr{V}^{ev}}$ and $\bar{H}_w = E_v \cap \widetilde{X}(\bar{E})$ are p–cuts of $\bar{E}$ in $\widetilde{X}(\bar{E})$. In particular, the resolution $\bar{\phi}$ of $(\bar{X}, o)$ satisfies the end curve condition with distinguished end curve functions inherited from $\{f_e\}_{e\in\mathscr{V}^e}$.*

Proof Let m_e be the vanishing order of f_e along E_v. Then for $\bar{\phi}$, $f_v|_{\widetilde{X}(\bar{E})}$ is a cut function for $\bar{H}_w$, and $\bar{f}_e = (f_e^{m_v}/f_v^{m_e})|_{\widetilde{X}(\bar{E})}$ is a cut function for H_e, $e \in \mathscr{V}^{ev}$. □

Let $i : \bar{L} \hookrightarrow L$ be the natural lattice embedding $\bar{E}_u \mapsto E_u$, $u \in \mathscr{V} \setminus v$. Its dual $R : L' \to \bar{L}'$ is defined by $(R(l'), \bar{l}) = (l', i(\bar{l}))$ (or by $R(E_v^*) = 0$ and $R(E_u^*) = \bar{E}_u^*$ for $u \in \mathscr{V} \setminus v$), and one also has

$$R(E_v) = -\bar{E}_w^* \text{ and } R(E_u) = \bar{E}_u \text{ for } u \in \mathscr{V} \setminus v. \tag{8.5.1}$$

Lemma 8.5.11 *The restriction of any natural line bundle of $\widetilde{X}$ to $\widetilde{X}(\bar{E})$ is natural. In fact, the restriction of $\mathscr{O}_{\widetilde{X}}(-l')$ is isomorphic to $\mathscr{O}_{\widetilde{X}(\bar{E})}(-R(l'))$.*

Proof We need to show that some power of the restriction has the form $\mathscr{O}_{\widetilde{X}(\bar{E})}(\bar{l})$ for some $\bar{l} \in \bar{L}$, cf. 6.2.4. Hence, by taking $|H|$-power of $\mathscr{O}_{\widetilde{X}}(-l')$, the proof reduces

to the cases $\mathscr{O}_{\widetilde{X}}(E_u)|\widetilde{X}(\bar{E})$. But this, for $u \neq v$ is $\mathscr{O}_{\widetilde{X}(\bar{E})}(\bar{E}_u)$, and for $u = v$ is $\mathscr{O}_{\widetilde{X}(\bar{E})}(\bar{H}_w)$, which equals $\mathscr{O}_{\widetilde{X}(\bar{E})}(-\bar{E}^*_w)$ by 8.5.10 and 8.5.5; all of them are natural. Since $\mathscr{O}_{\widetilde{X}}(-l')|_{\widetilde{X}(\bar{E})}$ is natural, it is identified by its Chern class. □

Here, a word of warning is necessary. Consider the restriction map

$$R^0 : H^0(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(-l')) \longrightarrow H^0(\widetilde{X}(\bar{E}), \mathscr{O}_{\widetilde{X}}(-l')|_{\widetilde{X}(\bar{E})}).$$

Although $\mathscr{O}_{\widetilde{X}}(-l')|_{\widetilde{X}(\bar{E})}$ and $\mathscr{O}_{\widetilde{X}(\bar{E})}(-R(l'))$ are isomorphic, the isomorphism connecting them might not send the restriction of a monomial section of $\mathscr{O}_{\widetilde{X}}(-l')$ to a monomial section of $\mathscr{O}_{\widetilde{X}(\bar{E})}(-R(l'))$.

The following technical lemma is proved in [75] and [484].

Lemma 8.5.12 *Let $\phi : \widetilde{X} \to X$ be a good resolution which satisfies the end curve condition, and $H^1(\widetilde{X}, \mathbb{Z}) = 0$. Let $\mathscr{L} = \mathscr{O}_{\widetilde{X}}(-l')$ ($l' \in L'$) be a natural line bundle on $\widetilde{X}$. Then:*

1. *For some $l \in L_{\geq 0}$ consider the vector space*

$$V := H^0(\widetilde{X}, \mathscr{L})/H^0(\widetilde{X}, \mathscr{L}(-l)).$$

Assume that a subset of $H^0(\widetilde{X}, \mathscr{L})$ has the following property: for any non-zero class in V of a monomial section of $\mathscr{L}$ the set contains an element with the same divisor. Then the classes of the elements of the set generate V as a vector space.

2. *Assume that $|\mathscr{V}| > 1$, $v \in \mathscr{V}^e$ and $(c_1(\mathscr{L}), E_v) \geq 0$. Set E_w and $\widetilde{X}(\bar{E})$ as in 8.5.9. Then the restriction map induces an isomorphism*

$$R^\flat : \frac{H^0(\widetilde{X}, \mathscr{L})}{H^0(\widetilde{X}, \mathscr{L}(-E_w))} \longrightarrow \frac{H^0(\widetilde{X}(\bar{E}), \mathscr{L}|_{\widetilde{X}(\bar{E})})}{H^0(\widetilde{X}(\bar{E}), \mathscr{L}|_{\widetilde{X}(\bar{E})}(-\bar{E}_w))}. \tag{8.5.2}$$

Proof We prove both statements by a simultaneous induction on the number of vertices of the resolution graph. For this we will use the notations 8.5.9 and 8.5.10. The cut functions $\{\bar{f}\}_{e\in\mathscr{V}^{ev}}$ and $f_v|_{\widetilde{X}(\bar{E})}$ induce sections in $H^0(\widetilde{X}(\bar{E}), \mathscr{O}_{\widetilde{X}(\bar{E})}(-\bar{E}^*_e))$, resp. in $H^0(\widetilde{X}(\bar{E}), \mathscr{O}_{\widetilde{X}(\bar{E})}(-\bar{E}^*_w))$, denoted by $\bar{z}_e$ for $e \in \mathscr{V}^{ev}$, resp. by y_w (cf. 8.5.5 and 8.5.10). Clearly, if $\kappa_w > 2$, then $w \notin \bar{\mathscr{V}}^e$ and $\bar{\mathscr{V}}^e = \mathscr{V}^{ev}$. Hence the new monomial sections have the form $\prod_{e\in\mathscr{V}^{ev}} \bar{z}_e^{\alpha_e}$ ($\alpha_e \geq 0$); a product of type $\prod_{e\in\mathscr{V}^{ev}} \bar{z}_e^{\alpha_e} \cdot y_w^{\beta}$ ($\beta \geq 0$) is an extended monomial section. If $\kappa_w = 2$, then $\bar{\mathscr{V}}^e = \mathscr{V}^{ev} \cup w$, hence all $\{\bar{H}_e\}_{e\in\mathscr{V}^{ev}}$ and $\bar{H}_w$ are p–cuts of $\widetilde{X}(\bar{E})$, and the new monomial sections have the form $\prod_{e\in\mathscr{V}^{ev}} \bar{z}_e^{\alpha_e} \cdot y_w^{\beta}$ ($\alpha_e \geq 0$, $\beta \geq 0$). If $\kappa_w = 1$ then for $\widetilde{X}(\bar{E})$ we preserve both (old and new) p–cuts H_w and $\bar{H}_w$, and the new monomial sections have the form $\bar{z}_w^{\alpha} y_w^{\beta}$.

The induction starts with a one-vertex graph with two p–cuts H_1 and H_2, in which case we have to check only part 1. In this case the picture is very explicit.

Set $n = -E^2$. Then $E^* = (1/n)E$ and $H = \mathbb{Z}_n$. The universal abelian covering X_a is $\mathbb{C}^2$, $\mathscr{O}_{X_a,o} = \mathbb{C}\{z_1, z_2\}$, where z_1 and z_2 are the two end curve sections. Moreover, $\widetilde{Z}$ is the blow up of X_a at o with irreducible exceptional curve E_a, and $p^*(E) = nE_a$. Set $\mathscr{L} = \mathscr{O}_{\widetilde{X}}(-kE^*)$ for some $k \in \mathbb{Z}$. Then $H^0(\widetilde{X}, \mathscr{L}) = H^0(\widetilde{Z}, \mathscr{O}(-kE_a))_{\theta([l'])}$ which equals the subspace of $\mathbb{C}\{z_1, z_2\}$ generated by monomials of degree $\deg \geq k$ and $\deg \equiv k \pmod n$. Therefore, if $l = mE$, the monomial sections z^α of $\mathscr{L}$ with $\alpha_1 + \alpha_2 = k + in$ $(0 \leq i < m)$ are linearly independent and the space spanned by them projects bijectively onto V. Since $div_{\mathscr{L}}(z^\alpha) = \alpha_1 H_1 + \alpha_2 H_2 + iE$, the vanishing orders of all these divisors at the intersection point $H_1 \cap E$ are all distinct. Hence, any set of section of $\mathscr{L}$ with the same cardinality and divisors projects into a linearly independent set in V, which necessarily form a basis of V.

Now, we consider a resolution $\phi : \widetilde{X} \to X$ whose graph has $|\mathscr{V}| > 1$ vertices. We fix a vertex $v \in \mathscr{V}^e$, and we will use the notations of 8.5.9. By induction, we assume that the statements of the lemma are true for $\bar{\phi}$.

First we prove statement 2 for ϕ. Since $R^\flat$ is injective, we have to show its surjectivity. For this consider all the monomial sections of $\mathscr{L}|_{\widetilde{X}(\bar{E})}$ with non-zero class in the target of $R^\flat$. They have the form $\bar{M} = \prod_{e \in \mathscr{V}^{ev}} \bar{z}_e^{\alpha_e} \cdot y_w^\beta$ (where $\alpha_e \geq 0$ and $\beta \geq 0$, and $\beta = 0$ if $w \notin \mathscr{V}^e$) such that

$$\sum_{e \in \mathscr{V}^{ev}} \alpha_e \bar{E}_e^* + \beta \bar{E}_w^* + R(c_1(\mathscr{L})) = \bar{l}$$

for some $\bar{l} \in \bar{L}_{\geq 0}$ and $\bar{l} \not\geq \bar{E}_w$. In particular, $\bar{l}$ is supported on the closure of $\bar{E} \setminus \bar{E}_w$, hence $(i(\bar{l}), E_v) = 0$. Since $R(\sum_{e \in \mathscr{V}^{ev}} \alpha_e E_e^* - \beta E_v + c_1(\mathscr{L}) - i(\bar{l})) = 0$, one has

$$\sum_{e \in \mathscr{V}^{ev}} \alpha_e E_e^* - \beta E_v + c_1(\mathscr{L}) - i(\bar{l}) = -\alpha_v E_v^* \tag{8.5.3}$$

for some $\alpha_v \in \mathbb{Z}$. Since $\beta E_v + i(\bar{l}) \in L_{\geq 0}$, and $\alpha_v = -\beta E_v^2 + (c_1(\mathscr{L}), E_v) \geq 0$, we get that $z^\alpha = \prod_{e \in \mathscr{V}^e} z_e^{\alpha_e}$ is a monomial section of $\mathscr{L}$, such that the divisors of $R^0(z^\alpha)$ and $\bar{M}$ coincide. By the inductive step (part 1 applied for $\widetilde{X}(\bar{E})$ and for the classes of $\{R^0(z^\alpha)\}_\alpha$, i.e. for the set $\{R^\flat(z^\alpha)\}_\alpha$), the collection $\{R^\flat(z^\alpha)\}_\alpha$ is a generator set of the target of $R^\flat$. In particular, $R^\flat$ is onto proving part 2.

Next, part 2 is used to prove a special case of part 1. Indeed, if we replace above each section z^α by another section $\mathfrak{s}_\alpha$ of $\mathscr{L}$ with the same divisor, then by the same inductive argument as above the set $\{R^\flat(\mathfrak{s}_\alpha)\}_\alpha$ form a basis of the target of $R^\flat$. But, since $R^\flat$ is an isomorphism (already proved by part 2), the classes of $\mathfrak{s}_\alpha$ necessarily form a basis too. This proves part 1 for $l = E_w$ and any $\mathscr{L}$ with $(c_1(\mathscr{L}), E_v) \geq 0$.

Now we turn to statement 1 for general $\mathscr{L}$ and $l \in L_{\geq 0}$. First we consider the following infinite computation sequence $\{x_n\}_{n\geq 0}$, $x_n \in L_{\geq 0}$, such that $x_0 = 0$, $x_{n+1} = x_n + E_{u(n)}$, where $u(n) \in \mathscr{V}$ is provided by the following principle:

(a) If $(-c_1(\mathscr{L}) + x_n, E_u) \leq 0$ for all $u \in \mathscr{V}$ then $u(n) = w$.
(b) If $(-c_1(\mathscr{L}) + x_n, E_u) > 0$ for some u, then take one of them for $u(n)$.

By 2.1.19 both steps occur infinitely many times. Moreover, at any step when (1) applies, $-c_1(\mathscr{L}) + x_n \in \mathscr{S}'$. Since $\mathscr{S}' \cap \{l' \in L' : l' \not\geq l - c_1(\mathscr{L})\}$ is finite (cf. 2.1.19), we get that $x_k \geq l$ for a sufficiently large k.

Next we analyze the steps of $\{x_n\}_{n=0}^k$ and we show by induction on n the validity of part 1 for $V_n := H^0(\mathscr{L})/H^0(\mathscr{L}(-x_n))$. In case (b), the natural projection $V_{n+1} \to V_n$ is an isomorphism and the divisors of the monomials can also be identified (cf. 8.5.8). In case (a), one has the exact sequence

$$0 \to \frac{H^0(\mathscr{L}(-x_n))}{H^0(\mathscr{L}(-x_n - E_w))} \to V_{n+1} \to V_n \to 0.$$

Part 1 is valid for V_n by the inductive hypothesis, and for the first term of the exact sequence too (by the particular case $l = E_w$ already proved). Hence, it works for the middle term as well. This ends the induction showing 1 for V_k.

Finally, consider the projection $V_k \to V = H^0(\mathscr{L})/H^0(\mathscr{L}(-l))$. Any set of $H^0(\mathscr{L})$ which satisfy the assumption of part 1 for V can be completed (by adding monomial sections whose classes in V are zero) to a set which satisfies the assumption of 1 for V_k. Hence, since V_k satisfies the statement of 1, so does V. □

Corollary 8.5.13 *Let $\phi : \widetilde{X} \to X$ be as in Lemma 8.5.12, $\mathscr{L} = \mathscr{O}_{\widetilde{X}}(-l')$ $(l' \in L')$ and $l \in L_{\geq 0}$. Then $H^0(\widetilde{X}, \mathscr{L})/H^0(\widetilde{X}, \mathscr{L}(-l))$ is generated as a vector space by the classes of monomial sections of $H^0(\widetilde{X}, \mathscr{L})$.*

Proof Apply part 1 for the subset of monomial sections. □

8.5.14 For further references we state a more general version of 8.5.10.

Lemma 8.5.15 *Assume that a resolution $\widetilde{X} \to X$ satisfies the end curve condition. Let Γ' be a connected subgraph of Γ, $\widetilde{X}'$ be a small neighbourhood of $E' := \cup_{v\in\mathscr{V}(\Gamma')} E_v$, and let $\widetilde{X}' \to X'$ be the contraction of E'. Then the resolution $\widetilde{X}' \to X'$ satisfies the end curve condition too. Furthermore, the natural line bundles of $\widetilde{X}$ restricted to $\widetilde{X}'$ are natural.*

Proof Use inductively 8.5.10 and 8.5.11, or modify accordingly their proofs. □

8.5.A *The Identity $P(\mathbf{t}) = Z(\mathbf{t})$ and the Monomial Filtration*

Theorem 8.5.16 ([484]) *Assume that (X, o) admits a resolution ϕ, which satisfies the end curve condition, and $H^1(\widetilde{X}, \mathbb{Z}) = 0$. Then $P(\mathbf{t}) = Z(\mathbf{t})$.*

Proof By the 'change of variables' $\{x_u = \mathbf{t}^{E_u^*}\}_{u\in\mathcal{V}}$, and using (8.2.15) and the support restriction (8.2.14), the identity $P(\mathbf{t}) = Z(\mathbf{t})$ is equivalent to

$$\sum_{k_u\geq 0} \mathfrak{p}_{\mathscr{O}(-\sum_u k_u E_u^*)} \prod_{u\in\mathcal{V}} x_u^{k_u} = \prod_{u\in\mathcal{V}} (1-x_u)^{\kappa_u-2}. \tag{8.5.4}$$

The proof runs by induction on the number of vertices (where we will also use the facts that the restriction preserves the class of singularities satisfying the end curve condition, and also their natural line bundles, cf. 8.5.10 and 8.5.15). The statement was already verified in Example 8.2.16 for a graph with only one vertex. For a graph Γ with $|\mathcal{V}| > 1$, let v, w, $\bar{\Gamma}$ be as in 8.5.9, and denote the restriction of $\mathscr{M} = \mathscr{O}(-\sum_u k_u E_u^*) \in \mathrm{Pic}(\widetilde{X})$ to $\widetilde{X}(\bar{E})$ by $\bar{\mathscr{M}}$ (where each $k_u \geq 0$). Then in the formula (8.2.13) of $\mathfrak{p}_{\mathscr{M}}$ we group the subsets in pairs $(I, I\cup\{w\})_{I\not\ni w}$. Hence

$$\mathfrak{p}_{\mathscr{M}} = \sum_{I\not\ni w} (-1)^{|I|} \frac{H^0(\mathscr{M}(-E_I))}{H^0(\mathscr{M}(-E_I - E_w))}. \tag{8.5.5}$$

Since $(c_1(\mathscr{M}), E_v) = k_v \geq 0$ and $-(E_I, E_v) \geq 0$, 8.5.12(2) can be applied for each $\mathscr{L} = \mathscr{M}(-E_I)$. Next, we separate the cases when I contains v or not and we reorganize them in expressions as (8.5.5) for $\mathfrak{p}_{\bar{\mathscr{M}}}$ and $\mathfrak{p}_{\bar{\mathscr{M}}(\bar{E}_w^*)}$. Then (using (8.5.1) too), we get

$$\mathfrak{p}_{\mathscr{M}} = \mathfrak{p}_{\bar{\mathscr{M}}} - \mathfrak{p}_{\overline{\mathscr{M}(-E_v)}} = \mathfrak{p}_{\bar{\mathscr{M}}} - \mathfrak{p}_{\bar{\mathscr{M}}(\bar{E}_w^*)}.$$

Notice that, by (8.2.14), $\mathfrak{p}_{\bar{\mathscr{M}}(\bar{E}_w^*)} = 0$ whenever $k_w = 0$. Therefore

$$P_{X,o} = P_{\bar{X},o} \cdot \sum_{k_v\geq 0} x_v^{k_v} \cdot (1-x_w) = P_{\bar{X},o} \cdot \frac{1-x_w}{1-x_v}.$$

But the right hand side of (8.5.4) satisfies the same inductive formula. □

Corollary 8.5.17 *Assume that (X, o) admits a resolution ϕ, which satisfies the end curve condition, and $H^1(\widetilde{X}, \mathbb{Z}) = 0$. Then $h^1(\mathscr{O}_{\widetilde{X}}(l'))$ is topological for any $l' \in L'$.*

Indeed, write $Z(\mathbf{t}) = \sum_{l'\in\mathscr{S}'} \mathfrak{z}(l')\mathbf{t}^{l'}$. Then, after the identification $P(\mathbf{t}) = Z(\mathbf{t})$, the formulae from 8.2.13 read as follows:

1. For $l' \in -K_{\widetilde{X}} + \mathscr{S}'$

$$\sum_{[\tilde{l}']=[l'],\ \tilde{l}'\not\geq l'} \mathfrak{z}(\tilde{l}') = \chi_{K_{\widetilde{X}}+2r_h}(l' - r_h) + h^1(\mathscr{O}_{\widetilde{X}}(-r_h)); \tag{8.5.6}$$

2. More generally, for $\mathscr{L} = \mathscr{O}_{\widetilde{X}}(-l')$ with arbitrary $l' \in L'$,

$$h^1(\mathscr{O}_{\widetilde{X}}(-l')) = -\sum_{[\tilde{l}']=[l'],\ \tilde{l}' \not\geq l'} \mathfrak{z}(\tilde{l}') + \chi_{K_{\widetilde{X}}+2r_h}(l'-r_h) + h^1(\mathscr{O}_{\widetilde{X}}(-r_h)). \tag{8.5.7}$$

8.5.18 In fact, the converse of 8.5.16 is also true:

Proposition 8.5.19 *Assume that the singularity (X, o) satisfies $H^1(\widetilde{X}, \mathbb{Z}) = 0$, and we fix one of its good resolutions ϕ. If associated with ϕ one has $P(\mathbf{t}) = Z(\mathbf{t})$, then the 'end curve condition' for ϕ is also satisfied.*

Proof If $Z = P$ then $\mathfrak{p}(E_e^*) = 1$ for any $e \in \mathscr{V}^e$. Hence, by 8.3.4, $A(E_e^*) \setminus \cup_v A_v(E_e^*) \neq \emptyset$ and there exists a global section $z_e \in H^0(\mathscr{O}_{\widetilde{X}}(-E_e^*))$ without fixed component. Thus, by its intersection properties, its divisor is necessarily a p–cut of E_e. □

8.5.20 Comparison of the Divisorial and Monomial Filtrations Besides the divisorial multi-filtration $\{\mathscr{F}(l')\}_{l'\in L'}$ there exists another filtration on $\mathscr{O}_{X_a,o}$, indexed by the very same set L'. One starts with the *monomial filtration* on $\mathbb{C}\{z\} = \mathbb{C}\{z_1, \dots, z_{|\mathscr{V}^e|}\}$ defined by the degrees of monomials:

$$\mathscr{G}_{\mathbb{C}\{z\}}(l') := \Big\{ \sum_{\alpha} a_\alpha z^\alpha \in \mathbb{C}\{z\} : \sum_{e\in\mathscr{V}^e} \alpha_e E_e^* \geq l' \ \text{ whenever } a_\alpha \neq 0 \Big\}. \tag{8.5.8}$$

Let $\Phi : \mathbb{C}\{z\} \to \mathscr{O}_{X_a,o}$ send the variable z_i to the corresponding end curve section (denoted by z_i too, cf. 8.5.5). By definition, the *monomial filtration* $\{\mathscr{G}(l')\}_{l'}$ of $\mathscr{O}_{X_a,o}$ is the image of $\{\mathscr{G}_{\mathbb{C}\{z\}}(l')\}_{l'}$ by Φ (compare with the construction from 8.1.22). By definitions $\mathscr{G}(l') \subset \mathscr{F}(l')$.

Theorem 8.5.21 *$\mathscr{G}(l') = \mathscr{F}(l')$ for all $l' \in L'$. In particular (take $l' = 0$), Φ is surjective.*

Proof Let $\mathfrak{m}$ and $\mathfrak{m}_{X_a,o}$ be the maximal ideal of $\mathbb{C}\{z\}$ and $\mathscr{O}_{X_a,o}$ respectively.

In the ring $\mathbb{C}\{z\}$ the $\mathfrak{m}$-adic filtration and the monomial multi-filtration $\{\mathscr{G}(l')\}_{l'}$ have bounded difference (use 2.1.19). Hence they determine the same completion. The same is true in $\mathscr{O}_{X_a,o}$ too: the $\mathfrak{m}_{X_a,o}$-adic filtration and the divisorial filtration determines the same completion, cf. 6.6.10.

Applying Corollary 8.5.13 to the eigenspaces, we obtain

$$\Phi(\mathscr{G}_{\mathbb{C}\{z\}}(l')) + \mathscr{F}(l'+l) = \mathscr{F}(l') \ \text{ for any } l \in L_{\geq 0}.$$

Therefore, at the level of $\mathfrak{m}$-adic completions, $\widehat{\Phi} : \widehat{\mathscr{G}_{\mathbb{C}\{z\}}(l')} \to \widehat{\mathscr{F}(l')}$ is onto for any l'. Then, by a theorem of Houzel [281], $\Phi : \mathscr{G}_{\mathbb{C}\{z\}}(l') \to \mathscr{F}(l')$ is onto too. □

For additional comments regarding the results of this section see 8.5.36.

Theorem 8.5.21 generalizes a result of Okuma (for splice-quotients) [533, (3.3)], where the filtrations are associated with only one node. For a similar discussion see [75, (8.2.1)] too. A similar identity (of the divisorial and a combinatorial Poincaré series), valid for *affine toric varieties*, was verified by Lemahieu in [383].

8.5.B Topological Characterization of $\mathscr{S}'_{an}$ and $\mathscr{S}_{an}$

Recall that $\mathscr{V}^e$ denotes the set of end-vertices of Γ and any $\{\alpha_e\}_{e\in\mathscr{V}^e}$ ($\alpha_e \in \mathbb{Z}_{\geq 0}$) provides a monomial cycle $D(\alpha) = \sum_e \alpha_e E_e^* \in L'$.

Theorem 8.5.22 (Combinatorial Characterization of $\mathscr{S}'_{an}$ [484]) *Assume that (X, o) admits a good resolution ϕ, which satisfies the end curve condition, and $H^1(\widetilde{X}, \mathbb{Z}) = 0$. Then for any $l' \in \mathscr{S}'$ the following facts are equivalent:*

1. *$l' \in \mathscr{S}'_{an}$;*
2. *for each $v \in \mathscr{V}$ there exists an effective cycle $l_v \in L_{\geq 0}$ such that:*
   - *(a) $E_v \not\subset |l_v|$,*
   - *(b) $(l_v, E_u) = -(l', E_u)$ for any $u \in \mathscr{V} \setminus \mathscr{V}^e$,*
   - *(c) $(l_v, E_u) \leq -(l', E_u)$ for any $u \in \mathscr{V}^e$;*
3. *there exists finitely many monomial cycles $\{D(\alpha_{(k)})\}_k \in l' + L$ so that $l' = \min_k D(\alpha_{(k)})$.*

Proof (1)⇒(2). If $l' \in \mathscr{S}'_{an}$ then $H^0(\mathscr{O}_{\widetilde{X}}(-l'))/H^0(\mathscr{O}_{\widetilde{X}}(-l' - E_v)) \neq 0$ for each $v \in \mathscr{V}$. Hence, by 8.5.13 (see also 8.5.8), there exists a monomial $z^{\alpha_{(v)}}$ so that

$$\sum_{e\in\mathscr{V}^e} \alpha_{(v),e}\, E_e^* - l' = l_v, \tag{8.5.9}$$

where $l_v \in L_{\geq 0}$ and $E_v \not\subset |l_v|$. Hence (2) follows. In order to prove (2)⇒(3), define the non-negative integers $\alpha_{(v),e} := -(l' + l_v, E_e)$ for all $e \in \mathscr{V}^e$. This means that (8.5.9) is satisfied. Since $\min_v D(\alpha_{(v)}) = \min_v(l' + l_v) = l'$, (3) follows too. For (3)⇒(1) notice that if $l' = \min_k D(\alpha_{(k)})$, then a generic linear combination of the monomials $z^{\alpha_{(k)}}$ is a holomorphic section of $\mathscr{O}_{\widetilde{X}}(-l')$, which is not zero along any of the components. □

The next result shows that the 'saturation' of the sub-monoid $\mathscr{S}_{an}$ is $\mathscr{S}$.

Corollary 8.5.23 *Let (X, o) be as above, then $m \cdot \mathscr{S}' \subset \mathscr{S}_{an}$ for some $m \in \mathbb{Z}_{>0}$.*

Proof For $v \in \mathscr{E}$, $E_v^* \in \mathscr{S}'_{an}$ by the definition. Next, take a vertex v with $\kappa_v \geq 2$ and adjacent vertices $\{w_k\}_k$ sitting in the branches (connected components of $\Gamma \setminus v$) $\{C_k\}_k$ of v. Regard C_k as a subgraph, let $\bar{E}_u^*$ be the C_k-dual of E_u in this subgraph ($u \in \mathscr{V}(C_k)$). Next, consider in C_k an arbitrary integral monomial cycle $\bar{l}_k$ of

type $\sum_{e\in\mathscr{V}^e_{C_k}} \alpha_{k,e}\bar{E}^*_e$, where $\mathscr{V}^e_{C_k}$ are the ends of Γ in C_k. Since $\alpha_{k,e}$ are positive $\bar{l}_k$ is effective. Let m_k be the multiplicity of E_{w_k} in $\bar{l}_k$. Then, one verifies (by intersecting both parts with all E_w, $w \in \mathscr{V}$) that $\sum_{e\in\mathscr{V}^e_{C_k}} \alpha_{k,e}E^*_e - m_k E^*_v = i(\bar{l}_k)$, hence it is effective, integral and supported on C_k. (Here $i : L(C_k) \hookrightarrow L$ is the inclusion of lattices.) Hence, by taking $m_v := \mathrm{lcm}_k\{m_k\}$, one has $m_v \cdot E^*_v = \min_k D(\alpha_k m_v/m_k) \in \mathscr{S}_{an}$. □

Remark 8.5.24

(a) In fact, if $\kappa_v \neq 2$ then $E^*_v \in \mathscr{S}'_{an}$, hence $o_v E^*_v \in \mathscr{S}_{an}$, where o_v is the order of $[E^*_v]$ in H.

Indeed, for end curves this is clear. For nodes the proof uses the *monomial condition* for Γ, cf. 5.4.10. This is a topological condition, which must be satisfied by Γ whenever (X, o) satisfies the *end curve condition*, for the proof see 8.5.D.

Let $\{C_k\}_{1\le k\le\kappa_v}$ be the branches of the node v. Then the monomial condition says that there exists a monomial cycle $D(\alpha_k)$ so that $D(\alpha_k) - E^*_v$ is effective, integral and supported on C_k. Hence $E^*_v = \min_k D(\alpha_k) \in \mathscr{S}'_{an}$ and $o_v E^*_v \in \mathscr{S}'_{an} \cap L = \mathscr{S}_{an}$.

(b) For v with $\kappa_v = 2$ it can happen that $o_v E^*_v \notin \mathscr{S}_{an}$ (hence $E^*_v \notin \mathscr{S}'_{an}$ neither). E.g., $o_v = 1$ and $E^*_v \notin \mathscr{S}_{an}$ in the case of E_1 in the right graph from 8.5.27.

(c) One of the motivations for the characterization of $\mathscr{S}_{an}$ is the Nash Problem. The components of the arc space can be separated using the ratios l_i/l_j of the coefficients of cycles $\sum_i l_i E_i \in \mathscr{S}_{an}$, see e.g. [574]. By 8.5.23, for splice-quotients, the set of these ratios remain the same if we replace $\mathscr{S}_{an}$ by $\mathscr{S}$.

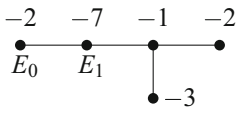
8.5.25 The Maximal Ideal Cycle We denote the (finitely generated) monoid of integral monomial cycles by

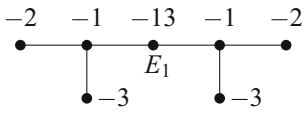
$$L_{end} := \Big\{ \sum_{e\in\mathscr{V}^e} \alpha_e E^*_e \,:\, \alpha_e \in \mathbb{Z}_{\ge 0} \Big\} \cap L.$$

Theorem 8.5.22 implies the following:

Corollary 8.5.26 *Let (X, o) be as in 8.5.22. Then $Z_{max} = \min\{l \,:\, l \in L_{end} \setminus \{0\}\}$.*

Example 8.5.27 Consider the following two graphs, both with $L' = L$, and both realized by splice quotient equations. The first one has equation $x^2 + y^3 + z^{13} = 0$, the second one is the complete intersection $z_1^2 + z_2^3 + z_3^2 z_4 = z_4^2 + z_3^3 + z_2^2 z_1 = 0$.





For the left graph $Z_{min} \in \mathscr{S}_{an}$, nevertheless $\mathscr{S}_{an} \neq \mathscr{S}$. In fact, $\mathscr{S} \setminus \mathscr{S}_{an} = \{E_1^*\}$, where E_1 is the -7-curve. To show that $E_1^* \notin \mathscr{S}_{an}$, choose E_0 in 8.5.22(2) for E_v. (Compare also with 7.2.100.)

The right graph is an example with $Z_{min} \notin \mathscr{S}_{an}$ and with infinite $\mathscr{S} \setminus \mathscr{S}_{an}$. Indeed, let E_1 denote the (-13)-curve and let E_e be any exceptional curve with valency 1. Then $E_1^* + kE_e^* \notin \mathscr{S}_{an}$ for every $k \geq 0$ (take in 8.5.22(2) for v the node adjacent to the end-vertex e). In this case $Z_{max} = 2Z_{min}$.

Example 8.5.28 Assume that (X, o) is as in 8.5.22, and the link is an integral homology sphere. Then $\mathscr{S}_{an}$ is a monoid generated by $\{E_e^*\}_{e \in \mathscr{V}^e}$ via the operators $+$ and min, where min is understood via the embedding into L. The presence of the min operator is necessary: if we take e.g. the E_8 singularity, then we have three generators $\{E_e^*\}_{e \in \mathscr{V}^e}$, however $\mathscr{S}_{an} = \mathscr{S}$ is an 8-dimensional affine semigroup.

Let $v \in \mathscr{V}$ be an arbitrarily chosen vertex, and let $\mathscr{S}_{an}|_{E_v}$ be the set of E_v-coefficients of elements of $\mathscr{S}_{an}$. Then $\mathscr{S}_{an}|_{E_v}$ is a numerical monoid of $\mathbb{N}$. By the above fact, it is generated by $\{-(E_e^*, E_v^*)\}_{e \in \mathscr{V}^e}$, via the operator $+$.

Let us consider the case of weighted homogeneous singularities. In this case, the link is an integral homology sphere exactly when (X, o) is a Brieskorn complete intersection of type $(\alpha_1, \dots, \alpha_\nu)$ with $\{\alpha_i\}_i$ pairwise relative prime, cf. 5.1.33. Let us fix the central vertex $E_v = E_0$. Then $-(E_e^*, E_v^*) = \alpha/\alpha_e$ for any $1 \leq e \leq \nu$. Hence $\mathscr{S}_{an}|_{E_0}$ is the numerical semigroup $G(a_1, \dots, a_\nu)$ generated by $a_i = \alpha/\alpha_i$.

We wish to comment this last statement. For any set of integers $(a_1, \dots, a_\nu)$ with $\gcd(a_1, \dots, a_\nu) = 1$ let us consider the numerical monoid $G = G(a_1, \dots, a_\nu)$ generated by them. Then the famous Diophantine Frobenius problem asks to find the Frobenius number $\mathrm{Frob}(G)$ of G, that is, the largest element of $\mathbb{N} \setminus G$. If $\nu = 2$ then $\mathrm{Frob}(G) = a_1a_2 - a_1 - a_2$ [673], however, in general the problem is still open, for more see [585]. In [112] the following upper bound is proved:

$$\mathrm{Frob}(G(a_1, \dots, a_\nu)) \leq (\nu - 1) \cdot \mathrm{lcm}(a_1, \dots, a_\nu) - \sum_i a_i. \tag{8.5.10}$$

Moreover, in [112] it is proved that in (8.5.10) the equality holds exactly when there exist pairwise relative prime integers $\{\alpha_i\}_{I=1}^{\nu}$ such that $a_i = (\prod_j \alpha_j)/\alpha_i$. From the above discussion G has this extreme property (equality in (8.5.10)) of the Frobenius number exactly when it coincides with the numerical semigroup $\mathscr{S}_{an}|_{E_0}$ associated with a Brieskorn complete intersection (as above). (Here, as in several cases in mathematics, the sharp bound is realized in algebraic geometry.) In the language of singularities, the formula from (8.5.10) reads as $\mathrm{Frob}(G) = \alpha + \mathfrak{t}$. This can be reproved by techniques from 5.1.29 (using $N(\mathfrak{t}) = -2$). For details (and for the case of a weighted homogeneous germ with rational homology sphere link) see [351].

8.5.C *Base Points of $\phi^*\mathfrak{m}_{X,o}$ and Multiplicity*

In general, the $\mathscr{O}_{\tilde{X}}$–module $\phi^*\mathfrak{m}_{X,o}$ (generated by elements $f \circ \phi$ with $f \in \mathfrak{m}_{X,o}$) has the form $\mathscr{O}_{\tilde{X}}(-Z_{max}) \otimes \otimes_{P\in\mathscr{B}}\mathscr{I}_P$, where $\mathscr{B}$ denotes the finite set of base-points and $\mathscr{I}_P$ is an $\mathfrak{m}_P$-primary ideal at $P \in \mathscr{B}$. Next we provide a combinatorial description of the ideals $\mathscr{I}_P$. We follow [484].

Assume that ϕ satisfies the end curve condition, we fix some p–cuts H_e for $e \in \mathscr{V}^e$ as in 8.5. Consider $D := E \cup (\cup_{e\in\mathscr{V}^e} H_e)$, and let $\mathrm{Sing}(D)$ denote the singular (double) points of D. Fix such a point $P \in \mathrm{Sing}(D)$ and order the two components of D containing it. Associate with each $D(\alpha) \in L_{end}$ the multiplicities (a_α, b_α) in $\sum_{e\in\mathscr{V}^e} \alpha_e(H_e + E_e^*) - Z_{max}$ of the two components of D containing P.

Corollary 8.5.29

1. $\mathscr{B} \subset \mathrm{Sing}(D)$. *In particular, if $H_e \cap E \in \mathscr{B}$ for some $e \in \mathscr{V}^e$, then the intersection point $H_e \cap E$ is independent of the choice of H_e (i.e. it is imposed by the analytic structure).*
2. *Fix $P \in \mathrm{Sing}(D)$. In some local coordinates (t, s) of P (so that $ts = 0$ is the local equation of D), $\mathscr{I}_P$ is the monomial ideal generated by $\{t^{a_\alpha}s^{b_\alpha}\}_{D(\alpha)\in L_{end}}$.*

Proof By 8.5.21 applied for Z_{max}, $\mathfrak{m}_{X,o}$ is generated by non–zero monomial sections. □

Example 8.5.30 On the left graph of 8.5.27, $\mathscr{B}$ consists of only one point, the intersection of the p–cut with E_0, the unique curve E_i with $(Z_{\min}, E_i) < 0$. In the case of the right graph $\mathscr{B} = \emptyset$. In the next example 8.5.32, $\mathscr{B}$ consists of one point, which is the intersection of two irreducible exceptional components.

For every $P \in \mathscr{B}$, the pairs $(a_\alpha, b_\alpha)_\alpha$ determine a (convenient) Newton diagram:

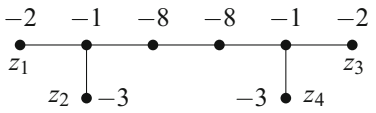
$$N_P^- = \mathbb{R}^2_{\geq 0} \setminus \text{convex closure}\Big\{ \bigcup_{D(\alpha)\in L_{end}} \big((a_\alpha, b_\alpha) + \mathbb{R}^2_{\geq 0}\big) \Big\}.$$

Theorem 8.5.31 ([484]) *The multiplicity* $\mathrm{mult}(X, o)$ *of* (X, o) *is topological:*

$$\mathrm{mult}(X, o) = -Z_{max}^2 + 2 \cdot \sum_{P\in\mathscr{B}} \mathrm{area}(N_P^-).$$

Proof For each P let ϕ_P and ψ_P be two plane curve singularities with Newton diagram N_P^- and generic coefficients. Then (see e.g. the proof of 6.7.9) $\mathrm{mult}(X, o)$ equlas $-Z_{max}^2 + \sum_{P\in\mathscr{B}}(\phi_P, \psi_P)_P$, where $(.,.)_P$ denotes the intersection multiplicity at P. But $(\phi_P, \psi_P)_P = 2 \cdot \mathrm{area}(N_P^-)$, see e.g. [683], or [575, page 276]. □

Example 8.5.32 Consider the following graph.



It is the graph of the following splice quotient singularity. Let $\{z_i\}_{1\le i\le 4}$ be the end curve sections (as indicated in the picture), then a possible choice for the splice diagram equations is $z_1^2+z_2^3+z_3^5=z_3^2+z_4^3+z_1^5=0$. $H\simeq\mathbb{Z}_3$ is generated by $[E_2^*]$, and acts by $[E_2^*]*(z_1,z_2,z_3,z_4)=(z_1,\overline{\psi}z_2,z_3,\psi z_4)$, where $\psi=e^{2\pi i/3}$. The generators of the invariants are $z_1, z_3, a=z_2^3, b=z_4^3$ and $c=z_2z_4$, hence the equations of (X,o) are $z_1^2+a+z_3^5=z_3^2+b+z_1^5=ab-c^3=0$, or $c^3=(z_1^2+z_3^5)(z_3^2+z_1^5)$.

In this case $Z_{max}=Z_{min}$ with $Z_{max}^2=-2$. Let P be the intersection point of the two (-8)-curves. Then $\phi^*(\mathfrak{m}_{X,o})=\mathscr{O}_{\widetilde{X}}(-Z_{max})\otimes\mathfrak{m}_P$, hence $\mathrm{mult}(X,o)=3$.

8.5.D The Monomial Condition and End Curve Theorem

In order to prove that a singularity with 'end curve condition' is a splice quotient, first we have to prove the 'monomial condition' [533] (see its definition in 5.4.10). We recall that the requirement of this condition is the following: if $\{C_k\}_{1\le k\le\kappa_v}$ are the branches of a node v then for each k there exists a monomial cycle $D(\alpha_k)$ such that $D(\alpha_k)-E_v^*$ is effective, integral and supported on C_k.

In this section we always assume $H^1(\widetilde{X},\mathbb{Z})=0$ and the proofs follow [75].

Proposition 8.5.33 *Assume that (X,o) admits a good resolution ϕ, which satisfies the 'end curve condition'. Then the 'monomial condition' is also satisfied.*

Proof Let v be a node and C one of its branches. Let w be the vertex in C adjacent to v. We concentrate on the connected full subgraph Γ_C spanned by v and C. Let $\bar{E}_u$ and $\bar{E}_u^*$ be the base elements and their duals in $L(\Gamma_C)$ and $L'(\Gamma_C)$. Then $\bar{E}_v$ in Γ_C inherits several p–cuts as the restrictions of the other branches of v, cf. 8.5.10 and 8.5.15. We select two such p–cuts (which clearly intersect $\bar{E}_v$ in different points), and let s_1 and s_2 be the corresponding two end curve sections. They are global sections of $\mathscr{O}(-\bar{E}_v^*)$ with divisors the corresponding p–cuts, hence they are not in $H^0(\mathscr{O}(-\bar{E}_v^*-\bar{E}_w))$. In particular, their images under the embedding

$$H^0(\mathscr{O}(-\bar{E}_v^*))/H^0(\mathscr{O}(-\bar{E}_v^*-\bar{E}_w))\hookrightarrow H^0(\mathscr{O}_{\bar{E}_w}(-\bar{E}_v^*))\simeq\mathbb{C}$$

are nonzero. Hence $s_1 - c \cdot s_2 \in H^0(\mathscr{O}(-\bar{E}_v^* - \bar{E}_w))$ for some nonzero constant c. Since the divisors of s_1 and s_2 intersect $\bar{E}_v$ in different points, $s_1 - cs_2$ is nonzero along $\bar{E}_v$ and its class is nonzero in $V := H^0(\mathscr{O}(-\bar{E}_v^* - \bar{E}_w))/H^0(\mathscr{O}(-\bar{E}_v^* - \bar{E}_w - \bar{E}_v))$. Since V embedds into $H^0(\mathscr{O}_{\bar{E}_v}(-\bar{E}_v^* - \bar{E}_w)) = \mathbb{C}$, we get $V = \mathbb{C}$, with the class of $s_1 - cs_2$ as its generator. Hence, by 8.5.13 there is a monomial divisor $D(\alpha)$ of Γ_C whose class is nonzero in V. This means, cf. 8.5.8, that $D(\alpha) = \bar{E}_v^* + \bar{E}_w + l$, with $l \in L_{\geq 0}$ but $\bar{E}_v$ is not in the support of l. This shows that $(\bar{E}_v^* + \bar{E}_w + l, \bar{E}_v) \geq 0$, hence in $D(\alpha)$ there is no contribution from end curves supported by $\bar{E}_v$, that is all the contributions come from the end curves of Γ sitting in C. (Hence $(l, \bar{E}_v) = (D(\alpha), \bar{E}_v) = 0$ too, showing that the support of l does not contain $\bar{E}_w$ either). Then $D(\alpha) - \bar{E}_v^* = \bar{E}_w + l$ is the wished effective divisor supported on C. □

Next we present the End Curve Theorem of Neumann and Wahl.

Theorem 8.5.34 (End Curve Theorem [75, 517, 534]) *Assume that a fixed good resolution ϕ of (X, o) satisfies the 'end curve condition' and $H^1(\widetilde{X}, \mathbb{Z}) = 0$. Then (X, o) is splice quotient associated with the graph of ϕ. More precisely, in the splice quotient equations, one can take for the coordinates of $(\mathbb{C}^{|\mathscr{V}^e|}, 0)$ the end curve functions $\{z_e\}_{e \in \mathscr{V}^e}$, and for the equations one can make a choice of certain Γ-combinatorial equations in variables $(\mathbb{C}^{|\mathscr{V}^e|}, 0)$. (For definitions see 5.4.10–5.4.11.)*

Proof We already constructed in 8.5.20 a ring homomorphism $\Phi : (\mathbb{C}\{z\}, 0) \to \mathscr{O}_{X_a, o}$, and in 8.5.21 we proved that it is surjective. Hence $\ker(\Phi)$ is a prime ideal of depth 2. Since *any* ideal generated by Γ-combinatorial equations is also prime of depth 2, cf. 5.4.12, it is enough to show that $\ker(\Phi)$ contains a set of Γ-combinatorial equations.

We fix our resolution and we choose a collection of end curve sections $\{z_e\}_{e \in \mathscr{V}^e}$. For any node v we find the equations as follows. By Proposition 8.5.33 the graph satisfied the monomial condition, hence it is possible to choose admissible monomials $M_{v,C}$ for all the branches of v. The admissible monomials are elements of $H^0(\mathscr{O}(-E_v^*))$. On the other hand, the vector space of power series with higher v-degree and with the same eigenspace is $H^0(\mathscr{O}(-E_v^* - E_v))$. Therefore, we need to show that the admissible monomials of v satisfy $\kappa_v - 2$ linear equations in $V := H^0(\mathscr{O}(-E_v^*))/H^0(\mathscr{O}(-E_v^* - E_v))$, whose matrix of coefficients has full rank (that is, the subspace generated by all the admissible monomials in V is generated by any two of the admissible monomials). For this, take the injection

$$H^0(\mathscr{O}(-E_v^*))/H^0(\mathscr{O}(-E_v^* - E_v)) \hookrightarrow H^0(\mathscr{O}_{E_v}(-E_v^*)) \simeq \mathbb{C}^2. \tag{8.5.11}$$

Any nonzero global section in $H^0(\mathscr{O}_{E_v}(-E_v^*))$ has exactly one zero point on E_v, and any two such sections having different zero points form a basis of it. Since the zero of $M_{v,C}$ is $C \cap E_v$ (hence they are all different), V is 2-dimensional, and any two admissible monomials form a basis of it. □

Corollary 8.5.35 *Fix a good resolution with graph Γ of a normal surface singularity (X, o), which satisfies $H^1(\widetilde{X}, \mathbb{Z}) = 0$. Then the following facts are equivalent:*

(a) (X, o) satisfies the 'end curve condition' (associated with Γ);
(b) (X, o) is splice quotient (associated with Γ, in the sense of 5.4.11);
(c) $Z(\mathbf{t}) = P(\mathbf{t})$ (both series associated with Γ).

Proof For *(a)⇔(b)* use comment from 5.4.13 and 8.5.34. For *(a)⇔(c)* use 8.5.16 and 8.5.19. □

8.5.36 The monomial Filtrations Revisited Assume that (X, o) is a splice quotient singularity given by Γ-combinatorial equations as in 5.4.11–5.4.12. We wish to analyze deeper the monomial filtration $\{\mathcal{G}(l')\}_{l'}$.

Recall that the monomial L'–multigrading of $\mathbb{C}\{z\}$ is defined in such a way that the variable z_e ($e \in \mathcal{V}^e$) has multidegree E_e^*. This means that the multidegree $w(z^\alpha)$ of the monomial z^α is $D(\alpha)$. In particular, the Poincaré series of $\mathbb{C}\{z\}$ is the expression $\prod_{e\in\mathcal{V}^e}(1-\mathbf{t}^{E_e^*})^{-1}$. The associated filtration is $\mathcal{G}_{\mathbb{C}\{z\}}$.

As usual, if $f = \sum a_\alpha z^\alpha$, then $w(f) := \inf_{a_\alpha\neq 0} D(\alpha)$ (see also 8.1.22).

Consider next the Γ-combinatorial equation $f_{v,j} = \sum_C a_{v,C,j} M_{v,C} + g_{v,j}$ ($j = 1, \dots, \kappa_v - 2$) associated with a node v, cf. 5.4.11. Since $M_{v,C}$ is an admissible monomial, there exists an integral effective cycle Z_C supported on the branch C such that $w(M_{v,C}) = E_v^* + Z_C$. In particular, $w\left(\sum a_{v,C,j} M_{v,C}\right) = \inf_C\{E_v^* + Z_C\} = E_v^*$.

Next,we analyze $g_{v,j}$: by the construction 5.4.11 it is a convergent power series of monomials of type z^α, each with multidegree $w(z^\alpha) = D(\alpha)$ and with v-degree $(\dagger) -(D(\alpha), E_v^*) > -(E_v^*, E_v^*)$. We claim that for each such monomial $D(\alpha) > E_v^*$ holds too. Indeed, take $D := D(\alpha) - E_v^*$ and write it as $D = D' + a_v E_v$, where the support of D' does not contain E_v. Then $(\dagger)$ just says that $a_v > 0$. Since $D(\alpha)$ is a monomial cycle, $(D, E_u) \leq 0$ for all $u \neq v$. Hence $(D', E_u) \leq 0$ too for all E_u in the support of D' implying $D' \geq 0$. Therefore, $w(f_{v,j}) = E_v^*$.

Since, the Γ-combinatorial equations define a complete intersection, the expectation for the Poincaré series of the *projected filtration* $\mathcal{G} := \Phi(\mathcal{G}_{\mathbb{C}\{z\}})$ of $\mathcal{O}_{(X_a,o)}$ is (cf. 8.1.2 and 8.1.23)

$$P_{\mathcal{G}}(\mathbf{t}) = \prod_{v\in\mathcal{V}} \left(1-\mathbf{t}^{E_v^*}\right)^{\kappa_v-2}.$$

This puts the combinatorial definition of $Z(\mathbf{t})$ in a new light: it gives a new reinterpretation for $Z(\mathbf{t})$. In fact, a combination of Theorems 8.5.16 ($P_{\mathcal{F}}(\mathbf{t}) = Z(\mathbf{t})$) and 8.5.21 ($\mathcal{G} = \mathcal{F}$) implies that, indeed,

$$P_{\mathcal{G}}(\mathbf{t}) = P_{\mathcal{F}}(\mathbf{t}) = Z(\mathbf{t}) = \prod_{v\in\mathcal{V}} \left(1-\mathbf{t}^{E_v^*}\right)^{\kappa_v-2}.$$

It is worth to mention that (conversely) the identity of the series $P_{\mathscr{G}} = P_{\mathscr{F}}$ would also imply $\mathscr{G} = \mathscr{F}$. Indeed, since for both filtrations the 'inversion' of type 8.1.19 holds, their Hilbert series must coincide: $H_{\mathscr{G}} = H_{\mathscr{F}}$. This together with $\mathscr{G}(l') \subset \mathscr{F}(l')$ for all l' imply $\mathscr{G} = \mathscr{F}$.

Example 8.5.37 For additional information regarding ECC and WECC properties of elliptic singularities see 7.2.70 above.

8.6 Reductions of Variables in the Series $P(\mathbf{t})$ and $Z(\mathbf{t})$

For any fixed resolution ϕ, in the definition of the series $P(\mathbf{t})$ and $Z(\mathbf{t})$ one takes a variable t_v for each exceptional divisor E_v of ϕ. However, in several situations and applications we realize that some of the variables are 'superfluous'. E.g., if the resolution is not minimal, the non-essential exceptional components (those which do not appear in the minimal resolution) carry less information. The same is valid even for some of the exceptional curves of the minimal resolution, e.g. for those with $\kappa_v = 2$. Or, from a different point of view, certain exceptional divisors might have some intrinsic geometric meaning, and sometimes we wish to concentrate only on them. Hence, one might focus only on the subset of vertices either corresponding to the nodes of the graph, or the ones corresponding to the essential exceptional divisors only, or WR– and SR–sets, or pullbacks/images of lattices provided by some geometric constructions.

In this part we reduce the variables corresponding to an arbitrary subset of $\mathscr{V}$, creating the theoretical background. First, we define the corresponding topological and analytical series with reduced number of variables. This procedure will be called 'reduction'. ♣

8.6.1 We fix (X, o) (as in 8.2.1) and the resolution ϕ. Let $\mathscr{I}$ be a non-empty subset of $\mathscr{V}$. Associated with it we consider formal series in variables $\{t_v\}_{v\in\mathscr{I}}$, denoted by $\mathbf{t}_{\mathscr{I}}$, and the projection $\pi_{\mathscr{I}} : L' \to L \otimes \mathbb{Q}$, $\pi_{\mathscr{I}}(\sum_{v\in\mathscr{V}} l'_v E_v) = \sum_{v\in\mathscr{I}} l'_v E_v$. We write

$$l'_{\mathscr{I}} := \pi_{\mathscr{I}}(l'), \quad \text{and} \quad \mathbf{t}^{l'}_{\mathscr{I}} = \prod_{v\in\mathscr{I}} t_v^{l'_v} = \mathbf{t}^{l'}|_{t_v=1 \text{ for all } v\notin\mathscr{I}}.$$

Here a word of warning is necessary. In the original case $\mathscr{I} = \mathscr{V}$, from a series $S(\mathbf{t}) = \sum_{l'} a_{l'}\mathbf{t}^{l'}$ we can recover its h-components S_h (cf. 8.2.3). Indeed, the monomial $a_{l'}\mathbf{t}^{l'}$ ($a_{l'} \neq 0$) belongs to S_h if and only if $[l'] = h$. However, this property will be lost when we reduce the variable: if $\mathscr{I} \subsetneq \mathscr{V}$ then from the information carried by $\pi_{\mathscr{I}}(l')$ usually one cannot recover $[l']$. Therefore, *the reduced h-components of a series $S(\mathbf{t})$ should be defined as the reductions of the original h-components $S_h(\mathbf{t})$ (and they cannot be recovered from the reduced S).*

8.6.2 The Topological Series $Z(\mathbf{t})$ is reduced as follows.

Definition 8.6.3 The reduced series of Z is defined as $Z_{\mathscr{J}}(\mathbf{t}_{\mathscr{J}}) := Z(\mathbf{t})|_{t_v=1 \text{ for all } v\notin\mathscr{J}}$. Similarly, for any $h \in H$, $Z_{h,\mathscr{J}}(\mathbf{t}_{\mathscr{J}}) := Z_h(\mathbf{t})|_{t_v=1 \text{ for all } v\notin\mathscr{J}}$. Equivalently,

$$Z_{h,\mathscr{J}}(\mathbf{t}_{\mathscr{J}}) := \frac{1}{|H|}\cdot\sum_{\rho\in\widehat{H}}\rho(h)^{-1}\cdot\prod_{v\in\mathscr{V}}(1-\rho([E_v^*])\mathbf{t}_{\mathscr{J}}^{E_v^*})^{\kappa_v-2}. \tag{8.6.1}$$

The substitutions $\{t_v = 1\}_{v\in\mathscr{J}}$ are well-defined since $Z(\mathbf{t})$ is supported on $\mathscr{S}'$, which has the special finiteness property 2.1.19.

8.6.4 Reducing Variables in Series $P(\mathbf{t})$ In the case of the analytic series $P(\mathbf{t})$ we can proceed, a priori, in two different ways (in 8.6.5 we will show that they agree). By the first one we reduce $P(\mathbf{t})$ 'blindly', as we did with $Z(\mathbf{t})$ in 8.6.3, via substitutions $t_v = 1$ for all $v \notin \mathscr{J}$. Again, this step is well-defined since P too is supported on $\mathscr{S}'$, cf. 8.2.8.

On the other hand, we can also repeat the original geometric definition of $P(\mathbf{t})$, as the multivariable Poincaré series associated with the divisorial filtration, however, at this time we will use the 'reduced set of divisors' indexed by $\mathscr{J}$. Let us examine next this second, more geometric approach.

For any $l'_{\mathscr{J}} \in \pi_{\mathscr{J}}(L')$ set

$$\mathscr{F}_{\mathscr{J}}(l'_{\mathscr{J}}) := \{f \in \mathscr{O}_{X_a,o} \mid \operatorname{div}(f\circ\phi_a) \geq p^*(l'_{\mathscr{J}})\}.$$

Furthermore, for any $h \in H$ and $l'_{\mathscr{J}} \in \pi_{\mathscr{J}}(r_h + L)$ let $\mathscr{F}_{h,\mathscr{J}}(l'_{\mathscr{J}})$ be the intersection of $\mathscr{F}_{\mathscr{J}}(l'_{\mathscr{J}})$ with the $\theta(h)$-eigenspace $(\mathscr{O}_{X_a,o})_{\theta(h)}$, and define

$$\mathfrak{h}_{h,\mathscr{J}}(l'_{\mathscr{J}}) = \dim(\mathscr{O}_{X_a,o})_{\theta(h)}/\mathscr{F}_{h,\mathscr{J}}(l'_{\mathscr{J}}).$$

Then set:

$$H_{h,\mathscr{J}}(\mathbf{t}_{\mathscr{J}}) := \sum_{l'_{\mathscr{J}}\in\pi_{\mathscr{J}}(r_h+L)}\mathfrak{h}_{h,\mathscr{J}}(l'_{\mathscr{J}})\cdot\mathbf{t}_{\mathscr{J}}^{l'_{\mathscr{J}}},$$

$$P_{h,\mathscr{J}}(\mathbf{t}_{\mathscr{J}}) := -H_{h,\mathscr{J}}(\mathbf{t}_{\mathscr{J}})\cdot\prod_{v\in\mathscr{J}}(1-t_v^{-1}).$$

If $l' = l + r_h$ ($l \in L$) then, in general, $l'_{\mathscr{J}} = (l + r_h)_{\mathscr{J}}$ is not an element of L'. However, $l_{\mathscr{J}} + r_h \in L'$ is the smallest element of $\{l' \in r_h + L : l' \geq l'_{\mathscr{J}}\}$. Furthermore, using the fact that $\operatorname{div}(f\circ\phi_a) \in p^*(r_h + L)$ for any $f \in (\mathscr{O}_{X_a,o})_{\theta(h)}$ (cf. 6.2.11*(b)*) we obtain

$$\mathscr{F}_{h,\mathscr{J}}((l+r_h)_{\mathscr{J}}) = \mathscr{F}(l_{\mathscr{J}}+r_h)\cap(\mathscr{O}_{X_a,o})_{\theta(h)}, \tag{8.6.2}$$

$$\mathfrak{h}_{h,\mathscr{J}}((l+r_h)_{\mathscr{J}}) = \mathfrak{h}_h(l_{\mathscr{J}}+r_h) = \mathfrak{h}(l_{\mathscr{J}}+r_h). \tag{8.6.3}$$

Proposition 8.6.5 *With the above notations,*

$$P_{h,\mathscr{I}}(\mathbf{t}_{\mathscr{I}}) = P_h(\mathbf{t})|_{t_v=1 \text{ for all } v\notin\mathscr{I}}. \tag{8.6.4}$$

In particular, $P_{h,\mathscr{I}}(\mathbf{t}_{\mathscr{I}})$ is supported on $\pi_{\mathscr{I}}(\mathscr{S}')$.

(The same statement is not true for $H_h(\mathbf{t})$: $\lim_{t_v\to 1} H_h(\mathbf{t})$ is not even convergent.)

Proof We run descending induction over the cardinality of $\mathscr{I}$. Set $\bar{\mathscr{I}} := \mathscr{I} \setminus v_0$ such that $\bar{\mathscr{I}} \neq \emptyset$. Then it is enough to verify

$$\Big(H_{h,\mathscr{I}}(\mathbf{t}_{\mathscr{I}}) \cdot \prod_{v\in\mathscr{I}} (1-t_v^{-1})\Big)\Big|_{t_{v_0}=1} = H_{h,\bar{\mathscr{I}}}(\mathbf{t}_{\bar{\mathscr{I}}}) \cdot \prod_{v\in\bar{\mathscr{I}}} (1-t_v^{-1}). \tag{8.6.5}$$

Define $S(\mathbf{t}_{\mathscr{I}}) := H_{h,\mathscr{I}}(\mathbf{t}_{\mathscr{I}}) \cdot \prod_{v\in\bar{\mathscr{I}}}(1-t_v^{-1})$, supported on $\pi_{\mathscr{I}}(L')$. Let $L(\mathscr{I})$ be the lattice generated by $\{E_v\}_{v\in\mathscr{I}}$. Then $S(\mathbf{t}_{\mathscr{I}}) \cdot \mathbf{t}_{\mathscr{I}}^{-r_h}$ has the form $\sum_{l\in L(\mathscr{I})} a_{h,\mathscr{I}}(l)\, \mathbf{t}^l_{\mathscr{I}}$. This can be rewritten as

$$\sum_{\bar{l}\in L(\bar{\mathscr{I}})} \mathbf{t}^{\bar{l}}_{\bar{\mathscr{I}}} \sum_{k\in\mathbb{Z}} a_{h,\mathscr{I}}(\bar{l}+kE_{v_0})\, t_{v_0}^k,$$

where

$$a_{h,\mathscr{I}}(\bar{l}+kE_{v_0}) = \sum_{I\subset\bar{\mathscr{I}}} (-1)^{|I|} \mathfrak{h}_{h,\mathscr{I}}(\bar{l}+kE_{v_0}+E_I).$$

By definition (cf. also (8.6.3)), for any fixed $\bar{l}$ and $I \subset \bar{\mathscr{I}}$, $\mathfrak{h}_{h,\mathscr{I}}(\bar{l}+kE_{v_0}+E_I) = \mathfrak{h}_{h,\bar{\mathscr{I}}}(\bar{l}+E_I)$ whenever $k \ll 0$. Hence $a_{h,\mathscr{I}}(\bar{l}+kE_{v_0}) = a_{h,\bar{\mathscr{I}}}(\bar{l})$ whenever $k \ll 0$. On the other hand, for $k \gg 0$, $a_{h,\mathscr{I}}(\bar{l}+kE_{v_0}+E_I) = 0$. Indeed, if $\bar{u} \in \bar{\mathscr{I}}$ is a fixed element of $\bar{\mathscr{I}}$, and $\bar{I}$ a subset of $\bar{\mathscr{I}} \setminus \bar{u}$, then

$$\mathfrak{h}_{h,\mathscr{I}}(\bar{l}+kE_{v_0}+E_{\bar{I}}) = \mathfrak{h}_{h,\mathscr{I}}(\bar{l}+kE_{v_0}+E_{\bar{I}}+E_{\bar{u}}).$$

(Recall that for any $l \in L$ there exists an integer n, such that for any $s \in \mathscr{S}'$ with $s \geq l$ and $(E_{v_0}, -s) \geq n$, one automatically has $s \geq l + E$.) Hence

$$\sum_{k\in\mathbb{Z}} a_{h,\mathscr{I}}(\bar{l}+kE_{v_0})\, t_{v_0}^k (1-t_{v_0}^{-1})\Big|_{t_{v_0}=1} = a_{h,\bar{\mathscr{I}}}(\bar{l}).$$

□

Corollary 8.6.6 *Assume that for a resolution ϕ and an element $h \in H$ the identity $P_h(\mathbf{t}) = Z_h(\mathbf{t})$ is valid. Then for the same ϕ and h and for any non-empty $\mathscr{I} \subset \mathscr{V}$ the 'reduced identity' $Z_{h,\mathscr{I}}(\mathbf{t}_{\mathscr{I}}) = P_{h,\mathscr{I}}(\mathbf{t}_{\mathscr{I}})$ (in $\mathbb{Z}[[t_v^{1/\det(I)}, v \in \mathscr{I}]]$) is valid too.*

8.6.7 We will use the following notation for the coefficients:

$$P_{h,\mathscr{I}}(\mathbf{t}_{\mathscr{I}}) = \sum_{l'_{\mathscr{I}} \in \pi_{\mathscr{I}}(r_h+L)} \mathfrak{p}_{h,\mathscr{I}}(l'_{\mathscr{I}})\mathbf{t}^{l'_{\mathscr{I}}}. \tag{8.6.6}$$

Corollary 8.6.8 *For any $l'_{\mathscr{I}} \in \pi_{\mathscr{I}}(r_h + L)$ one has*

$$\mathfrak{h}_{h,\mathscr{I}}(l'_{\mathscr{I}}) = \sum \mathfrak{p}_{h,\mathscr{I}}(\tilde{l}'_{\mathscr{I}}), \quad \textit{sum over } \tilde{l}'_{\mathscr{I}} \in \pi_{\mathscr{I}}(r_h + L),\ \tilde{l}'_{\mathscr{I}} \not\geq l'_{\mathscr{I}}.$$

Proof Write $l'_{\mathscr{I}} = (l + r_h)_{\mathscr{I}}$. Then by (8.6.3) and (8.2.10) one has

$$\mathfrak{h}_{h,\mathscr{I}}(l'_{\mathscr{I}}) = \mathfrak{h}_h(l_{\mathscr{I}} + r_h) = \sum_{\tilde{l} \not\geq l_{\mathscr{I}}} \mathfrak{p}_h(\tilde{l} + r_h).$$

Since P_h is supported on $\mathscr{S}'$, in this sum $\tilde{l} \not\geq l_{\mathscr{I}}$ can be replaced by $\tilde{l}_{\mathscr{I}} \not\geq l_{\mathscr{I}}$. Hence $\mathfrak{h}_{h,\mathscr{I}}(l'_{\mathscr{I}})$ equals the coefficient of $\mathbf{t}^{l'_{\mathscr{I}}}$ in $P_h(\mathbf{t})|_{t_v=1 \text{ for all } v \notin \mathscr{I}}$. Then use (8.6.4). □

Remark 8.6.9 If $|\mathscr{I}| = 1$, then all the coefficients of $P_{h,\mathscr{I}}(t_{\mathscr{I}})$ are nonnegative: $\mathfrak{p}_{h,\mathscr{I}}(l'_{\mathscr{I}}) = \dim \mathscr{F}_{h,\mathscr{I}}(l'_{\mathscr{I}})/\mathscr{F}_{h,\mathscr{I}}(l'_{\mathscr{I}} + E_{\mathscr{I}})$. However, this is not the case for the coefficients of $P(\mathbf{t})$. Take e.g. the singularity from 8.3.6. This germ is of splice type, hence $P(\mathbf{t}) = Z(\mathbf{t})$. Moreover, $H = 0$. Therefore, from the shape of $Z(\mathbf{t})$, cf. 8.4.2, for any node v the coefficient of $\mathbf{t}^{E_v^*}$ is -1 (note that all E_v^*'s are linearly independent).

Probably is worth to mention that in the case of $Z_{h,\mathscr{I}}(t_{\mathscr{I}})$, even if $|\mathscr{I}| = 1$, it can happen that it has some negative coefficients. For example, for a superisolated germ (and $\mathscr{I} = \{v_+\}$) we will show in 8.6.21 that $Z_{0,\mathscr{I}}(t) = N(t) + (1 - t^d)/(1 - t)^3$, where $N(t)$ is computed for several examples. E.g., for the last example from the table of 8.6.22 with $d = 5$ we have $N(t) = -8t$, hence $Z_0(t) = 1 - 5t + 6t^2 + \cdots$.

8.6.10 The Counting Function of $P_{h,\mathscr{I}}$ Usually, when we reduce variables, we loose some information. Nevertheless, when the reduced variables are indexed (say) by an SR–set, we still hope that the basic connections between $P(\mathbf{t})$ and certain numerical analytic invariants of the singularity will be preserved. E.g., we wish to have an analogous statement of the identity (8.2.26) in the reduces case as well. Recall that by (8.2.26) (see also 8.2.15) for any $l' \in -K + \mathscr{S}'$ the sum $\sum \mathfrak{p}(\tilde{l}')$ (over $\tilde{l}'$ with $[\tilde{l}'] = [l']$ and $\tilde{l}' \not\geq l'$) equals the quadratic function $\chi(l') - \chi(r_h) + h^1(\mathscr{O}_{\tilde{X}}(-r_h))$. In particular, $P(\mathbf{t})$ provides all the equivariant geometric genera via the free terms of these quadratic functions.

Our next goal is to prove a similar statement for $P_{h,\mathscr{I}}$.

Let (X, o) be a singularity with resolution ϕ. Let $\mathscr{I} = \overline{\mathscr{V}} \subset \mathscr{V}$ be an R–set. We write $\mathscr{V} = \overline{\mathscr{V}} \sqcup \mathscr{V}^*$ as in section 7.3.B. Let M be the bound required by the Unbalanced Vanishing Theorem 7.3.31. (We can assume that $M \geq 0$.)

As above, for any $l \in L$ abridge $\pi_{\mathscr{J}}(l)$ by $\bar{l}$ or $l_{\mathscr{J}}$. Next, by 7.3.24, associated with any $\bar{l} \in L(\overline{\mathscr{V}})$ and the representative r_h we have the universal element $x_{r_h}(\bar{l})$.

The assumption $l' \in -K + \mathscr{S}'$ of (8.2.26) will be replaced by the condition:

$$(a) \quad (x_{r_h}(\bar{l}) + r_h, E_v) \le -M \quad \text{for any } v \in \overline{\mathscr{V}}. \tag{8.6.7}$$

First note that *(a)* implies the next conditions as well:

$$\begin{aligned} &(b) \quad x_{r_h}(\bar{l}) + r_h \in \mathscr{S}'_h, \\ &(c) \quad h^1(\mathscr{O}_{\tilde{X}}(-x_{r_h}(\bar{l}) - r_h)) = 0, \\ &(d) \quad \mathfrak{h}(x_{r_h}(\bar{l}) + r_h) = \mathfrak{h}(\bar{l} + r_h). \end{aligned} \tag{8.6.8}$$

For *(b)* note that for $v \in \mathscr{V}^*$ one gets $(x_{r_h}(\bar{l}) + r_h, E_v) \le 0$ from the definition of $x_{r_h}(\bar{l})$, while for $v \in \overline{\mathscr{V}}$ one gets it from *(a)*. Part *(c)* follows from the Unbalanced Vanishing Theorem, while for *(d)* use $\bar{l} + r_h \le x_{r_h}(\bar{l}) + r_h \le s(\bar{l} + r_h)$ and (8.2.8).

Cycles $\bar{l}$ satisfying *(a)* always exist. Indeed, take $l \in L$ with the following two properties: *(i)* $\quad l + r_h \in \mathscr{S}'$, and *(ii)* $\quad (l + r_h, E_v) \le -M$ for any $v \in \overline{\mathscr{V}}$. Then $x_{r_h}(\bar{l})$ satisfies *(a)*. Indeed, $x_{r_h}(\bar{l}) \le l$ from the universal property of $x_{r_h}(\bar{l})$ and *(i)*. Hence $(x_{r_h}(\bar{l}) + r_h, E_v) \le (l + r_h, E_v) \le -M$ for $v \in \overline{\mathscr{V}}$ from *(ii)*.

Proposition 8.6.11 *For any $l_{\mathscr{J}} = \bar{l}$ satisfying (a) from (8.6.7) one has*

$$\begin{aligned} \sum_{\tilde{l} \in L,\ \tilde{l}_{\mathscr{J}} \not\ge l_{\mathscr{J}}} \mathfrak{p}_{h,\mathscr{J}}((\tilde{l} + r_h)_{\mathscr{J}}) &= \chi_{K+2r_h}(x_{r_h}(\bar{l})) + h^1(\mathscr{O}_{\tilde{X}}(-r_h)) \\ &= \chi(x_{r_h}(\bar{l}) + r_h) - \chi(r_h) + h^1(\mathscr{O}_{\tilde{X}}(-r_h)). \end{aligned} \tag{8.6.9}$$

Proof Use the identity (8.2.27) for $l' = x_{r_h}(\bar{l}) + r_h$, and the vanishing (8.6.8)*(c)*. Then via (8.6.8)*(d)*, (8.6.3) and Corollary 8.6.8 the sum in this expression is

$$\mathfrak{h}(x_{r_h}(\bar{l}) + r_h) = \mathfrak{h}(\bar{l} + r_h) = \mathfrak{h}_{h,\mathscr{J}}((l + r_h)_{\mathscr{J}}) = \text{left hand side of (8.6.9)}.$$

□

Note that from 7.3.25 one gets $x_{r_h}(l_{\mathscr{J}}) + r_h = x_{s_h}((l - \Delta)_{\mathscr{J}}) + s_h$, where $\Delta := s_h - r_h$. Therefore, in (8.6.9) via a shift $l \mapsto l - \Delta_{\mathscr{J}}$ we obtain the following analogue identity. Assume that $(x_{s_h}(\bar{l}) + s_h, E_v) \le -M$ for any $v \in \overline{\mathscr{V}}$. Then

$$\begin{aligned} \sum_{\tilde{l} \in L,\ \tilde{l}_{\mathscr{J}} \not\ge l_{\mathscr{J}}} \mathfrak{p}_{h,\mathscr{J}}((\tilde{l} + s_h)_{\mathscr{J}}) &= \chi_{K+2s_h}(x_{s_h}(\bar{l})) + h^1(\mathscr{O}_{\tilde{X}}(-s_h)) \\ &= \chi(x_{s_h}(\bar{l}) + s_h) - \chi(s_h) + h^1(\mathscr{O}_{\tilde{X}}(-s_h)). \end{aligned} \tag{8.6.10}$$

8.6.A *Examples. Is $P_{h,\mathscr{I}}$ Topological/Combinatorial?*

In this section we mostly assume that $\mathscr{I}$ *contains only one element*. We compute the analytic reduced series $P_{h,\mathscr{I}}$ for several families and we compare it with the topological series $Z_{h,\mathscr{I}}$, or with other series provided by certain weight filtrations. We show that in some special cases these filtrations might coincide, hence $P_{\mathscr{I}}$ can be described topologically/combinatorially. However, in general, the series $P_{\mathscr{I}}$ can be different from its topological/combinatorial candidates. ♣

We will analyze different filtrations $\{\mathscr{F}(l)\}_{l\geq 0}$ of the $\mathbb{C}$-algebra $\mathscr{O}_{X,o}$ (or $\mathscr{O}_{X_a,o}$). As usual, for any such filtration $\mathscr{F}$, we write $Gr_{\mathscr{F}}(l) := \mathscr{F}(l)/\mathscr{F}(l+1)$, and $P_{\mathscr{F}}(t) := \sum_{l\geq 0} \dim(Gr_{\mathscr{F}}(l))t^l$ for its Poincaré series.

Example 8.6.12 (Singularities Satisfying the End Curve Condition) Assume that the link is a rational homology sphere and (X,o) admits a resolution ϕ which satisfies the 'end curve condition'. We fix also a non-empty subset $\mathscr{I} \subset \mathscr{V}$.

Recall that we associated with a monomial $z^\alpha \in \mathbb{C}\{z\}$ the corresponding monomial cycle $D(\alpha) = \sum_{e\in\mathscr{V}} \alpha_e E_e^* \in L'$, cf. 5.4.10. For the definition of the monomial filtration of $\mathbb{C}\{z\}$ see (8.5.8). Its reduced version is defined as follows.

$$\mathscr{G}_{\mathbb{C}\{z\},\mathscr{I}}(l'_{\mathscr{I}}) := \{\sum_\alpha a_\alpha z^\alpha \ : \ D(\alpha)_{\mathscr{I}} \geq l'_{\mathscr{I}} \text{ whenever } \ a_\alpha \neq 0\},$$

and $\mathscr{G}_{\mathscr{I}}(l'_{\mathscr{I}}) := \Phi(\mathscr{G}_{\mathbb{C}\{z\},\mathscr{I}}(l'_{\mathscr{I}})) \subseteq \mathscr{O}_{X_a,o}$. We call $\mathscr{G}_{\mathscr{I}}$ the *monomial filtration of $\mathscr{O}_{X_a,o}$ associated with $\mathscr{I}$*.

Recall that $\mathbb{C}\{z\}$ has an H-eigenspace decomposition (induced by the action $h * z_e = \theta([E_e^*])(h)z_e$), namely $\mathbb{C}\{z\}_h = \{\sum_\alpha a_\alpha z^\alpha \ : \ [D(\alpha)] = h \text{ if } a_\alpha \neq 0\}$. Set $\mathscr{G}_{h,\mathscr{I}}((l+r_h)_{\mathscr{I}}) = \Phi(\mathscr{G}_{\mathbb{C}\{z\},\mathscr{I}}((l+r_h)_{\mathscr{I}}) \cap \mathbb{C}\{z\}_h)$. Then, similarly as in (8.6.2),

$$\mathscr{G}_{h,\mathscr{I}}((l+r_h)_{\mathscr{I}}) = \mathscr{G}_h(l_{\mathscr{I}} + r_h). \tag{8.6.11}$$

In particular, by Theorems 8.5.16 ($P = Z$) and 8.5.21 ($\mathscr{G} = \mathscr{F}$), the monomial and divisorial filtrations associated with $\mathscr{I}$ coincide, $\mathscr{G}_{h,\mathscr{I}} = \mathscr{F}_{h,\mathscr{I}}$, and their Poincaré series is given by the combinatorial series $Z_{h,\mathscr{I}}(\mathbf{t})$.

Example 8.6.13 (Weighted Homogeneous Germs) Assume that (X,o) is weighted homogeneous and its minimal good resolution is star-shaped with $\nu \geq 3$. We assume that $\mathscr{I} = \{\text{central vertex } v_0\}$ corresponding to the exceptional divisor E_0. For other notations see 5.1 and 8.1.B.

Our plan is to compare three filtrations and to show that they agree.

Firstly, the E_0-divisorial filtration coincides with the filtration given by the $\mathbb{C}^*$ action. Indeed, via the automorphic factor realization (cf. 5.1.25), which preserves both filtrations, it is enough to verify the identity for cones. For cones the coincidence is coded already in the definitions: a homogeneous element of degree ℓ (with respect to the $\mathbb{C}^*$ action), as a function from the total space of $\mathscr{L}^{-1}$ to $\mathbb{C}$, has

local form $s(x)u^{\ell}$, where x is a local coordinate of C and u is the fiber coordinate (cf. 5.1.3). Thus, their vanishing order along $E_0 = C$ is ℓ.

Assume next that $g = 0$, hence the universal abelian covering is well-defined, for its description see 5.1.32. Therefore, one has three equivariant $\mathbb{Z}$–filtrations of $\mathscr{O}_{X_a,o}$: the divisorial filtration $\mathscr{F}_{\mathscr{J}}$ associated with the central divisor E_0, the filtration/grading associated with the $\mathbb{C}^*$-action, and the monomial filtration $\mathscr{G}_{\mathscr{J}}$ associated with v_0.

The monomial filtration is defined as follows. Since the coefficient of E_0 in $D(\alpha)_{\mathscr{J}}$ is $\deg(z^{\alpha}) = \deg(D(\alpha)) = -(D(\alpha), E_0^*)$, the filtration is determined by this v_0–deg, namely $\deg(z_i) = \deg(E_{s_i}^*) = (\alpha_i |e|)^{-1}$ $(1 \le i \le \nu)$. The equations of the universal abelian covering (cf. 5.1.33) have v_0-degree $|e|^{-1}$.

This coincides exactly with the weights of the $\mathbb{C}^*$-action on (X_a, o) (see e.g. 5.1.33). In particular, the monomial filtration and the filtration induced by the $\mathbb{C}^*$-action agree. Since weighted homogeneous singularities satisfy the 'end curve condition' (see 5.1.39), by 8.6.12 the monomial and the divisorial filtrations agree too.

Hence, the three filtrations coincide. We claim that the (common) Poincaré series of the above filtrations agree with the topological series $Z_{h,\mathscr{J}}(t)$ (the variable t corresponds to v_0). This fact can be seen in many different ways (in the literature see e.g. [490, 506, 571]):

(i) The identity $P = Z$ was proved for any singularity which satisfies the end curve condition. Then the identity $P_{h,\mathscr{J}} = Z_{h,\mathscr{J}}$ follows from 8.6.6.

(ii) The Poincaré series of the E_0–monomial filtration (equivalently, of the $\mathbb{C}^*$-action grading) was computed in the first paragraphs of 8.1.B basically via $Z_{h,\mathscr{J}}$. This computation emphasizes that the topological series $Z(t)$ is suited basically for those situations when there is a complete intersection structure behind, cf. 8.5.36.

(iii) If $h = 0$ then the Poincaré series of the graded $\mathscr{O}_{X,o}$ was computed analytically via the Dolgachev–Pinkham–Demazure formalism, providing the same answer, cf. 5.1.26.

For any fixed $h \in H$, let $l'_h \in L'$ be one of its representatives. If $l'_h = a_0E_0^* + \sum_{ik} a_{ik}E_{ik}^*$, then $l'_{\mathrm{red}} := a_0E_0^* + \sum_{ik} a_{ik}n^i_{k+1,s_i}E^*_{is_i}$ is still a representative, and $\pi_{\mathscr{J}}(l') = \pi_{\mathscr{J}}(l'_{\mathrm{red}}) = \mathfrak{a} \in \frac{1}{\mathfrak{o}}\mathbb{Z}$, cf. 8.1.B. The rational number $\mathfrak{a}$ modulo $\mathbb{Z}$ is independent of the choice of the representative l'_h, it depends only on h (and any integral shift can be realized by different choices). In particular, $\pi_{\mathscr{J}}(L + r_h) = \mathfrak{a} + \mathbb{Z}$.

The common Poincaré series is given by

$$P_{h,\mathscr{J}}(t) = \sum_{\ell \in \mathbb{Z},\ \ell \ge -\mathfrak{a}} \max\left\{0\,,\ 1 + a_0 + \ell b - \sum_j \left\lceil \frac{\ell\omega_j - a_j}{\alpha_j} \right\rceil \right\} \cdot t^{\ell+\mathfrak{a}}.$$

With the choice $l'_h = r_h$ one has $\mathfrak{a} \in [0, 1)$.

This expression can also be compared with another expression obtained via a rather different construction, namely via the universal cycles $x(\ell)$ and their τ-

function, cf. 11.3.14. In this way, the universal concatenated sequence γ from 7.3.D (see also 11.3.14) has the minimality property $S(\gamma) = \min_{\tilde{\gamma}\in\mathscr{P}^i} S(\tilde{\gamma})$, and its expression recovers the Dolgachev–Pinkham–Demazure formula for p_g.

Additionally, notice the next identity as well. For simplicity let us assume that $h = 0$. Recall that $P_{0,\mathscr{J}}(t)$ determines in a unique way its 'polynomial part', denoted by P^+ in 8.1.9 and 8.1.10 (or P_{H^1} in (5.1.8)). Then $P^+(1) = p_g$.

Example 8.6.14 (Weight Filtrations and Hypersurface Singularities) We take a (not necessarily weighted homogeneous) isolated hypersurface singularity $(X, o) = \{f = 0\} \subset (\mathbb{C}^3, 0)$. Let $\Phi : \mathscr{O}_{\mathbb{C}^3,0} \to \mathscr{O}_{X,o}$ be the natural projection.

We fix a system of weights $\mathbf{w} = (w_1, w_2, w_3) \in \mathbb{Z}^3_{>0}$ as in 8.1.22. It defines a valuation $\mathfrak{v}_\mathbf{w}$ and filtration $\mathscr{F}_\mathbf{w}$ of $\mathscr{O}_{\mathbb{C}^3,0}$ and an induced order function $\mathfrak{v}_{X,\mathbf{w}}$ on $\mathscr{O}_{X,o}$,

$$\mathfrak{v}_{X,\mathbf{w}}(\Phi(g)) = \max_{h\in\mathscr{O}_{\mathbb{C}^3,0}} \mathfrak{v}_\mathbf{w}(g + fh).$$

Motivated by filtrations associated with Newton diagrams, we define the 'localized' version, a larger order function as well:

$$\mathfrak{v}^l_{X,\mathbf{w}}(\Phi(g)) = \max_{h\in\mathscr{O}_{\mathbb{C}^3,0}[x_1^{-1},x_2^{-1},x_3^{-1}]} \mathfrak{v}_\mathbf{w}(g + fh).$$

Consider the following filtrations on $\mathscr{O}_{X,o}$: $\mathscr{F}^{pr} = \Phi(\mathscr{F}_\mathbf{w})$ and $\mathscr{F}^o$ and $\mathscr{F}^{ol}$, induced by $\mathfrak{v}_{X,\mathbf{w}}$ and $\mathfrak{v}^l_{X,\mathbf{w}}$ respectively. Then, by definitions, $\mathscr{F}^{pr}(l) = \mathscr{F}^o(l) \subset \mathscr{F}^{ol}(l)$ for any l (see also 8.1.22). In particular, if $\mathfrak{v}(f) = d$, then the Poincaré series $P^{pr} = P^o$ of $\mathscr{F}^{pr} = \mathscr{F}^o$ is $P^o(t) = (1 - t^d)/\prod_i (1 - t^{w_i})$, cf. (8.1.16).

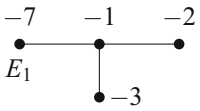
Note that the order functions $\mathfrak{v}_{X,\mathbf{w}}$ and $\mathfrak{v}^l_{X,\mathbf{w}}$ usually are not valuations.

However, in the next discussion we are interested in the existence of a filtration $\{\mathscr{F}(l)\}_l$ determined by a valuation $\mathfrak{v} : \mathscr{O}_{X,o} \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ centered at the maximal ideal with the property $\mathfrak{v}(z_i) = w_i$. Such a $\mathfrak{v}$ may appear as follows. Consider ϕ, the $\mathbf{w}$-weighted blow up of (X, o). Let E be that part of the the exceptional curve, which is not contained in the pull-back of the zero locus of the coordinates. Assume that $E \neq \emptyset$. Usually E is not irreducible. Let f_d be the degree d $\mathbf{w}$-homogeneous leading term of f, and write $f_d = \mathbf{x}^\alpha g$, where g is a polynomial not divisible by the coordinate functions. Then E is irreducible if and only if g is so. In that case we consider the divisorial valuation $\mathfrak{v}_E$ associated with E, and the corresponding divisorial filtration $\mathscr{F}_E$. Using local coordinates of the weighted blow up one verifies (using the irredicibility of g) that $\mathfrak{v}^l_{X,\mathbf{w}} = \mathfrak{v}_E$ and $\mathscr{F}^{ol} = \mathscr{F}_E$ (see also the next paragraph). Moreover, if f_d itself is irreducible then $\mathfrak{v}_{X,\mathbf{w}} = \mathfrak{v}_E$ and $\mathscr{F}^o = \mathscr{F}_E$.

Let Δ be a compact facet of the Newton diagram of f and let $\mathbf{w}$ be its (positive) primitive normal vector. Then in the above discussion the irreducibility of g is guaranteed by the Newton non-degeneracy of f along Δ. In this case the identity $\mathfrak{v}^l_{X,\mathbf{w}} = \mathfrak{v}_E$ is proved in [173].

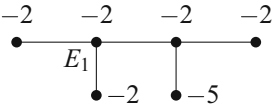
Example 8.6.15 Let us exemplify the previous subsection. Our goal is to determine the Poincaré series of the divisorial filtration $P_{\mathscr{I}}$ for certain $\mathscr{I}$ of cardinality one.

I. First assume that $f = z_1^2 + z_2^3 + z_3^7 + az_2z_3^5$, hence (X, o) is minimally elliptic, and also Newton non-degenerate. The minimal good resolution graphs is (cf. 5.5.23):



Set $\mathscr{I} = \{E_1\}$ (see the graph). Oka's algorithm 5.5.23 associates with the vertex of E_1 the vector $\mathbf{w} = (3, 2, 1)$ (hence $\mathbf{w}$–blow up creates E_1). Since $f_6 = z_1^2 + z_2^3$ is irreducible, $P_{\mathscr{I}}$ (of the E_1-divisorial filtration) is given by the filtrations $\mathscr{F}^o = \mathscr{F}^{pr}$ as well. Hence $P_{\mathscr{I}}(t) = (t^6-1)/((t^3-1)(t^2-1)(t-1))$. A direct computation gives that $Z_{\mathscr{I}}(t)$ is the same. (The identity $Z = P$ follows from 8.3.14 or 8.5.16 too.)

II. Next assume that $f = z_1^3 + z_2(z_3^2 + z_1^2z_2 + z_2^6)$. Its minimal resolution graph Γ is:

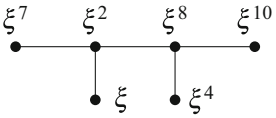


Γ can be obtained by Oka's algorithm 5.5.23. By this, there is a one-to-one correspondence between the faces of the Newton diagram of f and the nodes of Γ. By this correspondence, E_1 corresponds to the face Δ determined by the monomials of $z_2(z_3^2 + z_1^2z_2 + z_2^6)$. The positive primitive normal vector of this face is $\mathbf{w} = (5, 2, 6)$, whose entries agree with the vanishing orders of the three coordinates on E_1.

Our goal is to compare three series, namely $Z_{0,\mathscr{I}}(t)$, $P_{0,\mathscr{I}}(t)$ (for $h = 0$ and $\mathscr{I} = \{E_1\}$), and $P_{\mathscr{F}^o}(t)$ provided by the filtration $\mathscr{F}^o = \mathscr{F}^{pr}$ associated with $\mathbf{w}$.

First, notice that Γ is a minimally elliptic graph, and the resolution is minimal, hence $P(\mathbf{t}) = Z(\mathbf{t})$ by (8.3.14). In particular, $P_{0,\mathscr{I}} = Z_{0,\mathscr{I}}$ too.

$Z_{0,\mathscr{I}}$ can be read from the graph as follows. The group H is $\mathbb{Z}_{12}$, if ρ is a generator of $\widehat{H}$ then the possible values $\rho([E_i^*])$ are given as follows:



$(\xi^{12} = 1)$

Then $Z_{0,\mathscr{I}}$ is the Fourier–Dedekind sum

$$Z_{0,\mathscr{I}}(t^{12}) = \frac{1}{12}\sum_{\xi^{12}=1} \frac{(1-\xi^2 t^{52})(1-\xi^8 t^{40})}{(1-\xi^7 t^{26})(1-\xi t^{26})(1-\xi^{10} t^{20})(1-\xi^4 t^8)}.$$

By a computation

$$Z_{0,\mathscr{I}}(t) = \frac{(1-t^{15})(1-t^{12})}{(1-t^{13})(1-t^5)(1-t^2)(1-t^6)}.$$

Finally, notice that the series $P_{\mathscr{F}^o}$ provided by the face $\Delta:\ 5z_1+2z_2+6z_3=14$ is

$$P_{\mathscr{F}^o}(t) = \frac{1-t^{14}}{(1-t^5)(1-t^2)(1-t^6)}.$$

This also shows that the E_1-divisorial filtration $\mathscr{F}(l)$ (which agrees with $\mathscr{F}^{ol}(l)$ by the above discussion) is strictly larger than the filtration $\mathscr{F}^o(l)$ (for certain l's).

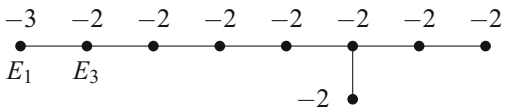
Let us analyze the case $l = 12$ and 13 more closely. In the case of $\mathscr{F}^o$ the weight 14 of the equation is higher, so $Gr_{\mathscr{F}^o}(l)$ is the same as the corresponding vector spaces computed from $\mathscr{O}_{\mathbb{C}^3,0}$: $\{z_3^2, z_3z_2^3, z_2^6, z_1^2z_2\}$ is a basis for $Gr_{\mathscr{F}^o}(12)$ and $\{z_1z_2z_3, z_1z_2^4\}$ for $Gr_{\mathscr{F}^o}(13)$. On the other hand, regarding $\mathscr{F}$, the expression $h := z_3^2 + z_1^2z_2 + z_2^6$ along E_1 equals z_1^3/z_2, whose vanishing order is 13. Hence h provides a relation in $Gr_{\mathscr{F}}(12) = Gr_{\mathscr{F}^o}(12)/(h=0)$, while h is a new element in $Gr_{\mathscr{F}}(13) = Gr_{\mathscr{F}^o}(13) \oplus \mathbb{C}\langle h\rangle$.

In particular, the 'naive' series $P_{\mathscr{F}^o} = P_{\mathscr{F}^{pr}}$ associated with a face Δ of a Newton diagram does not agree (in general) with the Poincaré series $P_{0,\mathscr{I}}$ provided by the valuation of the exceptional divisor corresponding to Δ (although they agree for weighted homogeneous singularities). Moreover, although both $P_{\mathscr{F}^o}$ and $Z_{0,\mathscr{I}}$ are topological/combinatorial, and both are very natural, and one might hope that they should coincide, in general, this is not the case. In general, $Z_{0,\mathscr{I}}$ contains more information about the Newton boundary of f (from the complement of Δ too).

Finally notice that the 'leading term' $z_2(z_3^2+z_1^2z_2+z_2^6)$ is not irreducible, causing the anomaly $\mathscr{F}^o \neq \mathscr{F}$, $\mathfrak{v}_{X,\mathbf{w}} \neq \mathfrak{v}^l_{X,\mathbf{w}}$.

Example 8.6.16 '$P \neq Z$'. We provide a general method to find singularities with $P \neq Z$, and by this discussion we wish to emphasize that the identity $P = Z$ can hold only for very special, rigid analytic structures (compare with 8.5.35).

Consider the following resolution graph Γ. It is realized e.g. by $\{z_1^2+z_2^3+z_3^{11} = 0\}$, but one might take any analytic structure supported by Γ. Notice that $H = 1$.



Let Γ^b be a non-minimal graph obtained from Γ by blowing up a point of $E_1 \setminus E_3$, and let E_2 be the new exceptional divisor. From topological point of view, all the possible choices of the center of this blow up are equivalent. But this is not the case from analytic point of view. Indeed, since the singularity is minimally elliptic with $Z_{min}^2 = -1$, in the resolution with graph Γ one has $Z_{max} = Z_{min}$ and $\phi^*(\mathfrak{m}_{X,o}) = \mathcal{O}(-Z_{min}) \cdot \mathfrak{m}_Q$ for a certain regular point Q of E_1, cf. 7.2.36.

Let ϕ^{gen} (respectively ϕ^Q) be the resolution obtained from ϕ by blowing up a generic point ($\neq Q$) of E_1 (respectively Q). In the first case $Z_{max} = Z_{min}$, while in the second case $Z_{max} = Z_{min} + E_2$. Notice that the multiplicity of E_2 in Z_{min} is 1.

Therefore, considering the reduced situation for the graph Γ^b and $\mathscr{I} = \{E_2\}$, we get the following: the coefficient of t in the series $P_{\phi^{gen},\mathscr{I}}(t)$ is *non-zero*, while in $P_{\phi^Q,\mathscr{I}}(t)$ is 0. Since the combinatorial computation in Γ^b gives $Z_{\mathscr{I}}(t) = (t^6 - 1)/((t^3 - 1)(t^2 - 1)^2)$, we get that $Z_{\phi^{gen},\mathscr{I}} \neq P_{\phi^{gen},\mathscr{I}}$.

In the previous discussion we focussed on the base points of $\mathcal{O}_{\widetilde{X}}(-Z_{max})$. However, there are many more subtle analytic pieces of information (e.g. base points of other linear systems) which may obstruct the identity $P = Z$.

Remark 8.6.17 Let us analyse the above examples from the point of view of splice quotient singularities. In the resolution ϕ any minimally elliptic structure of (X, o) is in fact a splice quotient, constructed from the graph Γ of ϕ. On the other hand, on the topological type of the resolution space $\widetilde{X}^{gen}$ of ϕ^{gen} (i.e., on Γ^b obtained from Γ by blow up E_1), there are two analytic structures (both minimally elliptic), which behave differently with respect to Γ^b. The first one, $\widetilde{X}^{gen}$, is the pullback of the splice quotient analytic structure constructed using the graph Γ (where the center of the blow up is a generic point). The other one is the splice quotient structure constructed using the graph Γ^b. The point is that the first structure, $\widetilde{X}^{gen}$, is not a splice quotient associated with its dual graph (e.g., it does not admit an 'end curve function' on E_2 by its very construction). The splice quotient analytic structure associated with Γ^b is the same with a pulled back splice quotient if and only if the center of the blow up is the base point (a very special, unique point).

Hence, by blowing up a splice quotient singularity at a *wrong point* we can get a non splice quotient singularity. The same is true for the property $P = Z$.

Example 8.6.18 Assume that (X, o) is a hypersurface singularity in $(\mathbb{C}^3, 0)$ with Newton non-degenerate principal part. Let us consider the filtrations $\mathscr{F}^{pr}$ and $\mathscr{F}^o$ associated with the facets of the diagram, or, equivalently, with the nodes of the (minimal good) resolution (given by Oka's algorithm). Then, if $N\Gamma$ is bi–stellar then $\mathscr{F}^{pr} = \mathscr{F}^o$, cf. [384], see also 8.1.24. Additionally, assume that (X, o) is a suspension Newton non-degenerate germs and let $\mathscr{F}^{\mathscr{N}}$ be the divisorial filtration reduced to the set of nodes. Then $\mathscr{F}^{\mathscr{N}} = \mathscr{F}^{pr} = \mathscr{F}^o$, cf. [643] (see also [173]).

In fact, Sigurðsson proves the following statement for any Newton non-degenerate hypersurface germ with rational homology sphere link: there exists a top-dimensional polyhedral cone $\mathscr{C}_{\mathbb{R}}$ in $\mathscr{S}_{\mathbb{R}}$ such that $\mathscr{F}^{\mathscr{N}}(l) = \mathscr{F}^{pr}(l) = \mathscr{F}^{o}(l)$ for any $l \in \mathscr{C}_{\mathbb{R}} \cap L$.

8.6.B *Example. P_0 and Z_0 for Superisolated Singularities*

We compute the one-variable $\{v_+\}$–reduced series P_0 and Z_0 for superisolated singularities associated with an irreducible curve C, and we formulate geometric properties and conjectures about their difference. Such properties might serve as combinatorial criteria for the existence of the rational cuspidal curve C with given topology. ♣

8.6.19 Assume that (X, o) is a superisolated singularity, where C is irreducible and the link is a rational homology sphere, cf. 5.2. Let ϕ be its minimal good resolution described in 5.2.4(a). For notations see 3.5.C and 5.2. We set $\mathscr{I} = \{v_+\}$ (the vertex corresponding to the curve) and $h = 0$.

Set $\Delta(t) := \prod_i \Delta_i$. Then $\Delta(1) = 1$ and $d\Delta/dt(1) = \delta$, where $\delta = \sum_i \delta_i = (\sum_i \mu_i)/2 = (d-1)(d-2)/2$ is the sum of delta-invariants. Hence, Δ can be written as $\Delta(t) = 1+(t-1)\delta+(t-1)^2Q(t)$ for an integral polynomial $Q(t) = \sum_{j=0}^{2\delta-2} \alpha_j t^j$ (see 5.2.2). For $v = 1$ one has $Q(t) = \sum_{s \notin \mathscr{S}_{C,p_1}} (1 + t + \cdots + t^{s-1})$ (cf. (5.2.3)), hence

$$\alpha_j = \#\{s \notin \mathscr{S}_{C,p_1} \,:\, s > j\} \quad (\text{if } v = 1). \tag{8.6.12}$$

Since $s \notin \mathscr{S}_{C,p_1}$ if and only if $2\delta - 1 - s \in \mathscr{S}_{C,p_1}$, we get

$$\alpha_{(d-3-j)d} = \#\{s \in \mathscr{S}_{C,p_1} \,:\, s \le jd\} \quad (\text{if } v = 1,\ 0 \le j \le d-3). \tag{8.6.13}$$

8.6.20 We wish to compute $P_{0,\mathscr{I}}(t)$ and $Z_{0,\mathscr{I}}(t)$ and to compare them.

By 8.6.14, one has $P_{0,\mathscr{I}}(t) = (1-t^d)/(1-t)^3$.

By the definition of $Z_{0,\mathscr{I}}$, and from A'Campo's formula 4.1.22 (and using the fact that $H = \mathbb{Z}_d$ is generated by $[E_+]$), we obtain

$$Z_{0,\mathscr{I}}(t) = \frac{1}{d} \sum_{\xi^d=1} \frac{\Delta(\xi t^{1/d})}{(1-\xi t^{1/d})^2}\,.$$

Lemma 8.6.21 ([198]) *The difference*

$$N(t) := Z_{0,\mathscr{I}}(t) - P_{0,\mathscr{I}}(t) = \frac{1}{d} \sum_{\xi^d=1} \frac{\Delta(\xi t^{1/d})}{(1-\xi t^{1/d})^2} - \frac{1-t^d}{(1-t)^3} \tag{8.6.14}$$

has the following properties:

(a) $N(0) = 0$, *and* $N(t)$ *is a symmetric polynomial:* $N(t) = t^{d-3} \cdot N(1/t)$.
(b)

$$N(t) = \sum_{j=0}^{d-3} \left(\alpha_{(d-3-j)d} - \frac{(j+1)(j+2)}{2} \right) t^{d-3-j}.$$

Proof *(a)* Use the symmetry of Δ_i's. *(b)* Use

$$\frac{1}{d} \sum_{\xi^d=1} \frac{1}{(1-\xi t^{1/d})^2} = \frac{1+(d-1)t}{(1-t)^2}, \quad \frac{1}{d} \sum_{\xi^d=1} \frac{1}{1-\xi t^{1/d}} = \frac{1}{1-t}.$$

□

Example 8.6.22 ([198]) The classification of combinatorial types of the cuspidal rational curves with degree $d \leq 5$ is given in [459], and of $d = 6$ in [193]; see 5.2.8 too.

If $d = 3$, then C has a unique singularity of type [2]. If $d = 4$, then there are four possibilities; the corresponding multiplicity sequences of the singular points are [3]; $[2_3]$; $[2_2]$, [2] and [2], [2], [2]. In all these cases $N(t) \equiv 0$. For $d = 5$ and $d = 6$ the next table shows all the possible multiplicity sequences together with $N(t)$.

d	ν	Type of cusp	$-N(t)$
5	1	[4]	0
5	1	$[2_6]$	0
5	2	[3, 2], $[2_2]$	0
5	2	[3], $[2_3]$	$2t$
5	2	$[2_2]$, $[2_4]$	$2t$
5	3	[3], $[2_2]$, [2]	$2t$
5	3	$[2_2]$, $[2_2]$, $[2_2]$	$6t$
5	4	$[2_3]$, [2], [2], [2]	$8t$

d	ν	Type of cusp	$-N(t)$
6	1	[5]	0
6	1	$[4, 2_4]$	0
6	1	$[3_3, 2]$	0
6	2	$[3_3]$, [2]	0
6	2	$[3_2, 2]$, [3]	0
6	2	$[3_2]$, [3, 2]	0
6	2	$[4, 2_3]$, [2]	0
6	2	$[4, 2_2]$, $[2_2]$	0
6	2	[4], $[2_4]$	$t + t^2$
6	3	[4], $[2_3]$, [2]	0
6	3	[4], $[2_2]$, $[2_2]$	0

For $d = 7$ there is no complete classification (known by the author). The next table shows some examples (the first column provides either a possible equation, or the reference where the corresponding curve has been constructed).

Reference	d	ν	type of cusp	$-N(t)$
x^6z+y^7	7	1	$[6]$	0
$x^5z^2+y^7$	7	2	$[5,2_2]$, $[2_3]$	0
$x^4z^3+y^7$	7	2	$[4,3]$, $[3_2]$	0
[193]	7	2	$[5]$, $[2_5]$	$2t+2t^2+2t^3$
[193]	7	2	$[4,2_3]$, $[3_2]$	$t+2t^2+t^3$
[193]	7	2	$[4,2_2]$, $[3_2,2]$	$2t+2t^2+2t^3$
[193]	7	2	$[4]$, $[3_3]$	$2t+2t^2+2t^3$
[203]	7	3	$[4,2_2]$, $[3_2]$, $[2]$	$3t+4t^2+3t^3$
[202]	7	3	$[5]$, $[2_4]$, $[2]$	$3t+3t^3$
[202]	7	3	$[5]$, $[2_3]$, $[2_2]$	$3t+3t^3$

Examples with arbitrarily large d and with non-vanishing $N(t)$ exist. E.g., if C has two cusps of types $[d-2]$, $[2_{d-2}]$ (for its realization see e.g. [193]), and d is even, then

$$-N(t)=\sum_{k=1}^{\frac{d}{2}-2} k(t^k+t^{d-3-k}).$$

The above examples supported the following (hasty) conjecture, formulated in [198].

Conjecture 8.6.23 ([198]) All the coefficients of $N(t)$ are non-positive for any rational cuspidal curve.

Assume that $\nu=1$. Then 8.6.21*(b)* combined with (8.6.13) says that the Semigroup Distribution Property 5.2.9(IV) guarantees the vanishing of $N(t)$. In particular, the conjecture follows from [68] with $N(t)\equiv 0$.

If $\nu=2$ then the Conjecture is true again, it follows from the Semigroup Distribution Property and certain lattice cohomology formulae of the link of superisolated singularities; the method even provides a conceptual meaning of the coefficients of $-N(t)$ in terms of ranks of certain first lattice cohomology groups. See 11.5.B for a detailed discussion.

However, the conjecture fails for certain curves with $\nu=3$.

Example 8.6.24 (Counterexample to Conjecture 8.6.23 [63]) Consider the rational three-cuspidal plane curve of degree 8 and with multiplicity sequences $[6]$, $[2_4]$ and $[2_2]$, respectively. (It is $C_{8,2}$ in the list 5.2.8(V).)

A computation shows that the coefficients $\alpha_{(d-3-j)d}-(j+1)(j+2)/2$ of $N(t)$ for $0\le j\le 5$ are $0,\ 1,\ -1,\ -1,\ 1,\ 0$ (for more details see 11.5.37). In fact, the smallest degree where Conjecture 8.6.23 fails among currently known rational cuspidal curves is exactly 8.

A 'weaker' version of Conjecture 8.6.23 was formulated in [63], it is a numerical inequality (instead of a polynomial one); in fact, it is more in the spirit of the

motivation of the original conjecture 8.6.23, since it is a reformulation of an inequality between the geometric genus of a superisolated singularity and the normalized Seiberg–Witten invariant of the link (see again 11.5.B for the complete discussion).

Conjecture 8.6.25 ([63]) $N(1) \leq 0$ for any rational cuspidal curve.

Note that by Lemma 8.6.21*(b)* one has:

$$N(1) = \sum_{j=0}^{d-3} \alpha_{(d-3-j)d} - \frac{d(d-1)(d-2)}{6} = -p_g + \sum_{j=0}^{d-3} \alpha_{(d-3-j)d}. \qquad (8.6.15)$$

Clearly, Conjecture 8.6.23 implies this second one, hence by the above discussion Conjecture 8.6.25 for $\nu \leq 2$ is also true. Moreover, in [63] a case-by-case verification provides its validity for all the curves from 5.2.8(V)-(VI) (which, conjecturally, provide all the possible combinatorial types with $\nu \geq 3$).

In fact, the computation from [63] shows that in all cases $N(1)$ is an even integer.

Recall that the link of a superisolated singularity is a surgery 3–manifold, cf. 5.2.5. The polynomial $N(t)$ can be considered for any surgery 3–manifold too, e.g. by the expression from 8.6.21*(b)*, see also the discussion from 11.5.B. The point is that, for surgery 3–manifolds which are not superisolated singularity links, it can happen that $N(1)$ is odd and positive. For example, $N(1) = 1$ for $d = 5$ and cusp types [3, 2], [2], [2] (see also 11.5.38). Hence, the facts that $N(1) \leq 0$, and $N(1)$ is even, are not topological statements regarding the link, they are consequences of the algebraic realizability of the cuspidal curve $C \subset \mathbb{CP}^2$ with given cusps (i.e., the realizability of the link as a superisolated singularity link). For more see 11.5.B.

8.6.C *The Ring* $\oplus_{n\geq 0}\, \phi_*(\mathscr{O}_{\widetilde{X}}(-nl))_o$

For a fixed resolution $\widetilde{X}$ and cycle $l \in \mathscr{S}$ one can ask whether the ring $\oplus_{n\geq 0}\, \phi_*(\mathscr{O}_{\widetilde{X}}(-nl))_o$ is finitely generated. Similarly, one can ask whether $P(\mathbf{t})$ (or some reduction of it) is a rational function (since $Z(\mathbf{t})$ is tautologically rational). Examples show that the answer to both questions is: sometimes yes, sometimes not. ♣

8.6.26 Fix a resolution $\phi : \widetilde{X} \to X$ of (X, o) as above. There is another way to reduce the multivariable Hilbert or Poincaré series to a one-variable series. Namely, for any fixed $l \in L$ and any $n \in \mathbb{Z}_{\geq 0}$ one can consider the following objects: $I_n := \phi_*(\mathscr{O}_{\widetilde{X}}(-nl))_o$, the filtration $F = \{I_n\}_{n\geq 0}$ of $\mathscr{O}_{X,o}$, the associated graded (Rees) ring $R = R(l) := \oplus_{n\geq 0} I_n$, (or, written differently, $R = \oplus_{n\geq 0} I_n t^n \subset \mathscr{O}_{X,o}[t]$), and the associated graded ring $\mathrm{gr}_F(\mathscr{O}_{X,o}) := \oplus_{n\geq 0} I_n/I_{n+1}$ of $\mathscr{O}_{X,o}$.

We will assume that $l \in \mathscr{S} \setminus \{0\}$.

Proposition 8.6.27 *Assume that (X, o) is rational and $l \in \mathscr{S}$. Then*

$$\phi_*(\mathscr{O}_{\widetilde{X}}(-l))_o^{\otimes n} \to \phi_*(\mathscr{O}_{\widetilde{X}}(-nl))_o$$

is onto. In particular R is finitely generated by elements of degree 1.

Proof By 7.1.11 $\mathscr{O}_{\widetilde{X}}(-l)$ is basepoint free, hence Theorem 6.5.8 applies. □

In general, both assumptions of Theorem 6.5.8 (the basepoint freeness and the h^1–vanishing) are obstructed. In some cases weaker assumptions imply a weaker statement (e.g. finite generation by elements of degree $\leq k$ for certain $k \geq 1$).

Take e.g. a minimally elliptic singularity, its minimal resolution, and $l = Z_{min}$. Then, usually $\mathscr{O}_{\widetilde{X}}(-Z_{min})$ is not basepoint free. However, when it is, then the vanishing $H^1(\mathscr{O}_{\widetilde{X}}(-Z_{min})) = 0$ holds (by 6.4.3), hence 6.5.8 applied for $\mathscr{L} = \mathscr{O}_{\widetilde{X}}(-Z_{min})$ and $\mathscr{F} = \mathscr{O}_{\widetilde{X}}(-2Z_{min})$ gives that R is generated in degrees ≤ 2. If $\mathscr{O}_{\widetilde{X}}(-Z_{min})$ is not basepoint free, then $\mathscr{O}_{\widetilde{X}}(-2Z_{min})$ is basepoint free (see [469, 5.51]), hence taking $\mathscr{L} = \mathscr{O}_{\widetilde{X}}(-2Z_{min})$ and $\mathscr{F} = \mathscr{O}_{\widetilde{X}}(-3Z_{min})$ we obtain that R is generated in degrees ≤ 4. Hence, R is finitely generated always.

However, if we take more general singularities, or non-minimal resolutions, or arbitrary cycles $l \in \mathscr{S}$, then the fact that R is finitely generated might be obstructed. In the next example $\mathrm{gr}_F(\mathscr{O})$ $(= R/\oplus_{n\geq 0} I_{n+1}t^n)$ is not finitely generated, hence R is not finitely generated either.

Example 8.6.28 Consider the minimal good elliptic graph from Example 7.2.99. It is unimodular, and the length of the elliptic sequence is 2. We fix an analytic structure such that E_1^* *is realized* by an analytic function, while E_2^* *is not*, cf. 7.2.99. Thus $p_g = 1$. We show that $\mathrm{gr}_F(\mathscr{O})$ **is not finitely generated** for $l := Z_{min}$.

Let C be the elliptic cycle, hence $E_2^* = l = C + E_2$, and $E_1^* = l + C$. Then

(a) $h^1(\mathscr{O}_{\widetilde{X}}(-nl - C)) = 0$ for $n \geq 1$;
(b) $H^0(\mathscr{O}_{\widetilde{X}}(-nl - C)) = H^0(\mathscr{O}_{\widetilde{X}}(-nl))$;
(c) $\mathscr{O}_C(-nl) \in \mathrm{Pic}^0(C)$ is a non-trivial bundle.

Part (a) follows from 7.2.34(b). (b) for $n = 1$ it follows from the facts that $l \notin \mathscr{S}_{an}$ and $l + C$ is the smallest element of $\mathscr{S}$ strict greater than l. (In fact, $l + C = Z_{max}$, cf. 7.2.99.) Next, consider

$$0 \to \mathscr{O}_{\widetilde{X}}(-nl - C) \to \mathscr{O}_{\widetilde{X}}(-nl) \to \mathscr{O}_C(-nl) \to 0. \tag{8.6.16}$$

For $n = 1$, using (a) we get that $H^0(\mathscr{O}_C(-l)) = 0$, hence (c) (for $n = 1$) follows from 7.2.77. Then (c) holds for an arbitrary n, since $\mathrm{Pic}^0(C)$ is torsion free. In particular, $H^0(\mathscr{O}_C(-nl)) = 0$ using 7.2.77 again. Finally, (b) for any n follows from this vanishing and (8.6.16).

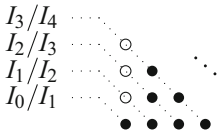
Since $\phi_*(\mathscr{O}(-l))_o = \mathfrak{m}_{X,o}$, we get $\dim I_0/I_1 = 1$.

From $0 \to \mathscr{O}_{\widetilde{X}}(-(n+1)l - C) \to \mathscr{O}_{\widetilde{X}}(-nl - C) \to \mathscr{O}_l(-nl - C) \to 0$ we get for $n \geq 1$ that $I_n/I_{n+1} = H^0(\mathscr{O}_l(-nl - C))$ of dimension $\chi(\mathscr{O}_l(-nl - C)) = n$.

Using the vanishing $H^*(\mathscr{O}_C(-(n+1)l)) = 0$ guaranteed by (c), in the cohomology exact sequence of $0 \to \mathscr{O}_C(-(n+1)l) \to \mathscr{O}_l(-nl-C) \to \mathscr{O}_{E_2}(-nl-C) \to 0$ we get $I_n/I_{n+1} = H^0(\mathscr{O}_{E_2}(-nl-C)) = H^0(\mathscr{O}_{\mathbb{P}^1}(n-1))$. But

$$H^0(\mathscr{O}_{E_2}(-nl-C)) \otimes H^0(\mathscr{O}_{E_2}(-ml-C)) \to H^0(\mathscr{O}_{E_2}(-(n+m)l-C))$$

cannot be onto, since it factorizes through $H^0(\mathscr{O}_{E_2}(-(n+m)l-2C))$, which has codimension 1. Here is a schematic picture of the multiplication in $R \simeq 1 \cdot \mathbb{C} \oplus x \cdot \mathbb{C}[x,y]$, as embedded as a subalgebra into $\oplus_{n\geq 0} H^0(\mathscr{O}_{E_2}(-nl)) \simeq \mathbb{C}[x,y]$, where I_n/I_{n+1} corresponds to the n-homogeneous elements:



For discussions and examples of similar nature see [129].

Example 8.6.29 There is a special interest in cycles of type $l = \mathfrak{o}_v E_v^* \in L$, where $\mathfrak{o}_v$ is the order of E_v^* in H (or a multiple of it). Set $p_v := m_v(\mathfrak{o}_v E_v^*)$, the E_v–multiplicity of l. We claim that $s(p_v E_v) = l$. [Indeed, by the universal property of the s-operator we have $s(p_v E_v) \leq l$. Furthermore, $(l - s(p_v E_v), E_u) \geq 0$ for any u, hence $l - s(p_v E_v) \leq 0$ too.] Similarly, $s(np_v E_v) = nl$ for $n \geq 1$. Hence, $I_n = \phi_*(\mathscr{O}_{\widetilde{X}}(-nl))_o = \phi_*(\mathscr{O}_{\widetilde{X}}(-np_v E_v))_o$. In particular,

$$\dim(I_0/I_n) = \mathfrak{h}_{0,\mathscr{I}}(np_v), \quad \text{where } \mathscr{I} = \{v\}.$$

Remark 8.6.30 By Lemma 6.6.25(b) and discussion from 6.6.26 we get that $\sum_{n\geq 0} \dim(I_n/I_{n+1})\, t^n$ is a rational function. This implies that $\sum_{n\geq 0} \dim(I_0/I_n)\, t^n$ is rational as well. However, the rationality of

$$\sum_{n_1,n_2,n_3\geq 0} \dim(\mathscr{O}_{X,o}/\phi_*(\mathscr{O}_{\widetilde{X}}(-n_1 l_1 - n_2 l_2 - n_3 l_3))_o)\, t_1^{n_1} t_2^{n_2} t_3^{n_3}$$

is not guaranteed in general [129]. Hence $P(\mathbf{t})$ is not rational in general.

8.7 The Periodic Constant of One-Variable Series

Assume that counting function $n \mapsto \sum_{i<n} a_i$ of a one variable series $S(t) = \sum_{i\geq 0} a_i t^i$ is a quasipolynomial for $n \gg 0$. Then, we associate — as a new

invariant — a number to $S(t)$, called the 'periodic constant of $S(t)$'. It measures the difference between the counting function and its quasipolynomial.

The main geometric application is provided by 'additivity formulae' (see 8.7.A). Consider a resolution $\widetilde{X} \to X$, a vertex $v \in \mathcal{V}$ (which satisfies a technical analytic assumption), a free $\mathcal{O}_{\widetilde{X}}$-module $\mathscr{M}$, and the resolutions $\{\widetilde{X}_j\}_j$ associated with the connected components of the subgraph $\Gamma \setminus \{v\}$. Then, it turns out that the 'additivity correction term' $h^1(\widetilde{X}, \mathscr{M}) - \sum_j h^1(\widetilde{X}_j, \mathscr{M}|_{\widetilde{X}_j})$ is given by the periodic constant of a series associated with these data.

In the case of $\mathscr{M} = \mathcal{O}_{\widetilde{X}}(-r_h)$ the needed series is $P_{h,\{v\}}(t)$.

Such 'additivity formulae' serve as inductive steps in the determination of invariants of type $h^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-l'))$ for splice quotient singularities (formulae initiated by Okuma), or for singularities with AR graphs.

The statements will serve as models for topological surgery formulae, where $P(\mathbf{t})$ will be replaced by $Z(\mathbf{t})$ and $\{h^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-r_h))\}_{h\in H}$ by the Seiberg–Witten invariant of the (rational homology sphere) link (cf. the next chapters). ♣

Definition 8.7.1 ([493, 3.9], [533]) Let $F(t) = \sum_{i\geq 0} a_i t^i$ be a formal power series. Suppose that there exist a positive integer p and a polynomial $\mathfrak{P}_p(t)$ such that $\sum_{0\leq i<pn} a_i = \mathfrak{P}_p(n)$ for every $n \in \mathbb{Z}_{>0}$. We call the constant term $\mathfrak{P}_p(0)$ the *periodic constant* of F and we denote it by $\mathrm{pc}(F)$. We refer to the integer p as the 'period'.

Furthermore, we extend the above definition to expressions of type $t^r \cdot F(t)$ via $\mathrm{pc}(t^r F(t)) := \mathrm{pc}(F(t))$, where F is a power series as above and $r \in \mathbb{Q} \cap [0, 1)$.

If the periodic constant exists then it is independent of the choice of the period p. (In the presence of two periods, compare them via their least common multiple.)

If the period p and the polynomial $\mathfrak{P}_p$ satisfy a weaker condition, namely $\sum_{0\leq i<pn} a_i = \mathfrak{P}_p(n)$ for every $n \geq n_0$, then $\sum_{0\leq i<pn_0n} a_i = \mathfrak{P}_p(n_0n)$ for every $n \geq 1$, hence $\mathrm{pc}(F(t))$ still exists (take $n \mapsto \mathfrak{P}_p(n_0n)$ with period pn_0) and it equals $\mathfrak{P}_p(0)$.

Furthermore, if F_1 and F_2 admit periodic constants, then the same is true for the series $F_1 + F_2$, cF_1 (where $c \in \mathbb{C}$), $F_1(t^m)$ (where $m \in \mathbb{Z}_{>0}$). Moreover, $\mathrm{pc}(F_1 + F_2) = \mathrm{pc}(F_1) + \mathrm{pc}(F_2)$, $\mathrm{pc}(cF_1) = c \cdot \mathrm{pc}(F)$, $\mathrm{pc}(F_1(t^m)) = \mathrm{pc}(F_1(t))$.

However, even if both $F(t)$ and $t^k F(t)$ admit periodic constants (for certain $k \in \mathbb{Z}_{>0}$), usually $\mathrm{pc}(F(t)) \neq \mathrm{pc}(t^k F(t))$.

If $F(t)$ is a finite sum (i.e. it is a polynomial), then $\mathrm{pc}(F)$ exists and equals $F(1)$.

For certain rational functions, one has the following equivalent description. (Here, we identify a rational function R with its Taylor expansion at the origin.) Clearly, any rational function can be written in a unique way as $R = R^+ + R^-$, where R^+ is a polynomial and R^- is a rational function of negative degree.

Lemma 8.7.2 ([77]) *Let R be a rational function having poles only at infinity or at certain roots of unity. Then R admits a periodic constant and* $\mathrm{pc}(R) = R^+(1)$.

Proof Write R as the finite sum

$$R(t) = R^+(t) + \sum_{\substack{k\geq 0\\ 0\leq j<p}} a_{kj}\frac{t^j}{(1-t^p)^{k+1}} \quad (a_{kj}\in\mathbb{C}).$$

But $t^j(1-t^p)^{-(k+1)} = \sum_{m\geq 0}\binom{k+m}{k}t^{mp+j}$ admits a periodic constant, which is 0. Indeed, the constant term of $n\mapsto \sum_{i<np} a_i = \sum_{m=0}^{n-1}\binom{k+m}{k} = \binom{k+n}{k+1}$ is 0. □

Note that if $R = R^+ + R^-$ is as above and $r\in[0,1)$, then t^rR^- still has a negative (rational) degree, and $t^rR^+(t)$ is a finite sum. Moreover, $\mathrm{pc}(t^rR(t)) = (t^rR^+)(1)$.

Before we list several examples related with surface singularities, let us emphasize the main point in the definition by comparing the coefficients of a Hilbert *function* with those provided by their Hilbert *polynomial*.

Example 8.7.3 Let X be a smooth projective variety of dimension d and $\mathscr{L}$ an ample line bundle on it. We wish to determine the periodic constant of $F(t) := \sum_{i\geq 0} h^0(X,\mathscr{L}^i)t^i$. Since $\mathscr{L}$ is ample, for $i\gg 0$ one has $h^0(X,\mathscr{L}^i) = \chi(\mathscr{L}^i)$. By Hirzebruch–Riemann–Roch Theorem (see e.g. [258, App. A]), $\chi(\mathscr{L}^i)$ is the d-coefficient of $\mathrm{Td}_X\cdot e^{ic_1(\mathscr{L})}$, where $\mathrm{Td}_X = \mathrm{Td}_d+\cdots+\mathrm{Td}_0$ is the Todd class of the tangent sheaf of X. This is $\sum_{k=0}^d i^k(c_1(\mathscr{L})^k\cdot \mathrm{Td}_{d-k},[X])/(k!)$.

Hence, if we replace the original series $F(t)$ by the 'regularized' series $F^{reg}(t) := \sum_{i\geq 0}\chi(\mathscr{L}^i)t^i$, we get that F^{reg} is a linear combination of series of type $\sum_{i\geq 0} i^kt^i$. All of them admit periodic constant, and they are 0. On the other hand, $F-F^{reg}$ is a polynomial. Hence, F itself admits a periodic constant, and $\mathrm{pc}(F) = \mathrm{pc}(F-F^{reg}) = (F-F^{reg})(1)$. This shows that the periodic constant is not the output of the 'Hilbert polynomial' $i\mapsto\chi(\mathscr{L}^i)$, but, on the contrary, it measures the difference between the series F and its regularized series F^{reg} provided by the Hilbert polynomial.

This is the general principle followed in the local case as well.

Example 8.7.4 Consider a subset $\mathscr{S}\subset\mathbb{Z}_{\geq 0}$ with finite complement. Then $S(t) = \sum_{s\in\mathscr{S}}t^s$ rewritten is $1/(1-t)-\sum_{s\notin\mathscr{S}}t^s$, hence $\mathrm{pc}(S) = -|\mathbb{Z}_{\geq 0}\setminus\mathscr{S}|$. In particular, if $\mathscr{S}$ is the semigroup of a local irreducible complex plane curve singularity, then $-\mathrm{pc}(S)$ is the delta-invariant of the corresponding germ.

In the next discussions we fix a resolution of a normal surface singularity (X,o).

Example 8.7.5 For any ideal $I\subset\mathscr{O}_{X,o}$ consider the series $S(t) = \sum_{n\geq 0}\dim(I^n/I^{n+1})t^n$. Then the Hilbert–Samuel function $n\mapsto\dim(I^n/I^{n+1})$ admits a Hilbert–Samuel polynomial of type $n\mapsto \mathrm{mult}(I)n+\bar{e}_1$. Therefore, similarly as in the above global case 8.7.3, $\mathrm{pc}(S)$ exists and equals $\sum_{n\geq 0}\big(\dim(I^n/I^{n+1}) - \mathrm{mult}(I)n-\bar{e}_1\big)$. Note that $\mathrm{pc}(S)$ cannot be read only from $n\mapsto\mathrm{mult}(I)n+\bar{e}_1$.

However, $\mathrm{pc}(S) = e_2(I)$, where $\mathrm{mult}(I)n^2/2 + e_1(I)n + e_2(I)$ is the Hilbert–Samuel polynomial of $n \mapsto \dim(\mathscr{O}_{X,o}/I^n)$.

As the presence of the Todd class in the previous global case suggests, in the local case too the term $e_2(I)$ must be related to the geometric genus of (X, o).

In fact, cf. [181], $e_2(I) = p_g$ for any Hironaka ideal $I \subset \mathscr{O}_{X,o}$, where I is Hironaka if blowing up (X, o) along I one gets a resolution. For example, if (X, o) is a hypersurface given by a homogeneous equation of degree d, and $I = \mathfrak{m}_{X,o}$, then $S(t) = (1 - t^d)/(1 - t)^3$, and its periodic constant is $e_2(I) = p_g = d(d - 1)(d - 2)/6$. Furthermore, $0 \le e_2(I) \le p_g$ for any I, provided that $\mathrm{Gr}_{I^n}(\mathscr{O}_{X,o})$ is Cohen–Macaulay for $n \gg 0$. (Here $\mathrm{Gr}_{I^n}(\mathscr{O}_{X,o})$ is the associated graded ring of $\mathscr{O}_{X,o}$ with respect to I^n.)

On the other hand, the strict inequality $e_2(I) < p_g$ might occur frequently; take e.g. $(X, o) = (\{x^2 + y^3 + z^7 = 0\}, 0) \subset (\mathbb{C}^3, 0)$ with $p_g = 1$ and set $I = \mathfrak{m}_{X,o}$. Then $S(t) = (1-t^2)/(1-t)^3$, hence, being a rational function with negative degree, $e_2(I) = \mathrm{pc}(S) = 0$.

Example 8.7.6 Fix a resolution $\widetilde{X} \to X$ and $l \in L$ such that $(l, E_v) < 0$ for all v. Set $S = \sum_{n\ge 0} \dim(I_n/I_{n+1})t^n$, where $I_n := \phi_*(\mathscr{O}_{\widetilde{X}}(-nl))_o$. Then, for $n \gg 0$, $h^1(\mathscr{O}(-nl)) = 0$, hence $\dim(\mathscr{O}_{X,o}/I_n) = \chi(nl) + p_g$. In particular, $\mathrm{pc}(S)$ exists and equals p_g.

Remark 8.7.7 Fix a resolution graph and a vertex $v \in \mathscr{V}$. Consider the reduced one-variable series $Z_{h,\mathscr{I}}(t)$ associated with $\mathscr{I} := \{v\} \subset \mathscr{V}$ and any $h \in H$. Then $Z_{h,\mathscr{I}}$ satisfies the requirements of Lemma 8.7.2. The denominator of Z is $\prod_{e\in\mathscr{V}_e}(1-\mathbf{t}^{E_e^*})$. This can be replaced by $Q = \prod_{e\in\mathscr{V}_e}(1 - \mathbf{t}^{\mathfrak{o}_e E_e^*}) \in \mathbb{Z}[[L_{\ge 0}]]$, where $\mathfrak{o}_e$ is the order of E_e^* in H. Hence Z_h has the form $\bar{P}_h/Q$, where $\bar{P}_h$ might have rational exponents. In particular, $Z_{h,\mathscr{I}}$ can be written as $t^r P_{h,\mathscr{I}}(t)/Q_{\mathscr{I}}(t)$, where $P_{h,\mathscr{I}}$ and $Q_{\mathscr{I}}$ are polynomials, $Q_{\mathscr{I}}$ is a product of factors of type $(1 - t^M)$, and $r \in \mathbb{Q} \cap [0, 1)$. (Here r equals the E_v-entry of r_h, namely $-(r_h, E_v^*)$.)

Example 8.7.8 Recall that for **cyclic quotients** (with $s > 1$) $Z(\mathbf{t}) = (1 - \mathbf{t}^{E_1^*})^{-1}(1 - \mathbf{t}^{E_s^*})^{-1}$, which equals also $P(\mathbf{t})$, cf. 8.2.17. We fix $\mathscr{I} = \{v_1\}$ and $h = e^{2\pi i a/n}$ $(0 < a < n)$. Then $Z_{h,\mathscr{I}}$ equals $P(\mathbb{C}[z_1, z_2]^{\chi_0^a}; t_1)$ of (8.1.3), or $t^{a/n} \cdot \sum_{m\ge 0} H^0(\widetilde{X}, \mathscr{O}(-r_h))_m t^m$, where the grading is given by the vanishing order along E_1, cf. 6.6.19. By any of these results one has $Z_{h,\mathscr{I}}(t) = t^{a/n} \cdot \sum_{m\ge 0}(1 + \lfloor (a + nm)/q \rfloor)t^m$.

For the period it is convenient to take q, and one can check that $\mathrm{pc}(Z_{h,\mathscr{I}}) = 0$.

Example 8.7.9 Fix a **weighted homogeneous germ** with $g = 0$ and the representative r_h. Take $\mathscr{I}$ consisting of the central vertex E_0. Then, with the notations of 8.1.B (where $\mathfrak{a} \in [0, 1)$ stays for $-(r_h, E_0^*)$)

$$P_{h,\mathscr{I}}(t) = Z_{h,\mathscr{I}}(t) = P_h(R(X_a), t) = \sum_{\ell\ge 0} \max\{0, 1 + N_{r_h}(\ell)\}t^{\ell+\mathfrak{a}}.$$

Here the coefficient of $t^{\ell+\mathfrak{a}}$ coincides also with $h^0(\widetilde{X}, \mathscr{O}(-r_h))_\ell$, cf. (6.6.27). Then, by 8.1.9, $P_h^+(R(X_a), t) = \sum_{\ell\geq 0} \max\{0, -1 - N_{r_h}(\ell)\}t^{\ell+\mathfrak{a}}$. Thus, by (6.8.7),

$$\mathrm{pc}(P_{h,\mathscr{I}}(t)) = \mathrm{pc}(Z_{h,\mathscr{I}}(t)) = \sum_{\ell\geq 0} \max\{0, -1 - N_{r_h}(\ell)\} = h^1(\mathscr{O}_{\widetilde{X}}(-r_h)).$$

The same statement follows from the expression (8.1.7) too.

Example 8.7.10 Consider the Brieskorn singularity $(X, o) = \{x^a + y^b + z^c = 0\}$ when a, b, c are pairwise relative prime. Then the Poincaré series of its local graded ring is $P(t) = (1-t^{abc})/((1-t^{ab})(1-t^{bc})(1-t^{ca}))$, cf. 8.1.2. By 8.7.9 its periodic constant is p_g. Hence, by 5.5.4, $\mathrm{pc}(P(t))$ is the number of lattice points in the open (a, b, c)–tetrahedron, namely $\#\{(i, j, k) \in \mathbb{Z}^3_{>0},\ i/a + j/b + k/c < 1\}$.

More generally, one has the following fact. Consider positive integers $w_1, \ldots, w_{n+1}$ $(n \geq 0)$ and d, and the rational function $P(t) := (t^d - 1)/(\prod_i (t^{w_i} - 1))$. Then $P^+(t) = \sum t^{d-\sum_i k_i w_i}$, where the sum is over the lattice points of the polyhedron $T := \{\mathbf{k} = (k_1, \ldots, k_{n+1}) \in (\mathbb{Z}_{>0})^{n+1},\ \sum_i k_i w_i \leq d\}$. (The elementary proof runs as follows: write $d = w_1 p + r$ with $0 \leq r < w_1$ and observe that modulo a negative degree rational function, $P(t)$ equals $(t^d - t^r)/\prod_i (t^{w_i} - 1) = \sum_{k=0}^{p-1} t^{r+kw_i}/\prod_{i>1}(t^{w_i} - 1)$; then use induction.) In particular, $P^+(1)$ is the number of lattice points in T.

If $P(t)$ is the Poincaré series of a weighted homogeneous isolated hypersurface singularity f (see 5.1.15) then $P_{M(f)_{(0,1]}}(t) = \sum_{\mathbf{k}\in T} t^{(\sum_i k_i w_i)/d}$. Hence there exists an identification of $P^+(t)$ with $\sum t^{s_i}$, where $\{s_i\}_i$ are the Hodge spectral numbers in $(0, 1]$ (cf. 5.1.15(g)).

8.7.11 More generally, assume that the **graph is almost rational** (AR), cf. 7.3.21. Assume that $\mathscr{I} = \{v\}$ is an SR–set. For any fixed $h \in H$ consider the series with reduced variables, cf. 8.6.6,

$$P_{h,\mathscr{I}}(t) = \sum \mathfrak{p}_{h,\mathscr{I}}((l + r_h)_{\mathscr{I}}) \cdot t^{(l+r_h)_{\mathscr{I}}}.$$

Since it is supported on $\pi_{\mathscr{I}}(\mathscr{S}')$, in the sum we can assume that $l_{\mathscr{I}} \in \mathbb{Z}_{\geq 0}$.

We wish to compute the periodic constant $\mathrm{pc}(P_{h,\mathscr{I}}(t))$.

Let $\mathfrak{o}_v > 0$ be the order of E_v^*, hence $l := \mathfrak{o}_v E_v^* \in L$. Write also $p := (\mathfrak{o}_v E_v^*)_{\mathscr{I}}$.

Since $nl + s_h \in \mathscr{S}'$, by the universality property one has $x_{s_h}(np) + s_h \leq nl + s_h$, and $nl - x_{s_h}(np)$ is supported on $\mathscr{V}^*$, hence $(x_{s_h}(np) + s_h, E_v) \leq (nl + s_h, E_v) = (s_h, E_v) - n\mathfrak{o}_v$. In particular, $(x_{s_h}(np) + s_h, E_v)$ tends to $-\infty$ as $n \to \infty$.

Since by 7.3.25 $x_{r_h}(np) + r_h = x_{s_h}(np - pr_{\mathscr{I}}(\Delta)) + s_h$, the same is true for $(x_{r_h}(np) + r_h, E_v)$. Therefore, Proposition 8.6.11 can be applied for nl with $n \gg 0$.

Furthermore, by 7.3.25*(b)*, $x_{r_h}(np) = x_{r_h}(0)+nl$. Note also that $\pi_{\mathscr{I}}(r_h) \in [0, 1)$. Therefore, 8.6.11 for $n \gg 0$ reads as follows

$$\sum_{\tilde{l}_{\mathscr{I}} < pn} \mathfrak{p}_{h,\mathscr{I}}((\tilde{l} + r_h)_{\mathscr{I}}) = \chi(x_{r_h}(np) + r_h) - \chi(r_h) + h^1(\mathscr{O}_{\widetilde{X}}(-r_h))$$
$$= \chi(x_{r_h}(0) + nl + r_h) - \chi(r_h) + h^1(\mathscr{O}_{\widetilde{X}}(-r_h)).$$

Taking $n = 0$ in the right hand side, we get

Theorem 8.7.12 *If $\mathscr{I} = \{v\}$ is an SR–set (hence Γ is AR) then*

$$\mathrm{pc}(P_{h,\mathscr{I}}(t)) = \chi(x_{r_h}(0) + r_h) - \chi(r_h) + h^1(\mathscr{O}_{\widetilde{X}}(-r_h)).$$

Note that the assumption is topological, while the statement is analytical.

Example 8.7.13 Assume the notations of Theorem 8.7.12.

(a) If $h = 0$ then $\mathrm{pc}(P_{0,\mathscr{I}}(t)) = p_g$. (Use 7.3.33.)
(b) By Lemma 7.3.26 $\chi(x_{r_h}(0)+r_h) \le \chi(r_h)$, hence $\mathrm{pc}(P_{h,\mathscr{I}}(t)) \le h^1(\mathscr{O}_{\widetilde{X}}(-r_h))$.
(c) For an arbitrary h, in some cases it might happen that $\chi(x_{r_h}(0) + r_h) = \chi(r_h)$, e.g. in the case of a star-shaped graph (cf. 8.7.9, or 8.7.28 for a direct argument). However, this is not always the case. Consider e.g. the following graph (left picture).

It is not rational, but it is AR with $\{v\}$ as an SR–set. Choose r_h as indicated on the right picture. Then $\chi(x_{r_h}(0) + r_h) = \chi(r_h) - 2$.

8.7.A Okuma's Additivity Formula

We discuss several versions of the h^1–additivity formula, with a focus on the correction term realized as a periodic constant. ♣

8.7.14 The Setup Consider a normal surface singularity (X, o) and fix one of its resolutions $\phi : \widetilde{X} \to X$ with exceptional curve E and graph Γ. We fix a vertex $v \in \mathscr{V}$. Let $\cup_{j\in J}\Gamma_j$ be the connected components of the graph obtained from Γ by deleting v and its adjacent edges. Assume that v is connected to each Γ_j by exactly one edge. Let X' be the space obtained from $\widetilde{X}$ by contracting (via τ) all irreducible exceptional curves to normal points except E_v. It has $|J|$ normal singular points

$\{o_j\}_j$, which are the images of the connected components of $E \setminus E_v$. Let X_j be a small Stein neighbourhood of o_j in X', and $\widetilde{X}_j = \tau^{-1}(X_j)$ its pre-image via the contraction $\tau : \widetilde{X} \to X'$. We denote the local singularities by (X_j, o_j). They are resolved by $\widetilde{X}_j$ with dual graphs Γ_j. Set $\tau(E) = E' \subset X'$. The resolution $\phi : \widetilde{X} \to X$ and the contraction $\tau : \widetilde{X} \to X'$ induce an analytic modification $\phi' : X' \to X$ with (irreducible) exceptional curve E'.

We say that the Assumption (C) is satisfied if

$$(C) \qquad nE' \subset X' \text{ is a Cartier divisor for a certain } n > 0.$$

If (C) is satisfied for some n, then it is satisfied by any positive multiple of it.

Next, fix an integer $\mathfrak{o}_v > 0$ such that $l := \mathfrak{o}_v E_v^* \in L$. For any $n \in \mathbb{Z}_{\geq 0}$ write $\mathscr{L}_n := \mathscr{O}_{\widetilde{X}}(-nl)$ and $I_n := (\phi_* \mathscr{L}_n)_o$. Define G as the graded ring $\oplus_{n \geq 0} I_n / I_{n+1}$ and $P_G(t)$ as its Poincaré series $\sum_{n \geq 0} \dim(I_n / I_{n+1}) t^n$.

Theorem 8.7.15 ([533] Additivity for $h^1(\mathscr{O}_{\widetilde{X}})$. First Version) *If Assumption (C) is satisfied then $P_G(t)$ admits a periodic constant and*

$$p_g(X, o) = \mathrm{pc}(P_G(t)) + \sum_j p_g(X_j, o_j).$$

This theorem is the prototype of a series of results, analytic and topological ones. In 8.7.22 we will prove its more general version, when we will replace the structure sheaf $\mathscr{O}_{\widetilde{X}}$ by an arbitrary locally free $\mathscr{O}_{\widetilde{X}}$-module. For other generalizations or topological analogues see e.g. 9.2.6, 9.6.5, 10.5.5.

Remark 8.7.16 Assumption (C) in Theorem 8.7.15 cannot be (entirely) omitted. Take e.g. the example from 7.2.99 and 8.6.28 (see also Remark 7.2.101). Here we contract the elliptic cycle (hence $|J| = 1$). Then $p_g(X, o) = p_g(X_1, o_1) = 1$. On the other hand, by 8.6.28, $P_G(t) = 1 + t/(1-t)^2$ with $\mathrm{pc}(P_G) = 1$. Hence the above additivity does not hold. In particular, Assumtion (C) is not satisfied in this case.

Remark 8.7.17 The validity of Assumption (C) can be tested as follows. Let $E(\Gamma_j)$ be the union of the irreducible exceptional divisors corresponding to the subgraph Γ_j. If for every $j \in J$ there exists a holomorphic function $f_j \in \mathscr{O}_{X,o}$ such that its strict transform in $\widetilde{X}$ does not intersect $E(\Gamma_j)$, then Assumption (C) is satisfied. Indeed, f_j provides a local equation of $m_v(f_j)E'$ at o_j.

For example, if nE_v^* is realized for a certain $n > 0$ as the divisor of a function restricted to E, then (C) is satisfied. This happens whenever $\mathscr{S} = \mathscr{S}_{an}$ (e.g. for rational singularities, or for minimal resolutions of minimally elliptic singularities).

Similarly, if $H^1(\widetilde{X}, \mathbb{Z}) = 0$ and $\widetilde{X}$ satisfies the 'end curve condition' (cf. 8.5), then again (C) holds. This happens e.g. for weighted homogeneous singularities.

8.7.18 In some cases the series $P_G(t)$ is easily computable, see e.g. Example 8.7.23. In general, it can be related with the reduced Hilbert and Poicaré series of the divisorial filtration associated with the subset $\mathscr{I} = \{v\}$.

Theorem 8.7.19 (Additivity for $h^1(\mathscr{O}_{\widetilde{X}})$. Second Version) *If Assumption (C) is satisfied then $P_{0,\mathscr{I}}(t)$ admits a periodic constant and*

$$p_g(X, o) = \mathrm{pc}(P_{0,\mathscr{I}}(t)) + \sum_j p_g(X_j, o_j).$$

Proof From 8.6.29

$$\sum_{i<np_v} \mathfrak{p}_{0,\mathscr{I}}(i) = \mathfrak{h}_{0,\mathscr{I}}(np_v) = \dim(I_0/I_n) \quad (\text{where } p_v := m_v(l)).$$

Since $P_G(t)$ admits a periodic constant (cf. first version or Theorem 8.7.22), the same is true for $P_{0,\mathscr{I}}(t)$, and $\mathrm{pc}(P_G(t)) = \mathrm{pc}(P_{0,\mathscr{I}}(t))$. □

Example 8.7.20

(a) Assume that Γ is rational. Then 8.7.15 gives $\mathrm{pc}(P_{0,\{v\}}(t)) = 0$ for any vertex v. Since $P = Z$ (cf. 8.5.A) one also has $\mathrm{pc}(Z_{0,\{v\}}(t)) = 0$.
(b) Similarly, if Γ is a minimal graph of a minimally elliptic singularity then for any v one obtains $\mathrm{pc}(P_{0,\{v\}}(t)) = \mathrm{pc}(Z_{0,\{v\}}(t)) = 1$.
(c) If Γ is a star-shaped graph and v is the central vertex then $\mathrm{pc}(P_{0,\{v\}}(t)) = \mathrm{pc}(Z_{0,\{v\}}(t)) = p_g$; for the compatibility of this formula with older ones see 8.7.9.

8.7.21 Consider the situation from 8.7.14. Let $\mathscr{M}$ be a locally free $\mathscr{O}_{\widetilde{X}}$-module. Let $\mathscr{M}_n := \mathscr{M} \otimes \mathscr{L}_n = \mathscr{M} \otimes \mathscr{O}_{\widetilde{X}}(-nl)$ and $J_n = (\phi_* \mathscr{M}_n)_o$. Consider the graded G-module $\oplus_{n\geq 0} J_n/J_{n+1}$ and its Poincaré series $P_{\mathscr{M}}(t) = \sum_{n\geq 0} \dim(J_n/J_{n+1}) \cdot t^n$.

Theorem 8.7.22 (Additivity for $\mathscr{M}$) *Assume that (C) is satisfied. Then $P_{\mathscr{M}}(t)$ admits a periodic constant and*

$$h^1(\widetilde{X}, \mathscr{M}) = \mathrm{pc}(P_{\mathscr{M}}(t)) + \sum_j h^1(\widetilde{X}_j, \mathscr{M}|_{\widetilde{X}_j}).$$

Proof From the exact sequence $0 \to \mathscr{M}_n \to \mathscr{M} \to \mathscr{M} \otimes \mathscr{O}_{nl} \to 0$ we obtain

$$\dim(J_0/J_n) = \chi(\mathscr{M}|_{nl}) - h^1(\widetilde{X}, \mathscr{M}_n) + h^1(\widetilde{X}, \mathscr{M}).$$

By Leray's spectral sequence associated with τ we have

$$0 \to H^1(X', \tau_* \mathscr{M}_n) \to H^1(\widetilde{X}, \mathscr{M}_n) \to H^0(X', R^1\tau_* \mathscr{M}_n) \to 0.$$

Since $\mathscr{O}_{X'}(-nm_v(l)E')$ is invertible by (C), and $\tau^*(-nm_v(l)E') = -nl$, we get

$$R^k\tau_*\mathscr{M}_n = R^k\tau_*(\mathscr{M} \otimes \tau^*\mathscr{O}_{X'}(-nm_v(l)E')) = (R^k\tau_*\mathscr{M}) \otimes \mathscr{O}_{X'}(-nm_v(l)E').$$

We claim that $h^1(X', \tau_*\mathscr{M}_n) = 0$ for $n \gg 0$. Indeed, by the above identity for $k = 0$, $h^1(X', \tau_*\mathscr{M}_n) = h^1(X', \mathscr{M}')$, where $\mathscr{M}' := (\tau_*\mathscr{M}) \otimes \mathscr{O}_{X'}(-nm_v(l)E'))$. By the normality of X' (and from sheaf theoretical version of Zariski's Main Theorem), $\tau_*\mathscr{M}$ is a free $\mathscr{O}_{X'}$-module too. Hence $h^1(\mathscr{M}') = 0$ follows since $-E'$ is an ample $\mathbb{Q}$–Cartier divisor via (C).

Next, note that $R^1\tau_*\mathscr{M}_n$ is supported at the points $\{o_j\}_j$. Using Assumption (C) for certain $n_0 \gg 0$ (which satisfies the previous vanishing as well) and the local trivialization of $\mathscr{O}_{X'}(-n_0m_v(l)E'))$ near o_j, we obtain that $\mathscr{O}_{X'}(-n_0m_v(l)E'))_{o_j} = \mathscr{O}_{X',o_j}$. Hence $R^1\tau_*\mathscr{M} \otimes \mathscr{O}_{X'}(-nm_v(l)E')$ is isomorphic to $\oplus_j(R^1\tau_*\mathscr{M})_{o_j}$. In particular, $h^1(\widetilde{X}, \mathscr{M}_{n_0}) = \sum_j h^1(\widetilde{X}_j, \mathscr{M})$. These facts show that for a certain $n_0 \gg 0$, and for all its multiples nn_0 ($n \geq 1$) the dimension $\dim(J_0/J_{n_0n})$ equals

$$\sum_{i<n_0n} \dim(J_i/J_{i+1}) = \chi(\mathscr{M}|_{n_0nl}) + h^1(\widetilde{X}, \mathscr{M}) - \sum_j h^1(\widetilde{X}_j, \mathscr{M}). \tag{8.7.1}$$

On the other hand, by Riemann–Roch Theorem,

$$\chi(\mathscr{M}|_{n_0nl}) = n_0n \cdot \deg(\det \mathscr{M}|_l) + \operatorname{rank}(\mathscr{M}) \cdot \chi(n_0nl). \tag{8.7.2}$$

Since $\chi(\mathscr{M}|_{nn_0l})$ from (8.7.2) is a quadratic polynomial in n, the periodic constant of $P_{\mathscr{M}}(t)$ exists and it can be obtained from the right hand side of (8.7.2) via the $n = 0$ substitution. But $\chi(\mathscr{M}|_{nn_0l})|_{n=0} = 0$. □

As a first application we obtain the following result of Tomari [689].

Corollary 8.7.23 *Let (X, o) be an isolated hypersurface singularity with multiplicity d and $X' \to X$ be the modification obtained by blowing up the origin $o \in X$. If X' has only isolated singularities $\{o_j\}_j$ then*

$$p_g(X, o) = d(d-1)(d-2)/6 + \sum_j p_g(X', o_j).$$

Proof Assume first that the d–homogeneous term of f is irreducible (in particular, the exceptional set E_v of the blow up is irreducible) and that the above theorem can be applied (i.e. in the good resolution graph Γ each component Γ_j of $\Gamma \setminus \{v\}$ is connected to v by one edge). Then Theorem 8.7.15 applies directly.

Indeed, the isolated singularities of X' are normal, hence the blow up is exactly the modification ϕ' of the theorem. Moreover, the strict transform of the generic linear function lifted to a good resolution intersects only E_v, hence Assumption (C) is satisfied (cf. 8.7.17). Furthermore, $\mathscr{O}_{X'}(-E') = \mathfrak{m}_{X,o}\mathscr{O}_{X'}$ (which has no base

points and it is ϕ'–ample). Hence, by 8.6.14, $\mathfrak{m}^n_{X,o} = \phi_*(\mathscr{O}_{X'}(-nE'))_o$, $I_n/I_{n+1} = \mathfrak{m}^n_{X,o}/\mathfrak{m}^{n+1}_{X,o}$, and $H(t) = (1-t^d)/(1-t)^3$. Its periodic constant is $\binom{d}{3}$ (cf. 8.7.10).

If the above assumptions are not satisfied then the proof repeats the arguments of the proof of 8.7.22, with the modification that E' is be replaced by the maximal ideal cycle of X'. (For more see [689].) □

The above Corollary can be applied e.g. for hypersurface superisolated singularities. In that case $p_g(X', o_j) = 0$ for all j, hence $p_g = \binom{d}{3}$ (cf. 5.2.4(d)).

8.7.24 Additivity for Line Bundles Assume that $H^1(\widetilde{X}, \mathbb{Z}) = 0$. Fix $h \in H$ and set $\mathscr{M} = \mathscr{O}_{\widetilde{X}}(-r_h)$. Then $\mathscr{M}_n = \mathscr{O}_{\widetilde{X}}(-r_h - nl)$, where $l = \mathfrak{o}_v E_v^*$ as above. By (8.2.7)

$$\dim(J_0/J_n) = \dim H^0(\mathscr{O}_{\widetilde{X}}(-r_h))/H^0(\mathscr{O}_{\widetilde{X}}(-nl - r_h)) = \mathfrak{h}(nl + r_h).$$

Note that $nl + r_h \geq nm_v(l)E_v + r_h$. On the other hand we claim that

$$s\big(nm_v(l)E_v + r_h\big) \geq nl + r_h.$$

Indeed, there exists a computation sequence as in 6.6.3 connecting $nm_v(l)E_v + r_h$ with $s(nm_v(l)E_v + r_h)$ (filling first the vertices from $\mathscr{V}^*$), which has $x_{r_h}(nm_v(l)E_v) + r_h$ as an intermediate term. Therefore, $s(nm_v(l)E_v + r_h) \geq x_{r_h}(nm_v(l)E_v) + r_h$. The right hand side of this is $x_{r_h}(0) + n\mathfrak{o}_v E_v^* + r_h$ by 7.3.25, part *(b)*. But $x_{r_h}(0) \geq 0$ by 7.3.33, hence the inequality follows.

In particular, by (8.2.8) we have

$$\mathfrak{h}(nl + r_h) = \mathfrak{h}(nm_v(l)E_v + r_h).$$

Using (8.6.3) and 8.6.8, the right hand side of this identity is

$$\mathfrak{h}_{h,\{v\}}(nm_v(l)E_v + m_v(r_h)) = \sum_{i < nm_v(l)} \mathfrak{p}_{h,\{v\}}(i + m_v(r_h)).$$

Since $m_v(r_h) \in [0, 1)$, this sum is the counting function of the coefficients of the series $P_{h,\{v\}}(t) = \sum_i \mathfrak{p}_{h,\{v\}}(i + m_v(r_h))t^{i+m_v(r_h)}$. In particular, $\mathrm{pc}(P_{\mathscr{M}}(t)) = \mathrm{pc}(P_{h,\{v\}}(t))$. This fact combined with Theorem 8.7.22 gives the following.

Theorem 8.7.25 *Set $\mathscr{I} = \{v\}$ and fix $h \in H$. Under the Assumption (C)*

$$h^1(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(-r_h)) = \mathrm{pc}(P_{h,\{v\}}(t)) + \sum_j h^1(\widetilde{X}_j, \mathscr{O}_{\widetilde{X}}(-r_h)|_{\widetilde{X}_j}).$$

Remark 8.7.26 In general, the restriction $\mathscr{O}_{\widetilde{X}}(-r_h)|_{\widetilde{X}_j}$ is *not* a natural line bundle on $\widetilde{X}_j$. If the restriction is not a natural line bundle then it is hard to identify as an element of $\mathrm{Pic}(\widetilde{X}_j)$, and to use the above theorem as an inductive step in

the computation of $h^1(\mathscr{O}_{\widetilde{X}}(-r_h))$. The same observation holds for Theorem 8.7.22 applied for any natural line bundle $\mathscr{M} = \mathscr{O}_{\widetilde{X}}(-l')$.

However, if $\mathscr{O}_{\widetilde{X}}(-r_h)|_{\widetilde{X}_j}$ (respectively $\mathscr{O}_{\widetilde{X}}(-l')|_{\widetilde{X}_j}$) are isomorphic with certain natural line bundles on $\widetilde{X}_j$, then Theorems 8.7.22 and 8.7.25 provide powerful inductive procedures. A few particular cases are analyzed in the next examples.

Example 8.7.27 (The Case of AR Graphs) Assume, that $\{v\}$ is an R–set (hence an SR–set too, cf. 7.3.22). Since each $\widetilde{X}_j$ is the resolutions of a rational singularity, $\mathscr{O}_{\widetilde{X}}(-r_h)|_{\widetilde{X}_j}$ is natural, say $\mathscr{O}_{\widetilde{X}_j}(-l'_j)$. (In particular, by projection formula, (†) $(l, r_h) = (l, \sum_j l'_j)$ for any cycle l supported on $\cup_{u\neq v} E_u$.) Then, by Corollary 7.1.12, on $\widetilde{X}_j$ one has $h^1(\mathscr{O}_{\widetilde{X}_j}(-l'_j)) = \chi(l'_j) - \chi(s_j(l'_j))$. Here $l'_j = R_j(r_h)$, where R_j is the cohomological restriction (see e.g. 9.3.8) and s_j is the s–operator in Γ_j (cf. 6.6.3).

We have the following key observation. One can identify the steps of the following two algorithms: the algorithm from 6.6.3 (which provides $\sum_j s_j(l'_j)$ from $\sum_j l'_j$) and the algorithm 7.3.26 (which provides the cycle $x_{r_h}(0)$). That is, $\sum_j (s_j(l'_j) - l'_j) = x_{r_h}(0)$. Therefore, $\sum_j (\chi(l'_j) - \chi(s_j(l'_j))) = \chi(r_h) - \chi(x_{r_h}(0) + r_h)$ (use the fact that $x_{r_h}(0)$ is supported on $\cup_{u\neq v} E_u$ and the projection formula (†)). In particular,

$$h^1(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(-r_h)) = \mathrm{pc}(P_{h,\{v\}}(t)) + \chi(r_h) - \chi(x_{r_h}(0) + r_h) \tag{8.7.3}$$

compatibly with Theorem 8.7.12. (In the proof of (8.7.3) we used Assumption (C), however in the proof of 8.7.12 it was not needed.)

This formula should be compared with its topological analogue from 10.5.8.

Example 8.7.28 Assume that (X, o) is **weighted homogeneous** and $\widetilde{X}$ is its minimal good resolution as in 8.7.9, and v is the central vertex.

Then we claim that $h^1(\widetilde{X}_j, \mathscr{O}_{\widetilde{X}}(-r_h)|_{\widetilde{X}_j}) = 0$ for all j. In particular (cf. 8.7.12 and (8.7.3)) $\chi(x_{r_h}(0) + r_h) = \chi(r_h)$ (see 8.7.13 too).

Indeed, using the notations of 8.2.A regarding the universal abelian covering, it is enough to show that $h^1(\widetilde{X}_j, p_*\mathscr{O}_{\widetilde{Z}}|_{\widetilde{X}_j}) = 0$, or that $h^1(p^{-1}(\widetilde{X}_j)) = 0$. But, this is true since all the components of $p^{-1}(\widetilde{X}_j)$ are resolutions of rational singularities (cf. 5.1.33).

Example 8.7.29 (The Case of Splice Quotient Singularities) Assume that $H^1(\widetilde{X}, \mathbb{Z}) = 0$ and the resolution $\widetilde{X}$ of (X, o) satisfies the end curve condition. Choose $v \in \mathscr{V}$ arbitrarily. Define the resolution of singularities $\{\widetilde{X}_j\}_j$ as above. Then, by 8.5.15, each resolution $\widetilde{X}_j$ of (X_j, o_j) satisfies the end curve condition, and the restriction $\mathscr{O}_{\widetilde{X}}(-r_h)|_{\widetilde{X}_j}$ is a natural line bundle on $\widetilde{X}_j$.

Furthermore, $P(\mathbf{t}) = Z(\mathbf{t})$ (see 8.5.A), hence

$$h^1(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(-r_h)) = \mathrm{pc}(Z_{h,\{v\}}(t)) + \sum_j h^1(\widetilde{X}_j, \mathscr{O}_{\widetilde{X}}(-r_h)|_{\widetilde{X}_j}).$$

This type of argument will be applied later to show inductively that $h^1(\widetilde{X}, \mathscr{L})$ of any natural line bundle $\mathscr{L}$ is topological (see 9.7 for the Seiberg–Witten Invariant Conjecture, especially 9.7.15 for this case).

For example, for $\mathscr{L} = \mathscr{O}_{\widetilde{X}}$ we get $h^1(\mathscr{O}_{\widetilde{X}}) = \mathrm{pc}(Z_{0,\{v\}}(t)) + \sum_j h^1(\mathscr{O}_{\widetilde{X}_j})$. Recall that contracting the exceptional curve of $\widetilde{X}_j$ we get a singularity satisfying the end curve condition. Therefore, by induction, we obtain that the geometric genus of a singularity satisfying the end curve condition is topological.

Remark 8.7.30 The above additivity formulae can be compared with the one established in (6.8.13). Let Ω_v be the subspace of $\Omega(\emptyset) := H^0(\widetilde{X} \setminus E, \Omega^2_{\widetilde{X}})/H^0(\widetilde{X}, \Omega^2_{\widetilde{X}})$ generated by (classes of) differential forms, which have no poles along the generic points of E_v. Then $h^1(\mathscr{O}_{\widetilde{X}}) - \sum_j h^1(\mathscr{O}_{\widetilde{X}_j}) = \mathrm{codim}(\Omega_v \subset \Omega(\emptyset))$.

Note that this last identity holds without any additional (topological or analytical) assumptions. (The two formulae are compared in [452].)

Remark 8.7.31

(a) In chapter 10 we will define the periodic constant for multivariable series as well. In that language (8.2.23) will be reinterpreted as follows: the multivariable series $P_0(\mathbf{t})$ admit a periodic constant, which is p_g.

(b) In order to continue the comparisons between $P(\mathbf{t})$ and $Z(\mathbf{t})$, we will establish the analogs—valid for $Z(\mathbf{t})$—of the statements proved for $P(\mathbf{t})$ above. The topological side will be strongly related to the low-dimensional topology of the link, and will be discussed in the next Chapter.

Chapter 9
Topological Invariants. The Seiberg–Witten Invariant

9.1 The Casson Invariant

The Casson invariant assigns an integer to any oriented connected integral homology 3-sphere. In this book, instead of the original definition of Casson, we will use as starting definition a surgery formula valid for negative definite plumbed 3-manifolds. Starting from this expression, we will prove the needed properties of the Casson invariant for normal surface singularity links.
In particular, we will prove the additivity with respect to splice decomposition, and the behaviour of the Casson invariant with respect to rational surgery along an algebraic knot. We establish a formula valid for Seifert 3-manifolds in terms of the Seifert invariants (involving Dedekind sums). For the Brieskorn singularity link $\Sigma = \Sigma(\alpha_1, \alpha_2, \alpha_3)$ we relate it to a certain lattice point count, and also to the number of irreducible SU_2-representations of $\pi_1(\Sigma)$. ♣

9.1.1 The Setup In this section we assume that Γ is a connected plumbing graph with negative definite intersection form. We denote its plumbed 3-manifold by $M(\Gamma)$. We also assume that $\det(\Gamma) = 1$, hence $M(\Gamma)$ is an integral homology sphere. Sometimes we replace Γ by its splice diagram $\mathfrak{G}$ (cf. 5.3.10).

Recall that in 3.5.11 we defined the surgery 3-manifolds $S^3_{p/q}(K)$. The construction can be generalized to the case of any connected oriented integral homology sphere M and knot $K \subset M$ (that is, $K \simeq S^1$). Indeed, for any $p/q \in \mathbb{Q} \cup \{\infty\}$ one defines the Dehn surgery 3-manifolds $M_{p/q}(K)$, in the very same way as $S^3_{p/q}(K)$ was defined. It has $H_1(M_{p/q}(K)) = \mathbb{Z}_p$, hence $M_{p/q}(K)$ is an integral homology sphere if and only if $p = 1$. The surgery with coefficient ∞ is trivial.

9.1.2 Alexander Polynomial Let K be an oriented link in the integral homology sphere M. Let F be an oriented Seifert surface of K (cf. 3.2.8). The orientation provides a positively oriented normal vector field along the interior F° of F. If b

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is any 1-cycle in F°, then let b^+ be its 'positive push off' by this vector field into $M \setminus F$. Then the Seifert form $S : H_1(F) \otimes H_1(F) \to \mathbb{Z}$ of $K \subset M$ associated with the choice of F is $S_F(a,b) := \ell n(a,b^+)$ (compare with 3.2.15). Then the Alexander polynomial (associated with F) is $\Delta_F(t) = \det(S_F - tS_F^*)$, where S_F^* is the transposed matrix. If F and F' are two Seifert surfaces then $\Delta_{F'}(t) = \pm t^n \Delta_F(t)$ for a certain $n \in \mathbb{Z}$. In particular, $\Delta(t) := \Delta_F(t)$ is well-defined only up to a unit of $\mathbb{Z}[t, t^{-1}]$.

If we change the orientation of all the components of K simultaneously, then $\Delta(t)$ remains unchanged (up to a unit). In particular, if K is a knot, then $\Delta(t)$ is independent of the orientation of K.

One sees that $\Delta_F(t^{-1}) = (-t)^{-r}\Delta_F(t)$ (†), where $r = \operatorname{rank} H_1(F)$. If K is a knot then r is even; let $\Delta_s(t)$ be the symmetrized Alexander polynomial of $K \subset M$, that is, Δ_s equals Δ_F up to a unit and it is normalized by $\Delta_s(1) = 1$ and $\Delta_s(t^{-1}) = \Delta_s(t)$ (by (†) this exists). (Note that $\Delta(1) = 0$ if K has more components.)

If $K \subset M$ is the binding of an open book with page F and algebraic monodromy h_1 (and compatible orientations), then the characteristic polynomial $\Delta_1(t)$ of h_1 equals $\Delta_F(t)$ (up to a unit). (This follows from the identities from 3.2.15.)

We denote by $\Delta_s''(t)$ the second derivative of $\Delta_s(t)$.

9.1.3 Rochlin Invariant Let M be an oriented integral homology 3-sphere. Then (cf. 6.10.10) M admits a spin structure, which is unique by 6.10.8. Furthermore, since the spin cobordism group is trivial in dimension 3, cf. [668], there exists a spin 4-manifold N whose spin boundary is M. Then one defines the Rochlin invariant $\mu(M)$ as the class of $\sigma(N)/8$ in $\mathbb{Z}_2$. [Note that the existence of the spin structure of N implies that $w_2(TN) = 0$, cf. 6.10.8. Hence, by Wu's formula (cf. 6.10.6(b)), the intersection form on N is even. Then, by linear algebra, $8|\sigma(N)$, cf. [283, 630].] The definition is independent of the choice of N. Indeed, by Rochlin's Theorem [594] the signature of any compact spin oriented 4-manifold is multiple of 16. In the presence of two fillings, this fact applied for $N_1 \cup (-N_2)$, combined with Novikov's additivity $\sigma(N_1 \cup -N_2) = \sigma(N_1) - \sigma(N_2)$, show the independence of N.

By another version of Rochlin Theorem, the signature of any simply connected compact oriented 4-manifold with even intersection form is multiple of 16. Hence, if M is as above, and N is a simply connected 4-manifold with even intersection from and $\partial N = M$, then $\mu(M) = \sigma(N)/8 \pmod 2$ also works.

One has the identities $\mu(-M) = \mu(M)$ and $\mu(M_1 \# M_2) = \mu(M_1) + \mu(M_2)$ (here # means the 'connected sum').

9.1.4 Casson proved the following facts.

Theorem 9.1.5 ([12]) *There is a unique invariant $M \rightsquigarrow \lambda(M) \in \mathbb{Z}$, of connected oriented integral homology spheres, which satisfies the following two properties:*

(a) $\lambda(S^3) = 0$,

(b) $\lambda(M_{1/q+1}(K)) - \lambda(M_{1/q}(K)) = \frac{1}{2}\Delta_s''(1)$ *for any knot $K \subset M$ as in 9.1.1.*

Additionally, it has the following properties too:

(c) $\lambda(-M) = -\lambda(M)$,
(d) *if* $\lambda(M) \neq 0$, *then* $\pi_1(M)$ *has a nontrivial* SU_2*-representation,*
(e) $\lambda(M)$ *modulo 2 is the Rohlin invariant* $\mu(M)$ *of* M,
(f) $\lambda(M_1 \# M_2) = \lambda(M_1) + \lambda(M_2)$.

The integer $\lambda(M)$, which satisfies the above properties is called the *Casson invariant* of the oriented integral homology sphere M.

The original definition of Casson is based on a Heegaard splitting $W_1 \cup W_2$ of M, and on the study of the space of conjugacy classes of SU_2-representations of the fundamental groups of M, W_1, W_2, $W_1 \cap W_2$, cf. [12], see also [200].

9.1.6 In this book we will adopt a specific formula of $\lambda(M)$ as starting definition, valid for any $M(\Gamma)$ as in 9.1.1. It was proved in the PhD thesis of A. Ratiu (Paris VII), and it follows also from the surgery formulae from Lescop's book [385].

Starting from this formula we will prove all the needed expressions and properties what our presentation needs (in particular, some properties from 9.1.5 as well).

Definition 9.1.7 With the standard notations associated with Γ we define

$$-24 \cdot \lambda(M) = \sum_{v \in \mathcal{V}} (E_v^2 + 3) + \sum_{v \in \mathcal{V}} (2 - \kappa_v)(E_v^*, E_v^*). \tag{9.1.1}$$

We extend the definition of λ by the very same expression for non-connected graphs as well, (that is, for connected sums of plumbed 3-manifolds). Note that via this definition the property 9.1.5*(f)* is guaranteed automatically.

9.1.8 The right hand side of (9.1.1) is an expression in terms of Γ, and it is not very transparent that this expression depends only on $M(\Gamma)$, that is, it is stable to the plumbing calculus of negative definite plumbing graphs. The direct verification of this fact is left to the reader (for a similar statement see 7.3.5).

A computation for the (-1)-curve shows that $\lambda(S^3) = 0$, cf. 9.1.5*(a)*.

Example 9.1.9 If M is a Seifert 3-manifold, then using 3.5.2 and (2.2.7) one has

$$-24 \cdot \lambda(\Sigma(\alpha_1, \dots, \alpha_v)) = \frac{1}{e}\Big(2 - v + \sum_{j=1}^{v} \frac{1}{\alpha_j^2}\Big) + e + 3 - 12 \cdot \sum_{j=1}^{v} \mathbf{s}(\omega_j, \alpha_j). \tag{9.1.2}$$

From 5.1.17, ω_j is determined from $\{\alpha_k\}_k$ by $\omega_j \cdot (\prod_k \alpha_k)/\alpha_j \equiv -1 \pmod{\alpha_j}$. Hence $\mathbf{s}(\omega_j, \alpha_j) = \mathbf{s}(\omega_j', \alpha_j) = -\mathbf{s}((\prod_k \alpha_k)/\alpha_j, \alpha_j)$. Note also that $e^{-1} = -\prod_k \alpha_k$.

Using Dedekind reciprocity law (2.2.6) one shows that for any $1 < j < v - 1$

$$\lambda(\Sigma(\alpha_1, \dots, \alpha_v)) = \lambda(\Sigma(\alpha_1, \dots, \alpha_j, \alpha_{j+1} \cdots \alpha_v)) + \lambda(\Sigma(\alpha_1 \cdots \alpha_j, \alpha_{j+1}, \dots, \alpha_v)). \tag{9.1.3}$$

In particular, the computation of $\lambda(\Sigma(\alpha_1,\dots,\alpha_\nu))$ can be reduced to the case $\nu = 3$.

Using either (9.1.1) or (9.1.2) one obtains $\lambda(\Sigma(2,3,5)) = \lambda(\Sigma(2,3,7)) = -1$.

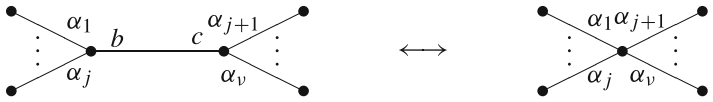
Additionally to the properties 9.1.5, Akbulut and McCarthy, Boyer and Nicas [73], Fukuhara and Maruyama [207] proved that the Casson invariant is additive with respect to the splice decomposition. For plumbed 3-manifolds (starting from the definition 9.1.7) the verification can be done independently.

Theorem 9.1.10 *Assume that M is obtained from M_1 and M_2 by splicing (as in 5.3.7). Then*

$$\lambda(M) = \lambda(M_1) + \lambda(M_2). \tag{9.1.4}$$

Proof Use the same strategy and identities as in the proof of 6.3.15. □

Remark 9.1.11 Formula (9.1.3) can also be interpreted as additivity in the sense of (9.1.4), if we allow splice diagrams with vanishing edge-determinants (and the additional splice operation described below). In this generalized sense, the following two splice diagrams determine the same 3-manifold (where $b = \prod_{k>j}\alpha_k$ and $c = \prod_{k\le j}\alpha_k$), cf. [178, Theorem 8.1]:



Example 9.1.12 $\lambda(\Sigma(\alpha_1,\alpha_2,\alpha_3))$ **and the Signature of the Milnor Fiber**

Let $\sigma = \sigma(\alpha_1,\alpha_2,\alpha_3)$ be the signature of the Brieskorn singularity $\sum_{k=1}^{3} z_k^{\alpha_k}$, cf. 5.3.6(c), see also 5.1.15–5.1.17. Recall that the α_k's are pairwise relative prime.

In 9.2.5 we will prove the next identity, as part of the Casson Invariant Conjecture

$$\lambda(\Sigma(\alpha_1,\alpha_2,\alpha_3)) = \sigma(\alpha_1,\alpha_2,\alpha_3)/8.$$

Via this identity we can provide several other reinterpretations of $\lambda(\Sigma(\alpha_1,\alpha_2,\alpha_3))$. First recall (see 6.9.14) that $\sigma = S_0 - S_1 + S_2$, where

$$S_i := \#\{(j_1,j_2,j_3) \mid 0 < j_k < \alpha_k\ (1\le k\le 3),\ \ i < \textstyle\sum_k j_k/\alpha_k < i+1\}.$$

First we give another interpretation of σ in terms of the number of certain lattice points. Define the cube $C = \{0 \le x_i \le \alpha_i,\ i = 1,2,3\}$, let C° be its interior. Each vertex v of C determines a tetrahedron T_v, whose vertices consist of v and the three other vertices of C adjacent to v. Let $\#_v$ be the number of lattice points in $T_v \cap C^\circ$. Then $\#_v$ is independent of v: e.g. the identity $\#_{(0,0,0)} = \#_{(\alpha_1,0,0)}$ is realized by $(j_1,j_2,j_3) \leftrightarrow (\alpha_1 - j_1, j_2, j_3)$. Let us denote this common value by $\#$.

Let T be the tetrahedron defined by the vertices $(\alpha_1, 0, 0)$, $(0, \alpha_2, 0)$, $(0, 0, \alpha_3)$, $(\alpha_1, \alpha_2, \alpha_3)$, and let $\#(T \cap C^\circ)$ be the number of lattice points in $T \cap C^\circ$. Then

$$\sigma = S_0 - S_1 + S_2 = 2S_0 + 2S_2 - \prod(\alpha_i - 1) = 4\# - \prod(\alpha_i - 1) = -\#(T \cap C^\circ).$$

In particular, one also has

$$\lambda(\Sigma(\alpha_1, \alpha_2, \alpha_3)) = -\#(T \cap C^\circ)/8. \tag{9.1.5}$$

Example 9.1.13 [200] $\lambda(\Sigma(\alpha_1, \alpha_2, \alpha_3))$ **and** SU_2**-Representations**

We will show that the number of conjugacy classes of irreducible SU_2 representations of $\pi_1 := \pi_1(\Sigma(\alpha_1, \alpha_2, \alpha_3))$ is finite and it is $\#(T \cap C^\circ)/4$ (cf. 9.1.12).

For a presentation of π_1 see 3.7.5. Let $\rho : \pi_1 \to \mathrm{SU}_2$ be an irreducible representation. Since γ_0 is central and ρ is irreducible one has $\rho(\gamma_0) = \pm 1$. Recall that $\mathrm{SU}_2/\pm 1 = \mathrm{SO}_3$. Therefore, ρ induces a representation $\bar{\rho} : \pi_1/\gamma_0 \to \mathrm{SO}_3$. The point is that this correspondence realizes a bijection between the irreducible representations $\pi_1 \to \mathrm{SU}_2$ and $\bar{\pi}_1 = \pi_1/\gamma_0 \to \mathrm{SO}_3$. We show this fact in two steps. First we identify the irreducible representations $\pi_1 \to \mathrm{SO}_3$ and $\bar{\pi}_1 = \pi_1/\gamma_0 \to \mathrm{SO}_3$. This follows, since SO_3 has no nontrivial central element, hence for any $\bar{\rho} : \pi_1 \to \mathrm{SO}_3$ one has $\bar{\rho}(\gamma_0) = 1$. Next, we have to lift any irreducible representation $\bar{\rho} : \pi_1 \to \mathrm{SO}_3$ to $\rho : \pi_1 \to \mathrm{SU}_2$. Since $\mathrm{SU}_2 = \mathrm{Spin}_3$ this is equivalent to the fact that any flat SO_3 bundle of M is spin. This is obstructed by the second Stiefel–Whitney class $w_2 \in H^2(\Sigma, \mathbb{Z}_2)$ of the bundle (similarly as in 6.10.7, cf. [368]). But since $H_1(\Sigma, \mathbb{Z}) = 0$, this obstruction vanishes.

Consider now an irreducible SO_3-representation $\bar{\rho}$ of the polygonal group $\bar{\pi}_1 = \langle x_1, x_2, x_3 \,|\, x_1^{\alpha_1} = x_2^{\alpha_2} = x_3^{\alpha_3} = x_1x_2x_3 = 1\rangle$, cf. 3.7.5.

Each $\bar{\rho}(x_i)$ is a rotation of S^2 of order α_i with fixpoints $P_i^{\pm}$. These fixpoints $\{P_i^{\pm}\}_i$ determine 8 triangles of S^2, and four pairs of triangles with respect to the symmetry $P \leftrightarrow -P$. If a triangle of S^2 has angles $(j_1\pi/\alpha_1, j_2\pi/\alpha_2, j_3\pi/\alpha_3)$ then we say that it is of type (j_1, j_2, j_3). The relation $x_1x_2x_3 = 1$ guarantees that the types of the four (non-symmetric) triangles of S^2 determined by $\bar{\rho}$ are

$$(j_1, j_2, j_3),\ (\alpha_1 - j_1, \alpha_2 - j_2, j_3),\ (\alpha_1 - j_1, j_2, \alpha_3 - j_3),\ (j_1, \alpha_2 - j_2, \alpha_3 - j_3).$$

Furthermore, the area of a triangle is $> \pi$, which translates into the fact that the four lattice points above are in $T \cap C^\circ$. In fact, on the lattice points of $T \cap C^\circ$ one has the natural free $\mathbb{Z}_2 \times \mathbb{Z}_2$ action whose orbit consists of the four triples above, corresponding to the four pairs of triangles. Such an orbit determines the representation uniquely, hence the number of irreducible SO_3 representations is $\#(T \cap C^\circ)/4$.

This fact together with (9.1.5) give that $\lambda(\Sigma)$ is

$$-\tfrac{1}{2}\cdot\{\text{number of irreducible } SU_2\text{-representations of } \pi_1(\Sigma) \text{ up to conjugation}\}.$$

9.1.14 From any of the following two examples we obtain that $\lambda(\Sigma(\alpha_1,\alpha_2,\alpha_3)) < 0$ (whenever all $\alpha_1,\ \alpha_2,\ \alpha_3 \geq 2$), hence via (9.1.3) and (9.1.4) we have:

Corollary 9.1.15 *For any singularity link M one has $\lambda(M) \leq 0$, and $\lambda(M) = 0$ if and only if $M = S^3$.*

9.1.16 Next, we prove the surgery formula 9.1.5*(b)*. We start with the $q = -1$ case.

We assume that $M = M(\Gamma)$ is given by the connected negative definite plumbing graph Γ, and the knot K is represented by an arrow supported by the vertex v. Set

$$d := \det(\Gamma \setminus v) \tag{9.1.6}$$

and consider the new graph $\Gamma^e(-k)$ obtained from Γ by adding a new vertex, say u, decorated by Euler number $-d-k$, and connected by a new edge to v ($k \in \mathbb{Z}$).

Lemma 9.1.17 *(a)* $\det(\Gamma^e(-k)) = k$*; (b)* $\Gamma^e(-k)$ *is negative definite provided that* $k > 0$*; and (c)* $M(\Gamma^e(-k)) = M_{-k}(K)$.

Proof *(a)* follows from 2.2.5(6) (where G is the edge uv) and (9.1.6). Part *(b)* follows from *(a)* and from the fact that Γ is negative definite. Next we prove *(c)*.

Consider the plumbed 4-manifolds $P = P(\Gamma)$ and $P(\Gamma^e(-k))$, cf. 3.3.5. Let $\mathscr{D}_v$ and $\mathscr{D}_u$ be the disc bundles with cores E_v and E_u, and let $D_v \times D_u$ be the bidisc obtained by the identification of $\pi_v^{-1}(D_v)$ with $\pi_u^{-1}(D_u)$. Then $K = 0 \times \partial D_u$ in $M = \partial P$, and its meridian μ in M is $\partial D_v \times 1$. Let us denote $1 \times \partial D_u$ by x. Then the linking number of x and K in M is $-I_{vv}(\Gamma) = d$. Since the intersection of x with μ in the torus $\partial D_v \times \partial D_u$ is 1, we get that in this torus $x = d\mu + \lambda$. Next, we take a C^∞ section in $\mathscr{D}_u$ which is $1 \times D_u$ in $D_v \times D_u$. By the Euler obstruction its intersection with E_u is $-d-k$, hence $x-(d+k)\mu$ generates $\ker(H_1(\partial D_v \times \partial D_u) \to H_1(\partial \mathscr{D}_u \setminus \partial D_v \times D_u))$, thus $d\mu+\lambda-(d+k)\mu$ and $p\mu+q\lambda$ are proportional. Thus $p/q = -k$. □

In the sequel we write $\Gamma^e := \Gamma^e(-1)$, hence $M(\Gamma^e) = M_{-1}(K)$.

Theorem 9.1.18 *If Δ_s is the symmetrized Alexander polynomial of $K \subset M$, then*

$$\lambda(M) - \lambda(M_{-1}(K)) = \tfrac{1}{2}\cdot\Delta_s''(1).$$

Proof We can assume that v is an end-vertex of Γ (if not, we blow up the supporting point of the arrow). Let w be a vertex of Γ, $w \neq v$. We wish to compare $-(E_w^*, E_w^*)(\Gamma) = \det(\Gamma \setminus w)$ with the corresponding value computed in Γ^e and we apply (9.1.1). Let m_w be the multiplicity of the arrow of K (considered with

multiplicity one) along the vertex w. Then we claim that

$$\det(\Gamma^e \setminus w) - \det(\Gamma \setminus w) = m_w^2.$$

Indeed, let $\{\Gamma_i\}_{i=0}^k$ be the connected components of $\Gamma \setminus w$, with $v \in \mathcal{V}(\Gamma_0)$. Let Γ_0^e be the connected components of $\Gamma^e \setminus w$ which contains Γ_0. Set $D := \prod_{i>0} \det(\Gamma_i)$. Using 2.2.5(6) (applied for Γ_0^e with G being the edge (u, v)), we get $\det(\Gamma_0^e) = (d+1)\det(\Gamma_0) - \det(\Gamma_0 \setminus v)$. Therefore,

$$\det(\Gamma^e \setminus w) - \det(\Gamma \setminus w) = D \cdot \big(d \det(\Gamma_0) - \det(\Gamma_0 \setminus v)\big).$$

On the other hand, 2.2.5(6) applied for Γ and $G = \Gamma_0 \setminus v$, gives

$$\det(\Gamma_0 \setminus v) = d \det(\Gamma_0) - D \cdot \det(\Gamma_0 \setminus \mathrm{path}(v, w))^2.$$

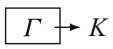
Hence $\det(\Gamma^e \setminus w) - \det(\Gamma \setminus w) = \big[D \cdot \det(\Gamma_0 \setminus \mathrm{path}(v, w))\big]^2 = m_w^2$ (cf. 5.3.19), which proves the claim. Note also that $\det(\Gamma \setminus v) = d$ and $\det(\Gamma^e \setminus u) = 1$. All these put together give

$$-24\lambda(M_{-1}) = -24\lambda(M) + 1 - \sum_{w \neq v} (2 - \kappa_w) m_w^2.$$

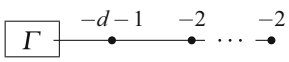
But $\Delta(t) = (1-t)\prod_{w \neq v}(1 - t^{m_w})^{\kappa_w - 2}$ (cf. 4.1.22), hence by 4.1.23 we conclude. □

This proves 9.1.5*(b)* for $q = -1$. For a more general q we proceed as follows.

9.1.19 First recall that if $M_{p/q}$ is an integral homology sphere then $p = 1$. Moreover, if M is represented by a negative definite graph Γ and we wish to have the same property for $M_{1/q}$, then necessarily $q < 0$. In fact, the analogous statements of 9.1.17 (with analogous proofs) are the following. For $q < 0$ and for the link $K \subset M$ represented by



we define the graph $\Gamma^e(1/q)$ as



where $d = \det(\Gamma \setminus v)$ as in 9.1.16, and outside the box there are $-q$ vertices. Then

(a) $\det(\Gamma^e(1/q)) = 1$ and $\Gamma^e(1/q)$ is negative definite,
(b) $M(\Gamma^e(1/q)) = M_{1/q}(K)$,

(c) Let K_q be the knot in $M(\Gamma^e(1/q))$ represented by an arrow supported on the right end (-2)-vertex. Then $\Delta_s\big(K_q \subset M_{1/q}(K)\big)$ is independent of q, and it is $\Delta_s(K \subset M)$. (In the proof use the fact that the multiplicities needed in the A'Campo's formula are q-independent.)

(d) $(M_{1/q})_{-1}(K_q) = M_{1/(q-1)}$ (for any $q < 0$).

Hence, (c)–(d) applied inductively provides 9.1.5*(b)* and the next expression too:

$$\lambda(M) - \lambda(M_{1/q}(K)) = \frac{-q}{2}\Delta_s''(1).$$

9.1.20 Generalization The above construction can be generalized as follows.

Assume that we have ν connected negative definite plumbing graphs Γ_i $(1 \le i \le \nu)$, $M_i = M(\Gamma_i)$, and for each i a knot $K_i \subset M_i$ represented by an arrow of Γ_i supported by the vertex $v_i \in \mathcal{V}(\Gamma_i)$. Set $d_i := \det(\Gamma_i \setminus v_i)$. Then consider the new graph Γ^e obtained from $\sqcup_i \Gamma_i$ by adding a new vertex u decorated by Euler number $-1-\sum_i d_i$ and connected to each v_i by a new edge. Then Γ^e is connected, negative definite, $\det(\Gamma^e) = 1$, and $M(\Gamma^e) = (\#_i M_i)_{-1}(\#_i K_i)$. If Δ is the product of all the Alexander polynomials $\Delta(K_i \subset M_i)$, then

$$\sum_i \lambda(M_i) - \lambda(M(\Gamma^e)) = \tfrac{1}{2}\cdot\Delta_s''(1) = \tfrac{1}{2}\cdot\sum_i \Delta(K_i \subset M_i)_s''(1).$$

This can be proved either similarly as the above $\nu = 1$ case, or one notices that the splice diagram $\mathfrak{G}(\Gamma^e)$ decomposes into splice components

$$\mathfrak{G}\Big(\boxed{\Gamma_i} \overset{-d_i-1}{\text{—}\bullet}\Big),$$

and another (non-minimal) splice component which has a central vertex and ν legs all decorated by 1 (representing S^3 with ν Hopf link-components in it). Then the formula follows from the splice formula 9.1.10.

9.1.21 Casson Invariant and Cyclic Coverings Assume that the integral homology 3-sphere M is represented by the splice diagram $\mathfrak{G}$, and the knot $K \subset M$ by an arrow of $\mathfrak{G}$, cf. 5.3.18. Assume also the n-fold cyclic covering of M branched along K, denoted by $M(K, n)$, is an integral homology sphere too. For a characterization of this fact, and for the splice diagram of $M(K, n)$ see 5.3.22. Let us define also $\ell \mapsto \eta(\ell)$ from the graph of M as in Proposition 6.9.16 (this can be done from the plumbing graph and also from the equivalent splice diagram). Equivalently, η can be defined in the following way as well.

Consider the homological package of the (Milnor) fibration $M \setminus K \to S^1$ with page F. (The binding determines uniquely the open book decomposition, cf. 3.4.10; and, in fact, the fibration can be realized even analytically, cf. 2.5.8.)

Let b be the intersection form on $H_1(F, \mathbb{C})$. Then the monodromy h_1, and also its power h_1^n, have no eigenvalue one (and preserve b). (Indeed, since $M(K, n)$ has a natural open book decomposition with the same page as M with monodromy h_1^n, cf. 4.1.D, one can apply 3.2.A or 3.4.) Fix an integer ℓ such that h_1^ℓ has no eigenvalue one. Then consider the generalized eigenspace decomposition $\oplus_\lambda (H_1(F, \mathbb{C})_\lambda, b_\lambda)$ of $(H_1(F, \mathbb{C}), b)$ associated with the operator h_1^ℓ. Then for each eigenvalue $\lambda = e^{2\pi i c}$ with $0 < c < 1$ of h_1^ℓ define $\eta(\ell)_\lambda$ as $(1 - 2c) \cdot \text{signature}(ib_\lambda)$. Finally, set $\eta(\ell) := \sum_\lambda \eta(\ell)_\lambda$. (This definition agrees with the one considered in 6.9.16, cf. [474]. The term $(1 - 2c)$ can also be compared with the Dedekind symbol $((c))$ from 2.2.A, or with the eta invariant of Atiyah, Patodi and Singer associated with the circle and the flat bundle over S^1 with monodromy λ and the hermitian form ib_λ. For more see e.g. [34, page 411] or [504] and the references therein.) Then, by [474],

$$\lambda(M(K, n)) - n \cdot \lambda(M) = \frac{1}{8} \cdot \big(\eta(n) - n \cdot \eta(1)\big). \tag{9.1.7}$$

In particular, $n \mapsto \lambda(M(K, n))$ is a sum of a linear and a periodic function.

For a slightly more general situation see again [474]. Here it is also conjectured the validity of the formula even if (M, L) has no splice representation.

The formula should be compared with a similar expression valid for the signature of the Milnor fiber, cf. 6.9.16. This similarity will be exploited and explained in the next section. For more regarding (9.1.7) see also 9.2.10 and the references therein.

9.2 The Casson Invariant Conjecture of Neumann–Wahl

Let (X, o) be an ICIS with integral homology sphere link L_X and Milnor fiber F. Then the Casson Invariant Conjecture (CIC), formulated by Neumann and Wahl, predicts that $\lambda(L_X) = \sigma(F)/8$. If this identity holds then it follows that $\sigma(F)$, $\chi_{top}(F)$ and $p_g(X, o)$ are all computable from the abstract link L_X.

In this section we verify CIC for Brieskorn complete intersections, suspension hypersurfaces and splice singularities. (The general case is still open.) ♣

9.2.1 Based on the computation of Fintushel and Stern valid for $\Sigma = \Sigma(\alpha_1, \alpha_2, \alpha_3)$, which identifies the irreducible SU_2-representations of $\pi_1(\Sigma)$ with Brieskorn formula for the signature of the Milnor fiber (cf. [200], or see examples 9.1.12 and 9.1.13 here), Neumann and Wahl formulated the following conjecture, which generated a fertile activity in the field.

Conjecture 9.2.2 (Casson Invariant Conjecture (CIC) [512]) Assume that (X, o) is an isolated complete intersection singularity of dimension two, whose link L_X is an integral homology sphere. Let $\sigma(F)$ be the signature of its Milnor fiber F. Then $\lambda(L_X) = \sigma(F)/8$.

The divisibility of $\sigma(F)$ by 8 can be deduced as follows. Since the intersection form on the Milnor fiber is even, cf. 6.10.9, and $H_1(\partial F, \mathbb{Z}) = 0$ implies the unimodularity of the intersection form, the signature is multiple of 8 by [630, p. 53].

The conjecture would imply (via formulae of Durfee and Laufer 6.9.6–6.9.7) that $\chi_{top}(F)$ and $p_g(X, o)$ can also be computed from the abstract link.

Remark 9.2.3 Since the intersection form of F is even (see above, cf. 6.10.9) and F is simply connected (cf. [252], see 3.2.13 here), the Rochlin invariant of L_X can be computed as $\mu(L_X) \equiv \sigma(F)/8 \pmod 2$, cf. 9.1.3. This, via Theorem 9.1.5*(e)* reads as $\lambda(L_X) \equiv \sigma(F)/8 \pmod 2$. Hence, CIC says that this identity lifts from $\mathbb{Z}_2$ to $\mathbb{Z}$. [In gauge theory and low-dimensional topology there is an expectation of a universal construction of a 4-manifold P from the geometry of a 3-manifold M with $\partial P = M$ and $\sigma(P)/8 = \lambda(M)$. CIC suggests that for the link $M = L_X$ this universal filling should be the Milnor fiber; for some speculation see [512].]

9.2.4 Neumann and Wahl supported the conjecture by its verification for Brieskorn–Hamm singularities and suspensions. Since for these germs the Casson invariant is known (see 9.1), any formula which provides the signature (or the Milnor number, or the geometric genus) in terms of the link can serve for the verification of the conjecture. (Nevertheless, usually these formulae are rather arithmetical, hence the verifications of the needed identity involve certain arithmetical manipulations).

9.2.5 Proof of CIC for Brieskorn–Hamm Germs Based on Signature Formula For an arbitrary Brieskorn–Hamm complete intersection (even if the link is not integral homology sphere) Hirzebruch in [272] obtained the following formula for the signature σ of the Milnor fiber. Below we adopt the notations of 5.1.17.

$$\sigma = -\sum_{\zeta^a=-1} \operatorname{res}_{t=\zeta} \left(\left(\frac{t^a-1}{t^a+1} \right)^{n-2} \cdot \frac{t+1}{t-1} \cdot \prod_{i=1}^{n} \left(\frac{t^{\gamma_i}+1}{t^{\gamma_i}-1} \right) \right) \frac{dt}{2t}.$$

In [512, (1.13)] this formula by a residue computation is transformed into

$$\sigma = -1 + \frac{A}{3a^2}\left(1 - (n-2)a^2 + \sum_{i=1}^{n} \gamma_i^2\right) - 4 \cdot \sum_{i=1}^{n} q_i \cdot \mathbf{s}(\gamma_i, \alpha_i). \tag{9.2.1}$$

[This formula determines σ in terms of the coefficients $\{a_i\}_i$ of the Brieskorn equations. Usually, if $H_1(\Sigma, \mathbb{Q}) \neq 0$, σ *cannot* be determined from (the Seifert invariants of) the link, e.g. for $(2, 7, 14)$ and $(3, 4, 12)$ the links agree but the signatures not. If the link is an integral homology sphere then $A = a$, $e = -1/A$, and one also has $\gamma_i \equiv -\omega_i' \pmod{\alpha_i}$, hence -3σ has an identical expression such as -24λ in (9.1.2).

9.2.6 CIC in Terms of the Milnor Number Recall that for any ICIS $3\sigma + 2\mu + K^2 + |\mathscr{V}| = 0$ holds (combine formulae of Durfee and Laufer 6.9.6–6.9.7). Therefore, the CIC can be rewritten as

$$2\mu = -24\lambda(L_X) - (K^2 + |\mathscr{V}|). \tag{9.2.2}$$

9.2.7 Proof of CIC for Brieskorn–Hamm Germs Based on Milnor Number Using the formulae (9.1.2) for $\lambda(L_X)$ and 6.3.12 for $K^2 + |\mathscr{V}|$, the CIC for Brieskorn–Hamm singularities is equivalent to

$$2\mu + 2 = -\frac{1}{e} \cdot \Big[(2 - n + \sum_i \frac{1}{\alpha_i})^2 - (2 - n + \sum_i \frac{1}{\alpha_i^2}) \Big]. \tag{9.2.3}$$

On the other hand, the Euler characteristic $\chi = 1 + \mu$ of the Milnor fiber is given by the following residue formula of Greuel and Hamm [234, Satz 3.7(c)]

$$\chi = \mathrm{res}_{t=0} \frac{1}{At^3 \cdot (1 + At)^{n-2}} \cdot \prod_{i=1}^{n} (1 + \frac{A}{\alpha_i} t) \quad (A = \prod_i \alpha_i).$$

Now, a direct computation of this residue gives (9.2.3).

9.2.8 The proof of CIC for Brieskorn–Hamm germs based on the Pinkham's formula of geometric genus will be given in 9.7.14 (see also 9.7.A), where we prove the more general Seiberg–Witten Invariant Conjecture for any weighted homogeneous singularity with rational homology sphere link.

9.2.9 The Proof of CIC for Suspensions We adopt the notation of 5.3.21. Let f (resp. g) be the irreducible plane curves singularities with splice decorations (a_i, p_i), $1 \le i \le s$ (resp. $1 \le i < s$), and let $\mathfrak{G}_{f+z^n}$ and $\mathfrak{G}_{g+z^n}$ be the splice diagrams of the corresponding 3-dimensional links. Then $\mu(f) = p_s \mu(g) + (a_s - 1)(p_s - 1)$ by A'Campo's formula 4.1.22. Hence (use 6.3.15 for $c(\Gamma) = (K^2 + |\mathscr{V}|)(\Gamma)$ too)

$$\mu(f + z^n) = p_s \mu(g + z^n) + (a_s - 1)(p_s - 1)(n - 1),$$

$$c(\mathfrak{G}_{f+z^n}) = c(\mathfrak{G}_{g+z^n}) + c(\Sigma(a_s, p_s, n)) - 2\mu(g + z^n)(p_s - 1).$$

Therefore, via $3\sigma + 2\mu + K^2 + |\mathscr{V}| = 0$ (cf. 6.9.6–6.9.7) one obtains $\sigma(f + z^n) = \sigma(g + z^n) + \sigma(\Sigma(a_s, p_s, n))$. Hence, inductively,

$$\sigma(f + z^n) = \sum_{i=1}^{s} \sigma(\Sigma(a_i, p_i, n)). \tag{9.2.4}$$

Then use the additivity in the splice formula 9.1.10 for λ and 9.1.12.

Remark 9.2.10 The case of suspensions 9.2.9 can be generalized as follows. Using identities of type (6.9.2) and (9.1.7) respectively one sees that CIC is stable with respect to cyclic branched coverings. For topological generalizations see also [122–124, 595, 612].

Theorem 9.2.11 ([493]) *CIC is valid for any splice (complete intersection) singularity.*

The proof will be given is Sect. 9.2.A.

Example 9.2.12 Consider the pair of examples from 5.3.22. Both of them are of splice type, cf. 5.4.3. Since the splice components are identical, we get that their Casson invariants are identical. By 9.2.11 their signatures are also identical, namely -2792. Nevertheless, several other numerical invariants are different. E.g., the invariants $K^2 + |\mathscr{V}|$, μ and p_g in the first case are -5440, 6908, 1029, while in the second case -5520, 6948, 1039.

Example 9.2.13 Consider the splice type complete intersection $\{x^2 = u^3 + v^2y,\ y^2 = v^3 + u^2x\} \subset (\mathbb{C}^4, 0)$ with link $S^3_{-1}(T_{2,3}\#T_{2,3})$ and graph given in 5.2.5. Then $\lambda = \sigma/8 = -2$, $K^2 + |\mathscr{V}| = -8$, $p_g = 3$.

9.2.A The Proof of CIC for Splice Type Singularities

9.2.14 In this subsection we follow [493].

Let (X, o) be a splice type singularity, whose minimal good resolution $\widetilde{X}$ satisfies the end curve condition. Assume also that the link L_X is an integral homology sphere. Γ will denote the graph associated with $\widetilde{X}$; $\mathscr{V}$ stays for its vertices.

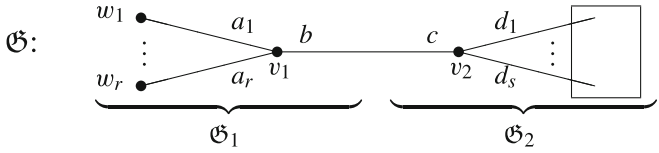
Using formulae of Durfee we can express σ in terms of geometric genus (via the link): $\sigma + 8p_g + K^2 + |\mathscr{V}| = 0$, cf. 6.9.7.

9.2.15 Assume that Γ has only one node. Then by 8.7.29 (applied for the central vertex) its geometric genus is $\mathrm{pc}(Z_{0,\{v\}}(t))$, which agrees with the geometric genus of the weighted homogeneous singularity with the same graph, cf. 8.7.9. Hence we can assume that (X, o) admits a good $\mathbb{C}^*$-action. In this case, by 5.1.33 it is of Brieskorn–Hamm type, hence the identity follows from the proofs listed in 9.2.

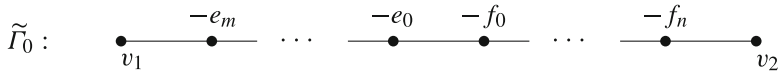
In the sequel we assume that Γ has at least two nodes.

Let v_1 be an 'end-node' of Γ (end vertex of the minimal subgraph which contains all the nodes), and v_2 be another node which is nearest to v_1.

Let $\mathfrak{G}$ be the splice diagram associated with Γ, cf. 5.3.10. We decompose $\mathfrak{G}$ into two splice diagrams $\mathfrak{G}_1$ and $\mathfrak{G}_2$, where $\mathfrak{G}_i \ni v_i$.



Let Γ_i be the minimal plumbing graph associated with $\mathfrak{G}_i$ $(i = 1, 2)$. The maximal string Γ_0 in Γ, which connects v_1 and v_2, is recovered from $\mathfrak{G}_1$ and $\mathfrak{G}_2$ as follows, cf. 5.3.10. Set $a := \prod_{i=1}^{r} a_i$ and $d := \prod_{i=1}^{s} d_i$, and set $a/b = [[e_0, \dots, e_m]]$ (with $e_0 \geq 1$ and $e_i \geq 2$ for $i \geq 1$) and similarly $d/c = [[f_0, \dots, f_n]]$. Next, consider the string:



This is not 'minimal', either $e_0 = 1$ or $f_0 = 1$. Then we successively blow down (-1)-vertices and at some moment inevitably a vertex with weight 0 appears; then we make a 0-absorption. In this way we obtain Γ_0. On the other hand, the corresponding maximal string (with determinant c) in Γ_2 has the form:

9.2.16 In our inductive procedure the following fact is crucial. For any splice type singularity with fixed (not necessarily minimal) good resolution graph Γ, consider any connected subgraph $\Gamma' \subset \Gamma$ (which is not necessarily unimodular). Let E' denote the reduced connected exceptional curve corresponding to Γ'. Then a neighborhood of E' satisfies the end curve condition (see 8.5.15) (hence Γ' satisfies the monomial condition too). This allows us to apply induction, and the inductive step 8.7.29 shows that $p_g(X, o)$ is topological. We write $p_g(X, o) = p_g(\Gamma)$.

In the proof we have to handle the following situation: for λ and $K^2 + |\mathcal{V}|$ the (topological) additivity formulae follow from the splice decomposition, while (the analytic) Okuma's additivity formula follows from the operation $\Gamma \rightsquigarrow \Gamma \setminus v$, and *it can be applied only for negative definite plumbing graphs, which satisfy the monomial condition*. Since the algorithm which describes the graph of a splicing (from the two components) involves non-negative definite plumbing calculus, the application of the geometric genus additivity needs some extra care.

9.2.17 Here are the additivity formulae what we will use. If $c(\Gamma)$ denotes $K^2+|\mathcal{V}|$, then $c(\Gamma) = c(\Gamma_1) + c(\Gamma_2) - 2\mu_1\mu_2$, where μ_i is the Milnor number of the knot $L_i \subset M(\mathfrak{G}_i)$ along which the splicing is performed, cf. 6.3.15. Next, $\lambda(\Gamma) =$

$\lambda(\Gamma_1)+\lambda(\Gamma_2)$, cf. 9.1.10. Hence, in order to prove CIC (that is, $8\lambda+8p_g+c(\Gamma)=0$) inductively, we need to verify

$$p_g(\Gamma) = p_g(\Gamma_1) + p_g(\Gamma_2) + \mu_1\mu_2/4. \tag{9.2.5}$$

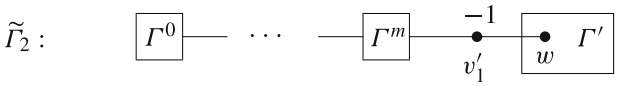
This will follow from the p_g-additivity, once we create the compatibility between Γ_i and several negative definite graphs obtained from Γ and Γ_i. [In this way we provide the prototype of a construction, which replaces the topological—'non-analytical'—splice decomposition with 'analytic' steps $\Gamma \rightsquigarrow \Gamma \setminus v$, involving negative definite plumbing graphs satisfying the monomial condition.]

9.2.18 Let $\Gamma' \subset \Gamma$ be that connected component of $\Gamma \setminus v_1$ which contains v_2, and denote by w that vertex in Γ' which is connected by v_1 in Γ. Let $Z^{\Gamma}_{\{v_1\}}(t)$ be the series appearing in the additivity formula 8.7.29. Since $p_g(\Gamma'') = 0$ for all other connected components Γ'' of $\Gamma \setminus v_1$ (since they are strings), by 8.7.29 we have

$$p_g(\Gamma) = \mathrm{pc}(Z^{\Gamma}_{\{v_1\}}(t)) + p_g(\Gamma'). \tag{9.2.6}$$

The graphs Γ' and Γ_2 can be compared as follows.

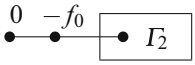
Lemma 9.2.19 *Consider the following graph $\widetilde{\Gamma}_2$ which satisfies* (1)–(4) *below:*



(1) the subgraph on the left hand side of w is a chain;
(2) Γ^0 consists of (e_0-1) vertices; the right-end has weight -3, all the others -2;
(3) Γ^m consists of (e_m-2) vertices, all of them with weight -2;
(4) for $0<i<m$, Γ^i consists of (e_i-2) vertices, the right-end has weight -3 and all the others -2.

Then $\widetilde{\Gamma}_2$ is negative definite, whose associated minimal graph is Γ_2.

Proof First replace in $\widetilde{\Gamma}_2$ the string Γ_0 of Γ' by $\widetilde{\Gamma}_0$. Hence, the above string of $\widetilde{\Gamma}_2$ from the left hand side of w is glued to $\widetilde{\Gamma}_0$. Then blow down repeatedly the (-1)-vertices and use the 0-absorption which identifies



with Γ_2. □

Corollary 9.2.20 *The (non-minimal) resolution graph $\widetilde{\Gamma}_2$ represents the splice diagram $\mathfrak{G}_2$, it has Γ' as a subgraph, and it supports a splice type singularity.*

Proof We only have to show that $\widetilde{\Gamma}_2$ satisfies the monomial condition. Let u be a node of Γ_2, let C'' be a branch of it (connected component of $\Gamma_2 \setminus u$), which contains the distinguished string involved in the splicing (otherwise the condition is trivial). Let C be the branch of u in Γ which contains C''. Then there exists an effective cycle A supported on C so that $E_u^* + A$ is monomial in Γ. Then the restriction $A|\Gamma'$ to Γ' has the property that $E_u^*(\Gamma') + A|\Gamma'$ is monomial and has negative intersection with E_w. Then $A|\Gamma'$ can be extended easily on the string considered in (9.2.19) to an effective cycle B on $\widetilde{\Gamma}_2$ such that $E_u^*(\widetilde{\Gamma}_2) + B$ is monomial in $\widetilde{\Gamma}_2$. □

Let $Z^{\widetilde{\Gamma}_2}_{\{v_1'\}}(t)$ be the topological series associated with v_1' in $\widetilde{\Gamma}_2$. Then by 8.7.29

$$p_g(\Gamma_2) = \mathrm{pc}(Z^{\widetilde{\Gamma}_2}_{\{v_1'\}}(t)) + p_g(\Gamma'). \tag{9.2.7}$$

Finally, the star-shaped Γ_1 automatically satisfies the monomial condition, hence

$$p_g(\Gamma_1) = \mathrm{pc}(Z^{\Gamma_1}_{\{v_1\}}(t)). \tag{9.2.8}$$

As a consequence, from (9.2.6), (9.2.7), (9.2.8) one gets:

$$p_g(\Gamma) - p_g(\Gamma_1) - p_g(\Gamma_2) = \mathrm{pc}(Z^{\Gamma}_{\{v_1\}}) - \mathrm{pc}(Z^{\widetilde{\Gamma}_2}_{\{v_1'\}}) - \mathrm{pc}(Z^{\Gamma_1}_{\{v_1\}}). \tag{9.2.9}$$

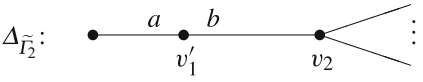
Proposition 9.2.21 $pc(Z^{\Gamma}_{\{v_1\}}) - pc(Z^{\widetilde{\Gamma}_2}_{\{v_1'\}}) - pc(Z^{\Gamma_1}_{\{v_1\}}) = \mu_1\mu_2/4.$

Proof We compute the corresponding series using the multiplicities $m_{uu'} = -I^{-1}_{uu'}$ expressed in terms of the decorations of the splice diagrams (cf. 5.3.19).

Let $\mathscr{V}_2$ be the set of vertices of Γ_2. Then in Γ one has $m_{v_1v_1} = ab$ and $m_{v_1w_i} = ab/a_i$. Consider the integers $p_i := m_{v_1w_i}/b = a/a_i$ and $q_u := m_{v_1u}/a$ for all $1 \le i \le r$ and $u \in \mathscr{V}_2$ respectively. Finally, set $g(t) = \prod_{u\in\mathscr{V}_2}(1 - t^{q_u})^{\kappa_u - 2}$. Then

$$Z^{\Gamma}_{\{v_1\}}(t) = \frac{g(t^a)(1-t^{ab})^{r-1}}{\prod_{i=1}^r (1-t^{p_ib})}, \quad Z^{\Gamma_1}_{\{v_1\}}(t) = \frac{(1-t^{ab})^{r-1}}{(1-t^a)\prod_{i=1}^r (1-t^{p_ib})}. \tag{9.2.10}$$

The non-minimal splice diagram associated with $\widetilde{\Gamma}_2$, having v_1' as a node, is



Since m_{v_1u} in Γ and $m_{v_1'u}$ in $\widetilde{\Gamma}_2$ are the same for any $u \in \mathcal{V}_2$,

$$Z^{\widetilde{\Gamma}_2}_{\{v_1'\}}(t) = \frac{g(t^a)}{1-t^b}. \tag{9.2.11}$$

We define the functions $Q_1(t)$ and $Q_2(t)$ by

$$Q_1(t) = \frac{(1-t^a)^{r-1}}{\prod_{i=1}^r (1-t^{p_i})} - \frac{1}{1-t}, \quad Q_2(t) = g(t) - \frac{1}{1-t}.$$

Then from (9.2.10) and (9.2.11) we have

$$Z^{\Gamma}_{\{v_1\}} - Z^{\widetilde{\Gamma}_2}_{\{v_1'\}} - Z^{\Gamma_1}_{\{v_1\}} = Q_1(t^b)Q_2(t^a) - \frac{1}{(1-t^a)(1-t^b)}. \tag{9.2.12}$$

On the other hand, by A'Campo's formula 4.1.22, the characteristic polynomials $\Delta_i(t)$ of the knots $L_i \subset M(\mathfrak{G}_i)$ can be related with the polynomial Q_i by $Q_i(t) = (\Delta_i(t) - 1)/(1-t)$. Since $\Delta_i(1) = 1$, Q_i's are polynomials. Hence $Q_i(1) = -\Delta_i'(1) = -\mu_i/2$ (cf. 4.1.23), and $\mathrm{pc}(Q_1(t^b)Q_2(t^a)) = Q_1(1)Q_2(1) = \mu_1\mu_2/4$. □

9.3 The Casson–Walker Invariant

The Casson invariant defined for integral homology spheres has an extension to rational homology spheres given by Walker [741]. We will refer to it as the Casson–Walker invariant. Similarly to the Casson invariant (cf. 9.1), we will adopt a working definition, valid for negative definite plumbed 3-manifolds, based on a surgery formula of [385]. ♣

Definition 9.3.1 Assume that $H = H_1(M(\Gamma), \mathbb{Z})$ is finite. We define

$$-\frac{24}{|H|} \cdot \lambda(M(\Gamma)) = \sum_{v \in \mathcal{V}} (E_v^2 + 3) + \sum_{v \in \mathcal{V}} (2 - \kappa_v)(E_v^*, E_v^*). \tag{9.3.1}$$

9.3.2 Again, cf. 9.1.8, a direct verification shows that the above expression depends only on $M(\Gamma)$, that is, it is independent of the choice of the graph Γ.

Example 9.3.3 If M is a Seifert 3-manifold with $\nu \geq 3$ then using 3.5.2 and (2.2.7) one has

$$-\frac{24}{|H|} \cdot \lambda(M) = \frac{1}{e}\Big(2 - \nu + \sum_{j=1}^{\nu} \frac{1}{\alpha_j^2}\Big) + e + 3 - 12 \cdot \sum_{j=1}^{\nu} \mathbf{s}(\omega_j, \alpha_j). \tag{9.3.2}$$

Example 9.3.4 For a lens space one has $\lambda(L(n,q)) = n \cdot \mathbf{s}(q,n)/2$ (use the definition and (2.2.7)).

Remark 9.3.5 The CIC identity $\lambda(L_X) = \sigma(F)/8$, expected in the case $H = 0$, does not extend in the same form to hypersurfaces with rational homology sphere links. For example, in the case of A_{n-1} germs, one has $\lambda(L_X) = \lambda(L(n, n-1)) = -(n-1)(n-2)/24$, while $\sigma/8 = -(n-1)/8$.

In the rational homology sphere case there is no natural splice decomposition and surgery operation, hence we will have no (immediate) analogues for the Casson–Walker invariant of the splice formula 9.1.10 and of surgery formula 9.1.18. However, we will have another type of 'additivity formula' (more in the spirit of Okuma's analytic additivity formulae). This is the subject of the next discussion.

9.3.A *Additivity Formulae for λ and $K^2 + |\mathscr{V}|$*

Assume that Γ is a connected negative definite plumbing graph with $M(\Gamma)$ rational homology sphere, $v \in \mathscr{V}$, and $\Gamma \setminus \{v\}$ has connected components $\{\Gamma_i\}_i$. Then for the invariants $I(\Gamma) = \lambda(\Gamma)$ and $I(\Gamma) = (K+2l')^2 + |\mathscr{V}|$ we determine the 'correction term' $I(\Gamma) - \sum_i I(\Gamma_i)$. ♣

9.3.6 Let Γ be a connected negative definite plumbing tree, such that $M(\Gamma)$ is a rational homology sphere (though some of the statements below can be generalized to a more general setup).

For $v, w \in \mathscr{V}$ we define $m_{vw} := -(E_v^*, E_w^*) = -(I^{-1})_{vw} \in \mathbb{Q}_{>0}$, and let κ_v be the valency of v in Γ as usual. Then for any fixed $v \in \mathscr{V}$ we set

$$\alpha_v := \sum_{w \in \mathscr{V}} (\kappa_w - 2) m_{vw}, \qquad \beta_v := \sum_{w \in \mathscr{V}} (\kappa_w - 2) m_{vw}^2. \tag{9.3.3}$$

Via the identities from 4.1.23, α_v and β_v can be expressed in terms of the first and second derivative of $(1-t)^2 \cdot Z_{\{v\}}(t)$ evaluated at $t = 1$, where $Z_{\{v\}}(t)$ is (by its definition) $\prod_w (1 - t^{m_{vw}})^{\kappa_w - 2}$.

9.3.7 For a fixed vertex v of Γ, we denote the connected components of $\Gamma \setminus v$ by Γ_i. Let v_i denote the unique vertex of Γ_i which is adjacent to v in Γ.

We indicate by a subscript i when we consider an invariant in Γ_i, instead of Γ. For example, we write L_i, L_i', $m_{uw,i}$, $E_{w,i}^* \in L_i'$, $\mathscr{V}_i := \mathscr{V}(\Gamma_i)$, and so on. We regard L_i as a sublattice of L via the natural inclusion $H_2(P(\Gamma_i)) \to H_2(P(\Gamma))$, $E_{w,i} \mapsto E_w$ for any $w \in \mathscr{V}_i$. It admits the natural extension $L_i \otimes \mathbb{Q} \to L \otimes \mathbb{Q}$.

Definition 9.3.8 The *restriction* $R_i : L' \to L'_i$ is the homomorphism induced by the inclusion $P(\Gamma_i) \hookrightarrow P(\Gamma)$ on second cohomology groups. In other words, $R_i(E^*_w) = E^*_{w,i}$ if $w \in \mathcal{V}_i$, and $R_i(E^*_w) = 0$ otherwise.

By projection formula $(R_i(x), x_i)_{L'_i} = (x, x_i)_{L'}$ for any $x \in L'$ and $x_i \in L'_i$. Furthermore, for $x = \sum_w r_w E_w = \sum_w n_w E^*_w$, and its restrictions $x_i := R(x)$ one has

$$x_i := R_i(x) = \sum_{w \in \mathcal{V}_i} n_w E^*_{w,i} = -r_v E^*_{v_i,i} + \sum_{w \in \mathcal{V}_i} r_w E_w. \tag{9.3.4}$$

Example 9.3.9 R_i maps $\mathrm{Char}(\Gamma)$ into $\mathrm{Char}(\Gamma_i)$, and the canonical characteristic element K of Γ into the canonical characteristic element K_i of Γ_i.

Lemma 9.3.10 *Any $x \in L'$ and its restrictions $x_i := R_i(x)$ satisfy*

$$x - \sum_i x_i = -\frac{(x, E^*_v)}{m_{vv}} E^*_v, \qquad x^2 - \sum_i x_i^2 = -\frac{(x, E^*_v)^2}{m_{vv}}.$$

Proof For the first identity check that the intersection of both sides with E^*_v and the E_w for $w \neq v$ are equal. For the second one take the square of the first identity. □

Theorem 9.3.11 ([77]) *For any $l' = \sum_w r_w E_w \in L'$*

$$((K + 2l')^2 + |\mathcal{V}|) - \sum_i ((K_i + 2R_i(l'))^2 + |\mathcal{V}_i|) = 1 - \frac{(\alpha_v + 1 - 2r_v)^2}{m_{vv}}, \tag{9.3.5}$$

$$\frac{24}{|H|} \cdot \lambda - \sum_i \frac{24}{|H_i|} \cdot \lambda_i = -3 + \frac{1 - \beta_v}{m_{vv}}. \tag{9.3.6}$$

Proof Since $K + E = \sum_w (2 - \kappa_w) E^*_w$ one gets $(K, E^*_v) = \alpha_v + 1$. Then 9.3.10 applied for $x = K + 2l'$ gives (9.3.5).

For the second identity first we apply 9.3.10 to $x := E^*_w$ for some $w \neq v$. Then $x_i = E^*_{w,i}$ if $w \in \mathcal{V}_i$, and $x_i = 0$ otherwise. Hence, 9.3.10 reads as

$$-m_{ww} + m_{ww,i} = -m^2_{vw}/m_{vv}, \quad w \in \mathcal{V}_i.$$

Next, we apply 9.3.10 to $x := E_v$. Then $x_i = -E^*_{v_i,i}$ and $E_v^2 + \sum_i m_{v_i v_i, i} = -1/m_{vv}$. Then (9.3.6) follows by the definition of λ and the above identities. □

Example 9.3.12 Consider the **surgery 3-manifold** $M = S^3_{-d}(\#_i K_i)$ as in 3.5.12 with $d > 0$ and K_i algebraic knot with Alexander polynomial Δ_i. Let $\Delta(t) =$

$\prod_i \Delta_i(t)$ and $\mu = \sum_i \mu_i = 2\delta$ as in 8.6.B. Then $K^2 + |\mathscr{V}|$ can be directly computed by the adjunction formula in the minimal resolution, see 6.3.13.

Using (9.3.5), where v is the vertex v_+, one has the following data: $m_{vv} = 1/d$ (use 2.2.5(5)) and $\alpha_v = (\mu - 2)/d$ (use A'Campo's formula 4.1.22). One gets $K^2 + |\mathscr{V}| = 1 - (\mu - 2 + d)^2/d$. If $\mu = (d-1)(d-2)$, as in the case of superisolated singularity links, this reads as $K^2 + |\mathscr{V}| = 1 - d(d-2)^2$.

Next, we determine λ for $M = S^3_{-d}(\#_i K_i)$. By A'Campo's formula 4.1.22 one has $Z_{\{v\}}(t) \cdot (1 - t^{1/d})^2 = \Delta(t^{1/d})$. Hence, by (9.3.3) and by the second derivative formula 4.1.23 one gets $d^2\beta_v = 12\Delta_s''(1) - 2$. Therefore, (9.3.6) reads as

$$24 \cdot \lambda = (d-1)(d-2) - 12 \cdot \Delta_s''(1) = (d-1)(d-2) + 3\mu(\mu-2) - 12 \cdot \Delta''(1).$$

If $\mu = (d-1)(d-2)$ then this transforms into $24\lambda = \mu(3\mu - 5) - 12 \cdot \Delta''(1)$.

9.4 The Reidemeister–Turaev Torsion. Generalities

This section contains the general definition of the sign-refined torsion associated with Euler structures (equivalently, with spin^c-structures). The presentation is based on the books of Turaev and work of Nicolaescu and Ranicki, see [519, 703, 705] and the references therein. ♣

9.4.1 Torsion of Finite Chain Complexes We fix a field F. If in an F-vector space we have two ordered bases c and c', then the determinant of the transformation matrix (the vectors of c expressed as linear combinations of c') is denoted by $[c/c'] \in F^* := F \setminus \{0\}$. We say that the bases c and c' are equivalent if $[c/c'] = 1$.

Consider a chain complex with finite dimensional components and of finite length $C_* = (C_*, \partial_*) : C_m \to \cdots \to C_0$. We assume that each C_i admits a distinguished basis c_i. Similarly, we assume that each homology group $H_i(C_*)$ has a distinguished basis, let h_i be an ordered set of vectors in $\ker(\partial_{i-1} : C_i \to C_{i-1})$, which projects to it. Let b_i be a set of ordered vectors in C_i such that $\partial_{i-1}(b_i)$ is a basis in $\mathrm{Im}(\partial_{i-1})$. Then for every i the ordered set $\partial_i(b_{i+1})h_i b_i$ is a basis in C_i which can be compared with c_i. The *torsion of* C_* is defined as

$$\tau(C_*) := (-1)^{|C|} \cdot \prod_{i=0}^{m} [\partial_i(b_{i+1})h_i b_i / c_i]^{(-1)^{i+1}} \in F^*,$$

where $|C| = \sum_{i=0}^m (\sum_{j=0}^i \dim C_j)(\sum_{j=0}^i \dim H_j(C_*))$. It depends only on the equivalence classes of the chosen bases in C_i and $H_i(C_*)$, and it is independent of the choice of h_i and b_i. If C_* is acyclic, then $|C| = 0$, and $\tau(C_*)$ associated with the classes of $\{c_i\}_i$ is $\tau(C_*) = \prod_i [\partial_i(b_{i+1})b_i/c_i]^{(-1)^{i+1}}$.

9.4.2 The Reidemeister Torsion Next, we fix a finite connected CW-complex Y. Set $H_1 = H_1(Y, \mathbb{Z})$ and fix a ring homomorphism $\rho : \mathbb{Z}[H_1] \to F$. Let $Y_a \to Y$ be the universal abelian covering of Y with its natural CW-complex structure. H_1 acts on Y_a via covering transformations, which permute the cells. Hence the chain complex $C_*(Y_a, \mathbb{Z})$ becomes a complex of (left) $\mathbb{Z}[H_1]$-modules. We regard F as a (right) $\mathbb{Z}[H_1]$-module via $f \cdot h := f\rho(h)$, and we define $C_*^\rho(Y) = F \otimes_{\mathbb{Z}[H_1]} C_*(Y_a, \mathbb{Z})$. $H_*(C_*^\rho(Y))$ is called the *ρ-twisted homology of Y*.

A set of cells in Y_a is called *fundamental* if over each cell of Y lies exactly one cell of this set. Take a fundamental set of cells, order them and orient them in an arbitrary way. They constitute a free generating set of $C_*(Y_a, \mathbb{Z})$ over $\mathbb{Z}[H_1]$, and a basis of $C_*^\rho(Y)$ over F. If $C_*^\rho(Y)$ is acyclic then define its torsion $\tau(C_*^\rho(Y))$ (as in 9.4.1) using this basis. Different choices of the fundamental cells, orientations and orderings provide a multiplicative ambiguity with $\pm\rho(h)$, $h \in H_1$. The class (set) $\tau^\rho(Y) := \pm\rho(H_1)\tau(C_*^\rho(Y)) \in F^*/\{\pm\rho(H_1)\}$ is called the *Reidemeister* (or *Reidemeister–Franz*, or *Reidemeister–Franz–de Rham*, or just *R-*) *torsion* of Y associated with ρ. If $C_*^\rho(Y)$ is not acyclic then we set $\tau^\rho(Y) := 0$.

The torsion behaves invariantly with respect to cellular subdivision of Y.

Remark 9.4.3 (Torsion in Commutative Rings) In the above discussion the field F can be replaced by a commutative ring A (with invertible elements A^*), and ρ by a ring homomorphism $\mathbb{Z}[H_1] \to A$. In this case the R-torsion is a class in $A^*/\{\pm\rho(H_1)\}$.

Assume that corresponding to two representations $\rho_i : \mathbb{Z}[H_1] \to A_i$ $(i = 1, 2)$ we have two R-torsions $\tau^{\rho_i}(Y)$. If $\rho_2 = \psi \circ \rho_1$ for some ring homomorphism $\psi : A_1 \to A_2$, then one has the *functoriality property* $\psi(\tau^{\rho_1}(Y)) = \tau^{\rho_2}(Y)$.

9.4.4 The Universal (Maximal) Abelian Torsion [705, I.3] Consider Y and H_1 as in 9.4.2. Let $Q(H_1)$ denote the ring of fractions (localization of $\mathbb{Z}[H_1]$ by all non-zerodivisors). The monomorphism $\mathbb{Z}[H_1] \hookrightarrow Q(H_1)$ can be considered as a representation, denoted by ρ_a. Usually $C_*^{\rho_a}(Y)$ is not acyclic, hence one has to modify either the ring or the representation in order to guarantee the acyclicity of $C_*^{\rho_a}(Y)$.

For example, assume that $H_1 = \mathbb{Z}_n$, hence $\mathbb{Z}[H_1] = \mathbb{Z}[T]/(T^n - 1)$. Then $Q(H_1) = \mathbb{Q}[H_1] = \mathbb{Q}[T]/(T^n - 1)$, cf. [703, Corollary 12.7]. In this case $\mathbb{Q}[H_1]$ has several 'unpleasant' properties, e.g. the element $T - 1$ is a non-trivial zerodivisor, or, a (sub)complex of type $\mathbb{Q}[H_1] \xrightarrow{T-1} \mathbb{Q}[H_1]$ is not acyclic. This implies that in several geometric cases the corresponding chain complex is not acyclic. However, instead of $\mathbb{Q}[H_1]$ one can take the ring $\bar{Q}(H_1) = \mathbb{Q}[H_1]/N$, where $N := 1 + T + \cdots + T^{n-1}$, and the corresponding homomorphism $\bar{\rho}_a : \mathbb{Z}[H_1] \to \bar{Q}(H_1)$. The advantage of $\bar{Q}(H_1)$ is that in this ring $T - 1$ (and any $T^q - 1$ with $(n, q) = 1$) is invertible. (Indeed, $(T - 1)(1 + 2T + \cdots + nT^{n-1}) = n$ in $\bar{Q}(H_1)$). Hence the torsion in $\bar{Q}(H_1)$ can be non-zero even when it is zero in $Q(H_1)$. In fact $\mathbb{Q}[H_1]$ is the direct sum $\mathfrak{aug} \oplus \mathbb{Q}\langle N\rangle$ of two ideals, the augmentation ideal $\mathfrak{aug} = (T - 1)\mathbb{Q}[H_1]$ and the 1-dimensional $\mathbb{Q}\langle N\rangle$. Hence $\mathbb{Q}[H_1] = \bar{Q}(H_1) \oplus \mathbb{Q}$ (or $\mathfrak{aug} = \bar{Q}(H_1)$), and the second component is responsible for the non-acyclicity in $Q(H_1)$ in several cases.

Therefore, in the case of ring representations, we obtain a finer invariant if we try to decompose the ring in a direct sum and we take the corresponding torsion for each component independently.

In general, we proceed as follows. According to Turaev, $Q(H_1)$ splits as a finite direct sum of fields. First, consider the torsion part $H := \mathrm{Tors}(H_1)$. Each character $\chi : H \to \mathbb{C}^*$ extends to a $\mathbb{Q}$-linear ring homomorphism $\rho_\chi : \mathbb{Q}[H] \to \mathbb{C}$, whose image is a cyclotomic field $K_\chi \subset \mathbb{C}$. Two characters χ_1 and χ_2 are equivalent if $K_{\chi_1} = K_{\chi_2}$ and ρ_{χ_1} is a composition of ρ_{χ_1} by a Galois automorphism of K_{χ_1} over $\mathbb{Q}$. Then a complete family of representatives $\{\chi_i\}_{i=1}^m$ of equivalence classes provides an isomorphism $(\rho_{\chi_1}, \dots, \rho_{\chi_m}) \to \mathbb{Q}[H] \to \oplus_{i=1}^m K_{\chi_i}$. Thus, $Q(H) = \mathbb{Q}[H]$ too.

In general, if $Fr := H_1/H$ is the free part of H_1, then $\mathbb{Q}[H_1] = (\mathbb{Q}[H])[Fr] = \oplus_i K_{\chi_i}[Fr]$. Since $K_{\chi_i}[Fr]$ is an integral domain, an element of $\mathbb{Q}[H_1]$ is a non-zerodivisor if and only if all its projections to the summands $K_{\chi_i}[Fr]$ are non-zero. Hence, $Q(H_1) = \oplus_i F_i$, where F_i is the field of fractions of $K_{\chi_i}[Fr]$.

Denote by ρ_i the composition $\mathbb{Z}[H_1] \to Q(H_1) \to F_i$. Then the *universal (maximal) abelian torsion* is

$$\tau_a(Y) = \sum_i \tau^{\rho_i}(Y) \in (\oplus_i F_i^*)/\{\pm(\oplus_i \rho_i)(H_1)\}. \tag{9.4.1}$$

Example 9.4.5 (The Alexander Polynomial of a Link as Torsion) Assume that M is an oriented compact integral homology sphere and $K \subset M$ a knot in it. Set $Y := M \setminus K$, and $H_1 := H_1(Y, \mathbb{Z})$. Then $H_1 = \mathbb{Z}$, $Q(H_1) = \mathbb{Q}(t)$ and the complex $C_*^{\rho_a}$ is acyclic. In particular, the universal abelian torsion is well defined. Moreover, by [427], it equals $\Delta(t)/(t-1) \in \mathbb{Q}(t)$ (modulo $\pm\{t^n\}_{n\in\mathbb{Z}}$), where $\Delta(t)$ is the Alexander polynomial of $K \subset M$ (cf. 9.1.2).

Example 9.4.6 (The Torsion of Lens Spaces) Consider the lens space $L(n,q) = S^3/\mathbb{Z}_n$, where the generator T of $\mathbb{Z}_n$ acts on $S^3 = \{|z_1|^2+|z_2|^2 = 1\}$ by $T(z_1, z_2) = (\xi z_1, \xi^q z_2)$, $\xi = e^{2\pi i/n}$. $L(n,q)$ is a CW-complex with four cells, the images of the cells $c_i \subset S^3$ $(0 \le i \le 3)$ defined as follows: (i) $e_0 = (1,0)$, (ii) $e_1 = (e^{it}, 0)_t$, (iii) $e_2 = (z_1, \sqrt{1-|z_1|^2})_{z_1}$, (iv) $e_3 = (z_1, e^{it}\sqrt{1-|z_1|^2})_{z_1})_{t,z_1}$, where $0 < t < 2\pi/n$ and $|z_1| < 1$. The complex $C_*(L(n,q)_a, \mathbb{Z})$ is freely generated by the fundamental cells $\{e_i\}_i$ over $\mathbb{Z}[\mathbb{Z}_n]$. The boundary relations are: (i) $\partial e_1 = (T-1)e_0$, (ii) $\partial e_2 = (1+T+\cdots+T^{n-1})e_1$, (iii) $\partial e_3 = (T^{q'}-1)e_2$, where $qq' \equiv 1 \pmod n$.

Let us assume that the representation $\rho : \mathbb{Z}[\mathbb{Z}_n] \to \mathbb{C}$ sends T into a n-root of unity $\xi \neq 1$. Then $\{e_i\}_i$ in $C_*^\rho(L(n,q))$ form a basis over $\mathbb{C}$, and $\partial e_1 = (\xi-1)e_0$, $\partial e_2 = 0$, $\partial e_3 = (\xi^{q'}-1)e_2$, hence the complex is acyclic. Moreover,

$$\tau^\rho(L(n,q)) = (\xi-1)^{-1}\cdot(\xi^{q'}-1)^{-1} \pmod{\pm\xi^k}.$$

For the representation $\phi : \mathbb{Z}[\mathbb{Z}_n] \to \mathbb{C}$ with $\rho(T) = 1$ we have $\tau^\rho = 0$.

Corresponding to the representation $\bar{\rho}_a : \mathbb{Z}[\mathbb{Z}_n] \to \bar{Q}(\mathbb{Z}_n)$ we get

$$\tau^{\bar{\rho}_a}(L(n,q)) = (T-1)^{-1} \cdot (T^{q'}-1)^{-1} \pmod{\pm T^k}.$$

9.4.7 Classification of Lens Spaces If the lens spaces $L(n_1,q_1)$ and $L(n_2,q_2)$ are homotopic then their fundamental groups are isomorphic, hence $n_1 = n_2$. Whitehead in [750] proved that $L(n,q_1)$ and $L(n,q_2)$ are homotopic if and only if there exists $r \in \mathbb{Z}_n^*$ such that $q_2 \equiv \pm q_1 r^2 \pmod{n}$. Indeed, there exists a homotopy equivalence $h : L(n,q_1) \to L(n,q_2)$ with $h_*(T) = T^r$ if and only if there exists a $\Lambda := \mathbb{Z}[\mathbb{Z}_n]$-module chain equivalence $h_*(C_*(L(n,q_1)_a,\mathbb{Z}) \to C_*(L(n,q_2)_a,\mathbb{Z})$, that is

$$\begin{array}{ccccccc}
\Lambda & \xrightarrow{T^{rq_1'}-1} & \Lambda & \xrightarrow{N} & \Lambda & \xrightarrow{T^r-1} & \Lambda \\
\downarrow{\scriptstyle u_3} & & \downarrow{\scriptstyle u_2} & & \downarrow{\scriptstyle u_1} & & \downarrow{\scriptstyle u_0=1} \\
\Lambda & \xrightarrow{T^{q_2'}-1} & \Lambda & \xrightarrow{N} & \Lambda & \xrightarrow{T-1} & \Lambda
\end{array}$$

where $N = 1 + T + \cdots + T^{n-1}$, and u_i are invertible in Λ. Then necessarily $u_1 = u_2 = 1 + T + \cdots + T^{r-1}$, and u_3 must satisfy (from the first commutative square)

$$u_3(T-1)(T^{q_2'}-1) = (T^r-1)(T^{rq_1'}-1). \tag{9.4.2}$$

Lemma 9.4.8 (See e.g. [429]) *There exists a unit u in Λ satisfying the equation $u\prod_{i=1}^{k}(T^{a_i}-1) = \prod_{i=1}^{\ell}(T^{b_i}-1)$ if and only if $k = \ell$ and $\prod_i a_i \equiv \pm\prod_i b_i$ (mod n).*

Hence, (9.4.2) implies that $q_2' \equiv \pm r^2 q_1' \pmod{n}$. Since $q' \equiv (q')^2 q$ holds, one also has $q_2 \equiv \pm r^2 q_1 \pmod{n}$ for a certain r with $(r,n) = 1$.

This can be compared with the isomorphism of the linking forms (cf. 6.10.2). The linking form $b_{L(n,q)} : \mathbb{Z}_n \times \mathbb{Z}_n \to \mathbb{Q}/\mathbb{Z}$ is $(T^a, T^b) \mapsto abq/n \in \mathbb{Q}/\mathbb{Z}$. Therefore, $b_{L(n,q_1)}$ and $b_{L(n,q_2)}$ are isomorphic (by $T \mapsto T^r$, $(r,n) = 1$) if and only if $q_2 \equiv r^2 q_1 \pmod{n}$. Hence, the homotopy equivalence of lens spaces is the same as the isomorphism of their linking form up to orientation (that is, up to a sign $\pm$).

Two CW-complexes are combinatorially equivalent if they have (rectilinear) subdivisions which are (simplicially) isomorphic. The combinatorial classification of lens spaces was established by Reidemeister in [588].

Theorem 9.4.9 *The following facts are equivalent: (i) $L(n,q_1)$ and $L(n,q_2)$ are combinatorially equivalent; (ii) their Reidemeister torsions τ^{ρ} (as in 9.4.6) are equal; (iii) (9.4.2) is valid in Λ with u_3 of the form $\pm T^k$; (iv) $q_2 \in \{\pm q_1, \pm q_1'\}$.*

[The implication (iii)⇒(iv) follows by a lemma similar to 9.4.8, called 'Franz Independence Lemma', cf. [429, 12.6].]

By a result of Moise [437] two 3-dimensional manifolds are combinatorially equivalent if and only if they are homeomorphic, that is, they are topologically equivalent up to orientation. (Since in dimension 3 there are no exotic smooth structures, this is equivalent to the C^∞ equivalence up to orientation as well.) If in 9.4.9 we take into account the orientation as well we obtain: $L(n, q_1)$ and $L(n, q_2)$ are equivalent by an orientation preserving homeomorphism if and only if $q_2 \in \{q_1, q_1'\}$. For example, $L(7, 1)$ and $L(7, 2)$ are homotopic, but not homeomorphic (and they have different torsions).

9.4.10 Recall that the famous Hauptvermutung of Tietze conjectured that for any triangulable space any two triangulations have a common refinement. In different words, homeomorphic triangulable spaces are also combinatorially equivalent. This is true in dimension 3 but it is false in dimensions ≥ 6, cf. [426]. Milnor considered the finite simplicial complex Y_q obtained from $L(7, q) \times \sigma^n$ by adjoining a cone over $L(7, q) \times \partial\sigma^n$, where σ^n is the n-simplex, and $n \geq 3$. Then, according to Milnor, Y_1 is homeomorphic to Y_2, but no finite cell subdivision of the simplicial complex Y_1 is isomorphic to a cell subdivision of Y_2 (the argument is based on a 'relative torsion').

9.4.11 Turaev's Sign-Refined Torsion [705] Consider the situation from 9.4.2. The torsion $\tau^\rho(Y) = \pm\rho(H_1)\tau(C_*^\rho(Y)) \in F^*/\{\pm\rho(H_1)\}$ has two types of ambiguities: a sign ambiguity $\pm$, and the ambiguity given by multiplication by elements of type $\rho(h)$, $h \in H_1$. First we eliminate the first ambiguity using an additional structure, a *homology orientation of* Y, which is an orientation ω of the real vector space $\oplus_i H_i(Y, \mathbb{R})$. Take a fundamental set of cells, order them and orient them in an arbitrary way, and fix a homomorphism $\rho : \mathbb{Z}[H_1] \to F$. Then 9.4.1 provides a torsion $\tau(C_*^\rho(Y)$. Since the fundamental cells are in bijection with the cells of Y, they induce an order and orientation of the cells of Y, hence a bases of the chain complex $C_*(Y, \mathbb{R})$. Fix basis h_i in $H_i(Y, \mathbb{R})$ such that the basis $h_0, h_1, \ldots$ has the same orientation as ω. $C_*(Y, \mathbb{R})$ together with $\{h_i\}_i$ gives a torsion $\tau(C_*(Y, \mathbb{R})) \in \mathbb{R}^*$; let its sign be $\tau_0 \in \{\pm 1\}$. Then $\tau^\rho(Y, \omega) := \tau_0\tau(C_*^\rho(Y) \in F^*$ is well-defined up to multiplication by $\rho(H_1)$. The class $\tau^\rho(Y, \omega) \in \rho(H_1)F^*$ is called the sign-refined torsion.

If M is an oriented manifold with $\partial M = \emptyset$ and of odd dimension m, then any fixed orientation induces automatically a homological orientation ω. It is determined by an arbitrary basis of $\oplus_{i<m/2} H_i(M, \mathbb{R})$ followed by the Poincaré dual basis in $\oplus_{i>m/2} H_i(M, \mathbb{R})$. In the sequel, for any 3-manifold with fixed orientation we consider this canonical ω in all sign-refined torsion considerations.

9.4.12 Turaev's Torsion Associated with Euler Structures [705] The $\rho(H_1)$ ambiguity can be eliminated by the help of an (any) Euler structure.

Additionally to the notations of 9.4.2 we fix a homology orientation ω as in 9.4.11 and an Euler structure $[u]$ as in 6.10.24. We assume that the Euler structure is realized by the choice of a set of fundamental cells of Y_a. Then using the very same set in both constructions 9.4.2 and 9.4.11 we obtain the torsion $\tau^\rho(Y, \omega, [u]) :=$

$\tau^\rho(Y,\omega) \in F^*$, which is independent of any choices of the fundamental set (whenever it determines $[u]$), and of the order and orientation. In this way, for any homology orientation ω and representation ρ we get a function $\tau^\rho : \mathrm{Eul}(Y) \to F^*$.

9.4.13 The Case of 3-Manifolds Assume that M is a closed connected 3-manifold without boundary with a fixed orientation, which determines a homological orientation ω as in 9.4.11. We assume that $H = H_1(M,\mathbb{Z})$ is finite, hence $Q(H) = \mathbb{Q}[H]$, cf. 9.4.4. Consider any triangulation of M (the next torsion invariant is independent of this choice by 9.4.10). Identify $\mathrm{Eul}(M)$ with $\mathrm{Spin}^c(M)$ as in 6.10.A. Consider the 'universal abelian representation' $\rho_a : \mathbb{Z}[H] \to Q(H) = \mathbb{Q}[H]$ and the corresponding 'universal abelian sign-refined torsion' (the analogue of (9.4.1) without the $\pm(\oplus_i \rho_i)(H)$-ambiguity, see 9.4.11 and 9.4.12):

$$\tau : \mathrm{Spin}^c(M) \to \mathbb{Q}[H]; \quad \sigma \mapsto \tau_\sigma = \sum_h \mathcal{T}_\sigma(h)h \quad (\mathcal{T}_\sigma(h) \in \mathbb{Q}). \tag{9.4.3}$$

Theorem 9.4.14 ([705]) *The torsion $\sigma \mapsto \tau_\sigma$ has the following properties:*

(a) ***Duality:*** *Consider the involution $\mathbb{Q}[H] \to \mathbb{Q}[H]$, given by $x = \sum_h a(h)h \mapsto \bar{x} := \sum_h a(h)h^{-1}$. Then $\tau_{\overline{\sigma}} = \overline{\tau_\sigma}$, or $\mathcal{T}_{\overline{\sigma}}(h^{-1}) = \mathcal{T}_\sigma(h)$.*

(b) ***H-equivariance:*** *$\tau_{h\sigma} = h\tau_\sigma$; that is, for any $g,h \in H$ one has $\mathcal{T}_{g\sigma}(gh) = \mathcal{T}_\sigma(h)$. In particular, for fixed $\sigma_0 \in Spin^c(M)$ the coefficients $\{\mathcal{T}_{\sigma_0}(h)\}_h$, or, for fixed $h_0 \in H$, the coefficients $\{\mathcal{T}_\sigma(h_0)\}_\sigma$, determine the whole τ.*

(c) ***τ determines the linking form*** *b_M: For any σ one has*

$$\mathcal{T}_\sigma(1) - \mathcal{T}_\sigma(h) - \mathcal{T}_\sigma(g) + \mathcal{T}_\sigma(hg) = -b_M(h,g) \ (mod\ \mathbb{Z}).$$

(Here H is written multiplicatively.) Hence, for any $\sigma \in Spin^c(M)$ the map

$$q_\sigma : H \to \mathbb{Q}/\mathbb{Z}, \quad q_\sigma(h) = \mathcal{T}_\sigma(1) - \mathcal{T}_\sigma(h) \ (mod\ \mathbb{Z})$$

is a quadratic function associated with b_M.

(d) ***τ determines the quadratic functions*** *[489]: The correspondence $Spin^c(M) \to Q^c(M)$ given by $\sigma \mapsto q_\sigma$ is a bijection and it agrees with the correspondence $\mathfrak{q}^c$ defined in 6.10.18.*

(e) ***Augmentation:*** *Let $\mathfrak{aug} : \mathbb{Q}[H] \to \mathbb{Q}$ be the augmentation $\sum_h a(h)h \mapsto \sum_h a(h)$. Then, for any σ one has $\mathfrak{aug}(\tau_\sigma) = 0$. Equivalently,*

$$\sum_\sigma \mathcal{T}_\sigma(h) = 0 \ \text{ for any } h.$$

Let us verify *(e)*. Consider the notations of 9.4.4. If 1 denotes the trivial character of H then $K_1 = \mathbb{Q}$, and the projection $\mathbb{Q}[H] \to K_1$ is exactly the augmentation map. Since $C_*^{\rho_1}(M)$ is not acyclic, $\tau^{\rho_1}(M) = 0$. Hence $\mathfrak{aug}(\tau) = \tau^{\rho_1} = 0$.

9.4.A *The Fourier Transform*

We need a dual description of the torsion invariants in terms of Fourier transform as well. Let H be a finite abelian group and let $\widehat{H} = \mathrm{Hom}(H, S^1)$ be its Pontryagin dual (the group of characters). If $\chi \in \widehat{H}$ then $\bar{\chi}$ denotes its conjugate: $\bar{\chi}(h) = \overline{\chi(h)}$.

The Fourier transform $\widehat{f} : \widehat{H} \to \mathbb{C}$ of a function $f : H \to \mathbb{C}$ satisfies

$$\widehat{f}(\chi) = \sum_{h \in H} f(h)\bar{\chi}(h), \quad f(h) = \frac{1}{|H|} \sum_{\chi \in \widehat{H}} \widehat{f}(\chi)\chi(h).$$

Example 9.4.15 For any σ set $f(h) := \mathcal{T}_\sigma(h)$. Then $\widehat{f}(1) = \widehat{\mathcal{T}_\sigma}(1) = \mathfrak{aug}(\tau_\sigma) = 0$.

Example 9.4.16 By 9.4.14*(a)–(b)* for any σ, χ, h one has

$$(a)\ \widehat{\mathcal{T}_\sigma}(\chi) = \widehat{\mathcal{T}_{\overline{\sigma}}}(\overline{\chi}), \quad (b)\ \widehat{\mathcal{T}_\sigma}(\chi) = \chi(h) \cdot \widehat{\mathcal{T}_{h\sigma}}(\chi). \tag{9.4.4}$$

Example 9.4.17 (The Fourier Transform of $h \mapsto \tau_\sigma(h)$ for Lens Spaces)

We will use the notations of 9.4.6 and 9.4.7. We wish to compute the coefficients $\mathcal{T}_\sigma(T^i)$ $(0 \le i < n)$ of τ_σ.

First consider the torsion in $\bar{Q}(H) = \mathbb{Q}[T]/N$, where $N = 1 + T + \ldots + T^{n-1}$. Corresponding to a fixed $\sigma \in \mathrm{Spin}^c(M)$ and orientation, we can assume that $\tau^{\bar{\rho}_a}$ is

$$\bar{\tau}_\sigma = \frac{T^s}{(1-T)(1-T^{q'})} \in \mathbb{Q}[T]/N \quad \text{for a certain } s = s_\sigma,\ 0 \le s < n.$$

We wish to determine $\tau_\sigma \in \mathbb{Q}[T]/(T^n - 1)$. Consider the natural projection $pr : \mathbb{Q}[T]/(T^n - 1) \to \mathbb{Q}[T]/N$. Then τ_σ is the unique element of $\mathbb{Q}[T]/(T^n - 1)$ such that $pr(\tau_\sigma) = \bar{\tau}_\sigma$ and $\mathfrak{aug}(\tau_\sigma) = 0$. E.g., if $\bar{\tau}_\sigma = \sum_{j=0}^{n-2} a_j T^j$, then $\tau_\sigma = \sum_{j=0}^{n-2} a_j T^j - aN$, where $an = \sum_j a_j$. Inverting $1 - T$ and $1 - T^{q'}$ in $\bar{Q}(H)$ one gets the expression

$$\bar{\tau}_\sigma = T^s \Big(\sum_{j=0}^{n-1} (j+1)T^j\Big)\Big(\sum_{j=0}^{n-1} (j+1)T^{q'j}\Big)/n^2,$$

hence the previous lifting for this expression applies. The coefficient $\mathcal{T}_\sigma(T^i)$ can be expressed via generalized Dedekind sums, cf. 2.2.11.

Let us compute the Fourier transforms $\widehat{\tau_\sigma}$. In general, for the trivial character, $\widehat{\tau_\sigma}(1) = 0$. For $\chi \in \widehat{H}$ with $\chi(T) = \xi \ne 1$ one has $\widehat{\tau_\sigma}(\chi) = \sum_{i=0}^{n-1} a_i \bar{\xi}^i$ whenever $\tau_\sigma = \sum_{i=0}^{n-1} a_i T^i$. Hence, $\widehat{\tau_\sigma}(\chi)$ is obtained from τ_σ by 'substitution' $T \mapsto \bar{\xi}$. The

above algebraic manipulation applied for T applies for ξ too, hence

$$\widehat{\tau_\sigma}(\chi) = \frac{\bar{\chi}(T^s)}{(1-\bar{\chi}(T))(1-\bar{\chi}(T^{q'}))} = \frac{\bar{\xi}^s}{(1-\bar{\xi})(1-\bar{\xi}^{q'})}, \quad \text{where } \chi(T) = \xi \neq 1.$$

Therefore, by Fourier inversion formula, the coefficient of T^i in τ_σ is

$$\mathcal{T}_\sigma(T^i) = \frac{1}{n} \cdot \sum_{\xi^n=1\neq\xi} \frac{\xi^{s-i}}{(1-\xi)(1-\xi^{q'})} \quad (s = s_\sigma).$$

This expression can be compared with generalized Dedekind sums from (2.2.11). This type of sums will be called Dedekind–Fourier sums. Furthermore, the above expressions can be compared with $Z(\mathbf{t})$ (consider e.g. (8.6.1) for a string graph); this relationship will be exploited fully in the next sections.

9.5 The Reidemeister–Turaev Torsion of Graph 3-Manifolds

Let M be an oriented rational homology sphere 3-manifold associated with a connected negative definite plumbing graph Γ.
In 9.5.7 we provide a combinatorial expression in terms of Γ for the refined Reidemeister–Turaev torsion. The equivalence of this expression with the original definition of the refined torsion is proved via a surgery formula in [489]; this proof will not be reproduced in this book. In this book this combinatorial expression will constitute the 'working definition' of the torsion, applied for negative definite plumbed 3-manifolds. All the needed properties of the torsion will be obtained starting from this expression. ♣

9.5.1 Preliminary Discussion We assume that $H_1(M,\mathbb{Q}) = 0$. We follow the standard notations of the plumbing graphs. First, let us guess what the shape of an appropriate definition should be. Let us assume that we have a function $\mathfrak{T}_\sigma : H \to \mathbb{C}$ indexed by the spin^c structures σ of M, $\sigma \mapsto \mathfrak{T}_\sigma$. Moreover, we fix a distinguished spin^c structure $\sigma_0 \in \text{spin}^c(M)$, and $\sigma \mapsto h_\sigma \in H$ is the bijection via the action $h_\sigma\sigma_0 = \sigma$. Motivated by 9.4.15 and the formulae from 9.4.17 we assume that the Fourier transform of $\mathfrak{T}_\sigma$ satisfies $\widehat{\mathfrak{T}_\sigma}(1) = 0$, and for $\chi \neq 1$ it has the following form

$$\widehat{\mathfrak{T}_\sigma}(\chi) = (\chi(h_\sigma))^\epsilon \cdot \prod_{v\in\mathcal{V}} (1-\chi([E_v^*]))^{\kappa_v-2}, \tag{9.5.1}$$

for a universal (χ and σ independent) choice of $\epsilon = \pm 1$. (Since $\sum_v(\kappa_v-2)$ is even, in the product the term $1-\chi([E_v^*])$ or $\chi([E_v^*])-1$ produce the same expression.)

We wish to find ϵ and σ_0 provided that $\mathfrak{T}$ satisfies the duality and the H-equivariance properties (9.4.4). By equivariance, $g+\epsilon h_{g\sigma} = \epsilon h_\sigma$, hence necessarily $\epsilon = -1$.

The duality imposes $\sum_v (\kappa_v - 2)[E_v^*] = h_\sigma + h_{\bar\sigma}$ in H. This can be interpreted in two different ways. First, since $\sum_v (\kappa_v - 2)E_v^* = -K - E$, we get $h_\sigma + h_{\bar\sigma} = -[K]$ in H. On the other hand, since $h_{\bar\sigma}\sigma_0 = \bar\sigma$ and $h_\sigma \sigma_0 = \sigma$, if we take the first Chern classes (cf. 6.10.16) we get $2(h_\sigma + h_{\bar\sigma} + c_1(\sigma_0)) = 0$. Therefore, it is natural to take $\sigma_0 = \sigma[K]$ (for notation see 6.10.21).

Let us try another definition too. Assume that

$$\widehat{\mathfrak{T}'_\sigma}(\chi) = (\chi(h_\sigma))^{\epsilon'} \cdot \prod_{v\in\mathcal{V}} (1 - \bar\chi([E_v^*]))^{\kappa_v - 2}, \tag{9.5.2}$$

for $\epsilon' = \pm 1$ and for $\sigma = h_\sigma \sigma'_0$, where σ'_0 is a certain distinguished spin^c structure. By similar computations as above we get $\epsilon' = -1$ and $\sigma'_0 = \sigma[-K]$.

We claim that with the above choices of ϵ, σ_0, ϵ' and σ'_0, the two possibilities provide the same function. Indeed, set $\sigma = h_\sigma \sigma'_0$, $h_\sigma = [l'_\sigma]$, $l'_\sigma \in L'$. Then

$$\begin{aligned}
\widehat{\mathfrak{T}'}_{\sigma[2l'_\sigma - K]}(\chi) = \widehat{\mathfrak{T}'_\sigma}(\chi) &= \chi(h_\sigma)^{-1} \prod_v (1 - \bar\chi([E_v^*]))^{\kappa_v - 2} \\
&= \chi(h_\sigma)^{-1} \prod_v \bar\chi([E_v^*])^{\kappa_v - 2} \prod_v (1 - \chi([E_v^*]))^{\kappa_v - 2} \\
&= \widehat{\mathfrak{T}}_{\sigma[2l'_\sigma + 2\sum(\kappa_v - 2)E_v^* + K]}(\chi) = \widehat{\mathfrak{T}}_{\sigma[2l'_\sigma - 2K + K]}(\chi).
\end{aligned} \tag{9.5.3}$$

Hence $\mathfrak{T}'_\sigma = \mathfrak{T}_\sigma$.

Above, tacitly, we assumed that $\chi([E_v^*]) \neq 1$ for any v with $\kappa_v \neq 2$. This property will be eliminated by a regularization procedure described below.

9.5.2 The Fourier Transform of $Z_{h,\mathcal{I}}(t)$ Assume that $\mathcal{I} = \{u\} \subset \mathcal{V}$ is a distinguished vertex, and for each $h \in H$ we consider the reduced series $Z_{h,\mathcal{I}}(t)$, where t is the variable corresponding to u. Set $m_{vu} := -(E_v^*, E_u^*) > 0$. From (8.4.2)

$$Z_{h,\{u\}}(t) = \frac{1}{|H|} \cdot \sum_{\chi \in \widehat{H}} \chi(h)^{-1} \cdot \prod_{v\in\mathcal{V}} \left(1 - \chi([E_v^*])t^{m_{vu}}\right)^{\kappa_v - 2}.$$

This shows that the Fourier transform of the map $h \mapsto Z_{h,\{u\}}(t)$ is

$$\widehat{Z_{\{u\}}(t)}(\bar\chi) = \prod_{v\in\mathcal{V}} \left(1 - \chi([E_v^*])t^{m_{vu}}\right)^{\kappa_v - 2}. \tag{9.5.4}$$

9.5.3 Character Values on Γ Since $\{[E_v^*]\}_v$ generate H, any character $\chi \in \widehat{H}$ is completely characterized by the values $\xi_v := \chi([E_v^*])$, $v \in \mathscr{V}$. These are roots of unity. When we wish to identify the character χ, we put its values $\{\xi_v\}_v$ as decorations on the vertices of the graph Γ. The collection $\{\chi([E_v^*])\}_{v,\chi}$ is a more subtle information than the abstract group $\widehat{H}$ itself: it shows the 'distribution along Γ' of the corresponding values of the characters as well. Since for any $v \in \mathscr{V}$ one has $e_v[E_v^*] + \sum_{(u,v)\text{ edge}}[E_u^*] = [-E_v] = 0$ in H (where $e_v = E_v^2$), for each χ one has

$$\xi_v^{e_v} \cdot \prod_{(u,v)\text{ edge}} \xi_u = 1. \tag{9.5.5}$$

Conversely, any collection of complex numbers $\{\xi_v\}_{v\in\mathscr{V}}$, $\xi_v \in S^1$, which satisfy (9.5.5) for any v, determines a character χ defined by $\chi([E_v^*]) = \xi_v$.

Furthermore, for any $\chi \in \widehat{H} \setminus \{1\}$, define the 'extended support' $\mathrm{supp}^e(\chi)$ of χ as the set of those vertices $v \in \mathscr{V}$ for which either $\chi([E_v^*]) \neq 1$, or v has an adjacent vertex w such that $\chi([E_w^*]) \neq 1$.

Lemma 9.5.4 *Fix a character $\chi \in \widehat{H} \setminus \{1\}$.*

(a) For an arbitrary vertex u the limit $\lim_{t\to 1} \widehat{Z_{\{u\}}(t)}(\chi)$ exists and it is finite.
(b) This limit is independent of u whenever $u \in \mathrm{supp}^e(\chi)$.

Proof The next terminology is needed: We say that a subgraph Γ' of the plumbing graph Γ satisfies the property (P) if $\sum_{\Gamma'}(\kappa_v - 2) \geq 0$, where the sum $\sum_{\Gamma'}$ runs over the vertices of Γ' (and κ_v is the valency of v in Γ). A subgraph Γ' of Γ is 'full' if any two vertices of Γ' adjacent in Γ are adjacent in Γ' as well. A full proper subgraph Γ' of Γ has property (C) if it has a unique vertex (say v_{end}), which is connected to a certain vertex v' of $\mathscr{V}(\Gamma) \setminus \mathscr{V}(\Gamma')$ by an edge of Γ. (The vertex v' is not necessarily unique.) Associated with our fixed $\chi \in \widehat{H} \setminus \{1\}$, let Γ^1 be the full subgraph of Γ with the set of vertices $\{v \in \mathscr{V} \mid \chi([E_v^*]) = 1\}$.

Proof of (a) We show that Γ^1 satisfies (P). Let $\Gamma^{1,c}$ be one of its connected components, and let $\kappa_v^{1,c}$ be the valency of v in $\Gamma^{1,c}$. Then $\sum_{\Gamma^{1,c}}(\kappa_v^{1,c} - 2) = -2$ since $\Gamma^{1,c}$ is a tree. Since $\Gamma^{1,c}$ is a proper subgraph of (the connected) Γ, there exists at least one edge of Γ, which is not an edge of $\Gamma^{1,c}$ and it has one of its end-vertices, say v^*, in $\Gamma^{1,c}$. But (9.5.5) shows that there are at least two such edges adjacent to v^*. Therefore, $\Gamma^{1,c}$ satisfies (P).

Now, *(a)* follows from the property (P) and the form of the function from (9.5.4).

Proof of (b) If $\sum_{\Gamma^{1,c}}(\kappa_v - 2) > 0$ for a component $\Gamma^{1,c}$, then the limit is zero (and it is independent of m_{vu}). Otherwise, for each $\Gamma^{1,c}$ one must have $\sum_{\Gamma^{1,c}}(\kappa_v - 2) = 0$. By the above discussion, this means that each $\Gamma^{1,c}$ has property (C). On the other hand, if Γ' is a full proper subgraph of Γ which satisfies (C), then the positive integers $\{m_{vu}\}_{v\in\mathscr{V}(\Gamma')}$ modulo a multiplicative constant are independent of the choice of u whenever $u \in (\mathscr{V}(\Gamma) \setminus \mathscr{V}(\Gamma')) \cup \{v_{end}\}$. This follows from 2.2.5.5. □

Remark 9.5.5 For $\chi = 1$, the Laurent expansion at 1 of the series $\widehat{Z_{\{u\}}(t)}(1)$ has a non-trivial principal part, cf. (9.5.22), hence $\lim_{t\to 1} \widehat{Z_{\{u\}}(t)}(1)$ is not finite.

9.5.6 In this book the torsion $\sigma \in \mathrm{Spin}^c(M) \mapsto \mathcal{T}_\sigma$, $\mathcal{T}_\sigma = \sum_h \mathcal{T}_\sigma(h)h \in \mathbb{Q}[H]$ is defined via the Fourier transform of $h \mapsto \mathcal{T}_\sigma(h)$ in the following way.

Definition 9.5.7 ([489])

(a) For the trivial character $\widehat{\mathcal{T}_\sigma}(1) = 0$.
(b) If $\chi([E_v^*]) \neq 1$ for every v with $\kappa_v \neq 2$, then we set

$$\widehat{\mathcal{T}_\sigma}(\chi) = (\chi(h_\sigma))^{-1} \cdot \prod_{v\in\mathcal{V}} (1 - \chi([E_v^*]))^{\kappa_v - 2}, \quad \sigma = h_\sigma \sigma[K].$$

(c) If $\chi \neq 1$, but the assumption from (b) does not hold, then the formula from (b) is regularised as follows:

$$\widehat{\mathcal{T}_\sigma}(\chi) = (\chi(h_\sigma))^{-1} \cdot \lim_{t\to 1} \prod_{v\in\mathcal{V}} (1 - \chi([E_v^*]t^{m_{vu}}))^{\kappa_v - 2} = (\chi(h_\sigma))^{-1} \cdot \lim_{t\to 1} \widehat{Z_{\{u\}}(t)}(\bar{\chi}),$$

for certain (any) $u = u_\chi \in \mathrm{supp}^e(\chi)$.

Theorem 9.5.8 ([489])

(a) $\sigma \mapsto \mathcal{T}_\sigma$ defined in 9.5.7 and the refined Reidemeister–Turaev torsion 9.4.13 coincide.
(b) $\mathcal{T}$ defined in 9.5.7 is independent of the choice of the resolution.

Proof For (a) see [489], (b) follows from (a) since it is true for the refined Reidemeister–Turaev torsion (as defined in 9.4.13). □

Remark 9.5.9

(a) By Fourier inversion

$$\mathcal{T}_\sigma(h) = \frac{1}{|H|} \cdot \sum_{\chi\in\widehat{H}\setminus\{1\}} \chi(h) \cdot (\chi(h_\sigma))^{-1} \cdot \lim_{t_{u_\chi}\to 1} \widehat{Z_{\{u_\chi\}}(t_{u_\chi})}(\bar{\chi}).$$

Similarly as in 9.5.1, one verifies that the properties (9.4.4) are valid, hence $\{\mathcal{T}_\sigma(h)\}_{\sigma,h}$ satisfy the duality and H-equivariance properties. Hence

$$\mathcal{T}_\sigma(1) = \mathcal{T}_{\overline{\sigma}}(1), \quad \text{and} \quad \mathcal{T}_\sigma(1) = \mathcal{T}_{h_\sigma\sigma[K]}(1) = \mathcal{T}_{\sigma[K]}(-h_\sigma). \tag{9.5.6}$$

In particular, $\mathcal{T}_{\sigma[K]}(h)h \in \mathbb{Q}[H]$ contains the same information as $\{\mathcal{T}_\sigma(1)\}_\sigma$.

(b) From part (a),

$$\mathcal{T}_\sigma(1) = \frac{1}{|H|} \cdot \sum_{\chi \in \widehat{H}\setminus\{1\}} (\chi(h_\sigma))^{-1} \cdot \lim_{t_{u_\chi}\to 1} \widehat{Z_{\{u_\chi\}}(t_{u_\chi})}(\bar{\chi}).$$

Usually, for different characters χ one needs different regularization vertex u_χ. However, if $\cap_{\chi\neq 1}\mathrm{supp}^e(\chi) \neq \emptyset$, then any $u \in \cap_{\chi\neq 1}\mathrm{supp}^e(\chi)$ might serve as a *common* regularization vertex (with a *common* variable $t = t_u$). In such a case, via $\widehat{Z_{\{u\}}(t)}(1) = Z_{\{u\}}(t) = \prod_v (1 - t^{m_{vu}})^{\kappa_v - 2}$,

$$\mathcal{T}_\sigma(1) = \lim_{t\to 1}\Big(\frac{1}{|H|}\cdot \sum_{\chi\in\widehat{H}\setminus\{1\}} (\chi(h_\sigma))^{-1}\cdot \widehat{Z_{\{u\}}(t)}(\bar{\chi})\Big) = \lim_{t\to 1}\Big(Z_{h_\sigma,\{u\}}(t) - \frac{1}{|H|}\cdot Z_{\{u\}}(t)\Big).$$

We rewrite $\{Z_{h,\{u\}}(t)\}_h$ equivariantly as $Z_{H,\{u\}}(t) := \sum_{h\in H} Z_{h,\{u\}}(t)h \in \mathbb{Q}[[t]][H]$, and we set $N := \sum_h h \in \mathbb{Q}[H]$. Then, via $\mathcal{T}_\sigma(1) = \mathcal{T}_{\sigma[K]}(-h_\sigma)$,

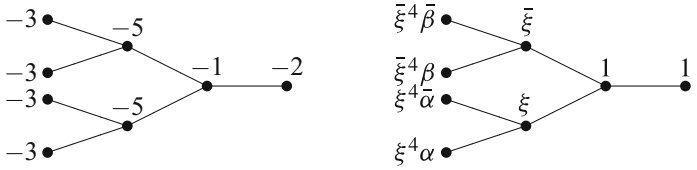
$$\mathcal{T}_{\sigma[-K]} = \overline{\mathcal{T}_{\sigma[K]}} = \lim_{t\to 1}\Big(Z_{H,\{u\}}(t) - Z_{\{u\}}(t)\cdot \frac{N}{|H|}\Big) \in \mathbb{Q}[H]. \tag{9.5.7}$$

The identity (9.5.7) is not true in general, i.e. when $\cap_{\chi\neq 1}\mathrm{supp}^e(\chi) = \emptyset$. For the formula valid in the general case see 9.5.23.

The above formula already shows in this special case that the principal (pole) part of the Laurent series at $t = 1$ of $Z_{h,\{u\}}(t)$ is independent of $h \in H$. This statement is true in general, even without the restriction $\cap_{\chi\neq 1}\mathrm{supp}^e(\chi) \neq \emptyset$, cf. 9.5.23.

(c) If Γ is star-shaped then the central vertex is an element of $\cap_{\chi\neq 1}\mathrm{supp}^e(\chi)$. Similarly, if H is cyclic, then again $\cap_{\chi\neq 1}\mathrm{supp}^e(\chi) \neq \emptyset$.

(d) Here is an example when $\cap_{\chi\neq 1}\mathrm{supp}^e(\chi) = \emptyset$. Consider the suspension singularity $\{(x^3+y^2)^2 + x^5 y = z^4\}$. Its resolution is shown on the left diagram; it has $H = \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{13}$. The distribution of the characters is shown on the right diagram, where $\alpha^3 = \beta^3 = \xi^{13} = 1$.



Example 9.5.10 (The Torsion of a Lens Space) We will consider the notations of 6.6.18. We fix $\sigma = h_\sigma\sigma[K] \in \mathrm{Spin}^c(L_X)$. Then for $\chi \neq 1$

$$\widehat{\mathcal{T}_\sigma}(\chi) = \chi(h_\sigma)^{-1}\cdot(1-\chi([E_s^*]))^{-1}(1-\chi([E_1^*]))^{-1}.$$

Assume that $h_\sigma = a[E_s^*]$ for some $0 \le a < n$. Set $\xi := \chi([E_s^*])$. Then, cf. (2.2.11),

$$\widehat{\mathcal{T}_\sigma}(\chi) = \frac{\xi^{-a}}{(1-\xi)(1-\xi^q)} \quad (\xi \neq 1),$$

$$\begin{aligned} \mathcal{T}_\sigma(1) &= \frac{1}{n} \cdot \sum_{\xi^n=1\neq\xi} \frac{\xi^{-a}}{(1-\xi)(1-\xi^q)} \\ &= -\mathbf{s}(q,n;\frac{a}{n},0) + \frac{n-1}{4n} - \frac{1}{2}\Big\{\frac{a}{n}\Big\} - \frac{1}{2}\Big(\Big(\frac{aq'}{n}\Big)\Big). \end{aligned} \tag{9.5.8}$$

Example 9.5.11 (Brieskorn–Hamm Complete Intersections of Type (i) [489])

Assume that $(X,o) = X_C(a_1,\dots,a_n)$ is a Brieskorn–Hamm complete intersection, with rational homology sphere link, such that $(a_1,\dots,a_n) = (kp_1, kp_2, p_3, \dots, p_n)$, where $k \ge 1$ and the integers $\{p_j\}_{j=1}^n$ are pairwise coprime, and $\gcd(k, p_j) = 1$ for any $j \ge 3$. We follow the notations of 5.1.17(i). Using the properties listed in 5.1.17 one verifies the following facts (see also [489]):

- There are n 'leg-groups', their cardinality is $q_1 = 1$, $q_2 = 1$, and $q_j = k$ for $j \ge 3$. The α-Seifert invariants are $\alpha_j = p_j$ for $j = 1,\dots,n$.
- $\alpha = \prod_{j=1}^n p_j$, and $-e\cdot\alpha = 1$, hence by (3.5.5) $|H| = \prod_{j\ge3} \alpha_j^{k-1}$ and $\mathfrak{o} = 1$.
- If h_{ij} $(3 \le j \le n,\ 1 \le i \le k)$ denote the classes $[E_v^*]$ of the end vertices with the corresponding indices, then

$$H \simeq \oplus_{j\ge3}\langle\, h_{ij},\ \ 1 \le i \le k \mid \alpha_j h_{ij} = 0 \text{ for all } i, \text{ and } \textstyle\sum_i h_{ij} = 0 \,\rangle \simeq \oplus_{j\ge3}(\mathbb{Z}_{\alpha_j})^{k-1}.$$

The character values of $\chi \neq 1$ at the central vertex and at the end vertices for $j < 3$ are 1, and at the other end vertices are ξ_{ij} with $\xi_{ij}^{\alpha_j} = 1$ and $\prod_i \xi_{ij} = 1$. Therefore,

$$\hat{\mathcal{T}}_{\sigma[K]}(\chi) = \lim_{t\to1} \frac{(1-t^\alpha)^{k(n-2)}}{(1-t^{\alpha/\alpha_1})(1-t^{\alpha/\alpha_2})\prod_{j\ge3}\prod_{i=1}^k (1-t^{\alpha/\alpha_j}\xi_{ij})}.$$

The limit is nonzero if and only if for exactly two pairs (i,j) (with the same j) $\xi_{ij} \neq 1$. Therefore

$$\mathcal{T}_{\sigma[K]}(1) = \frac{k(k-1)}{2|H|} \cdot \sum_{j\ge3} \frac{\alpha^{k(n-2)}}{\frac{\alpha}{\alpha_1}\frac{\alpha}{\alpha_2}(\frac{\alpha}{\alpha_3})^k\cdots(\frac{\alpha}{\alpha_j})^{k-2}\cdots(\frac{\alpha}{\alpha_n})^k} \cdot \sum_{\xi^{\alpha_j}=1\neq\xi} \frac{1}{(\xi-1)(\bar\xi-1)}.$$

Since $\sum_{\xi^p=1\neq\xi}(\xi-1)^{-1}(\bar\xi-1)^{-1} = (p^2-1)/12$ (cf. 2.2.A), one gets

$$\mathcal{T}_{\sigma[K]}(1) = \frac{\alpha\cdot k(k-1)}{24} \cdot \sum_{j\ge3}\Big(1-\frac{1}{\alpha_j^2}\Big).$$

Example 9.5.12 (Brieskorn–Hamm Complete Intersections of Type (ii) [489])
Assume that $(a_1, \dots, a_n) = (2^c p_1, 2p_2, 2p_3, p_4 \dots, p_n)$, where the integers $\{p_j\}_{j=1}^n$ are odd and pairwise coprime, and $c \geq 1$, cf. 5.1.17. The discussion is rather similar to the previous case, the only difference is that now h_0, the generator associated with the central vertex, is not trivial. We follow again the notations of 5.1.17, and we write $P := \prod_j p_j$. One has the following facts:

- $a = 2^c P$ and $A = 2^{c+2} P$. Moreover, $\alpha_1 = 2^{c-1} p_1$ and $\alpha_j = p_j$ for $j \geq 2$.
- $q_j = 2$ for $j \leq 3$ and $q_j = 4$ for $j \geq 4$. The number of legs is $\nu = 4n - 6$.
- $\alpha = 2^{c-1} P$, hence $-e^{-1} = 2^{c-2} P$. Therefore $\mathfrak{o} = 2$ and $|H| = 2^c P^3/(p_1 p_2 p_3)^2$.
- The self-intersection $-b_0$ of the central vertex is even (use the above data and (3.5.1)). In particular, $b_0 h_0 = 0$ in H, hence H equals (cf. 3.5.4)

$$\Big\langle h_{ij},\ \ 1 \leq j \leq n,\ \ 1 \leq i \leq q_j;\ h_0 \ \Big|\ \alpha_j h_{ij} = h_0 \text{ for all } i, j;\ \ \textstyle\sum_{i,j} \omega_j h_{ij} = 2h_0 = 0 \Big\rangle.$$

There are two types of non-trivial characters, since $\chi(h_0) = \pm 1$. If $\chi(h_0) = 1$, then the computation is similar as in the case (i) from 9.5.11. If $\chi(h_0) = -1$ then there is no need for regularization. The final answer is (cf. [489]):

$$\mathcal{T}_{\sigma[K]}(1) = \frac{\alpha}{8} + \frac{\alpha}{24} \cdot \sum_{j \geq 1} \frac{q_j(q_j - 1)}{2} \Big(1 - \frac{1}{\alpha_j^2}\Big).$$

Example 9.5.13 For additional computations of the torsion see (a) 9.7.A for the weighted homogeneous case, (b) 9.7.B and 11.5 for $S^3_{-d}(\#_i K_i)$, (c) [491] for the suspension case $f(x, y) + z^n$ with f irreducible.

9.5.A *Additivity Formula for the Torsion*

We fix a graph Γ such that $M(\Gamma)$ is a rational homology sphere. For a vertex $v \in \mathcal{V}$ of Γ let $\{\Gamma_i\}_i$ be the connected components of $\Gamma \setminus v$. For any $\sigma \in \mathrm{Spin}^c(M(\Gamma))$ we define in 9.5.20 its restrictions $\sigma_i \in \mathrm{Spin}^c(M(\Gamma_i))$. The main result determines $\mathcal{T}_\sigma(1)(M(\Gamma)) - \sum_i \mathcal{T}_{\sigma_i}(1)(M(\Gamma_i))$ in terms of certain numerical data associated with v. In this subsection we follow [77]. ♣

9.5.14 Notations Regarding Rational Functions We write any rational function R as $R^+ + R^-$, where R^+ is a polynomial and R^- is a rational function with negative degree. For R without pole at 0 we will refine it further: one writes R^- in a unique

way as a finite sum

$$R^-(t) = \sum_{\alpha\in\mathbb{C}^*} (\mathrm{L}_\alpha\, R)(t), \quad \text{where} \quad (\mathrm{L}_\alpha\, R)(t) = \sum_{k>0} \frac{a_{\alpha,k}}{(1-\alpha t)^k} \qquad (a_{\alpha,k}\in\mathbb{C}).$$

For any rational function $R(t)$ with Laurent expansion $\sum_{k\geq k_0} a_k(t-1)^k$ at $t=1$, we write $\mathrm{coef}_1^0\, R(t) := a_0$. If 1 is not a pole of R then $\mathrm{coef}_1^0\, R(t) = R(1)$.

The next identities are elementary and their proofs are left to the reader.

Lemma 9.5.15 *For any $0 \leq q < d$ one has*

$$\frac{1}{d}\sum_{\alpha^d=\epsilon} \frac{\alpha^{-q}}{1-\alpha t} = \frac{t^q}{1-\epsilon t^d} \qquad (\epsilon\neq 0), \tag{9.5.9}$$

$$\frac{1}{d}\sum_{\alpha^d=1} \frac{\alpha^{-q}}{(1-\alpha t)^2} = \frac{dt^q}{(1-t^d)^2} - \frac{(d-q-1)t^q}{1-t^d}, \tag{9.5.10}$$

$$\mathrm{coef}_1^0\left(\frac{1}{d}\sum_{\alpha^d=1} \frac{\alpha^{-q}}{1-\alpha t}\right) = \frac{d-1-2q}{2d}, \tag{9.5.11}$$

$$\mathrm{coef}_1^0\left(\frac{1}{d}\sum_{\alpha^d=1} \frac{\alpha^{-q}}{(1-\alpha t)^2}\right) = -\frac{(d-1)(d-5)}{12d} - \frac{q^2+2q-qd}{2d}. \tag{9.5.12}$$

9.5.16 Identities Regarding Determinants As above, write $d = \det(\Gamma) = |H|$ and $m_{uw} = -(E_u^*, E_w^*)$, and set $a_{uw} = dm_{uw} \in \mathbb{Z}_{>0}$ for any two vertices $u, w \in \mathscr{V}$ of Γ. Recall (cf. 2.2.5(5)) that if $\Gamma \setminus \overline{uw}$ denotes the subgraph of Γ obtained by deleting the path connecting u and w (including u and w), then

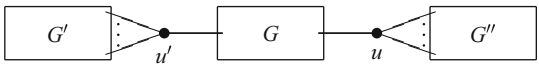
$$a_{uw} = \det(\Gamma \setminus \overline{uw}). \tag{9.5.13}$$

Lemma 9.5.17

1. For any $u \in \mathscr{V}$ one has

$$a_{uu} \cdot \prod_{w\in\mathscr{V}} a_{uw}^{\kappa_w - 2} = 1. \tag{9.5.14}$$

2. Assume that Γ has the form (compare also with 2.2.5(6)):



Then for any $S \subseteq \mathscr{V}(G'')$, one has:

$$\Big(\prod_{w\notin S} a_{wu'}^{\kappa_w-2}\Big)^{-1} = \det G' \cdot \det(G \cup G'' \cup u) \cdot (\det G' \cdot \det(G \setminus \overline{uu'}))^{\sum_{w\in S}(\kappa_w-2)} \cdot \prod_{w\in S} \det(G'' \setminus \overline{uw})^{\kappa_w-2}. \tag{9.5.15}$$

G is allowed to be empty. Also, u' and u are allowed to be the same, and in this case G is empty and one should write G'' instead of $G \cup G'' \cup u$ in the formula.

Proof (9.5.14) follows by induction. Indeed, let $\{\Gamma_i\}_i$ be the components of $\Gamma\setminus\{u\}$, and set $\Gamma_i' = \Gamma \setminus \cup_{j\neq i}\Gamma_j$. Let u_i be the vertex of Γ_i adjacent to u. Then the identity is valid for the pair (Γ, u) whenever it is true for each (Γ_i', u). Then compare the pairs (Γ_i', u) and (Γ_i, u_i) as well. Next we prove part 2. By (9.5.14), the left hand side of (9.5.15) is $a_{u'u'}\prod_{w\in S} a_{wu'}^{\kappa_w-2}$, then apply (9.5.13). □

9.5.18 Generalization of Characters Fix a vertex $u \in \mathscr{V}$. We will need to extend the expression $Z_{\{u\}}(t^{\vec{d}})(\bar{\chi}) = \prod_w (1-\chi([E_w^*])t^{a_{wu}})^{\kappa_w-2}$, valid for a character $\chi \in \widehat{H}$, for an arbitrary map $\psi : \mathscr{V} \to \mathbb{C}^*$. We write

$$\Pi_{\psi,\{u\}}(t) := \prod_{w\in\mathscr{V}} (1-\psi(w)t^{a_{wu}})^{\kappa_w-2}. \tag{9.5.16}$$

For such a map ψ, set $\mathscr{V}_{\psi=1} := \{w \in \mathscr{V} : \psi(w) = 1\}$ and $\mathrm{supp}(\psi) := \mathscr{V} \setminus \mathscr{V}_{\psi=1}$.

Lemma 9.5.19 *Let Γ be as above, fix $u \in \mathscr{V}$ and let $\psi : \mathscr{V} \to \mathbb{C}^*$ be any map. Then the least degree term of the Laurent series of $\Pi_{\psi,\{u\}}$ at 1 is*

$$\Pi_{\psi,\{u\}}(t) = \prod_{w\notin\mathscr{V}_{\psi=1}} (1-\psi(w))^{\kappa_w-2} \cdot \prod_{w\in\mathscr{V}_{\psi=1}} a_{uw}^{\kappa_w-2} \cdot (1-t)^n + O((1-t)^{n+1}),$$

where

$$n := \sum_{w\in\mathscr{V}_{\psi=1}} (\kappa_w - 2) = -2\#\{\text{components of } \mathscr{V}_{\psi=1}\} + \#\{\text{edges going out from } \mathscr{V}_{\psi=1}\}.$$

In particular, if every component of $\mathscr{V}_{\psi=1}$ has a vertex with at least two outgoing edges (e.g. ψ is a non-trivial character) then $n \geq 0$ with equality if and only if all components have exactly two outgoing edges.

9.5.20 Additivity Formula for the Torsion. Preparation

- Let $v \in \mathscr{V}$ be a fixed vertex and we will use the notations of 9.3.7: the connected components of $\Gamma \setminus v$ are Γ_i, and v_i is the unique vertex of Γ_i, which is adjacent

to v in Γ. Regarding graph-invariants, we indicate by a subscript i when we use invariants of Γ_i instead of Γ.

- Using (9.5.13) for every component Γ_i and vertex w of Γ_i one has

$$a_{vw} = a_{v_i w,i} \cdot \prod_{j \neq i} d_j \quad (\text{where } d_j = \det(\Gamma_j)). \tag{9.5.17}$$

- If $\mathscr{V}_i := \mathscr{V}(\Gamma_i)$, then

$$\prod_{w \in \mathscr{V} \setminus \mathscr{V}_i} a_{wv}^{\kappa_w - 2} = \frac{1}{d_i}. \tag{9.5.18}$$

[Use (9.5.17) for Γ and Γ_i and the identity $\sum_{w \in \mathscr{V}_i}(\kappa_w - 2) = -1$.]

- Fix an arbitrary map $\psi : \mathscr{V} \to \mathbb{C}^*$ as above. Then for any vertex $w \in \mathscr{V}$ we define $\mathrm{def}_w(\psi) := \psi(w)^{e_w} \cdot \prod_{(u,w)\ \text{edge}} \psi(u)$, cf. (9.5.5). The map ψ is called a *pseudo-character associated with the vertex* v if $\mathrm{def}_w(\psi) = 1$ for all $w \neq v$. Their collection will be denoted by $\widetilde{H}$. We set $\mathrm{def}(\psi) := \mathrm{def}_v(\psi)$. Notice that the pseudo-characters ψ with $\mathrm{def}(\psi) = 1$ are exactly the characters of H via the correspondence $\psi(w) = \psi([E_w^*])$. In fact, any $\psi : \mathscr{V} \to \mathbb{C}^*$ can be regarded as a character on L' (which does not necessarily descend to H): any $\psi \in \widetilde{H}$ gives a morphism $L' \to \mathbb{C}^*$ defined by $\psi\left(\sum_w m_w E_w^*\right) := \prod_w \psi(w)^{m_w}$.

  Write also $\psi_i := \psi|_{\mathscr{V}_i} : \mathscr{V}_i \to \mathbb{C}^*$ for the restrictions.
- We will use the following notation for a spin^c structure of $M(\Gamma)$ and its restrictions. We fix $l' = \sum_w r_w E_w \in L'$ such that $r_v \in [0, 1)$. Set $h_\sigma = [l']$ and $\sigma = \sigma[2l' + K] = h_\sigma \sigma[K] \in \mathrm{Spin}^c(M(\Gamma))$. Similarly, we set $\sigma_i = \sigma[R_i(2l' + K)] = [R_i(l')]\sigma[K_i] \in \mathrm{Spin}^c(M(\Gamma_i))$. (For the definition of R_i see 9.3.8.)

σ_i is the restriction of σ in the following sense. For any $\sigma = h_\sigma \sigma[K] \in \mathrm{Spin}^c(M(\Gamma))$ we can take a lifting $\widetilde{\sigma} \in \mathrm{Spin}^c(P(\Gamma))$ of the form $\widetilde{\sigma}(2l' + K))$ with $r_v \in [0, 1)$ and $[l'] = h_\sigma$. Each $P(\Gamma_i)$ can be considered as a submanifold of $P(\Gamma)$, hence one can take the restriction of $\widetilde{\sigma}$ to each boundary $\partial P(\Gamma_i) = M(\Gamma_i)$. They are exactly the spin^c structures σ_i (since the restriction of $K \in L'$ is $K_i \in L_i'$). They depend only on σ (and not on the choice of l'). Note that r_v depends also only on σ.

Theorem 9.5.21 *Set $l' = \sum_w r_w E_w$, $r_v \in [0, 1)$, $[l'] = h_\sigma$ as above. Recall also the notations (cf. 9.3.6)*

$$\alpha_v := \sum_{w \in \mathscr{V}} (\kappa_w - 2) m_{vw}, \quad \beta_v := \sum_{w \in \mathscr{V}} (\kappa_w - 2) m_{vw}^2.$$

Then

$$\mathcal{T}_\sigma(1)(M(\Gamma))-\sum_i \mathcal{T}_{\sigma_i}(1)(M(\Gamma_i)) = pc(Z_{h_\sigma,\{v\}}(t^d))+\frac{1-\beta_v}{24m_{vv}}-\frac{(\alpha_v+1-2r_v)^2}{8m_{vv}}.$$

Corollary 9.5.22 *$\mathcal{T}_\sigma(1)(M(\Gamma))$ is a rational number.*

Proof The right hand side of the above additivity formula is a rational number, then use induction. □

The remaining part of this subsection constitutes the proof of Theorem 9.5.21.

Proposition 9.5.23 (Breaking the Torsion)

(a) For any $\chi \in \widehat{H} \setminus \{1\}$ and $\chi_i = \chi|_{\mathcal{V}_i}$

$$\frac{1}{d}\widehat{\mathcal{T}_\sigma}(\chi) = \frac{1}{d}\chi(h_\sigma)^{-1}\cdot \Pi_{\chi,\{v\}}(1) + \begin{cases} \frac{1}{d_i}\widehat{\mathcal{T}_{\sigma_i}}(\chi_i) & \text{if } \chi|_{(\mathcal{V}\setminus\mathcal{V}_i)\cup\{v_i\}} = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{9.5.19}$$

(If $\chi(v) = 1$, e.g. as in the upper case of (9.5.19), then χ_i is a character of H_i.)

(b)

$$\mathcal{T}_\sigma(1) = \lim_{t\to 1}\left(Z_{h_\sigma,\{v\}}(t) - \frac{1}{d}\Pi_{1,\{v\}}(t^{1/d})\right) + \sum_i \frac{1}{d_i}\sum_{\substack{\rho_i\in\hat{H}_i\setminus 1\\ \rho_i(v_i)=1}} \widehat{\mathcal{T}_{\sigma_i}}(\rho_i). \tag{9.5.20}$$

In particular (since the limit exists), the principal (pole) part of the Laurent series at $t = 1$ of $Z_{h,\{v\}}(t)$ is independent of h. Moreover, the principal part of $Z_{h,\{v\}}(t^d)$ agrees with the principal part of $\Pi_{1,\{v\}}(t)$, (for its expression see (9.5.22)).

(Note also that $\Pi_{1,\{v\}}(t^{1/d})$ is identical to $Z_{\{v\}}(t)$.)

Proof

(a) The proof is a case-by-case verification.

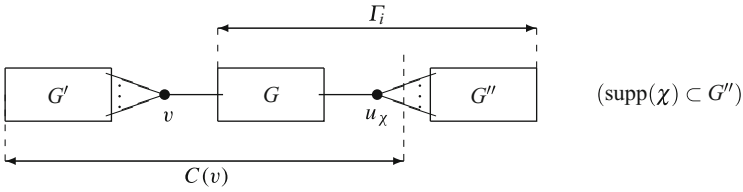
First, we consider the case when $v \in \mathrm{Supp}^e(\chi)$. Then we can choose $u_\chi := v$ in the regularization, hence the first two terms of (9.5.19) become equal. The third term is 0 because the lower case of the equation applies.

In the remaining cases, $\chi(v) = \chi(v_i) = 1$ for all i. By the second part of Lemma 9.5.19, all three terms of (9.5.19) are 0 (because $n > 0$) unless every component of $\mathcal{V}_{\chi=1}$ has exactly two outgoing edges. Hence the only remaining case is when every component of $\mathcal{V}_{\chi=1}$ has exactly two outgoing edges. By the character property, these two edges must be edges of the same vertex u_C of that

component C. Let $C(v)$ be that component which contains v. If $u_{C(v)} \in \mathcal{V}_i$, then for any $j \neq i$ one must have $\mathcal{V}_j \subset C(v)$. Hence, the upper case of (9.5.19) should hold.

Furthermore, by 9.5.19 with $n = 0$, $\sum_{w \in \mathrm{supp}(\chi)} (\kappa_w - 2) = -2$.

Choose u_χ as $u_{C(v)}$ and decompose Γ into subgraphs as shown in the next picture.



We express the terms of (9.5.19) using 9.5.19 and (9.5.15):

$$\Pi_{\chi,\{v\}}(1) = \frac{\det(G') \cdot \left(\det(G \setminus \overline{vu_\chi})\right)^2}{\det(G \cup G'' \cup u_\chi)} \cdot \prod_{w \in \mathrm{supp}(\chi)} \left(\frac{(1 - \chi([E_w^*]))}{\det(G'' \setminus \overline{u_\chi w})} \right)^{\kappa_w - 2},$$

$$\Pi_{\chi,\{u_\chi\}}(1) = \frac{\det(G' \cup G \cup v)}{\det(G'')} \cdot \prod_{w \in \mathrm{supp}(\chi)} \left(\frac{(1 - \chi([E_w^*]))}{\det(G'' \setminus \overline{u_\chi w})} \right)^{\kappa_w - 2},$$

$$\Pi_{\chi_i,\{u_\chi\}}(1) = \frac{\det(G)}{\det(G'')} \cdot \prod_{w \in \mathrm{supp}(\chi)} \left(\frac{(1 - \chi([E_w^*]))}{\det(G'' \setminus \overline{u_\chi w})} \right)^{\kappa_w - 2}.$$

Note that $\chi([l']) = \chi_i(R_i[l'])$. Hence, after we factor out the $\prod_{w \in \mathrm{supp}(\chi)}$ product, and we use $d_i = \det(G \cup G'' \cup u_\chi)$ and $d = \det(\Gamma)$, (9.5.19) reduces to 2.2.5(6).

(b) Use the definition of the torsion.

□

9.5.24 The Negative Part of $Z_{h_\sigma,v}(t)$ (For notations see 9.5.14 and 9.5.20.)

Lemma 9.5.25 *For every non-trivial pseudo-character ψ*

$$\mathrm{L}_1 \Pi_{\psi,\{v\}}(t) = \begin{cases} \frac{1}{d_i} \cdot \frac{\Pi_{\psi_i,\{v_i\}}(1) \cdot (1 - \psi_i(v_i))}{1 - t} & \text{if } \mathrm{supp}\psi \subseteq \mathcal{V}_i \text{ and } \psi(v_i) \neq 1, \\ 0 & \text{for all other } \psi \neq 1. \end{cases} \tag{9.5.21}$$

Proof We apply Lemma 9.5.19. By the pseudo-character relations, all components of $\mathcal{V}_{\psi=1}$ have at least two outgoing edges except possibly the component containing

v, which can have only one outgoing edge, which must start at v. Hence the lower degree of the Laurent expansion of $\Pi_{\psi,v}$ at 1 is at least -1 with equality if and only if all the components have the minimum number of outgoing edges declared above. In particular, if $\mathrm{L}_1\,\Pi_{\psi,\{v\}}(t) \neq 0$ then $\psi(v) = 1$, $\psi(v_i) \neq 1$ for exactly one i, and $\mathrm{supp}(\psi) \subset \mathscr{V}_i$. This proves the lower case of (9.5.21).

To prove the upper case, note that by (9.5.17) for any $w \in \mathscr{V}_i$

$$\Pi_{\psi,\{v\}}(t) = \Pi_{\psi_i,\{v_i\}}(t^{\prod_{j\neq i} d_j}) \cdot (1 - \psi_i(v_i)t^{a_{vv_i}}) \cdot \prod_{w\notin\mathscr{V}_i} (1 - t^{a_{wv}})^{\kappa_w - 2}.$$

$\Pi_{\psi_i,\{v_i\}}(t)$ is regular at 1 because ψ_i is a non-trivial character of H_i. Moreover, $\sum_{w\notin\mathscr{V}_i}(\kappa_w - 2) = -1$. Thus

$$(\mathrm{L}_1\,\Pi_{\psi,\{v\}})(t) = \frac{1}{1-t} \cdot \Pi_{\psi_i,\{v_i\}}(1) \cdot (1 - \psi_i(v_i)) \cdot \prod_{w\notin\mathscr{V}_i} a_{wv}^{\kappa_w - 2}.$$

Then for the last product use (9.5.18). □

For any character $\chi \in \widehat{H}$, all the poles α of $\Pi_{\chi,\{v\}}$ are roots of unity. Furthermore, for any fixed root of unity α, the map $w \mapsto \Lambda(w) := \alpha^{-a_{vw}}$ is a pseudo-character associated with v, with $\mathrm{def}(\Lambda) = \alpha^d$ (since $-I^{-1} = d(a_{uw})$). In particular, for any fixed α, the correspondence $\psi(w) := \chi(w) \cdot \alpha^{-a_{vw}}$ realizes a bijection between characters $\chi \in \widehat{H}$ and pseudo-characters $\psi \in \tilde{H}$ with $\mathrm{def}(\psi) = \alpha^d$. This reads as $\psi(l') = \chi([l']) \cdot \alpha^{d(E_v^*, l')}$. Therefore:

$$\begin{aligned} d \cdot Z_{h_\sigma,\{v\}}(t^d)^- &= \sum_{\alpha\in\mathbb{C}^*} \sum_{\chi\in\widehat{H}} \chi([l'])^{-1} (\mathrm{L}_\alpha\,\Pi_{\chi,\{v\}})(t) \\ &= \sum_{\alpha\in\mathbb{C}^*} \sum_{\substack{\psi\in\tilde{H} \\ \mathrm{def}(\psi)=\alpha^d}} \psi(l')^{-1}\alpha^{-dr_v} (\mathrm{L}_1\,\Pi_{\psi,\{v\}})(\alpha t). \end{aligned}$$

For $\psi \neq 1$, by 9.5.25 all such terms $\mathrm{L}_1\,\Pi_{\psi,\{v\}}$ are zero except when $\mathrm{supp}(\psi) \subset \mathscr{V}_i$, and $\alpha^d = \mathrm{def}(\psi) = \psi(v_i) \neq 1$. In particular, the restriction $\psi_i = \psi|_{\mathscr{V}_i}$ is a character of Γ_i. Conversely, any $\psi_i \in \widehat{H}_i$ with $\psi_i(v_i) = \alpha^d \neq 1$ can be extended (by 1) to a pseudo-character ψ with $\mathrm{def}(\psi) = \alpha^d \neq 1$. Hence, by 9.5.25 one has

$$\begin{aligned} d \cdot Z_{h_\sigma,\{v\}}(t^d)^- &= \sum_{\alpha^d=1} \alpha^{-dr_v} (\mathrm{L}_1\,\Pi_{1,\{v\}})(\alpha t) \\ &+ \sum_{\substack{\alpha\in\mathbb{C}^* \\ \alpha^d\neq 1}} \sum_i \frac{1}{d_i} \sum_{\substack{\psi_i\in\widehat{H}_i \\ \psi_i(v_i)=\alpha^d\neq 1}} \psi_i([l_i'])^{-1} \Pi_{\psi_i,\{v_i\}}(1)(1 - \psi_i(v_i)) \cdot \frac{\alpha^{-dr_v}}{1-\alpha t}. \end{aligned}$$

Using the definition of the torsion and of σ_i one has $\widehat{\mathcal{T}_{\sigma_i}}(\psi_i) = \psi_i([l_i'])^{-1}\Pi_{\psi_i,\{v_i\}}(1)$. Next, for $\epsilon \neq 1$, (9.5.9) for $q := dr_v \in [0, d)$ applies. Hence $Z_{h_\sigma,\{v\}}(t^d)^-$ equals

$$\frac{1}{d}\sum_{\alpha^d=1}\alpha^{-dr_v}(\mathrm{L}_1\,\Pi_{1,\{v\}})(\alpha t) + \sum_{\epsilon\in\mathbb{C}^*\setminus 1}\sum_i \frac{1}{d_i}\sum_{\substack{\psi_i\in\widehat{H_i}\\ \psi_i(v_i)=\epsilon\neq 1}} \widehat{\mathcal{T}_{\sigma_i}}(\psi_i)\cdot\frac{(1-\epsilon)t^{dr_v}}{1-\epsilon t^d}.$$

Since the last fraction has no pole at $t = 1$, its coef_1^0 is its value at 1. Hence

$$\mathrm{coef}_1^0\left(Z_{h_\sigma,\{v\}}(t^d)^-\right) = \mathrm{coef}_1^0\left(\frac{1}{d}\sum_{\alpha^d=1}\alpha^{-dr_v}(\mathrm{L}_1\,\Pi_{1,\{v\}})(\alpha t)\right) + \sum_i \frac{1}{d_i}\sum_{\substack{\psi\in\widehat{H_i}\\ \psi_i(v_i)\neq 1}}\widehat{\mathcal{T}_{\sigma_i}}(\psi_i).$$

Recall that (9.5.20) reads as

$$\mathcal{T}_\sigma(1) = \mathrm{coef}_1^0\left(Z_{h_\sigma,\{v\}}(t^d) - \frac{1}{d}\Pi_{1,\{v\}}(t)\right) + \sum_i \frac{1}{d_i}\sum_{\substack{\chi_i\in\hat{H}_i\setminus 1\\ \chi_i(v_i)=1}}\widehat{\mathcal{T}_{\sigma_i}}(\chi_i).$$

Since $\mathrm{coef}_1^0\left(Z_{h_\sigma,\{v\}}(t^d)^+\right) = \mathrm{pc}\big(Z_{h_\sigma,\{v\}}(t^d)\big)$, the last two identities combined provide:

$$\mathcal{T}_\sigma(1) - \sum_i \mathcal{T}_{\sigma_i}(1) = \mathrm{pc}\big(Z_{h_\sigma,\{v\}}(t^d)\big) + \frac{1}{d}\,\mathrm{coef}_1^0\Big(\sum_{\alpha^d=1}\alpha^{-dr_v}(\mathrm{L}_1\,\Pi_{1,\{v\}})(\alpha t) - \Pi_{1,\{v\}}(t)\Big).$$

The last coef_1^0 term depends only on the coefficients of terms with non-positive degree of the Laurent expansion of $\Pi_{1,\{v\}} = \prod_w (1-t^{a_{vw}})^{\kappa_w-2}$ at 1. (Indeed, $L_1\Pi_{1,\{v\}}$ is the principal part of this expansion.) This expansion (use also (9.5.14)) is:

$$\Pi_{1,\{v\}}(t) = \frac{1}{a_{vv}}\Big(\frac{1}{(t-1)^2} + \frac{1+d\alpha_v/2}{t-1} + \frac{3(d\alpha_v+1)^2 + d^2\beta_v - 1}{24} + R(t)\Big) \tag{9.5.22}$$

with $\lim_{t\to 1} R(t) = 0$. Hence (9.5.15) and a computation provide Theorem 9.5.21.

9.6 The Seiberg–Witten Invariant

In this section we fix a plumbed rational homology sphere 3-manifold M associated with a connected negative definite plumbing graph Γ. The Seiberg–Witten invariant of M, $\mathfrak{sw}$, associates to each spin^c structure $\sigma \in \mathrm{Spin}^c(M)$ of M

a rational number $\mathfrak{sw}_\sigma$. Here, based on [520], we 'define' it as the refined Turaev torsion modified by the Casson–Walker invariant. Based on the formulae of the previous sections, this provides $\mathfrak{sw}$ combinatorially from Γ. ♣

Definition 9.6.1 We define $\mathfrak{sw} : \mathrm{Spin}^c(M) \to \mathbb{Q}$, $\sigma \mapsto \mathfrak{sw}_\sigma$ by

$$\mathfrak{sw}_\sigma := \mathcal{T}_\sigma(1) - \lambda/|H|.$$

Example 9.6.2 If $H = 0$ then $\mathrm{Spin}^c(M)$ has only one element, and the corresponding Seiberg–Witten invariant is $-\lambda(M)$ (the negative of the Casson invariant).

Remark 9.6.3

(a) In the literature in some definitions the Seiberg–Witten invariant appears with opposite sign. Our choice 9.6.1 is motivated by the fact that in this way the Euler characteristic of the topological lattice cohomology $\mathbb{H}^*(M)$ equals $\mathfrak{sw}_\sigma$ (with our sign), and also we will have a sign-correction free comparison with the analytic theory. However, by 11.7.2 $\mathbb{H}^*(M)$ corresponds naturally to $HF^+(-M)$, and HF^+ is again a categorification of the Seiberg–Witten invarint. Hence $\mathfrak{sw}(M)$ in our sense is $\mathfrak{sw}(-M)$ in the sense of Heegaard Floer $+$-theory HF^+.

(b) Several topological/combinatorial expressions of this book can be extended to plumbed 3-manifolds associated with *not necessarily negative definite graphs*. Let us consider a plumbing graph Γ with intersection matrix I such that $M(\Gamma)$ is a $\mathbb{Q}HS^3$. In particular, Γ is a tree, $g_v = 0$ for all v and $\det(I) \neq 0$. We assume that $(E_u^*, E_v^*) = (I^{-1})_{uv} \neq 0$ for all u and v. Then the Casson–Walker invariant can be computed by the formula (9.3.1), the torsion can be recovered from the procedure of 9.5.7 [519]. Hence the Seiberg–Witten invariant (via 9.6.1) has the corresponding combinatorial expression as well.

9.6.4 Additivity Formula for the Seiberg–Witten Invariant We consider the situation from Sect. 9.3.A, see also 9.5.20 and 9.5.21 for notations.

Theorem 9.6.5 ([77]) *Set $l' = \sum_w l'_w E_w$, $l'_v \in [0, 1)$, as in 9.5.21. Let $\sigma \in \mathrm{Spin}^c(M(\Gamma))$ be defined as $[l']\sigma[K] = \sigma[K + 2l']$, and take also its restrictions $\sigma_i := [R_i(l')]\sigma[K_i] = \sigma[R_i(K + 2l')]$ too. Set $h_\sigma = [l']$. Then the following identities hold:*

$$\mathfrak{sw}_\sigma(M(\Gamma)) - \sum_i \mathfrak{sw}_{\sigma_i}(M(\Gamma_i)) = pc(Z_{h_\sigma,\{v\}}(t)) + \frac{1}{8} - \frac{(\alpha_v + 1 - 2r_v)^2}{8m_{vv}}.$$

and

$$\Big(\mathfrak{sw}_\sigma(M(\Gamma)) - \frac{(K+2l')^2+|\mathcal{V}|}{8}\Big) - \sum_i \Big(\mathfrak{sw}_{\sigma_i}(M(\Gamma_i)) - \frac{(K_i+2R_i(l'))^2+|\mathcal{V}_i|}{8}\Big)$$
$$= pc(Z_{h_\sigma,\{v\}}(t)).$$

Proof Combine Theorems 9.3.11 and 9.5.21 and use $\mathrm{pc}(S(t^d)) = \mathrm{pc}(S(t))$. □

This additivity formula should be compared with its 'analytic counterpart', namely with Okuma's additivity formula 8.7.25.

9.6.A *The Seiberg–Witten Invariant and the Series $Z(\mathbf{t})$*

We prove two key formulae for the Seiberg–Witten invariant of a rational homology sphere link. One of them identifies it with a weighted Euler characteristic of (shifted) weighted cubes in a large rectangle of $L \otimes \mathbb{R}$. The other one identifies it with the constant term of the counting function of the coefficients of $Z(\mathbf{t})$. The proofs are based on additivity formulae of the compared invariants.
The similarities with the analytic counterpart (the series $P(\mathbf{t})$ and the equivariant genera) are emphasized. In this section we follow [482]. ♣

9.6.6 In the next discussion we will use the weighted cubes introduced in 8.4.12. Let us fix an element h of H and write $L'_h = \{l' \in L' : [l'] = h\}$. Recall that the set of 'combinatorial' q-cubes (associated with h) consists of pairs $(l', I) \in L'_h \times \mathscr{P}(\mathcal{V})$, $|I| = q$ $(q \in \mathbb{Z}_{\geq 0})$. (l', I) will be identified with the vertices $\{l' + \sum_{v\in I'} E_v\}_{I'\subset I}$ of a 'Euclidean' cube in $L \otimes \mathbb{R}$. One defines the weight function $w : L' \to \mathbb{Q}$, $w(l') := \chi(l')$, and also the a weight of the q-cubes

$$w((l', I)) = \max_{I'\subset I} \Big\{ w(l' + \sum_{v\in I'} E_v) \Big\}.$$

Assume that a set $A \subset L \otimes \mathbb{R}$ has the following property: if an Euclidean cube (as above) is in A then any face of any dimension of that cube is in A. For such a set A one defines the 'weighted Euler characteristic'

$$Eu_\chi(A) := \sum_{(l',I)\in A} (-1)^{|I|+1} w((l', I)).$$

If A_1 and A_2 are two sets with the above property, then we might abridge the difference $Eu_\chi(A_1) - Eu_\chi(A_1 \cap A_2)$ by $Eu_\chi(A_1 \setminus A_2)$.

Such a set A might appear as follows. For the fixed class $h \in L'/L$ one takes two representatives l'_1, $l'_2 \in L'_h$ with $l'_2 \leq l'_1$. Then $R_h = R_h(l'_2, l'_1)$ consists of the

union of all combinatorial cubes (l', I), of any dimension, such that $[l'] = h$ and any vertex $l' + \sum_{v \in I'} E_v$ of (l', I) satisfies $l'_2 \le l' + \sum_{v \in I'} E_v \le l'_1$. Accordingly to the above identification, $R_h(l'_1, l'_2)$ will also denote the real rectangle $\{x \in L \otimes \mathbb{R} : l'_2 \le x \le l'_1\}$, or the union of all Euclidean cubes in this real rectangle.

Remark 9.6.7 In this book, for a fixed $h \in H$, we consider two types of rectangles and weighted q-cubes, depending on the geometric situation. First, in the context of lattice cohomology (see e.g. 11.1.B, and in its preparation 7.3.A) we take integral lattice points and rectangles $R(l_2, l_1)$ and cubes with vertices in the lattice L, but we twist the weight function: we take χ_k (which generates w_k) with $k = K + 2l'_h$, for some representative l'_h of h.

Second, in the context of 8.4.12–8.4.13 (and in the previous paragraph too), when we wish to relate the cubes with the coefficients of $Z(\mathbf{t})$, we take shifted rectangles $R_h = R_h(l'_2, l'_1)$ ($[l'_j] = h$) with cubes (l', I) of type $[l'] = h$ in them, together with the usual untwisted Riemann–Roch-function $\chi = \chi_K$.

The two approaches can be compared easily (see also 11.1.B). Indeed, if $k = K + 2l'_h$, $[l'_h] = h$, then for $l \in L$ we have $\chi(l + l'_h) = \chi_k(l) + \chi(l'_h)$. In particular, with the notation $l'_j = l_j + l'_h$ ($l_j \in L$), we have $R_h(l'_2, l'_1) = l'_h + R(l_2, l_1)$ as rectangles, and

$$Eu_\chi(R_h(l'_2, l'_1)) = Eu_{\chi_k}(R(l_2, l_1)) - \chi(l'_h).$$

9.6.8 Via the two incarnations of the weighted cubes (cf. 9.6.7) the next result is the 'pair' of the statements from Lemmas 7.3.9 and 7.3.12.

Lemma 9.6.9 *Fix a class h and take a representative l'_0 of h in $-K + \mathscr{S}'$.*

(a) For any $l' \in L'$, $[l'] = h$, $l' > l'_0$, there exists an E_v in the support of $l' - l'_0$ such that $w(l' - E_v) \le w(l')$.

(b) There exists a computation sequence $\{\ell_i\}_{i \ge 0}$, $\ell_i \in L$, with $\ell_0 = 0$, and $\ell_{i+1} = \ell_i + E_{v(i)}$ for some $v(i) \in \mathscr{V}$ when $i \ge 0$, satisfying:

(i) The coefficients of ℓ_i tend to infinity, that is $\lim_{i \to \infty}(\ell_i, -E^*_v) = \infty$ for all v.

(ii) For any $i \ge 0$ one has $w(l'_0 + \ell_i) \le w(l'_0 + \ell_{i+1})$.

(c) For any $l' < 0$, with $[l'] = h$, there exists $E_v \in |l'|$ such that $w(l' + E_v) \le w(l')$.

(d) For any representatives l'_1, l'_2 of h, such that $l'_1 \ge l'_0 > 0 \ge l'_2$, $Eu_\chi(R_h(l'_2, l'_1))$ is independent of the choice of l'_1 and l'_2. In particular, with such choices, $h \mapsto Eu_\chi(R_h(l'_2, l'_1))$ is a numerical invariant of $h \in H = L'/L$.

Proof *(a)* Assume that $w(l' - E_v) > w(l')$ for any $E_v \in |l' - l'_0|$. Then $(K + l', E_v) \ge 0$. Since $(K + l'_0, E_v) \le 0$, one gets $(l' - l'_0, E_v) \ge 0$ for any E_v, hence $(l' - l'_0)^2 \ge 0$, which is a contradiction. For *(b)* use *(a)*. Next we prove *(d)*.

For any $l'_1 > l'_0$ chose $E_v \in |l'_1 - l'_0|$ such that $w(l'_1 - E_v) \le w(l'_1)$. Then we claim that $Eu_\chi(R_h(l'_2, l'_1 - E_v)) = Eu_\chi(R_h(l'_2, l'_1))$. Indeed, for any l' with

$[l'] = h$ and $l'_2 \le l' \le l'_1$ and $m_{E_v}(l') = m_{E_v}(l'_1)$, we have $w(l' - E_v) \le w(l')$ too (use $(l', E_v) \le (l'_1, E_v)$). Hence, combinatorial cancelations of type $w(l' - E_v, I \cup v) = w(l', I)$, $I \not\ni v$, in the sum provide the identity. In particular, by induction, $Eu_\chi(R_h(l'_2, l'_1)) = Eu_\chi(R_h(l'_2, l'_0))$.

Statements regarding part *(c)* and l'_2 can be proved symmetrically. □

Definition 9.6.10 The invariant provided by 9.6.9(d) will be denoted by $\overline{\mathfrak{sw}}_h$.

9.6.11 Let $Z(\mathbf{t}) = \sum_{l' \in L'} \mathfrak{z}(l')\mathbf{t}^{l'}$ be the combinatorial series defined in Sect. 8.4. Since Z is supported on $\mathscr{S}'$, the next sum in (9.6.1) is finite by 2.1.19.

Theorem 9.6.12 *Fix $h \in H$. For any $l' \in -K + \mathscr{S}'$ with $[l'] = h$, the expression*

$$-\chi(l') + \sum_{l \in L,\, l \not\ge 0} \mathfrak{z}(l' + l) \tag{9.6.1}$$

depends only on the class h of l', and, in fact, it equals $\overline{\mathfrak{sw}}_h$ (defined in 9.6.10).

Proof In the proof we use the above rectangle notation (and several versions of it); for simplicity we will drop their h-index. Fix some $l'_2 \le 0$ with $[l'_2] = [l']$ and set

$$R' := \{l'' \in L \otimes \mathbb{R} : \ l'' \ge l'_2,\ l'' - l' = \sum c_v E_v \text{ so that } \exists\, c_v \le 0\}.$$

Although R' is not bounded, $R' \cap \mathscr{S}'$ is finite by 2.1.19. Fix some i_0 from Lemma 9.6.9*(b)* (applied for $l' = l'_0$) such that $R' \cap \mathscr{S}' \subset R(l'_2, l' + \ell_{i_0} - E)$, and define

$$\widetilde{R} = \widetilde{R}(\ell_{i_0}) := R' \cap R(l'_2, l' + \ell_{i_0}),$$

$$\partial_1 \widetilde{R} := \{l'' \in \widetilde{R} : \ l' \le l'',\ l'' - l' = \sum c_v E_v \text{ so that } \exists\, c_v = 0\},$$

$$\partial_2 \widetilde{R} := \{l'' \in \widetilde{R} : \ l'' - l' - \ell_{i_0} = \sum c_v E_v \text{ so that } \exists\, c_v = 0\}.$$

Then, by Theorem 8.4.13

$$\sum_{l \in L,\, l \not\ge 0} \mathfrak{z}(l' + l) = Eu_\chi\big(\widetilde{R}(\ell_{i_0})\big) - Eu_\chi\big(\partial_1 \widetilde{R}(\ell_{i_0}) \cup \partial_2 \widetilde{R}(\ell_{i_0})\big). \tag{9.6.2}$$

We claim that the right hand side of (9.6.2) equals with any similar expression whenever we replace $\widetilde{R}(\ell_{i_0})$ by any $\widetilde{R}(y)$ with $y \in L$, $y \ge \ell_{i_0}$. This follows from the fact that the identity (9.6.2) is true for any $\widetilde{R}(y)$ with $y \ge \ell_{i_0}$ (by the same argument

based on 8.4.13), and the left hand side of (9.6.2) is independent of the choice of y (since $Z(\mathbf{t})$ is supported in $\mathscr{S}'$, and in the sum the same coefficients appear).

Next, we prove that $Eu_\chi\big(\partial_2\widetilde{R}(y)\setminus\partial_1\widetilde{R}(y)\big)=0$. For this we distinguish for any $w\in\mathscr{V}$ the following part of $\partial_2=\partial_2\widetilde{R}(y)$:

$$\partial_{2,w}=\partial_{2,w}\widetilde{R}(y):=\{l''\in\widetilde{R}\ :\ \ l''-l'-\ell_{i_0}=\sum c_vE_v \text{ and } c_w=0\}.$$

First, we wish to choose a convenient y (which satisfies the above claim together with some additional properties). We start to fix a vertex $v_0\in\mathscr{V}$ and we denote by $d(v)$ the (graph-theoretical geodesic) distance from $v\in\mathscr{V}$ to v_0. Write $D:=\max_{w\in\mathscr{V}}\{d(w)\}$. Next we fix some $M>0$ such that $M>2(-E_v^2+1)$ for any $v\in\mathscr{V}$. Then we choose y such that the coefficients $\{m_u(y)\}_{u\in\mathscr{V}}$ of $l'+y$ satisfy

$$\begin{aligned}&(i)\quad m_u(y)>M\cdot m_v(y)\ \text{ whenever }\ d(u)>d(v);\\ &(ii)\quad m_{v_0}(y)/M^D \text{ is larger than any } E_v\text{-coordinate of } l'.\end{aligned}\tag{9.6.3}$$

Now fix some $u\in\mathscr{V}$, $u\neq v_0$, $d(u)=D$ and let v be its adjacent vertex with $d(v)=d(u)-1$. Then for any $x=m_uE_u+m_vE_v+\cdots\in\partial_{2,u}$ ($x\in L'_h$) one has $\chi(x+E_v)-\chi(x)=1-(x,E_v)\le 1-m_u-m_vE_v^2\le 0$ from (9.6.3)*(i)*. Here it is important that u and v are adjacent and the E_u-coefficient m_u of x is large compared to m_v. Hence by combinatorial cancellation—using pairs of cubes of type (x,I) and $(x,I\cup v)$, $I\not\ni v$—one has $Eu_\chi(\partial_{2,u}\setminus(\cup_{w\neq u}\partial_{2,w}\cup\partial_1))=0$. In similar way we eliminate all faces of type $\partial_{2,u}$ with $d(u)=D$, and we get $Eu_\chi(\partial_2\setminus(\cup_{w:\,d(w)<D}\partial_{2,w}\cup\partial_1))=0$.

Then we continue with vertices u with $d(u)$ one less. By induction, by this procedure we get that $Eu_\chi(\partial_2\setminus(\partial_{2,v_0}\cup\partial_1))=0$. Note that in this sequence of 'contractions' we used only part *(i)* of (9.6.3).

Next we eliminate the remaining cubes of ∂_{2,v_0} which are not in ∂_1. We consider a similar sequence of 'contractions', but now starting from v_0.

Let v be any vertex with $d(v)=1$. For an arbitrary element $x=m_{v_0}E_{v_0}+m_vE_v+\cdots\in\partial_{2,v_0}\setminus\partial_1$ ($x\in L_h$) we consider $\chi(x+E_v)-\chi(x)$, which is $\le 1-m_{v_0}-m_vE_v^2$.

We distinguish two cases according to the value of m_v. If $m_v<m_{v_0}/M$ (that is, if the ratio of m_v/m_{v_0} is 'small') then using $m_{v_0}/M\le(m_{v_0}-1)/(-E_v^2)$ guaranteed by *(i)*, we get $\chi(x+E_v)-\chi(x)\le 0$. Hence $w(x,I)=w(x,I\cup v)$ for any such x and $I\not\ni v$. After all such eliminations of pairs of cubes (x,I) and $(x,I\cup v)$, all the x-corners of the remaining cubes (x,J) satisfy $m_v\ge m_{v_0}/M$ (that is, the ratio of m_v/m_{v_0} is 'large'). This in particular shows, via (9.6.3)*(ii)*, that the E_v-coordinate of any such x is larger than the E_v-coordinate of l'.

With the remaining lattice points $\{\partial_{2,v_0}\setminus\partial_1\}\cap\{x\ :\ m_v\ge m_{v_0}/M\}$ we repeat the procedure for another vertex v' with $d(v')=1$ (which replaces the previous v). After we finish all such vertices we only have cubes (x,J) with x in

$$\{\partial_{2,v_0}\setminus\partial_1\}\cap\{x\ :\ m_v(x)\ge m_{v_0}(x)/M \text{ for all } v \text{ with } d(v)=1\}.\tag{9.6.4}$$

Any such x has the property that its E_v-coordinate is larger than the E_v-coordinate of l' for any v with $D(v) = 1$.

Next, chose v with $d(v) = 2$, and let u be its adjacent vertex with $d(u) = 1$. Take any $x = m_u E_u + m_v E_v + \cdots$ from the set (9.6.4). As above, we distinguish two cases. If $m_v < m_u/M$ then $m_u/M \le (m_u - 1)/(-E_v^2)$ guaranteed by *(i)* gives $\chi(x + E_v) - \chi(x) \le 0$. Hence $w(x, I) = w(x, I \cup v)$ for any such x and $I \not\ni v$. After all such eliminations of pairs of cubes (x, I) and $(x, I \cup v)$, all the x-corners of the remaining cubes (x, J) satisfy $m_v \ge m_u/M$. Since $m_u \ge m_{v_0}/M$, we also have $m_v \ge m_{v_0}/M^2$. Using *(ii)* we get that the E_v-coordinate of x is larger than the E_v-coordinate of l'. Then, we proceed with all the vertices with $d(v) = 2$. Finally, by induction, we continue in this order with all the vertices.

After all these eliminations, if a certain lattice point remains, then it must have the property that for any $v \in \mathcal{V}$ its E_v-coordinate is larger than the corresponding E_v-coordinate of l'. But there is no such lattice point in $\partial_2 \setminus \partial_1$. Hence, we proved that $Eu_\chi(\partial_2 \setminus \partial_1) = 0$, which via (9.6.2) implies

$$\sum_{l \in L,\, l \not\gtrless 0} \mathfrak{z}(l' + l) = Eu_\chi(\widetilde{R}(y)) \setminus \partial_1 \widetilde{R}(y)). \tag{9.6.5}$$

Finally, we claim that by a combinatorial cancelation in the sum, $Eu_\chi(\widetilde{R}(y)) = Eu_\chi(R)$, where $R = R(l_2', l')$. This follows by induction using a sequence of type $\{\ell_i\}_{0 \le i \le j_0}$ (as in 9.6.9) connecting l' with $l' + y$. Its existence follows from 9.6.9*(a)*. This provides the equality $Eu_\chi(R' \cap R(l_2', l' + \ell_{i+1})) = Eu_\chi(R' \cap R(l_2', l' + \ell_i))$. Indeed, for any I containing $v(i)$ and cube (l'', I) of $R' \cap (R(l_2', l' + \ell_{i+1})$ and not contained in $R(l_2', l' + \ell_i))$ one has $w((l'', I)) = w((l'' + E_{v(i)}, I \setminus v(i)))$ by 9.6.9. Similarly, by the same argument applied for $\partial_1 \widetilde{R}(y)$, one gets $Eu_\chi(\partial_1 \widetilde{R}(y)) = Eu_\chi(R(l', l')) = -w(l')$. Hence (9.6.5) reads as

$$-\chi(l') + \sum_{l \in L,\, l \not\gtrless 0} \mathfrak{z}(l' + l) = Eu_\chi(R) = \overline{\mathfrak{s}}_h. \tag{9.6.6}$$

□

Remark 9.6.13 The statement of theorem 9.6.12 can be (and should be) compared with Theorem 8.2.11 and its consequences, e.g. with (8.2.26). However, their proofs are very different. The main difference is in the weights of the 'analytic' and 'topological' cubes. For a fixed cube (l', I) the analytic invariant associated with (l', I) is $\mathfrak{h}(l' + E_I)$, which depends only on $l' + E_I$ (on the 'top' vertex). On the other hand, the topological weight $w((l', I))$ is a max of the χ-value running along all the vertices of the cube. (Compare also the formulae of $\mathfrak{p}(l')$ from (8.2.9) with the formula of $\mathfrak{z}(l')$ from (8.4.14).)

It is worth to mention also the following alterity. While the analytic linear subspace arrangement associated with some l' (cf. 8.3) is the projection into $H^0(\mathscr{O}_E(-l'))$ of a larger ideal arrangement of $\mathscr{O}_{X_a,o}$, a similar object at the

topological level (as lift of the topological linear subspace arrangement) does not exists.

9.6.14 The Additivity Formula for $\overline{\mathfrak{s}}$ Similarly as in the proof of Theorem 8.4.13, we change the variables of the series $Z(\mathbf{t})$. By setting $x_v := \mathbf{t}^{E_v^*}$ for all $v \in \mathcal{V}$, $Z(\mathbf{t})$ transforms into $z(\mathbf{x}) = \prod_{v\in\mathcal{V}}(1-x_v)^{\kappa_v-2}$, whose Taylor series at the origin is denoted by $\sum \mathfrak{z}^*(a)\mathbf{x}^a$, where $\mathbf{x}^a = x_1^{a_1}\cdots x_s^{a_s}$. Sometimes, in order to emphasize Γ, we write $z^\Gamma(\mathbf{x}) = z(\mathbf{x})$ and $\mathfrak{z}^{*,\Gamma}(a) = \mathfrak{z}^*(a)$. The exponents a_v are the coordinates of L' in the basis $\{E_v^*\}_v$, and $\mathbf{t}^{l'}$ transforms into $\mathbf{x}^a$. In particular, $\mathfrak{z}^*(a) = \mathfrak{z}(l')$. For any fixed $l' = \sum_v a_v E_v^*$ we define

$$h_a^\Gamma := \sum_{l\in L,\, l\not\geq 0} \mathfrak{z}(l'+l) = \sum_{b\in S^\Gamma(a)} \mathfrak{z}^{*,\Gamma}(b), \tag{9.6.7}$$

where $S^\Gamma(a) = \{b \in L'(\Gamma) : b = a + \sum n_v E_v,\ n_v \in \mathbb{Z}, \exists\, n_v < 0\}$. We assume that $|\mathcal{V}| \geq 2$ and we fix an *end-vertex* u of Γ. We also set

$$h_a^u := \sum \mathfrak{z}^{*,\Gamma}(b) \qquad (\text{sum over } b = a + \sum n_v E_v,\ n_v \in \mathbb{Z},\ n_u < 0).$$

The inclusion of the subgraph $\Gamma \setminus u$ induces $i : L(\Gamma\setminus u) \to L(\Gamma)$, $i(E_v) \mapsto E_v$ (usually we omit the symbol i), and its dual, the restriction $R : L'(\Gamma) \to L'(\Gamma\setminus u)$. Then

$$\begin{cases} R(E_v^{*,\Gamma}) = E_v^{*,\Gamma\setminus u} \text{ for } v \neq u, \text{ and } R(E_u^{*,\Gamma}) = 0; \\ R(E_v^{\Gamma}) = E_v^{\Gamma\setminus u} \text{ for } v \neq u, \text{ and } R(E_u^{\Gamma}) = -E_w^{*,\Gamma\setminus u}, \end{cases} \tag{9.6.8}$$

where w is the adjacent vertex of u. We abridge $R(\sum_v a_v E_v^*)$ by $R(a)$.

Proposition 9.6.15 *Assume that w, the adjacent vertex of u in Γ, has valency two. Then, if $a_u \gg 0$ (compared with the other entries $\{a_v\}_{v\neq u}$ of a), one has*

$$h_a^\Gamma = h_a^u + h_{R(a)}^{\Gamma\setminus u}.$$

Proof Since $\kappa_w = 2$, the series z^Γ and $z^{\Gamma\setminus u}$ have the form

$$z^\Gamma = \widetilde{z}\cdot\frac{1}{1-x_u};\quad z^{\Gamma\setminus u} = \widetilde{z}\cdot\frac{1}{1-x_w}, \tag{9.6.9}$$

where $\widetilde{z}$ does not depend on the variables x_w and x_u. Therefore, for any relevant $b \in S^\Gamma(a)$ (i.e. when $\mathfrak{z}^{*,\Gamma}(b) \neq 0$) one has: $b_w = 0$ and $b_u \geq 0$. Hence:

$$h_a^\Gamma - h_a^u = \sum_{b\in S_{\geq}^\Gamma(a)} \mathfrak{z}^{*,\Gamma}(b), \tag{9.6.10}$$

where $S^{\Gamma}_{\geq}(a) = \{b \in S^{\Gamma}(a) : b_w = 0,\ b_u \geq 0,\ n_u \geq 0, \exists\, n_v < 0\}$. Then taking $(\cdot, E_u)$ to the identity $b = a + \sum_v n_v E_v$, one gets (with $e_u := E_u^2$)

$$b_u = a_u - n_w - e_u n_u. \tag{9.6.11}$$

Sometimes it is convenient to write b as (b°, b_w, b_u), with $b^\circ = \sum_{v \neq w,u} b_v E_v^*$.

We wish to write the right hand side of (9.6.10) in terms of $\Gamma \setminus u$. Accordingly, we consider the expression $h^{\Gamma\setminus u}_{R(a)} = \sum \mathfrak{z}^{*,\Gamma\setminus u}(\bar{b})$, where the sum is over

$$S^{\Gamma\setminus u}(R(a)) := \{\bar{b} \in L'(\Gamma \setminus u),\ \bar{b} = R(a) + \sum_{v\neq u} m_v E_v,\ m_v \in \mathbb{Z},\ \exists\, m_v < 0\}.$$

By similar argument as above, based on (9.6.9), for any relevant $\bar{b} \in S^{\Gamma\setminus u}(R(a))$, one has $\bar{b}_w \geq 0$, hence it is enough to consider the subset $S^{\Gamma\setminus u}_{\geq}(R(a)) = \{\bar{b} \in S^{\Gamma\setminus u}(R(a)) : \bar{b}_w \geq 0\}$ in the computation of $h^{\Gamma\setminus u}_{R(a)}$. We write $\bar{b} = (\bar{b}^\circ, \bar{b}_w)$.

Since $n_u = -(E_u^{*,\Gamma}, b - a)$, for any fixed a one has a well-defined map

$$\Phi : S^{\Gamma}_{\geq}(a) \to S^{\Gamma\setminus u}_{\geq}(R(a)),\quad b = (b^\circ, b_w = 0, b_u) \mapsto (\bar{b}^\circ, \bar{b}_w) := (b^\circ, n_u).$$

In other words, $\Phi(b) = R(b) + n_u E_w^{*,\Gamma\setminus u}$ (use (9.6.8)). Moreover, again by (9.6.8), $\Phi(b) - R(a) = R(\sum_{v\neq u} n_v E_v^{\Gamma}) = \sum_{v\neq u} n_v E_v^{\Gamma\setminus u}$, hence the integers $\{n_v\}_{v\neq u}$ at the level of Γ are the same as the integers $\{m_v\}_{v\neq u}$ at $\Gamma \setminus u$ level. This fact, and (9.6.11) implies the injectivity of Φ. For the surjectivity, for any $(b^\circ, n_u) \in S^{\Gamma\setminus u}_{\geq}(R(a))$ set b_u defined by (9.6.11), then $(b^\circ, 0, b_u)$ satisfies automatically all the conditions of $S^{\Gamma}_{\geq}(a)$, except maybe one, namely $b_u \geq 0$. In order to guarantee this one too, we argue as follows: for $R(a)$ fixed, consider all the elements $(\bar{b}^\circ, \bar{b}_w) \in S^{\Gamma\setminus u}_{\geq}(R(a))$ (this is a finite set), and associate with them the maximum M of all the possible values $m_w + e_u \bar{b}_w$ (which equals under Φ to $n_w + e_u n_u$). M depends only on $R(a)$. Then, if we take $a_u \geq M$, then by (9.6.11) the inequality $b_u \geq 0$ is also satisfied.

Finally, notice that from (9.6.9), for any $b \in S^{\Gamma}_{\geq}(a)$ one has

$$\mathfrak{z}^{*,\Gamma}(b) = \mathfrak{z}^{*,\Gamma\setminus u}(\Phi(b)),$$

since both of them agree with the $\prod_{v\neq w,u} x^{b_v}$-coefficient of $\widetilde{z}$. □

Corollary 9.6.16 *Fix an end-vertex u as above and fix $l' = \sum_v a_v E_v^{*,\Gamma}$ too. Assume that every a_v is large (and a_u is large compared with the others). Then one has:*

$$h^u_a = \overline{\mathfrak{s}}^{\Gamma}_{[l']} + \chi^{\Gamma}(l') - (\overline{\mathfrak{s}}^{\Gamma\setminus u}_{[R(l')]} + \chi^{\Gamma\setminus u}(R(l'))).$$

With the notation

$$\mathfrak{s}^{\Gamma}_{[l']} := \overline{\mathfrak{s}}^{\Gamma}_{[l']} + (K^2 + |\mathscr{V}|)/8$$

one has

$$\begin{aligned} h^u_a &= \mathfrak{s}^{\Gamma}_{[l']} - \frac{(K_\Gamma + 2l')^2 + |\mathscr{V}|}{8} - \mathfrak{s}^{\Gamma\setminus u}_{[R(l')]} + \frac{(K_{\Gamma\setminus u} + 2R(l'))^2 + |\mathscr{V}\setminus u|}{8} \\ &= \overline{\mathfrak{s}}^{\Gamma}_{[l']} - \overline{\mathfrak{s}}^{\Gamma\setminus u}_{[R(l')]} - \frac{(l', E^*_u)\cdot(l' + K_\Gamma, E^*_u)}{2(E^*_u, E^*_u)} \\ &= \overline{\mathfrak{s}}^{\Gamma}_{[l']} - \overline{\mathfrak{s}}^{\Gamma\setminus u}_{[R(l')]} + \chi(k\cdot E^*_u), \quad \textit{where} \quad k = \frac{(l', E^*_u)}{(E^*_u, E^*_u)}. \end{aligned}$$

Proof Note that $K_{\Gamma\setminus u} = R(K_\Gamma)$. Therefore, if $l' \in -K + \mathscr{S}'$ in Γ, then $R(l') \in -K_{\Gamma\setminus u} + \mathscr{S}'(\Gamma\setminus u)$ as well. In particular, Theorem 9.6.12 applied for l' and $R(l')$, and Proposition 9.6.15 provide the first identity. Then use the identity

$$l'' = iR(l'') + \frac{(l'', E^*_u)}{(E^*_u, E^*_u)}\cdot E^*_u$$

for both $l'' = l'$ and $l'' = K_\Gamma$, and $(i(l), E^*_u) = 0$ for any $l \in L'(\Gamma\setminus u)$. □

9.6.17 Another Additivity Formula for $\mathfrak{s}_{[l']}$ Next we wish to establish the relationship between the coefficients h^u_a defined in (9.6.7) and the $[l']$-component $Z_{[l'],\{u\}}(t)$ of the series $Z_{\{u\}}(t)$, with reduced variables associated with the subset $\{u\} \subset \mathscr{V}$, and with the class $[l'] \in H$.

Theorem 9.6.18 *Consider the graph Γ and let u be one of its end-vertices.*

(a) Fix $l' = \sum_v a_v E^*_v = \sum_v l'_v E_v$ as in Corollary 9.6.16 and set $d = \det(\Gamma)$. Then

$$h^u_a = \sum_{i<dl'_u} \mathfrak{z}_i, \quad \textit{where} \quad Z_{[l'],\{u\}}(t^d) = \sum_{i\geq 0} \mathfrak{z}_i t^i.$$

(b) Take $\bar{l}' \in L'$ such that $(\bar{l}', E^*_u) \in (-1, 0]$. Then

$$\begin{aligned} &\mathrm{pc}(Z_{[\bar{l}'],\{u\}}(t^d)) = \\ &\quad \mathfrak{s}^{\Gamma}_{[\bar{l}']} - \frac{(K_\Gamma + 2\bar{l}')^2 + |\mathscr{V}|}{8} - \mathfrak{s}^{\Gamma\setminus u}_{[R(\bar{l}')]} + \frac{(K_{\Gamma\setminus u} + 2R(\bar{l}'))^2 + |\mathscr{V}\setminus u|}{8}. \end{aligned}$$

Proof Write $Z_{[l']}(\mathbf{t}) = \sum \mathfrak{z}(l'')\mathbf{t}^{l''}$. Then *(a)* follows from $Z_{[l'],\{u\}}(t^d) = \sum \mathfrak{z}(l'')t^{dl''_u}$ and

$$h_a^u = \sum_{b\,:\,n_u<0} \mathfrak{z}^{*,\Gamma}(b) = \sum_{l''\,:\,l''_u<l'_u} \mathfrak{z}(l'').$$

(b) For any fixed $\bar{l}'$ as in the assumption of *(b)*, take l' such that $[l'] = [\bar{l}']$, and $[R(l')] = [R(\bar{l}')]$ and $l' = \sum_v a_v E_v^* = \sum_v l'_v E_v$ with $a \gg 0$. Since $\bar{l}' - l' \in L$, we get that $l'_u + (\bar{l}', E_u^*) = (\bar{l}' - l', E_u^*) \in \mathbb{Z}$, hence $-(\bar{l}', E_u^*)$ is the fractional part of l'_u. Write $l'_u = \ell$ and let n be its integral part and $\{\ell\}$ its fractional part. For this l' we apply part *(a)* and the identity from Corollary 9.6.16.

In order to compute the periodic constant of $Z_{[l'],\{u\}}(t^d)$, notice that if $\mathfrak{z}(l'') \neq 0$ then $l'' - l' \in L$, hence $l''_u - l'_u = (E_u^*, l' - l'') \in \mathbb{Z}$. Therefore, if the coefficient $\mathfrak{z}_i$ of $Z_{[l'],\{u\}}(t^d)$ is nonzero, then $i - d\ell \in d\mathbb{Z}$. In particular, for $\mathfrak{z}_i \neq 0$, $i < d\ell$ if and only if $i \leq dn - 1$. This shows that one has to write h_a^u as $P(n)$ for some polynomial P, and then, the periodic constant of $Z_{[l'],\{u\}}(t^d)$ is $P(0)$. From the last identity of 9.6.16 we get that $P(n)$ is the right hand side of that identity after the substitution $(l', E_u^*) = -l'_u = -\{\ell\} - n$. Therefore, $\mathrm{pc}(Z_{[l'],\{u\}}(t^d))$ equals

$$\overline{\mathfrak{s}}^{\Gamma}_{[l']} - \overline{\mathfrak{s}}^{\Gamma\setminus u}_{[R(l')]} - \frac{\{\ell\}\cdot(\{\ell\} - (K_\Gamma, E_u^*))}{2(E_u^*, E_u^*)} = \overline{\mathfrak{s}}^{\Gamma}_{[l']} - \overline{\mathfrak{s}}^{\Gamma\setminus u}_{[R(l')]} - \frac{(\bar{l}', E_u^*)\cdot(\bar{l}' + K_\Gamma, E_u^*)}{2(E_u^*, E_u^*)}.$$

This provides *(b)* by a computation similar to the one from the proof of 9.6.16.

Strictly speaking, the above argument proves the additivity formula *(b)* only if $\kappa_w = 2$ (cf. the assumption of 9.6.15). In general we argue as follows: let u be an end-vertex, and let us blow up the unique edge adjacent to u, getting in this way Γ^b. Then the newly created vertex w has $\kappa_w = 2$. Hence in this situation we can apply the above proof for Γ^b and $\Gamma^b \setminus u$. Since blowing down the w-vertex in $\Gamma^b \setminus u$ we get $\Gamma \setminus u$, and all the involved invariants in *(b)* are stable with respect to blow up/down, the result follows. Indeed, $(K+2l')^2+|\mathscr{V}| = (K^b+2\pi^*(l'))^2+|\mathscr{V}^b|$, and $Z^{\{u\},[l']}(t)$ is stable since it depends only on the numbers $(E_{v_1}^*, E_{v_2}^*)$, where $\kappa_{v_i} \neq 2$.

The stability of the $\overline{\mathfrak{s}}$-terms will be proved later, when we show that they coincide with the Euler characteristic of the lattice cohomology of the link, which are independent of the choice of the plumbing graph, see 11.1.36 and 11.1.24. □

9.6.19 Identification of $\mathfrak{s}$ with the Seiberg–Witten Invariant Theorems 9.6.5 and 9.6.18 can now be compared. Since these theorems are true even if $\Gamma \setminus u$ is the empty graph, an induction on the number of vertices provides the following.

Corollary 9.6.20 *For any Γ and $[K + 2l'] \in Char$ one has*

$$\mathfrak{sw}_{\sigma[K+2l']}(M(\Gamma)) = \overline{\mathfrak{s}}_{[l']} + (K^2 + |\mathscr{V}|)/8 = \mathfrak{s}_{[l']},$$

or, with the notation of 9.6.9 and 9.6.10 (see also (9.6.6)),

$$Eu_\chi(R_{[l']}(l'_2, l'_1)) = \overline{\mathfrak{s}}_{[l']} = \mathfrak{sw}_{\sigma[K+2l']}(M(\Gamma)) - (K^2 + |\mathscr{V}|)/8. \tag{9.6.12}$$

Therefore, Theorem 9.6.12 reads as follows.

Theorem 9.6.21 *Assume that $l' \in -K + \mathscr{S}'$ and let $Z(\mathbf{t}) = \sum_{l' \in L'} \mathfrak{z}(l')\mathbf{t}^{l'}$ be the combinatorial series defined in Sect. 8.4. Then*

$$\sum_{[\tilde{l}']=[l'],\ \tilde{l}' \not\geq l'} \mathfrak{z}(\tilde{l}') = \mathfrak{sw}_{\sigma[K+2l']} - \frac{(K+2l')^2 + |\mathscr{V}|}{8}. \tag{9.6.13}$$

If we write $l' = r_h + l$ (where $h = [l']$ and $l \in L$), then (9.6.13) transforms into

$$\sum_{[\tilde{l}']=[l'],\ \tilde{l}' \not\geq l'} \mathfrak{z}(\tilde{l}') = \chi_{K+2r_h}(l) + \mathfrak{sw}_{\sigma[K+2r_h]} - \frac{(K+2r_h)^2 + |\mathscr{V}|}{8}. \tag{9.6.14}$$

In particular, in the chamber $l' = l + r_h \in -K + \mathscr{S}'$, the sum from the left hand side of the above identities is a multivariable quadratic function in l with constant term $\mathfrak{sw}_{\sigma[K+2r_h]} - ((K+2r_h)^2 + |\mathscr{V}|)/8$.

These formulae should be compared with those from (8.2.26) valid for the coefficients of the series P. The fact that in (8.2.26) (associated with the series P) the constant terms are the equivariant geometric genera, is rather natural. However, the fact that the constant terms in the above Theorem 9.6.21 (associated with Z, a rather 'simple' series) is the Seiberg–Witten invariant, is rather surprising. Nevertheless, the above identity provides a very natural, direct and conceptual explanation, how the Seiberg–Witten invariant might appear in the theory of singularity links.

Example 9.6.22 If Γ is numerically Gorenstein and $h = 0$ then (9.6.14) reads as

$$\sum_{l \in L,\ l \not\geq Z_K} \mathfrak{z}(l) = \mathfrak{sw}_{\sigma[K]} - \frac{K^2 + |\mathscr{V}|}{8}. \tag{9.6.15}$$

9.7 The Seiberg–Witten Invariant Conjecture/Coincidence

In this section we discuss a set of possible potential identities connecting the analytic invariants with the topological ones, namely, the equivariant geometric genera with the Seiberg–Witten invariants of the link. Whenever these identities are valid they provide a topological description of the equivariant geometric

genera. The identities are generalizations of the expectation of the Casson Invariant Conjecture to the case of singularities with rational homology sphere links. We verify their validity for cyclic quotients, Brieskorn–Hamm complete intersections, weighted homogeneous and splice quotient singularities.
Superisolated singularities in general do not satisfy SWIC, their case will be discussed in 9.7.B. ♣

9.7.1 Seiberg–Witten Invariant Conjecture/Coincidence (SWIC) In this section we assume that the link of (X, o) is a rational homology sphere, and we fix a resolution $\widetilde{X} \to X$, and we keep all the notations associated with it. We say that (X, o) satisfies SWIC(r_h) for a certain $h \in H$ if the following identity holds

$$h^1(\widetilde{X}, \mathscr{O}(-r_h)) = \mathfrak{sw}_{\sigma[K+2r_h]} - \frac{(K+2r_h)^2 + |\mathscr{V}|}{8}. \tag{9.7.1}$$

We say that (X, o) satisfies the *equivariant* SWIC if (9.7.1) holds for every $h \in H$.
We say that (X, o) satisfies the SWIC if it satisfies SWIC(0), that is, if

$$p_g(X, o) = \mathfrak{sw}_{\sigma[K]} - (K^2 + |\mathscr{V}|)/8. \tag{9.7.2}$$

The identity SWIC was formulated as a conjecture in [489] (while the equivariant case in [479]): the expectation was that it holds for any $\mathbb{Q}$-Gorenstein singularity. Although the conjecture can be verified for several subfamilies of singularities, since [399] we know that it is not true for the large class of $\mathbb{Q}$-Gorenstein singularities (see also 9.7.B for the treatment of superisolated singularities, a family which produces several counterexamples). But even in the case of families when it fails, it still indicates interesting 'virtual' properties (e.g., in the superisolated case it has lead to the Semigroup Distribution Property). The limits of the validity of the SWIC are not clarified at this moment. Having in mind the existence of cases when the identities do not hold, one might say that its name as SWI 'Conjecture' is not totally justified, although this was its name in the literature. Hence, the reader might read the abbreviation SWIC as SWI 'Coincidence' too.

Remark 9.7.2 One verifies that all the terms appearing in the SWIC are independent of the choice of the resolution. Indeed, $h^1(\widetilde{X}, \mathscr{O}(-r_h)) = p_g(X_a, o)_{\theta(h)}$, and $p_g(X_a, o)$, hence $p_g(X_a, o)_{\theta(h)}$ too, have resolution independent descriptions (see 6.8). The statement for $\mathfrak{sw}_{\sigma[K+2r_h]}$ follows from 9.3.2 and 9.5.8, while the independence of $\frac{(K+2r_h)^2+|\mathscr{V}|}{8}$ is a computation (if $\pi : \Gamma^b \to \Gamma$ is an elementary blow up with exceptional divisor E_{new}, then $r_h(\Gamma^b)$ is either $\pi^*(r_h(\Gamma))$ or $\pi^*(r_h(\Gamma)) - E_{new}$).

Example 9.7.3 Assume that (X, o) is Gorenstein and it admits a smoothing with smooth nearby (Milnor) fiber F. Then the signature satisfies $\sigma(F) + 8p_g + K^2 +$

$|\mathscr{V}| = 0$, hence the SWIC (for $h = 0$) reads as

$$-\sigma(F)/8 = \mathfrak{sw}_{\sigma[K]}. \tag{9.7.3}$$

In this case, usually, $\sigma(F)/8$ is not an integer, see the germ A_n.

Example 9.7.4 Assume that (X, o) is a complete intersection with integral homology sphere link. Then $\mathfrak{T}_{\sigma[K]}(1) = 0$, hence the SWIC reduces to the CIC (see 9.2):

$$\sigma(F)/8 = \lambda(L_X).$$

Example 9.7.5 The identity $P(\mathbf{t}) = Z(\mathbf{t})$ (that is, the topological description via Z of the Poincaré series associated with the divisorial filtration) implies the equivariant SWIC. In particular, the identity $P_0(\mathbf{t}) = Z_0(\mathbf{t})$ implies SWIC. Indeed, for any $l' \in -K + \mathscr{S}'$ with $l' = l + r_h$ ($l \in L$), from (8.2.26) one has

$$\sum_{[\tilde{l}']=[l'],\ \tilde{l}' \not\geq l'} \mathfrak{p}(\tilde{l}') = \chi_{K+2r_h}(l) + h^1(\mathscr{O}(-r_h)). \tag{9.7.4}$$

On the other hand, from (9.6.14),

$$\sum_{[\tilde{l}']=[l'],\ \tilde{l}' \not\geq l'} \mathfrak{z}(\tilde{l}') = \chi_{K+2r_h}(l) + \mathfrak{sw}_{\sigma[K+2r_h]} - \frac{(K+2r_h)^2 + |\mathscr{V}|}{8}. \tag{9.7.5}$$

For $l' \in -K + \mathscr{S}'$ and $l' = l + r_h$, we can regard the evaluation at $l = 0$ of the counting function $\sum_{[\tilde{l}']=[l'],\ \tilde{l}' \not\geq l'} \mathrm{coeff}(\tilde{l}')$ as an operator. It associates with any multivariable series its 'multivariable periodic constant', cf. 10.2. In this sense, the above identities say that the periodic constant of P_h is $h^1(\mathscr{O}(-r_h))$, while of Z_h is $\mathfrak{sw}_{[K+2r_h]} - ((K+2r_h)^2 + |\mathscr{V}|)/8$.

Hence, if $P_h(\mathbf{t}) = Z_h(\mathbf{t})$ then the SWIC(r_h) automatically holds as well.

In fact, in order to have the SWIC(r_h) we need the validity of the above identities for a certain $l' \in -K + \mathscr{S}'$ ($[l'] = h$) only. Indeed, if a certain $l'_0 \in -K + \mathscr{S}'$, $[l'_0] = h$, has the property that $P_h(\mathbf{t}) - Z_h(\mathbf{t})$ is supported on $\{\tilde{l}' : \tilde{l}' \geq l'_0\}$, then by the above identities applied for this l'_0 we obtain SWIC(r_h). In such a case, again by the identities (9.7.4)–(9.7.5), even if $P_h(\mathbf{t}) \neq Z_h(\mathbf{t})$, their counting functions $l' \mapsto \sum_{[\tilde{l}']=[l'],\ \tilde{l}' \not\geq l'} \mathrm{coeff}(\tilde{l}')$ in the whole chamber $l' \in -K + \mathscr{S}'$ coincide (independently of the position of l'_0 in this chamber).

For a fixed h, the identity $P_h = Z_h$ is much stronger than the SWIC(r_h): examples when $P_h \neq Z_h$ but the SWIC(r_h) holds can be constructed. E.g., in 8.6.16 we constructed two resolutions of the same singularity such that in one of them $P = Z$ while in the other one $P \neq Z$. The validity of the (resolution independent) SWIC identity can now be read from that resolution which satisfies $P = Z$. (See also 9.7.17.)

9.7.6 Extension to the Other Natural Line Bundles Recall that in 6.6.3 we proved that for any $l' \in L'$ there exists a unique minimal $s(l') \in \mathscr{S}'$ such that $s(l') - l' \in L_{\geq 0}$. We wish to compare $h^1(\mathscr{O}(-l'))$ and $h^1(\mathscr{O}(-s(l')))$ via the SWIC property.

We say that $l' \in L'$ satisfies the SWIC identity, denoted by SWIC(l'), if

$$\text{SWIC}(l'): \qquad h^1(\widetilde{X}, \mathscr{O}(-l')) = \mathfrak{sw}_{\sigma[K+2l']} - \frac{(K+2l')^2 + |\mathscr{V}|}{8}. \tag{9.7.6}$$

If this holds, then it obviously provides a topological description for $h^1(\widetilde{X}, \mathscr{O}(-l'))$.

By 6.6.5 one has

$$h^1(\widetilde{X}, \mathscr{O}(-s(l'))) - h^1(\widetilde{X}, \mathscr{O}(-l')) = \chi(s(l')) - \chi(l').$$

A computation shows that the right hand side of (9.7.6) behaves similarly. Hence

Proposition 9.7.7 (First Extension) *The SWIC(l') is valid if and only if SWIC($s(l')$) is valid. In particular, SWIC(r_h) is valid if and only if SWIC(s_h) holds.*

This shows that the validity of SWIC(r_h) implies the validity of SWIC(l') for all $l' \in L'_h$ with $s(l') = s_{[l']}$. This covers exactly those cycles $l' \in L'_h$ with $l' \leq s_{[l']}$ (including all cycles $l' = \sum_v l'_v E_v$ with $l'_v < 1$ for any v).

This topological characterization SWIC(l') of $h^1(\mathscr{O}(-l'))$ (modulo the validity of SWIC) in this 'negative' region $\{l' : l' \leq s_{[l']}\}$ can be compared with the vanishing $h^1(\mathscr{O}(-l')) = 0$ in the 'opposite positive' region $\{l' : l' \in -K + \mathscr{S}'\}$.

It is natural to ask the question: what can one say in the case of an arbitrary l', which sits outside of these two regions. The answer is hidden in the identity (8.2.17):

$$h^1(\mathscr{O}(-l')) = -\sum_{a \in L,\ a \not\geq 0} \mathfrak{p}(l'+a) - \text{const}_{[\mathscr{O}(-l')]} - \frac{(K+2l')^2 + |\mathscr{V}|}{8}. \tag{9.7.7}$$

Assume that for $h = [l']$ SWIC(r_h) holds. Then in (9.7.7) one has $\text{const}_{[\mathscr{O}(-l')]} = -\mathfrak{sw}_{\sigma[K+2l']}$. Hence, we obtain the following.

Proposition 9.7.8 (Second Extension) *If SWIC(r_h) holds then for any $l' \in L'_h$*

$$h^1(\mathscr{O}(-l')) = -\sum_{a \in L,\ a \not\geq 0} \mathfrak{p}(l'+a) + \mathfrak{sw}_{\sigma[K+2l']} - \frac{(K+2l')^2 + |\mathscr{V}|}{8}. \tag{9.7.8}$$

Additionally, if $P_h = Z_h$ (or, at least their counting functions coincide), then one has the following topological characterization of $h^1(\mathscr{O}(-l'))$:

$$h^1(\mathscr{O}(-l')) = -\sum_{a\in L,\ a\not\geq 0} \mathfrak{z}(l'+a) + \mathfrak{sw}_{\sigma[K+2l']} - \frac{(K+2l')^2 + |\mathscr{V}|}{8}. \qquad (9.7.9)$$

Remark 9.7.9 In general it is very hard to determine the possible values $h^1(\widetilde{X}, \mathscr{L})$, where $\mathscr{L}$ varies in $\mathrm{Pic}^{l'}(\widetilde{X})$, even if the analytic structure of (X, o) is 'nice' (e.g., even if the equivariant SWIC holds). In general we know that $\mathscr{L} \mapsto h^1(\widetilde{X}, \mathscr{L})$ behaves semicontinuously (cf. [452, 5.2.1]) with respect to $\mathscr{L} \in \mathrm{Pic}^{l'}(\widetilde{X})$, but definitely it is not constant (except for some very particular cases), hence it cannot be recovered from the Chern class l' and Γ. In fact, $\mathscr{L} \mapsto h^1(\widetilde{X}, \mathscr{L})$ determines a complicated stratification of $\mathrm{Pic}^{l'}(\widetilde{X})$ (the analogue of the Brill–Noether stratification considered in the context of projective smooth curves). For several properties of this stratification the reader might consult [452–456], where the Abel map for surface singularities is developed.

Remark 9.7.10 Assume that the equivariant SWIC is true for (X, o). Then, taking the sum of the identities SWIC(r_h) from (9.7.1), and using $\sum_\sigma \mathfrak{T}_\sigma(1) = 0$, we get the following expression for the geometric genus of the universal abelian covering (X_a, o) in terms of the graph Γ:

$$p_g(X_a, o) = -\lambda(M(\Gamma)) - |H| \cdot \frac{K^2 + |\mathscr{V}|}{8} + \sum_{h\in H} \chi(r_h).$$

Note that for the validity of this identity we do *not* require the validity of the SWIC(0) for (X_a, o). See also Sect. 9.7.C for behaviour of the Seiberg–Witten invariant with respect to coverings.

9.7.11 Starting from the next paragraph we discuss several families of singularities and we will prove/disprove the identity SWIC for them. (When SWIC is true we try to find even several independent arguments for its validity.) As a byproduct, we provide several closed fomulae for $\mathfrak{sw}_\sigma$ in the corresponding cases.

Example 9.7.12 (SWIC Is True for Cyclic Quotients [489]) In this case the link is $L(n, q)$, $H = \mathbb{Z}_n$ and the spin^c structures are indexed by $\sigma = \sigma[K + 2aE_s^*]$, where $a \in \mathbb{Z}$ and $0 \leq a < n$. Set also $h = a[E_s^*] \in H$. From 2.2.A and 9.5.10 we get

$$\mathfrak{T}_\sigma(1) = -\mathbf{s}(q, n) + \frac{n-1}{4n} - \frac{a}{2n} - \sum_{i=1}^{a} \left(\!\!\left(\frac{iq'}{n}\right)\!\!\right).$$

Since $\lambda/n = \mathbf{s}(q,n)/2$, cf. 9.3.4, we also have

$$\mathfrak{sw}_\sigma = -\frac{3}{2}\cdot \mathbf{s}(q,n) + \frac{n-1}{4n} - \frac{a}{2n} - \sum_{i=1}^{a} \left(\!\!\left(\frac{iq'}{n}\right)\!\!\right).$$

On the other hand, $(K+2r_h)^2 + |\mathcal{V}|)/8 = (K^2+|\mathcal{V}|)/8 - \chi(r_h)$ can also be computed explicitly. From 6.3.11 one has $(K^2+|\mathcal{V}|)/8 = (n-1)/4n - 3\mathbf{s}(q,n)/2$.

Furthermore, from 6.6.5 we have $h^1(\mathcal{O}(-s_h)) - h^1(\mathcal{O}(-r_h)) = \chi(s_h) - \chi(r_h)$. But $h^1(\mathcal{O}(-s_h)) = 0$ by the vanishing 6.4.3, while $h^1(\mathcal{O}(-r_h)) = p_g(X_a,o)_{\theta(h)} = 0$ (cf. 6.8.11) since the universal abelian covering (X_a,o) is smooth. Hence $\chi(r_h) = \chi(s_h)$, and its expression is computed in (6.6.11):

$$\chi(r_h) = \frac{a}{2n} + \sum_{i=1}^{a} \left(\!\!\left(\frac{iq'}{n}\right)\!\!\right).$$

In particular, the right hand side of SWIC(r_h) is zero, and the same is true for the left hand side because of the vanishing already mentioned.

Example 9.7.13 (SWIC for Brieskorn–Hamm Complete Intersections via Hirzebruch's Signature Formula [489]) Assume that $(X,o) = (X_C(a_1,\dots,a_n),o)$ is as in 5.1.17. The torsion of the link is computed in 9.5.11 and 9.5.12, while the Casson–Walker invariant in 9.3.3 (for the determination of the Seifert invariants in terms of $(a_1,\dots,a_n)$ see 9.5.11 and 9.5.12). This provides a concrete formula for $\mathfrak{sw}_{\sigma[K]}$ in terms of $(a_1,\dots,a_n)$. On the other hand, the signature of the Milnor fiber was also determined by Hirzebruch [272] (as a residue) and by Neumann–Wahl in [512] in terms of $(a_1,\dots,a_n)$, see (9.2.1) for the corresponding expression. It satisfies $\sigma(F) = -8\mathfrak{sw}_{\sigma[K]}$, hence SWIC holds (cf. 9.7.3).

Example 9.7.14 (The Equivariant SWIC Is True for Weighted Homogeneous Singularities [490]) We prove that SWIC(r_h) is true for any h. The next short proof does not use the precise expression of the involved ingredients, it is based on the analytic and topological additivity theorems. Let v be the central vertex, and note that $Z_{h,\{v\}}(t) = P_{h,\{v\}}(t)$, cf. 8.6.A. Then use the additivity formulae 8.7.25 for $h^1(\mathcal{O}(-r_h))$ and 9.6.5 for the Seiberg–Witten invariant applied for $l' = r_h$. Then note that for each h the restriction $R(r_h)$ (on any leg) has all the E-coefficients < 1 and each $\widetilde{X}_j$ is the resolution of a cyclic quotient singularity, a case for which SWIC(l') was established in 9.7.12. (See also the paragraph after 9.7.7.)

For a different proof based on the expression of all the involved invariants in terms of Seifert invariants (and for several additional properties) see 9.7.A.

Example 9.7.15 (The Equivariant SWIC Is True for Singularities Satisfying the End Curve Condition [492]) For such singularities $P(\mathbf{t}) = Z(\mathbf{t})$ by 8.5.16. Then for any $l' \in -K + \mathscr{S}'$ SWIC(l') is valid by 9.7.5, and even the identity (9.7.9) holds for any l' by 9.7.8. In particular, *all these identities hold for any rational*

and minimally elliptic singularity. (A slightly different proof, based on additivity formulae, follows the arguments from 9.7.14 and induction. The inductive step is guaranteed by 8.5.15—which says that the resolutions $\widetilde{X}_j$ determine singularities satisfying the end curve condition—, and also by the fact that the E-coefficients of $R(r_h)$ are < 1, cf. 9.7.7.)

9.7.16 A different proof of the equivariant SWIC for **rational** singularities appears in 11.1.38, and for **minimally elliptic** ones in 11.1.39. See also the proof of the (non-equivariant) SWIC for the **elliptic** case in 11.1.40.

Example 9.7.17 Consider the germ $\{Z^2 = (u^2 - w)(w^3 + u^{11})\}$ treated in 5.4.5. Note that its graph does not satisfy the monomial condition, hence $P(\mathbf{t}) \neq Z(\mathbf{t})$.

Let v be the (-4) vertex of Γ. Then each connected component of $\Gamma \setminus v$ supports a rational singularity. Furthermore, a Fourier–Dedekind sum computation provides $Z_{0,\{v\}}(t) = (t^4 + 1)/((t^2 - 1)(t - 1))$, with polynomial part $t + 1$ and periodic constant 2. Hence, from the additivity formula 9.6.5, $\mathfrak{sw}_{\sigma[K]} - (K^2 + |\mathscr{V}|)/8 = 2$. On the other hand, (X, o) is an elliptic singularity with $p_g = 2$. Hence the SWIC holds.

Example 9.7.18 (SWIC for Suspensions) In [491] it is proved that SWIC(0) is true for suspensions $\{z^n + f(x, y) = 0\}$, where f is irreducible.

9.7.A SWIC for Weighted Homogeneous Singularities

In this subsection we compute the torsion and the Seiberg–Witten invariant of any rational homology sphere Seifert 3-manifold (with $e < 0$) and we prove the equivariant SWIC for any weighted homogeneous singularity. (SWIC follows from $P = Z$ too, here we present a proof based on an independent computation, which provides an explicit formula for the torsion as well.) We follow [490]. ♣

9.7.19 We will use the standard notations, see e.g. Sect. 8.1.B. First we consider the topological series $Z_{\mathscr{J}}(t)$ reduced to the central vertex and with variable t. Its definition can be compared with the formula of the Poincaré series $P_h(t) := P_h(R(X_a), t)$ from (8.1.5), hence they are equal (see also 8.6.13). For fixed h set $(s_h)_{\mathrm{red}} = a_0 + \sum_j a_j E^*_{js_j}$. Then, according to 8.1.B,

$$Z_{h,\mathscr{J}}(t) = P_h(t) = \sum_{\ell \geq 0} \max\left\{0\,,\, 1 + N_{s_h}(\ell)\right\} \cdot t^{\ell + \mathfrak{a}(s_h)}. \tag{9.7.10}$$

(The choice of s_h is motivated by the fact that $\chi(s_h)$ is available in (6.6.19).)

Since the central vertex is a regularization vertex for all characters, by 9.5.9,

$$\mathcal{T}_\sigma(1) = \lim_{t\to 1}\Big(Z_{h,\mathscr{J}}(t) - \frac{1}{|H|}\cdot Z_{\mathscr{J}}(t)\Big). \tag{9.7.11}$$

$Z_{h,\mathscr{J}}$ has a sum-decomposition $Z^+_{h,\mathscr{J}} + Z^-_{h,\mathscr{J}}$ (cf. 8.1.9), where

$$Z^+_{h,\mathscr{J}}(t) = \sum_{\ell\geq 0} \max\big\{\,0\,,\,-1-N_{s_h}(\ell)\,\big\}\cdot t^{\ell+\mathfrak{a}}, \quad Z^-_{h,\mathscr{J}}(t) = \sum_{\ell\geq 0}(1+N_{s_h}(\ell))\cdot t^{\ell+\mathfrak{a}}.$$

Here $Z^+_{h,\mathscr{J}}(t)/t^{\mathfrak{a}}$ is a polynomial, therefore

$$\mathcal{T}_\sigma(1) = Z^+_{h,\mathscr{J}}(1) + \lim_{t\to 1}\Big(Z^-_{h,\mathscr{J}}(t) - \frac{1}{|H|}\cdot Z_{\mathscr{J}}(t)\Big).$$

From Eq. (8.1.9), and from the fact that each $0 \leq a_j < \alpha_j$, for a certain function $R(t)$ with $\lim_{t\to 1} R(t) = 0$ one has

$$\begin{aligned} Z^-_{h,\mathscr{J}}(t) = {} & \frac{-e}{(1-t)^2} - \frac{\chi/2+e}{t-1} + \frac{2-\chi}{4} - \mathfrak{a}\cdot\frac{\chi-e\mathfrak{a}}{2} + \frac{a_0}{2} \\ & - \sum_{j=1}^{\nu}\Big(\mathbf{s}(\omega_j,\alpha_j) + \sum_{i_j=1}^{a_j}\Big(\Big(\frac{i_j\omega'_j}{\alpha_j}\Big)\Big)\Big) + R(t). \end{aligned} \tag{9.7.12}$$

Note that the pole of $Z^-_{h,\mathscr{J}}(t)$ is independent of h, and by the existence of the limit, it should be the same as the pole of $Z_{\mathscr{J}}(t)$. Indeed, one has the following.

Lemma 9.7.20 *$Z_{\mathscr{J}}(t)/|H|$ has the following Laurent expansion:*

$$Z_{\mathscr{J}}(t)/|H| = \frac{-e}{(t-1)^2} - \frac{\chi/2+e}{t-1} + E + Q(t),$$

where $\lim_{t\to 1} Q(t) = 0$ *and E equals*

$$-\frac{(e+1)(e+5)}{12e} + \frac{1}{4}\sum_j\big(1-\frac{1}{\alpha_j}\big) + \frac{1}{12e}\sum_j\big(1-\frac{1}{\alpha_j}\big)\big(4+\frac{1}{\alpha_j}\big) - \frac{1}{4e}\sum_{i<j}\big(1-\frac{1}{\alpha_i}\big)\big(1-\frac{1}{\alpha_j}\big).$$

Indeed, in the proof use $\nu+2$ times in the expression

$$Z_{\mathscr{J}}(t) = \frac{(1-t^{1/|e|})^{\nu-2}}{\prod_j(1-t^{1/(\alpha_j|e|)})} = \frac{1}{(t-1)^2}\cdot\Big(\frac{t-1}{t^{1/|e|}-1}\Big)^2\cdot\prod_j\frac{t^{1/|e|}-1}{t^{1/(\alpha_j|e|)}-1}$$

the Taylor expansion

$$\frac{t^\gamma-1}{t^\tau-1}=\frac{\gamma}{\tau}+\frac{\gamma}{2\tau}(\gamma-\tau)\cdot(t-1)+\frac{\gamma}{\tau}(\gamma-\tau)\Big(\frac{2\gamma-\tau}{12}-\frac{1}{4}\Big)\cdot(t-1)^2+\cdots.$$

Corollary 9.7.21

$$\begin{aligned}\mathcal{T}_\sigma(1)&=Z^+_{h,\mathscr{J}}(1)-\frac{e\mathfrak{r}\mathfrak{a}}{2}+\frac{e\mathfrak{a}^2}{2}+\frac{a_0}{2}-\sum_{j=1}^{\nu}\Big(\mathbf{s}(\omega_j,\alpha_j)+\sum_{i_j=1}^{a_j}\Big(\Big(\frac{i_j\omega'_j}{\alpha_j}\Big)\Big)\Big)\\&\quad-\frac{(e+1)(e+5)}{12e}+\frac{1}{12e}\sum_j\big(1-\frac{1}{\alpha_j}\big)\big(4+\frac{1}{\alpha_j}\big)-\frac{1}{4e}\sum_{i<j}\big(1-\frac{1}{\alpha_i}\big)\big(1-\frac{1}{\alpha_j}\big)\\&=Z^+_{h,\mathscr{J}}(1)-\frac{1}{24e}\Big(2-\nu+\sum_j\frac{1}{\alpha_j^2}\Big)+\frac{1}{8e}\Big(2-\nu+\sum_j\frac{1}{\alpha_j}\Big)^2+\frac{e+6}{12}\\&\quad-\frac{e\mathfrak{r}\mathfrak{a}}{2}+\frac{e\mathfrak{a}^2}{2}+\frac{a_0}{2}-\sum_{j=1}^{\nu}\Big(\mathbf{s}(\omega_j,\alpha_j)+\sum_{i_j=1}^{a_j}\Big(\Big(\frac{i_j\omega'_j}{\alpha_j}\Big)\Big)\Big).\end{aligned}$$

Theorem 9.7.22 *SWIC(s_h) holds for weighted homogeneous singularities.*

Indeed, $Z^+_{h,\mathscr{J}}(1)=h^1(\mathscr{O}(-s_h))$ by (6.8.8), hence SWIC(s_h) follows from 9.7.21, (9.3.2), 6.3.12 and (6.6.19).

Remark 9.7.23

(a) **(The Milnor number)** Assume that $(X,0)$ is Gorenstein and it admits a smoothing. Then using the Laufer–Steenbrink [362, 655] formula $\mu=12p_g+K^2+|\mathscr{V}|$ for the Milnor number, one obtains

$$2(1+\mu)=24\cdot\mathcal{T}_{\sigma[K]}(1)+\frac{1}{e}\Big(2-\nu+\sum_i\frac{1}{\alpha_i^2}\Big)-\frac{1}{e}\Big(2-\nu+\sum_i\frac{1}{\alpha_i}\Big)^2.$$

(If $H_1(L_X,\mathbb{Z})=0$, then $\mathcal{T}_{\sigma[K]}(1)=0$ too, hence the formula simplifies.)

(b) **(Laurent expansion)** We already mentioned that the non-regular (Laurent) part of $Z^-_{h,\mathscr{J}}(t)$ is independent of h. It is $-e(1-t)^{-2}-(\chi/2+e)(t-1)^{-1}$, and it is completely determined by the virtual Euler number and virtual Euler characteristic e and χ. On the other hand, the coefficient of $(t-1)^0$ involves the torsion invariants (depending on h). Indeed, the above proof shows that

$$\lim_{t\to1}\Big(Z^-_{h,\mathscr{J}}(t)-\frac{1}{|H|}Z_{\mathscr{J}}(t)\Big)=\frac{\lambda}{|H|}+\frac{K^2+|\mathscr{V}|}{8}-\chi(s_h).$$

In particular (by subtractions and by (9.7.11))

$$\begin{aligned}\lim_{t\to 1}\Big(Z^-_{h,\mathscr{J}}(t)-Z^-_{0,\mathscr{J}}(t)\Big)&=-\chi(s_h)\\ \lim_{t\to 1}\Big(Z_{h,\mathscr{J}}(t)-Z_{0,\mathscr{J}}(t)\Big)&=\mathfrak{T}_{h\sigma[K]}(1)-\mathfrak{T}_{\sigma[K]}(1).\end{aligned}\tag{9.7.13}$$

We emphasize that $Z_{0,\mathscr{J}}(t)$ and $Z_{\mathscr{J}}(t)$ are also equal to the Poincaré series associated with the $\mathbb{C}^*$-gradings of (X,o) and (X_a,o) respectively.

Example 9.7.24 (Complete Intersections) Since $Z_{0,\mathscr{J}}(t)=P_{0,\mathscr{J}}(t)$ is the Poincaré series of the $\mathbb{C}^*$-grading too, cf. 8.6.13, in special cases it can be determined via different numerical invariants as well. Assume e.g. that $(X,o)\subset(\mathbb{C}^n,0)$ is a complete intersection with weights $w_1,\dots,w_n$ (where $\gcd(w_1,\dots,w_n)=1$) and degrees $d_1,\dots,d_{n-2}$. Then (similarly as in the proof of 9.7.20) we get the expansion

$$P_{0,\mathscr{J}}(t)=\frac{\prod_{i=1}^{n-2}(1-t^{d_i})}{\prod_{j=1}^{n}(1-t^{w_j})}=\frac{\prod_i d_i}{\prod_j w_j}\Big[\frac{1}{(t-1)^2}+\frac{\sum_i d_i-\sum_j w_j+2}{2(t-1)}+F+U(t)\Big],$$

where $\lim_{t\to 1}U(t)=0$ and F is the expression

$$\frac{1}{6}\sum_i d_i^2+\frac{1}{4}\sum_{i<j}d_id_j+\frac{1}{4}\sum_i d_i+\frac{1}{12}\sum_j w_j^2+\frac{1}{4}\sum_{i<j}w_iw_j-\frac{1}{4}\sum_j w_j$$
$$-\frac{1}{4}\sum_{i,j}d_iw_j+\frac{1}{12}.$$

This compared with 9.7.20 and (9.7.11) provides the next identities

$$-e=\textstyle\prod_i d_i/\prod_j w_j,\quad \mathfrak{r}=\sum_i d_i-\sum_j w_j\quad\text{and}\quad \mathfrak{T}_{\sigma[K]}(1)=-e\cdot F-E,$$

where F is given above and E in 9.7.20. (For the first two identities see e.g. [721], considered in the hypersurface case.)

Example 9.7.25 (Hypersurface Singularities) For weighted homogeneous hypersurface isolated singularities there is an additional subtle connection between the above series $Z_{\mathscr{J}}(t)$, $Z_{0,\mathscr{J}}(t)$ and the characteristic polynomial Δ of the monodromy on the second homology of the Milnor fiber.

Fix an integer $h>0$ and assume that $\phi(t)$ is a rational function of the form $\phi(t)=\prod_{m|h}(1-t^m)^{\chi_m}$. Saito in [605] defined a *dual* rational function (with respect to the integer h) by $\phi^*(t)=\prod_{k|h}(1-t^k)^{-\chi_{h/k}}$. Here we wish to eliminate the dependency of the duality on the integer h via the following principle. If $\phi(t)$ is a rational function of the form $\prod_m(1-t^m)^{\chi_m}$, then we take $h=h(\phi):=\operatorname{lcm}\{m:$

$\chi_m \neq 0\}$, and we define $\phi^*(t)$ using this $h = h(\phi)$. One verifies that if $\gcd\{m : \chi_m \neq 0\} = 1$ then $h(\phi^*) = h(\phi)$ and $\phi^{**}(t) = \phi(t)$.

Let L_X be the link of a weighted homogeneous isolated hypersurface singularity with Seifert invariants as above (all possible equations are listed in 5.1.15). Set

$$\psi(t) := (1-t)^{2-\nu} \prod_{j=1}^{\nu} (1 - t^{\alpha_j}) \quad \text{and} \quad \phi(t) := Z_{0,\mathscr{J}}(t) \cdot \psi(t).$$

In fact, via the above definition, $\psi^*(t) = Z_{\mathscr{J}}(t^{\mathfrak{o}})$. Note also that $\Delta(t)$ always can be written in the form $\prod_{m|h}(1-t^m)^{\chi_m}$ with $\gcd\{m : \chi_m \neq 0\} = 1$ (in fact, $\chi_1 \neq 0$, cf. [7]). The point is that Ebeling in [170] proved a new type of relationship (via concrete computation of the involved invariants):

$$\Delta^*(t) = \phi(t).$$

[In [170] it is not explicitly said that $h = h(\phi)$, but it can be verified using [432].]

This can also be combined with (9.7.11) providing for $\mathcal{T}_{\sigma[K]}(1)$ the expression

$$\lim_{t\to 1}\Big(\phi(t)\psi^{-1}(t) - Z_{\mathscr{J}}(t)/|H|\Big) = \lim_{t\to 1}\Big(\Big(\Delta^*(t) - Z_{\mathscr{J}}(t)\psi(t)/|H|\Big)\psi^{-1}(t)\Big).$$

Since $\lim_{t\to 1}\psi^{-1}(t)(1-t)^2 = |e|/|H|$, we get

$$\mathcal{T}_{\sigma[K]}(1) = \frac{|e|}{|H|} \cdot \lim_{t\to 1} \frac{\Delta^*(t) - Z_{\mathscr{J}}(t)\psi(t)/|H|}{(t-1)^2}. \tag{‡}$$

Similarly as in the proof of 9.7.20 the Taylor expansion $Z_{\mathscr{J}}\psi$ at $t = 1$ can be computed. This gives the first three terms of Δ^* as well. E.g., the first two terms are

$$\frac{\Delta^*(t)}{|H|} = 1 + \frac{1}{2}(\mathfrak{r} + 2 - \nu + \sum_j \alpha_j)(t-1) + \cdots \tag{9.7.14}$$

The term $e_{st} := -(2 - \nu + \sum_j \alpha_j)/\mathfrak{r}$ is Batyrev's stringy Euler characteristic [45] as it is generalized by Veys in [716]. Equation (9.7.14) can also be compared with the expansion

$$\frac{\Delta(t)}{|H|} = 1 + \frac{\mu}{2}(t-1) + \cdots,$$

where μ is the Milnor number of (X, o) (for the proof use e.g. A'Campo's formula).

9.7.B SWIC and Superisolated Singularities

We assume that (X, o) is a superisolated singularity associated with the irreducible projective rational cuspidal curve C of degree d, and we use the notations of 5.2 and 8.6.B. We prove that for a superisolated singularity the SWIC is valid if and only if $N(1) = 0$, a property which is not always true, cf. 8.6.B. Several concrete examples (and their universal abelian coverings) are discussed. ♣

9.7.26 Though in many cases (e.g. for weighted homogeneous singularities) we discuss the SWIC together with equivariant SWIC, this is not the case for the superisolated germs. The main obstruction is that in the superisolated case (though $p_g(X, o)$ and $P_{0,\{v_+\}}(t)$ are extremely simple), usually we have very little information about the analytic properties of the universal abelian covering, e.g. about its geometric genus $p_g(X_a, o)$. (The reader might consult [665] to see certain difficulties, and also some particular explicit examples. See comment from 11.5.47 as well.) Therefore, in this subsection we focus merely on the SWIC (for $h = 0$).

Let us list first the involved invariants.

9.7.27 From 9.3.12 we have $K^2 + |\mathcal{V}| = -d(d-2)^2 + 1$ and $24\lambda = \mu(3\mu - 5) - 12 \cdot \Delta''(1)$ ($\mu = 2\delta$). Since C is irreducible, by 8.6.14, the divisorial filtration associated with $\mathcal{J} = \{C\} = \{v_+\}$ agrees with the filtration associated with weights $(1, 1, 1)$, hence $P_{0,\mathcal{J}}(t) = (1 - t^d)/(1 - t)^3$ (see also 8.6.B). Since in the good resolution $\Gamma \setminus \{v_+\}$ supports only rational (in fact smooth) germs, by 8.7.25 $p_g(X, o) = \mathrm{pc}(P_{0,\mathcal{J}}(t))$ which is $d(d-1)(d-2)/6$ (see also 5.2.4).

9.7.28 The definition of $Z_{\mathcal{J}}(t)$ compared with A'Campo formula 4.1.22 gives

$$Z_{0,\mathcal{J}}(t) = \frac{1}{d} \sum_{\xi^d = 1} \frac{\Delta(\xi t^{1/d})}{(1 - \xi t^{1/d})^2} \qquad \text{and} \qquad Z_{\mathcal{J}}(t) = \frac{\Delta(t^{1/d})}{(1 - t^{1/d})^2}.$$

Since H is generated by $[E_+^*]$, the vertex v_+ (corresponding to C) is a regularization vertex for any character. Therefore, from 9.5.9

$$\mathcal{T}_{\sigma[K]}(1) = \lim_{t \to 1} \left(Z_{0,\mathcal{J}}(t) - \frac{1}{d} Z_{\mathcal{J}}(t) \right) = \frac{1}{d} \sum_{\xi^d = 1 \neq \xi} \frac{\Delta(\xi)}{(1 - \xi)^2}.$$

Following 8.6.B we also consider

$$N(t) := Z_{0,\mathcal{J}}(t) - P_{0,\mathcal{J}}(t) = \frac{1}{d} \sum_{\xi^d = 1} \frac{\Delta(\xi t^{1/d})}{(1 - \xi t^{1/d})^2} - \frac{1 - t^d}{(1 - t)^3}.$$

For some of its properties see 8.6.21 and the list of examples after it. Then

$$\lim_{t\to 1} N(t) = \mathcal{T}_{\sigma[K]}(1) + \lim_{t\to 1}\Big(\frac{1}{d}\cdot\frac{\Delta(t^{1/d})}{(1-t^{1/d})^2} - \frac{1-t^d}{(1-t)^3}\Big).$$

If we write $\Delta(t) = 1 + \delta(t-1) + Q(t)(t-1)^2$ as in 8.6.B, then the limit can be computed in terms of d and $Q(1) = \Delta''(1)/2$. The computation provides

Proposition 9.7.29 ([199])

$$N(1) = \mathfrak{sw}_{\sigma[K]} - \frac{K^2 + |\mathscr{V}|}{8} - p_g.$$

This combined with (8.6.15) gives (with $Q(t) = \sum_{j=0}^{\mu-2}\alpha_j t^j$*)*

$$\mathfrak{sw}_{\sigma[K]} - \frac{K^2 + |\mathscr{V}|}{8} = \sum_{j=0}^{d-3}\alpha_{jd}.$$

Corollary 9.7.30

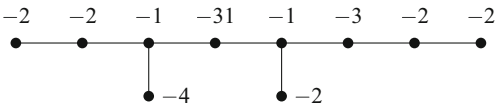
(a) SWIC for $h = 0$ *is equivalent to* $N(1) = 0$.
(b) The Conjecture 8.6.25 (which predicts that $N(1) \le 0$ *for any superisolated singularity) is equivalent to* $\mathfrak{sw}_{\sigma[K]} - \frac{K^2+|\mathscr{V}|}{8} \le p_g$.

Corollary 9.7.30 has the following consequences (for some of the arguments see the paragraphs after 8.6.23): via the 'Semigroup Distribution Property' 5.2.9, the SWIC (for $h = 0$) is valid whenever $\nu = 1$. In fact, in this case not only $N(1) = 0$, but even $N(t) \equiv 0$, i.e. $Z_{0,\mathscr{J}}(t) \equiv P_{0,\mathscr{J}}(t)$.

If $\nu = 2$ then the coefficients of $N(t)$ are non-positive, however, it can happen that $N(t) \neq 0$, see. e.g. several examples in the list 8.6.22. Hence, if $\nu = 2$ and $N(t) \neq 0$ then the SWIC fails and $\mathfrak{sw}_{\sigma[K]} - \frac{K^2+|\mathscr{V}|}{8} < p_g$. (The difference will be interpreted in terms of lattice cohomology in 11.5.B.)

Remark 9.7.31 Though till now we tried to convince the reader that the SWIC, for certain analytic types, is a 'natural' reality, the superisolated case suggests the opposite. Indeed, for such germs, p_g depends only on d, but the topological side depends in a subtle way on the local singularity types of C (see above the formulae of λ and $\mathcal{T}_{\sigma[K]}(1)$). Having in mind this subtle sensitivity to the local singularity data of C, the validity of SWIC (when it holds) is a true marvel.

Example 9.7.32 Let us analyse a particular case with more details. Assume $d = 5$, $\nu = 2$, and the two singularities have multiplicity sequence [3] and [2_3] (see the table from 8.6.22). The graph Γ is presented below, and $N(t) = -2t$, hence SWIC fails: $p_g = 10$, while $-\lambda = 21/2$ and $\mathcal{T}_{\sigma[K]}(1) = 2/5$, hence $\mathfrak{sw}_{\sigma[K]} - (K^2 + |\mathscr{V}|)/8 = 8$.



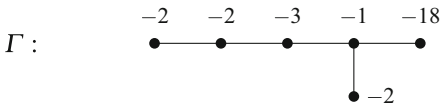
Supported on this topological type (given by the graph) we will consider two analytic structures. They are rather different, though both are very natural. The first is a superisolated hypersurface singularity, as analysed above.

On the other hand, this topological type supports also a splice quotient singularity which satisfies SWIC (cf. 9.7.15), hence it has $p_g = 8$. This singularity, by the general construction of splice quotients from 5.4, is the $\mathbb{Z}_5$-factor of the complete intersection $\{z_1^3+z_2^4+z_3^5z_4 = z_3^7+z_4^2+z_1^4z_2 = 0\} \subset (\mathbb{C}^4, 0)$ by the diagonal action $(\alpha^2, \alpha^4, \alpha, \alpha)$ $(\alpha^5 = 1)$.

For this topological type one can verify that the topological upper bound Path^i for any potential p_g (depending on analytic structures) is 10. Indeed, since $p_g = 10$ can be realized for an analytic type, using 6.8.B we obtain $\text{Path}^i \geq 10$. But a special path with $S(\gamma) = 10$ can be found (see e.g. the reduced χ-table in 11.4.12). In fact, in 11.6.11 we will prove that for any superisolated germ $p_g = \text{Path}^i$.

This means that the two analytic structures make their choice for p_g in two different ways: the choice of the superisolated germ is the optimal topological upper bound Path^i, while the choice of the splice quotient is the value predicted by SWIC. (It is hard to decide which analytic structure is 'nicer'!)

Example 9.7.33 (The Case $d = 4$ with Multiplicity Sequence $[2_3]$, and Its Universal Abelian Covering [399]) Such a C is projectively equivalent to $(zy - x^2)^2 = xy^3$, (see [459], page 146). Hence a possible choice for f is $f = (zy - x^2)^2 - xy^3 + z^5$. Furthermore, $p_g(X, o) = 4$ and the resolution graph Γ is



Since the graph is star-shaped, the same resolution graph can be realized by a weighted homogeneous singularity (X_w, o) as well. In the present case this is an isolated complete intersection in $(\mathbb{C}^4, 0)$ with two equations: $yz = x^2$ and $z^5 + t^2 - xy^3 = 0$. The corresponding weights of the coordinates (x, y, z, t) are: (16,18,14,35). By Dolgachev–Pinkham–Demazure formula one gets that $p_g(X_w, 0) = 4$ as well.

Now, one can use the result of Neumann and Wahl (cf. [512, (3.3)]), which guarantees that a Gorenstein singularity (with the same link as a weighted homogeneous singularity $(X_w, 0)$) is an equisingular deformation of $(X_w, 0)$ if and only if its p_g equals the number predicted by Dolgachev–Pinkham–Demazure formula.

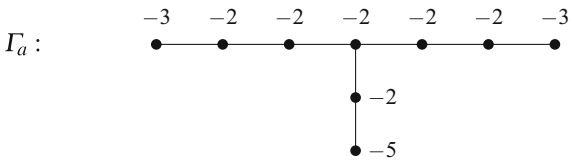
(In [loc. cit.] the result is stated for integral homology spheres links, but the proof works without modification for rational homology sphere links as well.)

In particular, the superisolated singularity (X, o) is an equisingular deformation of $(X_w, 0)$. Here, this deformation can be written easily: the pair of equations $yz - x^2 = \lambda t$, and $z^5 + t^2 - xy^3 = 0$—for the parameter $\lambda \neq 0$—is equivalent to (X, o). Note also that (X, o) and $(X_w, 0)$ *have the same multiplicity* 4, but clearly have *different embedding dimensions—hence different Hilbert-Samuel functions*.

Next, we wish to analyze the corresponding universal abelian coverings.

The universal abelian covering $(X_{w,a}, o)$ of $(X_w, 0)$ is easy: by 5.1.33 it is the hypersurface Brieskorn singularity $\{u^7 = v^{18} + w^2\}$. (The action of a generator of $H = \mathbb{Z}_4$ is $u \mapsto -u$, $v \mapsto iv$ and $w \mapsto -iw$.) Furthermore, $p_g(-u^7 + v^{18} + w^2) = 10$.

The resolution graphs of both universal abelian coverings are the same:



We claim that the universal abelian covering of (X, o) is not an equisingular deformation of $(X_{w,a}, o)$. In particular, using again the above result of Neumann–Wahl, we obtain that their geometric genera must be different. For details see [399].

In this case, for the graph Γ_a the construction of the splice equations is not obstructed, it admits a splice singularity, namely $(X_{w,a}, o)$. However, (X_a, o) with the same graph is not of splice type, since its geometric genus is different than the geometric genus of the splice type singularity $(X_{w,a}, o)$. Both (X_a, o) and $(X_{w,a}, o)$ are Gorenstein, with a common star-shaped graph, but $p_g(X_a, o) < 10 = p_g(X_{w,a}, o)$.

Example 9.7.34 (The Case C_2: $d = 5$, with Multiplicity Sequence $[2_6]$, and Its Universal Abelian Covering [399]) This provides a very similar scenario as 9.7.33. in this case again (X, o) is an equisingular deformation of $(X_w, 0)$, both with geometric genus 10. However their embedding dimensions are different (3 and 6), and their multiplicities are different (5 and 6).

The new phenomenon in this example is that the universal abelian covering graph Γ_a is an integral homology sphere, namely the graph of $\Sigma(2, 13, 31)$. Again, one verifies that (X_a, o) is not an equisingular deformation of the Brieskorn equation $(X_{w,a}, o)$. Hence $p_g(X_a, o) < p_g(X_{w,a}, o)$ and we obtain two Gorenstein singularities with the same *unimodular* star-shaped graph, but different p_g.

Example 9.7.35 Consider $M = S^3_{-5}(T_{2,3}\#T_{2,3}\#T_{2,5}\#T_{2,5})$ (i.e. the multiplicity sequences are [2], [2], $[2_2]$, $[2_2]$ and $d = 5$). A computation regarding $Q(t)$ gives $\alpha_0 = 6$, $\alpha_5 = -8$, $\alpha_{10} = 1$, hence $\mathfrak{sw}_{\sigma[K]} - (K^2 + |\mathcal{V}|)/8 = -1$, cf. (11.5.4).

Since this number is negative, it cannot equal the geometric genus of any analytic structure. That is, this topological type does not support any analytic types which satisfies SWIC(0).

9.7.C The Seiberg–Witten Invariant and Abelian Coverings

9.7.36 Assume that $\mathfrak{I}$ is a numerical invariant associated with pairs (M, h), where M is a connected oriented rational homology sphere 3-manifold and $h \in H_1(M, \mathbb{Z})$. We say that $\mathfrak{I}$ satisfies the *Covering Additivity Property* (CAP) if $\sum_h \mathfrak{I}(M, h) = \mathfrak{I}(M_a, 0)$, where M_a is the universal abelian covering of M (and we assume that it is a rational homology sphere as well). (Obviously, the definition can be extended to more general 3-manifolds, and also to arbitrary finite coverings.)

There is a similar analytic version, where $\mathfrak{I}$ is associated with pairs $((X, o), h)$. Here (X, o) is a normal surface singularity with rational homology sphere link M and $h \in H_1(M, \mathbb{Z})$. Here we ask for $\sum_h \mathfrak{I}((X, o), h) = \mathfrak{I}((X_a, o), 0)$, where (X_a, o) is the analytic universal abelian covering of (X, o). In this analytic context we have a rather immediate example. Let $\mathfrak{I}((X, o), h)$ be the equivariant geometric genus of (X, o), namely $h^1(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(-r_h)) = p_g(X_a, o)_{\theta(h)}$. By its very definition, see 6.8.11 or 6.2.9, $\sum_h p_g(X_a, o)_{\theta(h)} = p_g(X_a, 0)$. Hence, this analytic invariant satisfies CAP.

Having in mind the equivariant SWIC, we can ask whether the invariant

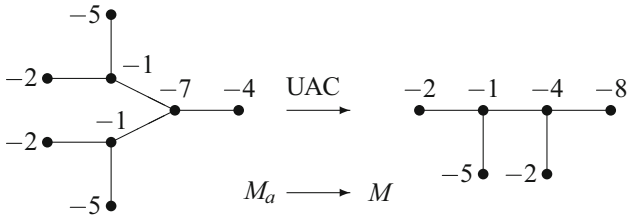
$$(M, h) \mapsto \overline{\mathfrak{sw}}(M, h) := \mathfrak{sw}_{\sigma[K+2r_h]} - \frac{(K + 2r_h)^2 + |\mathscr{V}|}{8}$$

satisfies CAP or not. Here M is a 3-manifold associated with a connected negative definite plumbing graph and we assume that both M and M_a are rational homology spheres. [This is a similar situation, what we witnessed in the case of the additivity formula 9.6.5. Once the analytic p_g-additivity formula of Okuma appeared in the literature, it was natural to ask, motivated by the SWIC, about the existence of a topological version, valid for any singularity link. The answer was positive and the solution provided a very useful topological tool.]

Clearly, if M is the link of a certain analytic singularity, which satisfies the equivariant SWIC, and its UAC $(X_a, 0)$ satisfies the SWIC too, then the answer is positive for $(M, h) \mapsto \overline{\mathfrak{sw}}(M, h)$. Since this is a rather large class of 3-manifolds, one can hope that it remains true for any rational homology sphere singularity link.

However, the answer in this general case is not true—not even when M is obtained by an integral surgery along algebraic knots in an integral homology sphere. Such an explicit example is given in [64, Example 5.0.2]. [If we replace the -8-vertex below by an arrowhead we get a knot K in an integral homology sphere $\mathfrak{S}$, and $M = \mathfrak{S}_{-2}(K)$ and $H_1(M, \mathbb{Z}) = \mathbb{Z}_2$.] The plumbing graphs are the

following:



Nevertheless, the class of manifolds for which CAP works is much larger then the class covered by the analytic (eq)SWIC mentioned above. Surprisingly, it is valid even for links of superisolated singularities (a family, which regularly produces counterexamples for the SWIC).

Theorem 9.7.37 ([64]) *Let $M = S^3_{-p/q}(\#_i K_i)$ be a negative rational Dehn surgery of S^3 along a connected sum of algebraic knots $K_1\# \cdots \#K_\nu$ ($p, q > 0$, $gcd(p, q) = 1$). Assume that M_a, the UAC of M, is a $\mathbb{Q}HS^3$. Then CAP for $\overline{\mathfrak{sw}}$ holds.*

For some other examples, which satisfy CAP see [64]; the author does not know the limits of CAP.

Remark 9.7.38 (Characteristic Numbers) Let us consider that version of the above question, when $\mathfrak{I}$ is independent of h. That is, we search (in a slightly more general context) for an invariant $\mathfrak{I}$ associated with 3-manifolds such that whenever $M' \to M$ is a regular covering of degree d then $\mathfrak{I}(M') = d \cdot \mathfrak{I}(M)$. Such invariants are called *characteristic numbers* by Milnor and Thurston in [434]. They list several characteristic numbers: the next one is relevant for singularity theory as well (they ascribe it to [39]).

Let M be any 3-manifold such that some d-fold covering M' of M fibers as a circle bundle with Euler number $e' \neq 0$ over a surface B'. Then $\mathfrak{I}(M) := \chi(B')^2/d|e'|$ is a well-defined characteristic number of M.

Note that if M is the link of a weighted homogeneous singularity, then using the covering from Theorem 5.1.23 and the notations/discussion from 5.1.24 we get that $\mathfrak{I}(M) = (2-2g')^2/\deg(\rho)\cdot|e'| = \chi^2/|e|$, where χ is the virtual Euler characteristic (cf. 5.1.21) and e is the virtual Euler number (cf. 3.5.1) of the singularity.

This characteristic number was generalized to the case of arbitrary singularity links by Wahl in [727]. Wahl's construction is the following. Let $\widetilde{X} \to X$ be a good resolution of a normal surface singularity. Set $l' := Z_K - E$, and consider the Zariski decomposition $-l' = \widetilde{e}_Z(l') - \widetilde{s}_Z(l')$ as in 6.6.23 (Here $\widetilde{e}_Z(l') \geq 0$ and $\widetilde{s}_Z(l') \in \mathscr{S}_{\mathbb{Q}}$.) (Usually this is written in the form $K + E = N + P$.) Then $\mathfrak{I}(M) := -(\widetilde{s}_Z(l'), \widetilde{s}_Z(l'))$ is a non-negative characteristic number associated with *singularity links*. From the negative definiteness of the form we get that $\mathfrak{I}(M) = 0$ if and only if $\widetilde{s}_Z(l') = 0$, or $Z_K \leq E$, i.e. (X, o) is numerically log-canonical (cf. 6.3.33). For weighted homogeneous singularity links $\mathfrak{I}(M)$ agrees with $\chi^2/|e|$.

[Its relationship with the m-th log-plurigenus is the following: $\lambda_m = m^2 \cdot \mathfrak{I}(M)/2 +$ lower order terms, cf. [727], see also 6.6.26 here. For another analytic application of $\mathfrak{I}(M)$ see [529].]

In fact, Wahl also proved that for any analytic finite map $(X', o) \to (X, o)$ of degree d one has $\mathfrak{I}(\mathrm{Link}(X')) \geq d \cdot \mathfrak{I}(\mathrm{Link}(X))$. Invariants satisfying such *covering inequality property* (in the presence of a branch covering, in either analytic or topological context) are more frequent.

As a sample let us mention the following related topological example from [296]. Let $f : M' \to M$ be a smooth map between Seifert homology spheres, and let $\mathbb{H}^0_{red}$ denote the reduced lattice cohomology, as categorification of the Seiberg–Witten (Casson) invariant (for details see Sect. 11). Then $\mathrm{rank}_{\mathbb{Z}}\mathbb{H}^0_{red}(M') \geq |\deg(f)| \cdot \mathrm{rank}_{\mathbb{Z}}\mathbb{H}^0_{red}(M)$. For similar statements see e.g. [387] and the references therein.

Chapter 10
Ehrhart Theory and the Seiberg–Witten Invariant

10.1 Introduction into Ehrhart Theory

This section is a brief introduction into the equivariant multivariable Ehrhart theory. The main steps are: the definition of the 'rational vector-dilated polytopes' and of the equivariant quasipolynomials associated with a fixed chamber (combinatorial type of the polytope). We also state the Ehrhart–Macdonald–Stanley reciprocity law. For more on Ehrhart theory see [42, 43, 47–51, 53, 114, 146]. The theory has several similarities and connections with the theory of partition functions, see e.g. [86, 669, 674]. ♣

10.1.1 The Classical Ehrhart Polynomial Fix an Euclidean space $\mathbb{R}^N$ and an embedded lattice M (e.g. $\mathbb{Z}^N$) in it. Assume that P is an n-dimensional compact convex polytope in $\mathbb{R}^N$ with all its vertices on M. In this case we say that P is an *integral polytope*. For any $t \in \mathbb{Z}_{>0}$ the polytope tP is still integral (tP is the homothety from $0 \in M$). Let $Eh(P,t)$ denote the cardinality $|tP \cap M|$ of $tP \cap M$. Ehrhart in [176, 177] proved that Eh is a rational polynomial of degree n in variable t, that is, there exist rational numbers $\{Eh_i(P)\}_{i=0}^n$ such that $Eh(P,t) = \sum_{i=0}^n Eh_i(P)t^i$ for all $t \in \mathbb{Z}_{>0}$. This statement was extended to rational polytopes (that is, when P is the convex hull of finitely many points from $M \otimes \mathbb{Q}$). In such a case $Eh(P,t)$ is a quasipolynomial.

Recall that $Q(t)$ is a quasipolynomial if there exists a *period* p and (rational) polynomials $Q^{(i)}$, $i = 0, \dots, p-1$, such that $Q(t) = Q^{(i)}(t)$, for $p|t-i$ and any $t \in \mathbb{Z}$.

If P° denotes the interior of P then by another result of Ehrhart $Eh(P^\circ,t) = |tP^\circ \cap M|$ is also a quasipolynomial, and he conjectured the *reciprocity law* $Eh(P^\circ,-t) = (-1)^n Eh(P,t)$ (as an identity of quasipolynomials). This was

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proved by Macdonald in [402], and the identity is called *Ehrhart–Macdonald reciprocity law*.

10.1.2 Reinterpretation in Terms of Toric Geometry Ehrhart theory in algebraic geometry appears usually in the context of counting functions and Riemann–Roch–Hirzebruch type formulae, and the Ehrhart–Macdonald reciprocity law is linked to duality theorems from algebraic geometry, like the Serre (or Gorenstein) dualities. (This will happen in our local singularity context as well.) Let us remind the toric prototype of this fact. We follow [134], see also [208, p. 90]. We will use the standard notations of toric geometry, see [208].

Let P be an n-dimensional convex polytope in $M \otimes \mathbb{R}$ (of rank n), with vertices in the lattice M. Then one can associate with P a fan, which provides a complete toric variety X, and also a T-Cartier divisor D on X. Furthermore, the line bundle $\mathscr{O}_X(D)$ is generated by its global sections, and $H^0(\mathscr{O}_X(D))$ is generated freely by monomials χ^m with $m \in P \cap M$. By refining the fan we can assume that X is smooth. Write also the canonical divisor in the form $\mathscr{O}_X(-\sum_i D_i)$. Consider also the exact sequence $0 \to \mathscr{O}_X(tD - \sum_i D_i) \to \mathscr{O}_X(tD) \to \mathscr{O}_{\sum_i D_i}(tD) \to 0$ ($t \in \mathbb{Z}_{>0}$). Then the higher cohomologies of all these three sheaves vanish, and one has the following reinterpretation of the h^0-ranks: $h^0(\mathscr{O}_X(tD)) = |tP \cap M|$, $h^0(\mathscr{O}_{\sum_i D_i}(tD)) = |\partial tP \cap M|$. (Here ∂ denoted the boundary of the corresponding polytope.) Hence by the exact sequence $h^0(\mathscr{O}_X(tD - \sum_i D_i)) = |tP^\circ \cap M|$ too. By Riemann–Roch–Hirzebruch (RRH) theorem the expression $|tP \cap M| = h^0(\mathscr{O}_X(tD)) = \chi(\mathscr{O}_X(tD))$ is a polynomial in t, say $RRH(t)$. On the other hand, again by RRH, $RRH(-t) = \chi(\mathscr{O}_X(-tD))$. This by Serre duality is $(-1)^n \chi(\mathscr{O}_X(tD - \sum_i D_i)) = (-1)^n |tP^\circ \cap M|$. Hence via Serre duality we recover the Ehrhart–Macdonald reciprocity law for $RRH(t) = Eh(P, t)$.

If we start with a rational polytope P then D is a $\mathbb{Q} - T$-divisor and we have to handle in the similar way the expression $\chi(\mathscr{O}_X(\lfloor tD \rfloor))$, which is a quasipolynomial.

10.1.3 By a similar argument as in 10.1.2, based on toric geometry, one can prove the following statement as well, cf. [134, 312]. Let $P_1, \ldots, P_k$ be n-dimensional convex compact polytopes with vertices in M ($\simeq \mathbb{Z}^n$). Let $Eh(\mathbf{P}, \mathbf{t})$ (resp. $Eh^\circ(\mathbf{P}, \mathbf{t})$) be the cardinality $|\sum_i t_i P_i \cap M|$ (resp. $|(\sum_i t_i P_i)^\circ \cap M|$), where $t_i \in \mathbb{Z}_{>0}$. Then both $Eh(\mathbf{P}, \mathbf{t})$ and $Eh^\circ(\mathbf{P}, \mathbf{t})$ are quasipolynomials and $(-1)^n \cdot Eh(\mathbf{P}, \mathbf{t}) = Eh^\circ(\mathbf{P}, -\mathbf{t})$.

10.1.4 Generalizations The statements from 10.1.1 have several generalizations. First, one can replace the one-variable t by several variables and the dilated polytope by a vector-dilated polytope. This means that if P is an n-dimensional convex rational polytope given by $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} \leq \mathbf{l}\}$ with $\mathbf{A}$ a $(m \times n)$ integral matrix and $\mathbf{l} \in \mathbb{Z}^m$, and m is the number of facets of P, then one can consider the vector-dilated polytopes $P_{(\mathbf{t})} := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} \leq \mathbf{t}\}$. The members of this family of polytopes can be combinatorially very different depending on the choice of $\mathbf{t}$: we say that $P_{(\mathbf{t})}$ and P are *combinatorially equivalent* if there exists a bijection between their faces that preserves the inclusion relation. For $\mathbf{t}$ staying in a connected component

of such fixed combinatorial type, Beck in [48] proved that $Eh(P, \mathbf{t}) = |P_{(\mathbf{t})} \cap M|$ and $Eh(P^\circ, \mathbf{t}) = |P^\circ_{(\mathbf{t})} \cap M|$ are multivariable quasipolynomials in $\mathbf{t} \in \mathbb{Z}^m$ and they satisfy (†) $Eh(P^\circ, -\mathbf{t}) = (-1)^n Eh(P, \mathbf{t})$. [Multivariable quasipolynomials are defined similarly: a finite sum $\sum_{\mathbf{k}} c_{\mathbf{k}}(\mathbf{t})\mathbf{t}^{\mathbf{k}}$ is a quasipolynomial if the coefficients $c_{\mathbf{k}}(\mathbf{t})$ are periodic functions in $\mathbf{t}$, with respect to a sublattice.]

In fact, if P is replaced by the union of convex polytopes $\cup_{k=1}^r P_k$, and $P_{(\mathbf{t})}$ by the dilated families $\cup_k P_{k,(\mathbf{t}_k)}$, then (†) still holds [48] (see also 10.1.3).

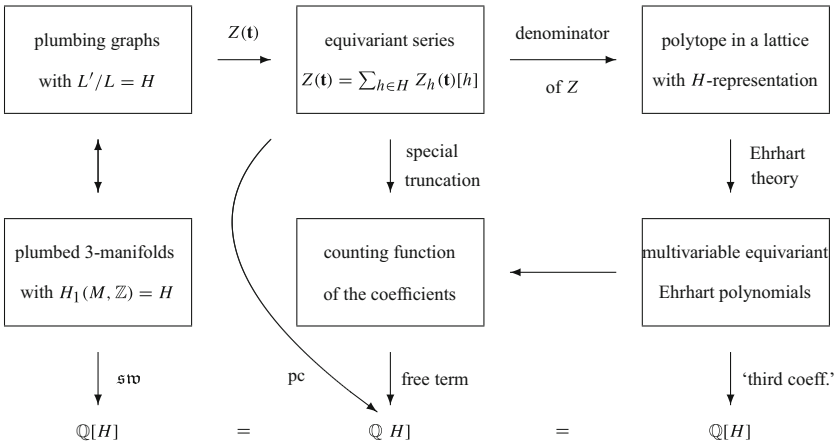
Stanley in [649] had the following generalization (see also [48, 86, 674] too), which will be crucial in our applications as well. Instead of considering the interior of P (that is, the open polytope with all facets removed), we can consider the case when we remove only some of the facets. Under the assumptions that $P_{(\mathbf{t})}$ varies in (a fixed connected component of) a fixed combinatorial equivalence class, let us denote the set of (closed) facets by $\mathscr{F}$. Let $\mathscr{T}$ be a subset of $\mathscr{F}$ and assume additionally that $\cup_{F\in\mathscr{T}} F$ is homeomorphic to an $(n-1)$-manifold. Then, for any $\mathscr{T}$, $Eh(P, \mathscr{T}, \mathbf{t}) = |(P_{(\mathbf{t})} \setminus \cup_{F\in\mathscr{T}} F) \cap M|$ is a quasipolynomial in $\mathbf{t}$, and $Eh(P, \mathscr{F} \setminus \mathscr{T}, -\mathbf{t}) = (-1)^n Eh(P, \mathscr{T}, \mathbf{t})$. This version is called *Stanley's reciprocity law*.

10.1.5 Chambers Results of Brion–Vergne [86], Clauss–Loechner [114], Sturmfels [669] and Szenes–Vergne [674] show that $\mathbb{Z}^s \otimes \mathbb{R}$ has a chamber decomposition in such a way that if $\mathbf{t}$ varies in one of the chambers then the combinatorial stability holds and the corresponding lattice point counting function $Eh(P, \mathscr{T}, \mathbf{t}) = |(P_{(\mathbf{t})} \setminus \cup_{F\in\mathscr{T}} F) \cap M|$ is a quasipolynomial. We will return to a possible constructions of the chambers later. In some statements not only the theoretical existence of the chambers is important, but their concrete description as well (cf. 10.6.3).

10.1.6 Coefficients Some of the coefficients $\{Eh_i(P)\}_{i=0}^n$ of $Eh(P, t) = \sum_{i=0}^n Eh_i(P)t^i$ have immediate geometric/combinatorial interpretations. First of all, in the context of 10.1.1, the constant coefficient $Eh_0(P)$ is 1. Next, the leading term $Eh_n(P)$ is the n-dimensional volume of P divided by the volume of the fundamental region of the lattice. Similarly, for any facets F let M_F be the lattice induced by M on the affine $(n-1)$-space determined by F, and consider the $(n-1)$-dimensional volume of F divided by twice the volume of the fundamental region of M_F. The sum of all these values summed over all the facets is $Eh_{n-1}(P)$.

However, already the next 'third' coefficient Eh_{n-2} is rather hard. Even for $n = 3$ it is an expression in terms of Dedekind sums, see e.g. Mordell's formula in 6.9.14. In this chapter we will connect this third coefficient (associated with certain polytopes constructed from a dual resolution graph) with the Seiberg–Witten invariants of the corresponding plumbed 3-manifold. For this we need to develop the equivariant version of the theory corresponding to the abelian group $H = L'/L$.

10.1.7 Here is a schematic picture of these connections and areas we target:



Above we assume $H_1(M(\Gamma), \mathbb{Q}) = 0$ and the 'right chamber' is the Lipman cone.

10.1.A Equivariant Multivariable Ehrhart Theory

10.1.8 In this section we generalize the classical Ehrhart theory to the equivariant multivariable version, involving non-convex polytopes. The presentation will fit in with the equivariant multivariable series provided by plumbing graphs. In the next sections we will follow closely the joint work with T. László [349].

10.1.9 The Equivariant Setup Let us start with an n-dimensional lattice $M \subset \mathbb{R}^n$ and a group homomorphism $\rho : M \to \mathfrak{H}$ to a finite abelian group $\mathfrak{H}$. We consider a *rational vector-dilated polytope* with parameter $\mathbf{l} = (\mathbf{l}_1, \dots, \mathbf{l}_r)$, $\mathbf{l}_v \in \mathbb{Z}^{m_v}$,

$$P^{(\mathbf{l})} = \bigcup_{v=1}^{r} P_v^{(\mathbf{l}_v)}, \quad \text{where} \quad P_v^{(\mathbf{l}_v)} = \{\mathbf{x} \in \mathbb{R}^n \ : \ \mathbf{A}_v\mathbf{x} \le \mathbf{l}_v\}, \tag{10.1.1}$$

with $\mathbf{A}_v \in M_{m_v,n}(\mathbb{Z})$ (integral $m_v \times n$ matrices). If $\{A_{v,ki}\}_{k,i}$ and $\{l_{v,k}\}_k$ are the entries of $\mathbf{A}_v$ and $\mathbf{l}_v$, then the inequality $\mathbf{A}_v\mathbf{x} \le \mathbf{l}_v$ in (10.1.1) reads as $\sum_{i=1}^n x_i A_{v,ki} \le l_{v,k}$ for any $k = 1, \dots, m_v$.

We will vary the parameter $\mathbf{l}$ in certain chambers (described below for the needed cases) such that the polytopes $P^{(\mathbf{l})}$ remain combinatorial stable whenever $\mathbf{l}$ belongs to the relative interior of a chamber. We also suppose that $P^{(\mathbf{l})}$ is homeomorphic to a n-dimensional manifold (possibly with boundary). Denote the set of all closed facets of $P^{(\mathbf{l})}$ by $\mathscr{F}$ and let $\mathscr{T}$ be a subset of $\mathscr{F}$, such that $\cup_{F^{(\mathbf{l})}\in\mathscr{T}} F^{(\mathbf{l})}$ is homeomorphic to a $(n-1)$-manifold (possibly with boundary). Then we have the following

generalization to the *equivariant version* of results of Stanley [669], McMullen [415] and Beck [47, 48]. The proof reduces the statement to the non-equivariant case, see [114, Theorem 2] and [349] for details.

Theorem 10.1.10 *For any $h \in \mathfrak{H}$ and $\mathscr{T} \subset \mathscr{F}$ let*

$$\mathscr{L}_h(\mathbf{A}, \mathscr{T}, \mathbf{l}) := \mid \left(P^{(\mathbf{l})} \setminus \cup_{F^{(\mathbf{l})} \in \mathscr{T}} F^{(\mathbf{l})}\right) \cap \rho^{-1}(h) \mid. \tag{10.1.2}$$

(a) If $\mathbf{l}$ moves in some (connected) region in such a way that $P^{(\mathbf{l})}$ stays combinatorially stable then the expression $\mathscr{L}_h(\mathbf{A}, \mathscr{T}, \mathbf{l})$ is a quasipolynomial in $\mathbf{l} \in \mathbb{Z}^{\sum_v m_v}$.

(b) For a fixed region and $\mathscr{T}$ (as in (a)), the quasipolynomials $\mathscr{L}_h(\mathbf{A}, \mathscr{T}, \mathbf{l})$ and $\mathscr{L}_{-h}(\mathbf{A}, \mathscr{F} \setminus \mathscr{T}, \mathbf{l})$ satisfy the Ehrhart–Macdonald–Stanley reciprocity law

$$\mathscr{L}_h(\mathbf{A}, \mathscr{T}, \mathbf{l}) = (-1)^n \cdot \mathscr{L}_{-h}(\mathbf{A}, \mathscr{F} \setminus \mathscr{T}, \mathbf{l})|_{\text{replace } \mathbf{l} \text{ by } -\mathbf{l}}. \tag{10.1.3}$$

Remark 10.1.11 In order to avoid any confusion regarding the expression from (10.1.3) we note the following. Under $\mathscr{L}_h(\mathbf{A}, \mathscr{T}, \mathbf{l})$ we understand the following: the value $\mathbf{l}$ determines a chamber, which contains $\mathbf{l}$, hence a combinatorial type of $P^{(\mathbf{l})}$ too. Furthermore, (in the presence of h and $\mathscr{T}$) the data h, $\mathscr{T}$ and this chamber determine a quasipolynomial—depending essentially on the chamber—which restricted to this chamber is $\mathbf{l}' \mapsto \mathscr{L}_h(\mathbf{A}, \mathscr{T}, \mathbf{l}')$. The quasipolynomial $\mathscr{L}_{-h}(\mathbf{A}, \mathscr{F} \setminus \mathscr{T}, \mathbf{l})$ is associated with $-h$, $\mathscr{F} \setminus \mathscr{T}$ and the very same chamber (i.e. which contains $\mathbf{l}$). (10.1.3) means that the first quasipolynomial agrees with the quasipolynomial obtained from the second one after the substitution $\mathbf{l} \mapsto -\mathbf{l}$ and multiplied by $(-1)^n$. Usually for $-\mathbf{l}$ the combinatorial type of $P^{(-\mathbf{l})}$ is different, hence the right hand side of (10.1.3) *does not equal* $(-1)^n \cdot \mathscr{L}_{-h}(\mathbf{A}, \mathscr{F} \setminus \mathscr{T}, -\mathbf{l})$. This last expression is the value at $-\mathbf{l}$ of that quasipolynomial, which is associated with the chamber which contains $-\mathbf{l}$ (and with $-h$ and $\mathscr{F} \setminus \mathscr{T}$).

Definition 10.1.12 The quasipolynomial $\mathscr{L}_h(\mathbf{A}, \mathscr{T}, \mathbf{l})$ considered in Theorem 10.1.10, associated with a fixed combinatorial type of $P^{(\mathbf{l})}$, is called the *equivariant multivariable quasipolynomial* associated with the corresponding data.

If we vary $\mathbf{l}$ in $\mathbb{Z}^{\sum m_v}$ (hence we allow the variation of the combinatorial type) we obtain the *equivariant multivariable piecewise quasipolynomial* $\mathscr{L}_h(\mathbf{A}, \mathscr{T}, \mathbf{l})$ (see also Theorem 10.2.8 and Corollary 10.2.9 below).

Remark 10.1.13

(a) Parallel to the collection $\{\mathscr{L}_h\}_h$, defined in (10.1.2), one can consider their Fourier transforms too: for any character $\xi \in \mathrm{Hom}(\mathfrak{H}, S^1)$ one sets

$$\mathscr{L}_\xi(\mathbf{A}, \mathscr{T}, \mathbf{l}) := \sum \xi^{-1}(\rho(\mathbf{x})), \quad \text{sum over } \mathbf{x} \in \left(P^{(\mathbf{l})} \setminus \cup_{F^{(\mathbf{l})} \in \mathscr{T}} F^{(\mathbf{l})}\right), \tag{10.1.4}$$

which satisfies $\mathscr{L}_\xi = \sum_h \mathscr{L}_h \cdot \xi^{-1}(h)$, and $|\mathfrak{H}| \cdot \mathscr{L}_h = \sum_\xi \mathscr{L}_\xi \cdot \xi(h)$. Hence, the above properties of $\mathscr{L}_h$ can be obtained from similar properties of $\mathscr{L}_\xi$ as well. Hence, Theorem 10.1.10 can be deduced from [86, § 4.3] too.

(b) In the sequel we will consider only those polytopes which will be associated with multivariable rational functions, whose denominators have the form $\prod_i (1-\mathbf{t}^{a_i})$, cf. 10.2. In order to avoid unnecessary technical details, the stability of the combinatorial type and the corresponding chamber decomposition will be considered only for these special polytopes, see 10.2.7.

10.2 Multivariable Rational Functions and Their Periodic Constants

We associate a dilated polytope to the denominator of a certain type of multivariable rational function (in the equivariant setup) and a piecewise quasipolynomial to the counting function of its coefficients. The combinatorial type of the polytope induces a chamber decomposition of the dilation parameter space. Then we define the (multivariable equivariant) periodic constant associated with each chamber. (For the periodic constant of the Taylor expansion at the origin of one-variable rational functions see 8.7.) We follow [349]. ♣

10.2.1 The Multivariable Setup We wish to extend the definition of the periodic constant to the case of Taylor expansions at the origin of multivariable rational functions of type

$$f(\mathbf{t}) = \frac{\sum_{k=1}^{r} \iota_k \mathbf{t}^{b_k}}{\prod_{i=1}^{n}(1-\mathbf{t}^{a_i})} \quad (\iota_k \in \mathbb{Z}). \tag{10.2.1}$$

Let us explain the notation. Let L be a lattice of rank s with fixed bases $\{E_v\}_{v=1}^s$. Let L' be an overlattice of it with the same rank, $L \subset L' \subset L \otimes \mathbb{Q}$. Write $H := L'/L$ and set $|H| = d$. Then, in (10.2.1), $\{b_k\}_{k=1}^r$, $\{a_i\}_{i=1}^n \in L'$ and for any $l' = \sum_v l'_v E_v \in L'$ we write $\mathbf{t}^{l'} = t_1^{l'_1} \dots t_s^{l'_s}$. We also assume that *all the coordinates* $a_{i,v}$ of a_i *are strictly positive*. Hence, in general, the coefficients l'_v are not integral, and the Laurent expansion $Tf(\mathbf{t})$ of $f(\mathbf{t})$ at the origin is

$$Tf(\mathbf{t}) = \sum_{l'} z(l')\mathbf{t}^{l'} \in \mathbb{Z}[[t_1^{1/d}, \dots, t_s^{1/d}]][t_1^{-1/d}, \dots, t_s^{-1/d}] := \mathbb{Z}[[\mathbf{t}^{1/d}]][\mathbf{t}^{-1/d}].$$

We also consider the natural partial ordering of $L \otimes \mathbb{Q}$ (via $\{E_v\}_v$, defined as in 2.1.15). If all vectors $b_k \geq 0$ then $Tf(\mathbf{t}) \in \mathbb{Z}[[\mathbf{t}^{1/d}]]$. Sometimes we will not make difference between f and Tf.

10.2.2 We consider the H-decomposition of Tf as usual:

$$\text{if} \quad Tf(\mathbf{t}) = \sum_{l'} z(l')\mathbf{t}^{l'} \quad \text{then for } h \in H \text{ set} \quad Tf_h(\mathbf{t}) := \sum_{[l']=h} z(l')\mathbf{t}^{l'}. \tag{10.2.2}$$

Hence Tf written equivariantly is $Tf^e = \sum_h Tf_h \cdot [h] \in \mathbb{Z}[[\mathbf{t}^{1/d}]][\mathbf{t}^{-1/d}][H]$.

The next warning is appropriate: We always start with this group $L'/L = H$, hence also with the decomposition $\sum_{h\in H} f_h$ of f. Nevertheless, some alterations will appear. In order to simplify the rational function we will eliminate some of its variables (e.g., we substitute $t_i = 1$ for certain indices i), i.e. we restrict f to a linear subspace V. Then, after this modification, the restricted function $f|_{t_i=1}$ will not determine anymore the restrictions $(f_h)|_{t_i=1}$ of the 'old' components f_h. That is, the new pair of lattices $(L_V, L'_V) = (L \cap V, L' \cap V)$ and the 'old group' $H = L'/L$ become rather independent. In such cases we will always keep the old group $H = L'/L$ (and the 'old' decomposition f_h) without asking any compatibility with L'_V/L_V.

10.2.3 Since all the coordinates $a_{i,v}$ of a_i are strictly positive, for any $Tf(\mathbf{t}) = \sum_{l'} z(l')\mathbf{t}^{l'}$ we get a well defined *counting function* of the coefficients,

$$L' \ni l' \mapsto Q(l') := \sum_{l'' \not\geq l'} z(l'') \in \mathbb{Z}.$$

If $Tf = \sum_h Tf_h$, then each Tf_h determines a counting function Q_h defined in the same way. That is, in the group ring $\mathbb{Z}[H]$,

$$\sum_{l'' \not\geq l'} z(l'') \cdot [l''] = \sum_{h\in H} Q_h(l') \cdot [h]. \tag{10.2.3}$$

The definitions are motivated by formulae from 8.2.12, 8.2.13, (9.6.1) and (9.6.13). The functions $Q_h(l')$ will be studied in the next subsections via Ehrhart theory.

10.2.4 Ehrhart Quasipolynomials Associated with Denominators of Rational Functions Let us fix a rational function as in (10.2.1). First we consider the case $n > 0$; the special case $n = 0$ (i.e. when $f(\mathbf{t}) = \sum_{k=1}^{r} \iota_k \mathbf{t}^{b_k}$) will be treated in 10.2.10.

10.2.5 First Step: The Polytope Associated with $\{a_i\}_{i=1}^n$ In order to run the Ehrhart theory we have to fix a lattice M and a representation $\rho : M \to \mathfrak{H}$, cf. 10.1.A. Here we set $M = \mathbb{Z}^n$, $\alpha : M \to L'$ given by $\alpha(\mathbf{x}) = \sum_{i=1}^n x_i a_i \in L'$, $\mathfrak{H} = H = L'/L$, $|H| = d$, and ρ is the composition $M \xrightarrow{\alpha} L' \to L'/L$. Set also $s := \operatorname{rank}(L)$.

Consider the matrix $\mathbf{A}$ with column vectors $d \cdot a_i$ and write $\mathbf{A}_v$ for its rows. Then the construction of Sect. 10.1.A can be repeated (eventually completing each $\mathbf{A}_v$ to assure the inequalities $x_i \geq 0$ as well). For $l \in \sum_v l_v E_v \in L$ consider

$$P_v^{\triangleleft} := \{\mathbf{x} \in (\mathbb{R}_{\geq 0})^n \ : \ d \cdot \sum_i x_i a_{i,v} < l_v\} \ \text{ and } \ P^{\triangleleft} := \bigcup_{v=1}^{s} P_v^{\triangleleft}. \tag{10.2.4}$$

The closure P_v of $P_v^{\triangleleft}$ is a dilated convex (simplicial) polytope depending on the one-dimensional parameter l_v. Moreover, $P^{\triangleleft}$ is described via the partial ordering of $L \otimes \mathbb{Q}$ as $\sum_i x_i a_i \not\geq l/d$. Since $L' \subset L/d$, we can restrict ourself to the lattice L' (preserving all the general results of Sect. 10.1.A). Hence for any $l' \in L'$ we set

$$P^{(l'),\triangleleft} := \{\mathbf{x} \in (\mathbb{R}_{\geq 0})^n \ : \ \sum_i x_i a_i \not\geq l'\}, \quad P^{(l')} = \text{closure of } (P^{(l'),\triangleleft}). \tag{10.2.5}$$

The combinatorial type of $P^{(l')}$ might vary with l'. Nevertheless, by definition, the facets will be grouped for all different combinatorial types by the same principle: we consider the coordinate facets $F_i^{(l')} := P^{(l')} \cap \{x_i = 0\}$, $1 \leq i \leq n$, and we denote by $\mathscr{T}$ the collection of all other facets. Hence $P^{(l'),\triangleleft} = P^{(l')} \setminus \cup_{F^{(l')} \in \mathscr{T}} F^{(l')}$. The construction is motivated by the summations from (9.6.1) and (9.6.13).

10.2.6 Second Step: The Counting Function Associated with $\{a_i\}_{i=1}^n$ The general theory from 10.1.A leads to the next counting function

$$\mathscr{L}^e(\mathbf{A}, \mathscr{T}, l') := \sum_{h \in H} \mathscr{L}_h(\mathbf{A}, \mathscr{T}, l') \cdot [h] := \sum 1 \cdot [\alpha(\mathbf{x})] \in \mathbb{Z}[H], \tag{10.2.6}$$

where the last sum runs over $\mathbf{x} \in \left(P^{(l')} \setminus \cup_{F^{(l')} \in \mathscr{T}} F^{(l')}\right) = P^{(l'),\triangleleft}$.

By the very construction, we have the following identity. Consider the equivariant Taylor expansion at the origin of the function determined by the *denominator of* f,

$$\widetilde{f}^e(\mathbf{t}) = \frac{1}{\prod_{i=1}^n (1 - [a_i]\mathbf{t}^{a_i})} = \sum_{l''} \widetilde{z}(l'')\mathbf{t}^{l''} \cdot [l''] \in \mathbb{Z}[[\mathbf{t}^{1/d}]]\,[H]. \tag{10.2.7}$$

Note that since all the $\{E_v\}$-coefficients of each a_i are strictly positive, for any $l' \in L'$ the set $\{l'' \ : \ \widetilde{z}(l'') \neq 0, l'' \not\geq l'\}$ is finite. Then, by the above construction,

$$\sum_{l'' \not\geq l'} \widetilde{z}(l'') \cdot [l''] = \mathscr{L}^e(\mathbf{A}, \mathscr{T}, l'). \tag{10.2.8}$$

We also define the corresponding non-equivariant counting function by

$$\mathscr{L}^{ne}(\mathbf{A}, \mathscr{T}, l') := \sum_{h \in H} \mathscr{L}_h(\mathbf{A}, \mathscr{T}, l') \in \mathbb{Z}.$$

10.2.7 Combinatorial Types, Chambers Next, we wish to make precise the *combinatorial stability* condition. The results from [86, 114, 669, 674], cf. 10.1.5, imply that $\mathscr{L}^e$ from (10.2.8) (that is, each $\mathscr{L}_h$) has an extension from L' to $L \otimes \mathbb{R}$, say $\mathscr{L}^e_{\mathbb{R}}$, which is a *piecewise quasipolynomial* on $L \otimes \mathbb{R}$: the parameter space $L \otimes \mathbb{R}$ decomposes into several chambers, the restriction of $\mathscr{L}^e_{\mathbb{R}}$ on each chamber is a quasipolynomial, and $\mathscr{L}^e_{\mathbb{R}}$ is continuous. Sometimes $\mathscr{L}^e_{\mathbb{R}}$ will be denoted by the same $\mathscr{L}^e$. A possible choice of chambers is described by an algorithm (principle) as follows.

Notice that the combinatorial type of $P^{(l')}$ in (10.2.5) vary in the same way as the combinatorial type of the closure of its *convex* complement in $\mathbb{R}^n_{\geq 0}$, namely $\{\mathbf{x} \in (\mathbb{R}_{\geq 0})^n : \sum_i x_i a_i \geq l'\}$, since both are determined by their common boundary $\mathscr{T}$. The inequalities in this last system can be viewed as a *vector partition* $\sum_i x_i a_i + \sum_v y_v(-E_v) = l'$, with $x_i \geq 0$ and $y_v \geq 0$. Hence, according to the above references, we have the following chamber decomposition of $L \otimes \mathbb{R}$.

Let $\mathbf{M}$ be the matrix with column vectors $\{a_i\}_{i=1}^n$ and $\{-E_v\}_{v=1}^s$. A subset $\mathfrak{b}$ of indices of columns is called *basis* if the corresponding columns form a basis of $L \otimes \mathbb{R}$; in this case we write $Cone(\mathbf{M}_{\mathfrak{b}})$ for the positive closed cone generated by them. Then the chamber decomposition is the polyhedral subdivision of $L \otimes \mathbb{R}$ provided by the (minimal) common refinement of the cones $Cone(\mathbf{M}_{\mathfrak{b}})$, where $\mathfrak{b}$ runs all over the bases as above. *A chamber is a closed cone of the subdivision whose interior is non-empty.* Usually we denote them by $\mathfrak{c}$, let their collection be $\mathfrak{C}$.

We will need the associated *disjoint* decomposition of $L \otimes \mathbb{R}$ with relative open cones as well. A typical element of this disjoint decomposition is the *relative interior of an intersection of type* $\cap_{\mathfrak{c} \in \mathfrak{C}'} \mathfrak{c}$, where $\mathfrak{C}'$ runs over the subsets of $\mathfrak{C}$. For such relative interior cones we use the notation $\mathfrak{c}_{op}$.

Each chamber $\mathfrak{c}$ determines an open cone, namely its interior. And, conversely, each top dimensional open cone determines a chamber $\mathfrak{c}$, namely its closure.

The next theorem is the direct consequence of [86, 4.4], [674, 0.2] and 10.1.10 using the summation of the Ehrhart quasipolynomial associated with suitable convex parts of $P^{(l')}$. (We state it for our specific facet–collection $\mathscr{T}$, the case which will be used later, but it is true for any other facet-decomposition of the boundary whenever $\cup_{F^{(l')} \in \mathscr{T}} F^{(l')}$ is homeomorphic to a $(n-1)$-manifold.)

Theorem 10.2.8

(a) For each relative open cone $\mathfrak{c}_{op}$ of $L \otimes \mathbb{R}$, $P^{(l')}$ is combinatorially stable, that is, the polytopes $\{P^{(l')}\}_{l' \in \mathfrak{c}_{op}}$ have the same combinatorial type. Therefore, for any fixed $h \in H$, the restrictions $\mathscr{L}_h^{\mathfrak{c}_{op}}(\mathbf{A}, \mathscr{T})$ and $\mathscr{L}_h^{\mathfrak{c}_{op}}(\mathbf{A}, \mathscr{F} \setminus \mathscr{T})$ to $\mathfrak{c}_{op} \cap L'$ of $\mathscr{L}_h(\mathbf{A}, \mathscr{T})$ and $\mathscr{L}_h(\mathbf{A}, \mathscr{F} \setminus \mathscr{T})$ respectively are quasipolynomials.

(b) *These quasipolynomials have a continuous extension to the closure of* $\mathfrak{c}_{op}$. *Namely, if* $\mathfrak{c}'_{op}$ *is in the closure of* $\mathfrak{c}_{op}$, *then the restrictions to* $\mathfrak{c}'_{op}$ *of* $\mathscr{L}_h(\mathbf{A}, \mathscr{T})$ *and* $\mathscr{L}_h^{\mathfrak{c}_{op}}(\mathbf{A}, \mathscr{T})$ *agree. (Similarly for* $\mathscr{L}_h^{\mathfrak{c}_{op}}(\mathbf{A}, \mathscr{F} \setminus \mathscr{T})$.*)*

In particular, for any chamber $\mathfrak{c}$ *one has a well defined quasipolynomial* $\mathscr{L}_h^{\mathfrak{c}}(\mathbf{A}, \mathscr{T})$*—defined as* $\mathscr{L}_h^{\mathfrak{c}_{op}}(\mathbf{A}, \mathscr{T})$, *where* $\mathfrak{c}_{op}$ *is the interior of* $\mathfrak{c}$ *—, which equals* $\mathscr{L}_h(\mathbf{A}, \mathscr{T})$ *for all points of* $\mathfrak{c}$.

Hence, for any two chambers $\mathfrak{c}_1$ *and* $\mathfrak{c}_2$ *one has the continuity property*

$$\mathscr{L}_h^{\mathfrak{c}_1}(\mathbf{A}, \mathscr{T})|_{\mathfrak{c}_1 \cap \mathfrak{c}_2} = \mathscr{L}_h^{\mathfrak{c}_2}(\mathbf{A}, \mathscr{T})|_{\mathfrak{c}_1 \cap \mathfrak{c}_2}. \tag{10.2.9}$$

(c) $\mathscr{L}_h^{\mathfrak{c}}(\mathbf{A}, \mathscr{T})$ *and* $\mathscr{L}_{-h}^{\mathfrak{c}}(\mathbf{A}, \mathscr{F} \setminus \mathscr{T})$, *as abstract quasipolynomials associated with a fixed chamber* $\mathfrak{c}$, *satisfy the reciprocity law*

$$\mathscr{L}_h^{\mathfrak{c}}(\mathbf{A}, \mathscr{T}, l') = (-1)^n \cdot \mathscr{L}_{-h}^{\mathfrak{c}}(\mathbf{A}, \mathscr{F} \setminus \mathscr{T}, -l').$$

We define the semi-open unit cube (associated with $L \subset L'$) by $\{l' = \sum_v l'_v E_v \in L' : 0 \le l'_v < 1\}$. For any $h \in H$ let $r_h \in L'$ be the unique representative of h in the semi-open unit cube. We have the following consequences regarding the counting function $l' \mapsto Q_h(l')$ of $f^e(\mathbf{t})$ defined in (10.2.3):

Corollary 10.2.9

(a) Q_h *is a piecewise quasipolynomial. Indeed, for any* $h \in H$ *and* $l' \in L'$

$$Q_h(l') = \sum_k \iota_k \cdot \mathscr{L}_{h-[b_k]}(\mathbf{A}, \mathscr{T}, l' - b_k). \tag{10.2.10}$$

In particular, the right hand side of (10.2.10) is independent of the representation of f *as in (10.2.1) (that is, of the choice of* $\{b_k, a_i\}_{k,i}$*), it depends only on the rational function* f.

(b) *Fix a chamber* $\mathfrak{c}$ *of* $L \otimes \mathbb{R}$ *and for any* $h \in H$ *define the quasipolynomial*

$$\overline{Q}_h^{\mathfrak{c}}(l') := \sum_k \iota_k \cdot \mathscr{L}_{h-[b_k]}^{\mathfrak{c}}(\mathbf{A}, \mathscr{T}, l' - b_k). \tag{10.2.11}$$

Then the restriction of $Q_h(l')$ *to* $\cap_k(b_k + \mathfrak{c})$ *is a quasipolynomial, namely*

$$Q_h(l') = \overline{Q}_h^{\mathfrak{c}}(l') \ \text{ on } \ \bigcap_k (b_k + \mathfrak{c}). \tag{10.2.12}$$

Moreover, there exists $l'_* \in \mathfrak{c}$ *such that* $l'_* + \mathfrak{c} \subset \cap_k(b_k + \mathfrak{c})$.

[Warning: $\mathscr{L}_{h-[b_k]}^{\mathfrak{c}}(\mathbf{A}, \mathscr{T}, l' - b_k) \neq \mathscr{L}_{h-[b_k]}(\mathbf{A}, \mathscr{T}, l' - b_k)$ *unless* $l' - [b_k] \in \mathfrak{c}$.*]*

(c) For any fixed $h \in H$, the quasipolynomial $\overline{Q}_h^{\mathfrak{c}}(l')$ satisfies the following property: for any $l' \in L'$ with $[l'] = h$, and any q from the semi-open unit cube one has

$$\overline{Q}_h^{\mathfrak{c}}(l'-q) = \overline{Q}_h^{\mathfrak{c}}(l'). \tag{10.2.13}$$

In particular, by taking $l' = q = r_h$:

$$\overline{Q}_h^{\mathfrak{c}}(0) = \overline{Q}_h^{\mathfrak{c}}(r_h). \tag{10.2.14}$$

Proof For *(a)* use (10.2.5) and the fact that $b_k + \sum x_i a_i \not\geq l'$ if and only if $\sum x_i a_i \not\geq l' - b_k$. Since the coefficients of the Taylor expansion depend only on f, the second sentence follows too. For *(b)* use part *(a)* and the fact that $\mathfrak{c} \cap (\cap_k (b_k + \mathfrak{c}))$ contains a set of type $l'_* + \mathfrak{c}$.

(c) Consider those values l' in some $l'_* + \mathfrak{c}$ for which all elements of type $l' - b_k$ and $l' - q - b_k$ are in $\mathfrak{c}$. For these values l', (10.2.13) follows from the identity $P^{(l'),\lhd} \cap \rho^{-1}(h) = P^{(l'-q),\lhd} \cap \rho^{-1}(h)$ whenever $[l'] = h$. This is true since for any l'' with $[l''] = [l']$, $l'' \geq l'$ is equivalent to $l'' \geq l' - q$. Indeed, taking $y = l'' - l'$, this reads as follows: for any $y \in L$, $y \geq 0$ if and only if $y \geq -q$.

Now, if two quasipolynomials agree on some $l'_0 + \mathscr{C}$ then they are equal. □

10.2.10 The $n = 0$ Case Though in this case the polytope constructed in 10.2.5 does not exist, we can look at $f(\mathbf{t}) = \sum_k \iota_k \mathbf{t}^{b_k}$ directly. Using the notation of (10.2.3) we set

$$\sum_{h \in H} Q_h(l') \cdot [h] = \sum_{l'' \not\geq l'} z(l'') \cdot [l''] = \sum_{\{k\,:\, b_k \not\geq l'\}} \iota_k [b_k].$$

The point is that these functions $L' \ni l' \mapsto Q_h(l')$ have no extension to $L \otimes \mathbb{R}$ as continuous piecewise quasipolynomials. However, we can define a chamber decomposition of $L \otimes \mathbb{R}$ imposed by $\{-E_v\}_{v=1}^s$ via the same principle as above. This provides two chambers: $\mathfrak{c}_0 := \mathbb{R}_{\geq 0}\langle -E_v \rangle_v$ and $\mathfrak{c}_1$, the closure of the complement of $\mathfrak{c}_0$ in $\mathbb{R}^s$. Then $Q_h(l') = \sum_{\{k\,:\,[b_k]=h\}} \iota_k$ on $\cap_k (b_k + \mathfrak{c}_{1,op})$ and 0 on $\cap_k (b_k + \mathfrak{c}_0)$. (Nevertheless, they have no common real continuous extension.)

10.2.11 The Multivariable Equivariant Periodic Constant of a Counting Function Corollary 10.2.9 motivates the following general definition. Fix a lattice L with fixed bases $\{E_v\}_{v=1}^s$ and an overlattice $L' \subset L \otimes \mathbb{R}$ of the same rank, and set $H := L'/L$. For any $h \in H$ denote by $r_h \in L'$ the unique representative of h in the semi-open unit cube $\sum_v [0,1)E_v$. We fix some multivariable series $S(\mathbf{t}) = \sum_{l'} z(l')\mathbf{t}^{l'} \in \mathbb{Z}[[L']]$, and we associate with its coefficients a 'counting function'. This depends on the choice of some 'summation procedure'. In many cases (of this book) this appears as follows: for any $l' \in L'$ we define a subset $R(l') \subset L'$ such that its intersection with the support $\{l' : z(l') \neq 0\}$ of $S(\mathbf{t})$ is

finite. Then we define the counting function $l' \mapsto Q^R(l')$ associated with the choice of $\{l' \mapsto R(l')\}_{l'}$ by $\sum_{l'' \in R(l')} z(l'') \cdot [l''] = \sum_{h \in H} Q_h^R(l')[h] \in \mathbb{Z}[H]$.

Definition 10.2.12 Let $\mathscr{K} \subset L' \otimes \mathbb{R}$ be a closed real cone whose affine closure $\mathrm{aff}(\mathscr{K})$ has positive dimension. For any $h \in H$ we assume that there exist

(*a*) $l'_* \in \mathscr{K}$
(*b*) a sublattice $\widetilde{L} \subset L$ of finite index, and
(*c*) a multivariable polynomial $\widetilde{l} \mapsto \mathfrak{Q}_h(\widetilde{l})$, defined on $\widetilde{L} \cap \mathrm{aff}(\mathscr{K})$ such that

$$Q_h^R(\widetilde{l} + r_h) = \mathfrak{Q}_h(\widetilde{l}) \quad \text{for any } \widetilde{l} \in \widetilde{L} \cap (l'_* + \mathscr{K}). \tag{10.2.15}$$

Then we say that the counting function of S admits an equivariant periodic constant in $\mathscr{K}$ and we define this equivariant periodic constant of Q^R associated with $\mathscr{K}$ by

$$\mathrm{pc}^{e,\mathscr{K}}(Q^R) = \sum_{h \in H} \mathrm{pc}_h^{\mathscr{K}}(Q_h^R) \cdot [h] := \sum_{h \in H} \mathfrak{Q}_h(0) \cdot [h] \in \mathbb{Z}[H]. \tag{10.2.16}$$

In a similar way, for any fixed $h \in H$ one can define the periodic constant of Q_h^R independently of the other classes. (Sometimes we will use the same notation for the real cone $\mathscr{K}$ and for its lattice points $\mathscr{K} \cap L'$ in L'.)

Remark 10.2.13

(a) The above definition is independent of the choice of the sublattice $\widetilde{L}$: any two sublattices $\widetilde{L}_1$ and $\widetilde{L}_2$ can be replaced by their intersection sharing the very same properties and the very same periodic constant. The advantage of 'small' sublattices is that convenient restrictions of Q_h^R might have nicer forms which are easier to compute. The choice of $\widetilde{L}$ corresponds to the choice of p in 8.7.1, and it is responsible for the name 'periodic' in the name of $\mathrm{pc}^{e,\mathscr{K}}$.
(b) Usually, the candidates for $\mathfrak{Q}$ are quasipolynomials, hence in *(c)* we might ask for the existence of a quasipolynomial. However, this is not really an extension, since the restriction of a quasipolynomial to a convenient sublattice is a polynomial.

Proposition 10.2.14 *Let f be as in (10.2.1) above with $n > 0$ and $b_k \geq 0$ for all k.*

(a) Consider the chamber decomposition of $L \otimes \mathbb{R}$ given by the denominator $\prod_i (1 - \mathbf{t}^{a_i})$ of f as in Theorem 10.2.8. Then Q defined in (10.2.3) admits a periodic constant $\mathrm{pc}_h^{\mathfrak{c}}(f) := \mathrm{pc}_h^{\mathfrak{c}}(Q)$ in each chamber $\mathfrak{c}$ and

$$\mathrm{pc}_h^{\mathfrak{c}}(f) = \overline{Q}_h^{\mathfrak{c}}(r_h) = \overline{Q}_h^{\mathfrak{c}}(0). \tag{10.2.17}$$

(b) If two functions f_1 and f_2 admit periodic constant in some cone $\mathscr{K}$, then the same is true for $\alpha_1 f_1 + \alpha_2 f_2$ and

$$\mathrm{pc}^{\mathscr{K}}(\alpha_1 f_1 + \alpha_2 f_2) = \alpha_1 \mathrm{pc}^{\mathscr{K}}(f_1) + \alpha_2 \mathrm{pc}^{\mathscr{K}}(f_2) \quad (\alpha_1,\ \alpha_2 \in \mathbb{C}).$$

(c) If f admits periodic constants in two (top dimensional) cones $\mathscr{K}_1$ and $\mathscr{K}_2$, and the interior $(\mathscr{K}_1 \cap \mathscr{K}_2)^\circ$ of $\mathscr{K}_1 \cap \mathscr{K}_2$ is non-empty, then $\mathrm{pc}^{\mathscr{K}_1}(f) = \mathrm{pc}^{\mathscr{K}_2}(f)$.

In particular, if $\{\mathfrak{c}_i\}_{i=1,2}$ are two chambers as in (a), and f admits a periodic constant in $\mathscr{K}$, and $(\mathfrak{c}_i \cap \mathscr{K})^\circ \neq \emptyset$ $(i = 1, 2)$, then $\mathrm{pc}^{\mathfrak{c}_1}(f) = \mathrm{pc}^{\mathfrak{c}_2}(f)$.

Proof For *(a)* use Corollary 10.2.9; *(b)* is clear. For *(c)* we can assume that $\mathscr{K}_2 \subset \mathscr{K}_1$ (by considering $\mathscr{K}_i$ and $\mathscr{K}_1 \cap \mathscr{K}_2$). Then if Q_h is quasipolynomial on $l_1' + \mathscr{K}_1$ (with $l_1' \in \mathscr{K}_1$), then $(l_1' + \mathscr{K}_2) \cap \mathscr{K}_2$ contains a set of type $l_2' + \mathscr{K}_2$ with $l_2' \in \mathscr{K}_2$, on which one can take the restriction of the previous quasipolynomial. □

Remark 10.2.15

(a) By Corollary 10.2.9*(b)–(c)* $Q_h(\widetilde{l} + r_h) = Q_h(\widetilde{l})$ on $\cap_k b_k + \mathfrak{c}$, hence in the definition of the periodic constant we can replace (10.2.15) by the assumption $Q_h(\widetilde{l}) = \mathfrak{Q}_h(\widetilde{l})$ for any $\widetilde{l} \in \widetilde{L} \cap (l_*' + \mathscr{K})$.

(b) Note that in the rational presentation of f we might assume that $a_i \in L$ for every i. Indeed, take $o_i \in \mathbb{Z}_{>0}$ such that $o_i a_i \in L$, and amplify the fraction by $\prod_i (1 - \mathbf{t}^{o_i a_i})/(1 - \mathbf{t}^{a_i})$. Therefore, for each h we can write $f_h(\mathbf{t})$ in the form

$$f_h(\mathbf{t}) = \mathbf{t}^{r_h} \sum_k \iota_k \cdot \frac{\mathbf{t}^{\overline{b}_k}}{\prod_i (1 - \mathbf{t}^{a_i})},$$

where $a_i,\ \overline{b}_k \in L$, hence $f_h(\mathbf{t})/\mathbf{t}^{r_h} \in \mathbb{Z}[[\mathbf{t}]][\mathbf{t}^{-1}]$. Then if we consider the nonequivariant periodic constant $\mathrm{pc}^{\mathfrak{c}}$ of $f_h(\mathbf{t})/\mathbf{t}^{r_h}$, (10.2.3), (10.2.12) and (10.2.17) imply that $\mathrm{pc}_h^{\mathfrak{c}}(f(\mathbf{t})) = \mathrm{pc}^{\mathfrak{c}}(f_h(\mathbf{t})/\mathbf{t}^{r_h})$ for all chambers $\mathscr{C}$ associated with $\{a_i\}_i$.

Example 10.2.16 Assume that $L = L' = \mathbb{Z}$ (with the basis $E_1 = 1_{\mathbb{Z}}$) and $\mathscr{K} = \mathbb{R}_{\geq 0}$, and consider $S(t)$ as in 8.7.1. If $S(t)$ admits a periodic constant in $\mathscr{K}$, then the definition of $\mathrm{pc}(S)$ from 10.2.12 is compatible with 8.7.1.

Example 10.2.17

(a) (The $n = 0$ case) Assume that $f(\mathbf{t}) = \sum_{k=1}^r \iota_k \mathbf{t}^{b_k}$. Then, using 10.2.10 (and its notation), $\mathrm{pc}^{e,\mathfrak{c}_0}(f) = 0$ and $\mathrm{pc}^{e,\mathfrak{c}_1}(f) = \sum_{k=1}^r \iota_k [b_k] \in \mathbb{Z}[H]$.

(b) Assume that the rank is $s = 2$ and $f(\mathbf{t}) = \mathbf{t}^b/(1 - \mathbf{t}^a)$, with both entries (a_1, a_2) of a positive. We assume that $a \in L$ while $b \in L'$. Again, for $h \neq [b]$ the counting function, hence its periodic constant too, is zero. Assume $h = [b]$, and write $b = (b_1, b_2)$. Then the denominator provides three chambers: $\mathfrak{c}_0 := \mathbb{Z}_{\geq 0}\langle -E_1, -E_2\rangle$, $\mathfrak{c}_1 := \mathbb{Z}_{\geq 0}\langle a, -E_2\rangle$, $\mathfrak{c}_2 := \mathbb{Z}_{\geq 0}\langle a, -E_1\rangle$. Then the

three quasipolynomials for $1/(1-\mathbf{t}^a)$ are $\mathscr{L}_0^{\mathfrak{c}_0}=0$ and $\mathscr{L}_0^{\mathfrak{c}_i}(n_1,n_2)=\lceil n_i/a_i\rceil$ $(i=1,2)$. Thus $\mathrm{pc}_h^{\mathfrak{c}_0}(f)=0$ and $\mathrm{pc}_h^{\mathfrak{c}_i}(f)=\lceil -b_i/a_i\rceil$ $(i=1,2)$, since $Q_h(n)=\mathscr{L}_0^{\mathfrak{c}_i}(n-b)=\lceil n_i-b_i/a_i\rceil$.

In particular, $\mathrm{pc}_h^{\mathfrak{c}}(f)$, in general, depends on the choice of $\mathfrak{c}$.

(c) Assume that $L=L'$ and $f(t)=\frac{t_1^{b_1}t_2^{b_2}}{(1-t_1t_2)(1-t_1^2t_2)}$. Then the chambers associated with the denominator are: $\mathfrak{c}_0:=\mathbb{R}_{\geq 0}\langle -E_1,-E_2\rangle$, $\mathfrak{c}_2:=\mathbb{R}_{\geq 0}\langle -E_1,(1,1)\rangle$, $\mathfrak{c}:=\mathbb{R}_{\geq 0}\langle (1,1),(2,1)\rangle$ and $\mathfrak{c}_1:=\mathbb{R}_{\geq 0}\langle (2,1),-E_2\rangle$. Then, by a computation,

$$\begin{aligned}
&\mathscr{L}^{\mathfrak{c}_0}=0; && \mathscr{L}^{\mathfrak{c}_2}(l_1,l_2)=\tfrac{l_2^2}{2}+\tfrac{l_2}{2};\\
&\mathscr{L}^{\mathfrak{c}}(l_1,l_2)=\tfrac{l_1^2}{2}+l_2^2+\tfrac{l_1}{2}-l_1l_2; && \mathscr{L}^{\mathfrak{c}_1}(l_1,l_2)=\tfrac{l_1^2}{4}+\tfrac{l_1}{2}+\tfrac{1+(-1)^{l_1+1}}{8}.
\end{aligned} \tag{10.2.18}$$

Hence, by 10.2.14 and (10.2.11), $\mathrm{pc}^{\mathfrak{c}_*}(f)=\mathscr{L}^{\mathfrak{c}_*}(-b_1,-b_2)$.

Example 10.2.18 (Normal Affine Monoids) Consider the following objects (cf. 10.2.1): a lattice L with fixed bases $\{E_v\}_{v=1}^n$ (hence $s=n$) and with induced partial ordering $\leq$, an overlattice $L'\subset L\otimes\mathbb{Q}$ with finite abelian quotient $H:=L'/L$ and projection $\rho:L'\to H$. Furthermore, let $\{a_i\}_{i=1}^n$ be ($\mathbb{R}$–) linearly independent vectors in L with all their $\{E_v\}$-coordinates positive. Let $\mathscr{K}$ be the positive real cone generated by the vectors $\{a_i\}_i$, and consider the Hilbert series of the pair $(\mathscr{K},L')$

$$f(\mathbf{t}):=\sum_{l'\in\mathscr{K}\cap L'}\mathbf{t}^{l'}.$$

Consider the monoid $M:=\mathbb{Z}_{\geq 0}\langle a_i\rangle$ and $B:=(\sum_{i=1}^d[0,1)a_i)\cap L'$ (cf. e.g. [88, 2.C]). Then the normal affine monoid $\mathscr{K}\cap L'$ is a module over M and by [88, Prop. 2.43] $\mathscr{K}\cap L'=\bigsqcup_{b\in B}b+M$. Thus, $f(\mathbf{t})$ equals $\sum_{b\in B}\mathbf{t}^b/\prod_{i=1}^n(1-\mathbf{t}^{a_i})$ and it has the form considered in 10.2.1 with every $b\geq 0$.

If the rank n is ≥ 3 then usually $\mathscr{K}$ is cut in more chambers. Indeed, take e.g. $n=3$, $a_i=(1,1,1)+E_i$ for $i=1,2,3$. Then $\mathscr{K}$ is cut in its barycentric subdivision. Nevertheless, if $n=2$ then $\mathscr{K}$ consists of a unique chamber and f admits a periodic constant in $\mathscr{K}$. Indeed, one has:

Lemma 10.2.19 *If $n=2$ then $\mathrm{pc}_h^{\mathscr{K}}(f)=0$ for every $h\in H$.*

Proof It is elementary to see that $\mathscr{K}$ is one of the chambers (use the construction from 10.2.7). Take $B=\{b_k\}_k$, and write $f=\sum_k f_k$, where $f_k=\mathbf{t}^{b_k}/(1-\mathbf{t}^{a_1})(1-\mathbf{t}^{a_2})$. The only relevant classes $h\in H$ are given by $\{[b_i]:b_i\in B\}$, otherwise already the Ehrhart quasipolynomials are zero (since $a_i\in L$). Fix such a class $h=[b_i]$. Let $\{\mathscr{L}_{h'}^{\mathscr{K}}(\mathscr{T})\}_{h'}$ be the quasipolynomial associated with the chamber $\mathscr{K}$ and the denominator of f. Then, by (10.2.17) and (10.2.11),

$\mathrm{pc}_h^{\mathscr{K}}(f_k) = \mathscr{L}^{\mathscr{K}}_{[b_i-b_k]}(\mathscr{T}, -b_k)$. This, by the Reciprocity Law 10.2.8(c) equals $\mathscr{L}^{\mathscr{K}}_{[b_k-b_i]}(\mathscr{F}\setminus\mathscr{T}, b_k)$. Again, since the denominator is a series in L, for $[b_k-b_i]\neq 0$ the series is zero; so we may assume that $[b_k - b_i] = 0$. But, since $b_k \in \mathscr{K}$, the value $\mathscr{L}_0^{\mathscr{K}}(\mathscr{F}\setminus\mathscr{T}, b_k)$ of the quasipolynomial carries its geometric meaning, it is the cardinality of the set consisting of $m = n_1a_1 + n_2a_2$ such that $n_1 > 0$, $n_2 > 0$, and the entries of $m - b_k$ cannot be both > 0. But since for any such m one has $m \geq a_1 + a_2$, and $a_1 + a_2 - b_k$ has both entries > 0, this set is empty. □

Example 10.2.20 (General Affine Monoids of Rank $n = 2$) Consider the situation of Example 10.2.18 with $n = 2$, and let N be a submonoid of $\widehat{N} = \mathscr{K}\cap L'$ of rank 2, and we also assume that $\widehat{N}$ is the normalization of N. Set

$$f(\mathbf{t}) := \sum_{l'\in N}\mathbf{t}^{l'}.$$

Then $f(\mathbf{t})$ is again of type (10.2.1). Indeed, by Bruns and Gubeladze [88, Prop. 2.35], $\widehat{N}\setminus N$ is a union of a finite family of sets of type (I) $b\in\widehat{N}$, or (II) $b + \mathbb{Z}\ell a_i$, where $b\in\widehat{N}$, $\ell\in\mathbb{Z}_{\geq 0}$, $i = 1$ or 2. Obviously, two sets of type (II) with different i-values might have an intersection point of type (I). In particular,

$$f(\mathbf{t}) = \sum_{l'\in\widehat{N}}\mathbf{t}^{l'} - \sum_i \frac{\mathbf{t}^{b_{i,1}}}{1-\mathbf{t}^{k_{i,1}a_1}} - \sum_j \frac{\mathbf{t}^{b_{j,2}}}{1-\mathbf{t}^{k_{j,2}a_2}} + \sum_k(\pm\mathbf{t}^{b_k}).$$

Note that the periodic constant of the first sum is zero by Lemma 10.2.19, and the others can easily be computed (even with closed formulae) via Example 10.2.17(a)–(b). The computation shows that the periodic constant carries information about the failure of normality of N (compare with the delta-invariant computation from 8.7.4).

10.3 Ehrhart Theory of $Z(\mathbf{t})$ Associated with a Plumbing Graph

As an application of the general theory, we associate with $Z(\mathbf{t})$ a dilated polytope and the piecewise quasipolynomials $\{\mathscr{L}_h\}_h$. The restriction of $\mathscr{L}_h$ to the real Lipman's cone $\mathscr{S}_{\mathbb{R}}$ is a quasipolynomial (though $\mathscr{S}_{\mathbb{R}}$ might be cut into several chambers). We follow [349]. ♣

10.3.1 The Construction Consider the topological setup of a surface singularity with *rational homology sphere link*. We fix a good resolution and let Γ be its graph.

The lattice L has a canonical basis $\{E_v\}_{v\in\mathscr{V}}$ corresponding to the vertices of the graph Γ. We investigate the periodic constant of the rational function $Z(\mathbf{t})$, defined in 8.4.2 from Γ. Since $Z(\mathbf{t})$ has the form (10.2.1) with every $b_k \geq 0$, all the results

of Sect. 10.2 can be applied. In particular, if $\mathcal{V}^e = \{v \in \mathcal{V} : \delta_v = 1\}$ denotes the set of *ends* of the graph, then $\mathbf{A}$ has column vectors $a_v = E_v^*$ for $v \in \mathcal{V}^e$. Hence, the rank of the lattice/space where the polytopes $P^{(l')} = \cup_v P_v$ sit is $n = |\mathcal{V}^e|$, and the closed convex polytopes $\{P_v\}$ (closures of $\{P_v^\triangleleft\}_v$) are indexed by $\mathcal{V}$. Furthermore, the dilation parameter l' of the polytopes runs in a $|\mathcal{V}|$-dimensional space $L \otimes \mathbb{R}$. In the sequel we will drop the symbol $\mathbf{A}$ from $\mathscr{L}_h^{\mathfrak{c}}(\mathbf{A}, \mathscr{T}, l')$.

In this construction, a crucial additional ingredient comes from singularity theory, it is Theorem 9.6.21 (in fact, this is the main starting point and motivation of the whole approach). This combined with facts from Sect. 10.2 give:

Corollary 10.3.2 *Let $\mathscr{S} = \mathscr{S}_{\mathbb{R}}$ be the (real) Lipman's cone $\{x \in \mathbb{R}^{|\mathcal{V}|} : (x, E_v) \leq 0$ for every $v\}$. Furthermore, as usual, set $l' = l + r_h$ with $h = [l']$ and $l \in L$.*

(a) The rational function $Z_h(\mathbf{t})$ admits a quasipolynomial in the cone $\mathscr{S}$, namely

$$\mathfrak{Q}_h(l) = \mathfrak{sw}_{\sigma[K+2r_h]}(M) - \frac{(K + 2r_h + 2l)^2 + |\mathcal{V}|}{8}.$$

In particular, $Z_h(\mathbf{t})$ admits periodic constant in $\mathscr{S}$, which equals the normalized Seiberg–Witten invariant

$$\mathrm{pc}^{\mathscr{S}}(Z_h) = \mathfrak{Q}_h(0) = \mathfrak{sw}_{\sigma[K+2r_h]}(M) - \frac{(K + 2r_h)^2 + |\mathcal{V}|}{8}. \tag{10.3.1}$$

(b) Consider the chamber decomposition associated with the denominator of $Z(\mathbf{t})$ as in Theorem 10.2.8, and let $\mathfrak{c}$ be a chamber such that the interior $(\mathfrak{c} \cap \mathscr{S})^\circ \neq \emptyset$. Then for $l' \in \cap_k (b_k + \mathfrak{c})$ one has

$$Q_h(l') = \sum_k \iota_k \cdot \mathscr{L}^{\mathfrak{c}}_{h-[b_k]}(\mathscr{T}, l' - b_k).$$

Furthermore, $Z_h(\mathbf{t})$ admits a periodic constant in $\mathfrak{c}$, which equals both $\mathrm{pc}^{\mathscr{S}}(Z_h)$ and

$$\mathrm{pc}^{\mathfrak{c}}(Z_h) = \sum_k \iota_k \cdot \mathscr{L}^{\mathfrak{c}}_{h-[b_k]}(\mathscr{T}, -b_k) = \sum_k \iota_k \cdot \mathscr{L}^{\mathfrak{c}}_{[b_k]-h}(\mathscr{F} \setminus \mathscr{T}, b_k). \tag{10.3.2}$$

Thus, $\mathrm{pc}^{\mathfrak{c}}(Z_h)$ does not depend on the choice of $\mathfrak{c}$ (under the above assumption).

Remark 10.3.3 The Newton diagrams of hypersurface singularities have certain similarities with the polytopes $P^{(l')}$. Indeed, cf. 5.5, the Newton polytope $N\Gamma_+(f) \subset (\mathbb{R}_{\geq 0})^3$ is a convex polytope (similarly as the closure of $(\mathbb{R}_{\geq 0})^{|\mathcal{V}^e|} \setminus P^{(l')}$), hence $N\Gamma_-(f)$ can be compared with $P^{(l')}$. However, their role is rather different. If f has a non-degenerate principal part (and, for simplicity, let us assume

that $N\Gamma$ is convenient as well) then $N\Gamma_-(f)$ is related with the equation of f, with the equisingularity type of f. In contrast, $P^{(l')}$ is determined from the abstract topological type of the link of $V(f)$. The first one sits in $(\mathbb{R}_{\geq 0})^3$, the second in $(\mathbb{R}_{\geq 0})^{|\mathcal{V}^e|}$.

On the other hand, in the view of Theorem 5.5.7, which recovers $N\Gamma_-(f)$ (up to equivalence) from the link, it would be a nice achievement to recover $N\Gamma_-(f)$ directly from $P^{(l')}$ for a certain l' (or from some of the $P_v^{(l')}$'s), e.g. as in 10.3.5.

In general, for a normal surface singularity, $N\Gamma_+(f)$ is not even defined, and even for a hypersurface with *degenerate* principal part, the amount of information what it carries is rather restrictive. In parallel, $P^{(l')}$ is always defined and it caries strong topological information about the link.

Remark 10.3.4 Corollary 10.3.2 shows that the Lipman's cone $\mathcal{S}$ admits a periodic constant, hence, from the point of view of the periodic constant, it behaves uniformly (as a chamber). However the next example shows that $\mathcal{S}$ can be cut in several chambers of the denominator of Z. This fact indicates that in the case of $Z(\mathbf{t})$, though the chamber decomposition algorithm from 10.2.7—valid for general rational functions of type (10.2.1)—presumably is optimal from the point of view of the combinatorial types of $P^{(l')}$, or for the quasipolynomials $\mathscr{L}_h^{\mathfrak{c}}(\mathcal{T}, l')$, however, in general it is not optimal for the quasipolynomials of the counting function $Q_h(l')$. In other words, several neighboring chambers might have the same quasipolynomial. (In fact, later in 10.4 we will modify this chamber decomposition, and in the new setup the Lipman's cone becomes one of the new chambers.)

Example 10.3.5 (Lipman's Cone Disjointed by Chambers) Consider the hypersurface singularity $z_1^2 + z_2^3 + z_3^7 = 0$, with triangle graph $D_{2,3,7}$ (cf. 7.2.20), central exceptional curve E_0, and three other curves E_1, E_2, E_3. The polytopes defined in (10.2.4), with parameter $l = (l_0, l_1, l_2, l_3) \subset \mathbb{Z}^4$, sit in $\mathbb{R}^3$. Let u_1, u_2, u_3 be the basis of $\mathbb{R}^3$. Then the polytopes are the next convex closures:

$$P_0^{(l)} = conv\{0, \frac{l_0}{21}u_1, \frac{l_0}{14}u_2, \frac{l_0}{6}u_3\}, \quad P_1^{(l)} = conv\{0, \frac{l_0}{11}u_1, \frac{l_0}{7}u_2, \frac{l_0}{3}u_3\}$$

$$P_2^{(l)} = conv\{0, \frac{l_0}{7}u_1, \frac{l_0}{5}u_2, \frac{l_0}{2}u_3\}, \quad P_3^{(l)} = conv\{0, \frac{l_0}{3}u_1, \frac{l_0}{2}u_2, l_0u_3\}.$$

Note that $(-e_i)E_i^* = E_0^* + E_i$ for $1 \leq i \leq 3$, hence the chamber decomposition is dictated by 4-tuples of the collection $\{E_0^* + E_i\}_{i=1}^3, -E_0, \{-E_i\}_{i=1}^3$. Since $E_0^* + E_1, -E_0, -E_1, -E_2$ is a possible basis, for $1 > \epsilon > 0$ the vector $E_0^* + \varepsilon(-E_0)$ is on the boundary of the corresponding cone. Since this $E_0^* + \varepsilon(-E_0)$ is in the interior of the (real) Lipman's cone, we get that the Lipman's cone is cut in several chambers. (The interested reader might check the variation of the combinatorial types as well.)

By 10.3.2 $Q_0(l)$ admits a (quasi)polynomial in the cone $\mathcal{S}$, namely $\mathfrak{Q}_0(l)$ from 10.3.2*(a)*. But for (X, o) the SWIC is true, hence $\mathfrak{sw}_{\sigma[K]}(M) - (K^2 + |\mathcal{V}|)/8 = p_g = 1$. Hence $\mathfrak{Q}_0(l) = 1 + \chi(l)$ on $\mathcal{S}$. Furthermore, if $\mathfrak{c}$ is a chamber with

$(\mathfrak{c} \cap \mathscr{S})^\circ \neq \emptyset$, then by part *(b)* of 10.3.2 $\mathscr{L}^{\mathfrak{c}}(l) - \mathscr{L}^{\mathfrak{c}}(l - E_0^*) = 1 + \chi(l)$. Though the right hand side is independent of $\mathfrak{c}$, $\mathscr{L}^{\mathfrak{c}}$ might depend on $\mathfrak{c}$.

The periodic constant of $Q(l)$ can be computed in any $\mathfrak{c}$, it is the periodic constant of $1+\chi(l)$, hence it is 1. In fact, in general, by the continuity of the quasipolynomials associated with the chambers, any quasipolynomial associated with a chamber which contains any ray in the Lipman's cone $\mathscr{S}_{\mathbb{R}}$ (even if it is situated at its boundary $\partial\mathscr{S}_{\mathbb{R}}$), provides the periodic constant. One such degenerated polytope provided by a ray from $\partial\mathscr{S}_{\mathbb{R}}$ is of special interest. Namely, if we take $l = \lambda E_0^* \in \mathscr{S}$ for $\lambda > 0$, then $P^{(l)} = \bigcup_{v=0}^{3} P_v^{(l)}$ is the same as $P_0^{(l)} = conv(0, 2\lambda u_1, 3\lambda u_2, 7\lambda u_3)$. (For a generalization see 10.3.A.) Moreover, if $\mathfrak{c}$ is any chamber which contains the ray $\mathbb{R}_{\geq 0}E_0^*$ at its boundary, then for any $l = \lambda E_0^*$ one has $\mathscr{L}^{\mathfrak{c}}(\mathbf{A}, \mathscr{T}, l) = \mathscr{L}(\widetilde{P}_0, \mathscr{T}, \lambda)$, where the last is the classical Ehrhart polynomial of the tethrahedron $\widetilde{P}_0 := conv(0, 2u_1, 3u_2, 7u_3)$. Here we witness an additional coincidence of $\widetilde{P}_0$ with the Newton polytope $N\Gamma_-$ of the equation $z_1^2 + z_2^3 + z_3^7$.

Let us compute $\mathscr{L}(\widetilde{P}_0, \mathscr{T}, \lambda)$. From Corollary 10.2.9, we get that (†) $Q_0(l) = \mathscr{L}(\widetilde{P}_0, \mathscr{T}, \lambda) - \mathscr{L}(\widetilde{P}_0, \mathscr{T}, \lambda - 1)$. Again, as above, from SWIC $Q_0(l) = 1 + \chi(l)$. A χ-computation gives $Q_0(l) = 1 + \chi(\lambda E_0^*) = 21\lambda^2 - \lambda + 1$. Since the free term of $\mathscr{L}(\widetilde{P}_0, \mathscr{T}, \lambda)$ is zero (since for $\lambda = 0$ the zero polytope with boundary conditions contains no lattice point), from (†) we get $\mathscr{L}(\widetilde{P}_0, \mathscr{T}, \lambda) = 7\lambda^3 + 10\lambda^2 + 4\lambda$.

From the point of view of the Newton diagram $N\Gamma_- = \widetilde{P}_0$, the formula

$$\mathscr{L}(\widetilde{P}_0, \mathscr{T}, \lambda) - \mathscr{L}(\widetilde{P}_0, \mathscr{T}, \lambda - 1) = p_g + \chi(\lambda E_0^*),$$

realises a bridge between χ (supplemented with the geometric genus) and the Ehrhart polynomial of the Newton diagram.

Example 10.3.6 More generally, if Γ is as in 10.3.1 and $v_0 \in \mathscr{V}$ is adjacent to *all* other vertices, then by a similar argument as above $E_0^* + \varepsilon(-E_0)$ $(0 < \varepsilon \ll 1)$ belongs to a chamber wall and also to the interior of a real Lipman's cone. In particular, the interior of $\mathscr{S}_{\mathbb{R}}$ is cut by the chamber decomposition.

Example 10.3.7 (The Case of Lens Spaces) This example determines $P^{(l')}$ for the minimal resolution of cyclic quotient singularities (when the Newton polytope is not even defined). For notations regarding the graph see 2.2.A and 2.3. In this case $H = \mathbb{Z}_n$, it is generated by $[E_s^*]$, cf. 6.6.18. For different invariants and properties see 2.2.A, 2.3, 6.3.11, 6.6.18, 6.6.19, 7.1.21, 9.3.4, 9.7.12.

We compare these results with Ehrhart theory. In this case $Z(\mathbf{t}) = (1 - \mathbf{t}^{E_1^*})^{-1}(1 - \mathbf{t}^{E_s^*})^{-1}$. The vectors $a_1 = E_1^*$ and $a_s = E_s^*$ determine the polytopes $P^{(l')}$ and a chamber decomposition. Since $nE_1^* = \sum_{v=1}^{s} n_{v+1,s}E_v$ and $nE_s^* = \sum_{v=1}^{s} n_{1,v-1}E_v$, for any $l' = \sum_v l'_v E_v \in \mathscr{S}'$, the (non-convex) polytopes are

$$P^{(l')} = \bigcup_{v=1}^{s} \left\{ (x_1, x_s) \in \mathbb{R}^2_{\geq 0} \,:\, x_1 n_{v+1,s} + x_s n_{1,v-1} \leq n l'_v \right\} \subset \mathbb{R}^2_{\geq 0}. \tag{10.3.3}$$

The representation $\mathbb{Z}^2 \xrightarrow{\rho} \mathbb{Z}_n = H$ is $(x_1, x_s) \mapsto (qx_1 + x_s)[E_s^*]$.

Though $P^{(l')}$ is a plane polytope, the direct computation of its equivariant Ehrhart multivariable polynomial is still non-trivial. Here we will rely again on Theorem 9.6.21. On a subset of type $l'_* + \mathscr{S}'$ the identity (9.6.13) provides the counting function. The right hand side of (9.6.13) depends on all the coordinates of l', hence all the triangles P_v contribute to $P^{(l')}$. Since this can happen only in a unique combinatorial way, we get that there is a chamber $\mathfrak{c}$ which contains the Lipman's cone. Let $\mathscr{L}^{e,\mathfrak{c}}$ be its quasipolynomial, and $\mathscr{L}^{e,\mathscr{S}}$ its restriction on $\mathscr{S}$. Since the numerator of $Z(\mathbf{t})$ is 1, $\overline{Q}_h^{\mathfrak{c}} = \mathscr{L}_h^{\mathfrak{c}}$. Since this agrees with the right hand side of (9.6.13) on a cone of type $l'_* + \mathscr{S}'$, and the Lipman's cone is in $\mathfrak{c}$, we get that

$$Q_h(l') = \overline{Q}_h^{\mathfrak{c}}(l') = \mathscr{L}_h^{\mathscr{S}}(\mathscr{T}, l') = \mathfrak{sw}_{\sigma[K+2l']}(M) - \frac{(K+2l')^2 + |\mathscr{V}|}{8} \tag{10.3.4}$$

for any $l' \in (r_h + L) \cap \mathscr{S}'$ and $h \in H$. Since the SWIC is true for cyclic quotients, from (9.7.1) one has $\mathfrak{sw}_{\sigma[K+2l']}(M) = ((K+2r_h)^2 + |\mathscr{V}|)/8$. Hence

$$\mathscr{L}_h^{\mathscr{S}}(\mathscr{T}, l') = \chi(l') - \chi(r_h) \quad \text{for any } \ l' \in (r_h + L) \cap \mathscr{S}'. \tag{10.3.5}$$

For an arbitrary $l' \in L'$ and $h \in H$ we proceed as follows. Note that for any fixed h and any l' there exists a unique $q_{l',h}$ in the semi-open unit cube such that $l' + q_{l',h} := l'' \in r_h + L$. Then (10.2.13) and (10.3.5) imply the general expresion

$$\mathscr{L}_h^{\mathscr{S}}(\mathscr{T}, l') = \mathscr{L}_h^{\mathscr{S}}(\mathscr{T}, l'') = \chi(l' + q_{l',h}) - \chi(r_h). \tag{10.3.6}$$

This formula emphasizes the periodic behavior of $l' \mapsto \mathscr{L}_h^{\mathscr{S}}(\mathscr{T}, l')$ as well.

If l' is an element of L then $q_{l',h} = r_h$, hence (10.3.6) gives in this case

$$\mathscr{L}_h^{\mathscr{S}}(\mathscr{T}, l) = \chi(l + r_h) - \chi(r_h) = \chi(l) - (l, r_h) \quad \text{for } l \in L \cap \mathscr{S}. \tag{10.3.7}$$

Hence, $\mathrm{pc}(\mathscr{L}_h^{\mathscr{S}}(\mathscr{T})) = \chi(r_h) - \chi(r_h) = 0$ (a fact compatible with $p_g((X_a, o)) = 0$).

The non-equivariant case looks also interesting. Let $\mathscr{L}^{ne,\mathscr{S}}(\mathscr{T}) = \sum_{h \in H} \mathscr{L}_h^{\mathscr{S}}(\mathscr{T})$. Then, (10.3.7) for $l \in L \cap \mathscr{S}$ gives

$$\mathscr{L}^{ne,\mathscr{S}}(\mathscr{T}, l) = n \cdot \chi(l) - (l, \textstyle\sum_h r_h). \tag{10.3.8}$$

$\sum_h r_h$ can explicitly be determined. Set $d_v = \gcd\{n, n_{1,v-1}\}$. Then from 6.6.19 $aE_s^* = \sum_v n_{1,v-1} \frac{a}{n} E_v$, $r_{[aE_s^*]} = \sum_v \left\{ n_{1,v-1} \frac{a}{n} \right\} E_v$ and $\sum_h r_h = \sum_v \frac{n-d_v}{2} E_v$.

The coefficients of the polynomial $\mathscr{L}^{ne,\mathscr{S}}(\mathscr{T}, l)$ can be compared with the coefficients given by general theory of Ehrhart polynomials applied for $P^{(l)}$. E.g., the leading coefficient shows that the area of $P^{(l)}$ is $-n \cdot (l, l)/2$.

Knowing that all the P_v's contribute to $P^{(l)}$, and it depends on s parameters, and the intersections of the boundaries of different P_v's are messy, the simplicity and conceptual form of (10.3.8) is rather remarkable.

10.3.A Reduction of the Variables of $Z(\mathbf{t})$

We show that the periodic constant (and certain restrictions of the counting function) of Z_h can be recovered from the series obtained from $Z_h(\mathbf{t})$ by the substitution $t_v = 1$ for every non-node vertex v. We follow [349]. ♣

10.3.8 Let $\mathcal{N}$ denote the set of nodes $\{v \in \mathcal{V} : \kappa_v \geq 3\}$, and we assume that $\mathcal{N} \neq \emptyset$. (The analogue statements, valid for $\mathcal{N} = \emptyset$, are presented in 10.3.7.) Let $\mathcal{S}_{\mathbb{R},\mathcal{N}}$ be the positive real cone $\mathbb{R}_{\geq 0}\langle E_n^*\rangle_{n\in\mathcal{N}}$ generated by the dual base elements indexed by $\mathcal{N}$, and let $V_{\mathcal{N}}^* := \mathbb{R}\langle E_n^*\rangle_{n\in\mathcal{N}}$ be its supporting linear subspace in $L \otimes \mathbb{R}$. Clearly $\mathcal{S}_{\mathbb{R},\mathcal{N}} \subset \mathcal{S}_{\mathbb{R}}$. Furthermore, consider $L_{\mathcal{N}} := \mathbb{Z}\langle E_n\rangle_{n\in\mathcal{N}}$ generated by the node base elements, and the projection $\pi_{\mathcal{N}} : L \otimes \mathbb{R} \to L_{\mathcal{N}} \otimes \mathbb{R}$ on the node coordinates.

10.3.9 We wish to prove that if $l' \in V_{\mathcal{N}}^*$, then the polytopes $\{P_v^{(l')}\}_{v\notin\mathcal{N}}$ become non-visible in $P^{(l')} = \cup_{v\in\mathcal{V}} P_v^{(l')}$, hence they can be eliminated. In particular, for such an l', the counting function $Q(l')$ depends only on $pr_{\mathcal{N}}(l')$. This fact will provide a remarkable simplification in the periodic constant computation as well.

Theorem 10.3.10

(a) The restriction of $\mathscr{L}_h(\mathbf{A}, \mathcal{T}, l')$ to $V_{\mathcal{N}}^*$ depends only on those coordinates of l', which are indexed by the nodes.

(b) The same is true for the counting function Q_h associated with $Z_h(\mathbf{t})$ as well. In other words, if we consider the restriction

$$Z_h(\mathbf{t}_{\mathcal{N}}) := Z_h(\mathbf{t})|_{t_v=1 \text{ for all } v\notin\mathcal{N}}$$

then for any $l' \in L_{\mathcal{N}}^*$, the counting functions $\sum_{l''\not\geq l'} \mathfrak{z}^{l''}[l'']$ of $Z_h(\mathbf{t})$ and $Z_h(\mathbf{t}_{\mathcal{N}})$ equal.

(c) Consider the chamber decomposition of $\mathcal{S}_{\mathbb{R},\mathcal{N}}$ by intersections of type $\mathfrak{c}_{\mathcal{N}} := \mathfrak{c} \cap \mathcal{S}_{\mathbb{R},\mathcal{N}}$, where $\mathfrak{c}$ denotes a chamber (of Z) such that $int(\mathfrak{c} \cap \mathcal{S}) \neq \emptyset$, and the intersection of $\mathfrak{c}$ with the relative interior of $\mathcal{S}_{\mathbb{R},\mathcal{N}}$ is also non-empty. Then

$$\mathrm{pc}^{\mathfrak{c}_{\mathcal{N}}}(Z_h(\mathbf{t}_{\mathcal{N}})) = \mathrm{pc}^{\mathfrak{c}}(Z_h(\mathbf{t})) = \mathfrak{Q}_h(0). \tag{10.3.9}$$

The theorem applies as follows. Assume that we are interested in the computation of $\mathrm{pc}^{\mathfrak{c}}(Z_h(\mathbf{t}))$ for some chamber $\mathfrak{c}$, which intersects the relative interior of both $\mathscr{S}_{\mathbb{R}}$ and $\mathscr{S}_{\mathbb{R},\mathscr{N}}$. Then, the restriction to $\mathfrak{c} \cap \mathscr{S}_{\mathbb{R},\mathscr{N}}$ of the quasipolynomial associated with $\mathfrak{c}$ has two properties: it still preserves sufficient information to determine $\mathrm{pc}^{\mathfrak{c}}(Z_h(\mathbf{t}))$ (via the periodic constant of the restriction, see (10.3.9)), but it also has the advantage that for these dilatation parameters l' the union $\cup_{v\in\mathscr{V}} P_v^{(l'),\triangleleft}$ equals the union of essentially much less polytopes, namely, it is $\cup_{n\in\mathscr{N}} P_v^{(l'),\triangleleft}$. For example, when we have only one node, one has to handle only one convex simplicial polytope instead of a union of $|\mathscr{V}|$ ones.

Proof *(a)* We show that for any $l' \in V^*_{\mathscr{N}}$ one has the inclusions

$$P_v^{(l'),\triangleleft} \subset \bigcup_{n\in\mathscr{N}} P_n^{(l'),\triangleleft} \text{ for any } v \notin \mathscr{N}. \tag{10.3.10}$$

We consider two cases. First we assume that v is on a leg (chain) connecting an end $e(v) \in \mathscr{V}^e$ with a node $n(v)$ (where $e(v) = v$ is also possible). Then, clearly, (10.3.10) follows from

$$P_v^{(l'),\triangleleft} \subset P_{n(v)}^{(l'),\triangleleft} \quad \text{for any } l' \in V^*_{\mathscr{N}}. \tag{10.3.11}$$

Let $E^*_{uv} = (E^*_u)_v = -(E^*_u, E^*_v)$ be the v-coordinate of E^*_u. Note that $E^*_{uv} = E^*_{vu}$. Using the definition of the polytopes, (10.3.11) is equivalent to the implication (cf. 10.2.5)

$$\Big(\sum_{e\in\mathscr{V}^e} x_e E^*_{ve} < l'_v\Big) \Longrightarrow \Big(\sum_{e\in\mathscr{V}^e} x_e E^*_{n(v)e} < l'_{n(v)}\Big) \quad \text{for any } l' \in V^*_{\mathscr{N}} \text{ and } x_e \geq 0. \tag{10.3.12}$$

Let $\mathscr{W}$ be the set of vertices on this leg (including $e(v)$ but not $n(v)$). Then, one verifies that there exist positive rational numbers α and $\{\alpha_w\}_{w\in\mathscr{W}}$, such that

$$E^*_v = \alpha\, E^*_{n(v)} + \sum_{w\in\mathscr{W}} \alpha_w E_w. \tag{10.3.13}$$

(See 10.3.16 and 10.3.17 for a similar situation with detailed arguments.) The numbers α and $\{\alpha_w\}_{w\in\mathscr{W}}$ can be determined from the linear system obtained by intersecting (10.3.13) with $\{E_w\}_w$ and $E_{n(v)}$. Intersecting (10.3.13) with E^*_e ($e \in \mathscr{V}^e$), we get that $E^*_{ve} = \alpha E^*_{n(v)e}$ for any $e \neq e(v)$, and $E^*_{v,e(v)} = \alpha E^*_{n(v),e(v)} + \alpha_{e(v)}$. Hence

$$\sum_{e\in\mathscr{V}^e} x_e E^*_{ve} = \alpha \sum_{e\in\mathscr{V}^e} x_e E^*_{n(v)e} + x_{e(v)}\alpha_{e(v)}. \tag{10.3.14}$$

On the other hand, intersecting (10.3.13) with E_n^*, for $n \in \mathscr{N}$, we get $E_{vn}^* = \alpha E_{n(v)n}^*$. Since l' is a linear combination of $\{E_n^*\}_{n\in\mathscr{N}}$'s, we get $(l', \sum_{w\in\mathscr{W}} \alpha_w E_w) = 0$ and

$$-l'_v = (l', E_v^*) = \alpha(l', E_{n(v)}^*) = -\alpha l'_{n(v)}. \tag{10.3.15}$$

Since $x_{e(v)}\alpha_{e(v)} \geq 0$, (10.3.14) and (10.3.15) imply (10.3.12). This ends this case.

Next, we assume that v is on a chain connecting two nodes $n(v)$ and $m(v)$. Let $\mathscr{W}$ be the set of vertices on this string (not including $n(v)$ and $m(v)$). Then

$$P_v^{(l'),\lhd} \subset P_{n(v)}^{(l'),\lhd} \cup P_{m(v)}^{(l'),\lhd} \text{ for any } l' \in V_{\mathscr{N}}^*. \tag{10.3.16}$$

This follows as above from the existence of positive rational numbers α, β and $\{\alpha_w\}_{w\in\mathscr{W}}$ with $E_v^* = \alpha\, E_{n(v)}^* + \beta\, E_{m(v)}^* + \sum_{w\in\mathscr{W}} \alpha_w E_w$.

(b) follows from *(a)* and from the fact that $b_k \in \mathscr{S}_{\mathbb{R},\mathscr{N}}$ for every k.

(c) If $\mathrm{pc}^{\mathfrak{c}}(Z_h(\mathbf{t}))$ is computed as $\mathfrak{Q}_h(0)$ for some quasipolynomial $\mathfrak{Q}_h$ defined on $\widetilde{L} \subset L$, then part *(b)* shows that $\mathrm{pc}^{\mathfrak{c}_{\mathscr{N}}}(Z_h(\mathbf{t}_{\mathscr{N}}))$ can be computed as $(\mathfrak{Q}_h|_{\widetilde{L}\cap\mathscr{S}_{\mathbb{R},\mathscr{N}}})(0)$, which equals $\mathfrak{Q}_h(0)$. □

Remark 10.3.11 Even after this reduction to $V_{\mathscr{N}}^*$, if $|\mathscr{N}| \geq 3$ then in general $\mathscr{S}_{\mathbb{R},\mathscr{N}}$ still can be divided into several chambers. For an example see [349, Ex. 5.4.12].

Example 10.3.12 (The One-Node Case, the Star-Shaped Plumbing Graphs)

We fix $h \in H$ and a representative l' with $\mathfrak{a} := -(E_0^*, l') \in [0, 1)$. For notations and several formulae see 8.1.B. As it is explained in 8.1.B, $\mathfrak{a}$ is uniquely defined under this assumption, and it is realized e.g. for $l' = r_h$. Recall that $Z_h(t)$ has the form $t^{\mathfrak{a}} \cdot \sum_{\ell\in\mathbb{Z}_{\geq 0}} \mathfrak{z}_\ell t^\ell$; here the exponents $\{\mathfrak{a}+\ell\}_{\ell\geq 0}$ are the possible E_0-coordinates of the elements $(r_h + L) \cap \mathscr{S}'$. Next we compute the counting function for $Z_h(t) = Z_h(\mathbf{t})|_{t_v=1,\, v\notin\mathscr{N}}$. We will rely on the computation of $P_h(t)$ from 8.1.B, since for weighted homogeneous singularities $Z(\mathbf{t}) = P(\mathbf{t})$. In particular, corresponding to the sum-decomposition $P_h = P_h^+ + P_h^-$ we have $Z_h = Z_h^+ + Z_h^-$ as well.

If $S(t) = \sum_r z_r t^r$ is a series, we also define $Q(S)(r') = \sum_{r<r'} z_r$ for $r' \in \mathbb{Q}_{\geq 0}$.

The next lemma follows from 8.1.9 (recall that $\mathfrak{r} := \chi/e$, cf. 5.1.29).

Lemma 10.3.13 *For any $n \in \mathbb{N}_{\geq 0}$ one has the following facts.*

(a) $Q(Z_h)(n) = Q(Z_h)(n + \mathfrak{a})$.

(b) $Q(Z_h^+)(n)$ *is a step function (hence piecewise quasipolynomial) with*

$$Q(Z_h^+)(n) = \mathrm{pc}(Z_h) \quad \textit{for } n > \deg(Z_h^+).$$

(c) $Q(Z_h^-)(n)$ *is a quasipolynomial:*

$$Q(Z_h^-)(n) = (1 - e\mathfrak{a})n - e \cdot \frac{n(n-1)}{2} - \sum_{i=1}^{\nu} \sum_{k_i=0}^{\alpha_i - 1} \left\{ \frac{a_i - \omega_i k_i}{\alpha_i} \right\} \left\lceil \frac{n - k_i}{\alpha_i} \right\rceil \tag{10.3.17}$$

$$= -\frac{en^2}{2} + \frac{en}{2}(\mathfrak{r} + 1 - 2\mathfrak{a}) - \sum_{i=1}^{\nu} \sum_{k_i=0}^{\alpha_i - 1} \left\{ \frac{a_i - \omega_i k_i}{\alpha_i} \right\} \left(\left\{ \frac{k_i - n}{\alpha_i} \right\} - \frac{k_i}{\alpha_i} \right).$$

In particular, if $n = m\alpha$ for $m \in \mathbb{Z}$, and $n > \deg(Z_h^+)$, then the double sum is zero, hence

$$Q(Z_h)(n) = -\frac{en^2}{2} + \frac{en}{2}(\mathfrak{r} + 1 - 2\mathfrak{a}) + \mathrm{pc}(Z_h). \tag{10.3.18}$$

This is compatible with the expression provided by Theorems 9.6.21 and 10.3.10. Indeed, let us fix any chamber $\mathfrak{c}$ such that $int(\mathfrak{c} \cap \mathscr{S}') \neq \emptyset$, and $\mathfrak{c}$ contains the ray $\mathscr{R} = \mathbb{R}_{\geq 0} \cdot E_0^*$. Since the numerator of $f(\mathbf{t})$ is $(1 - \mathbf{t}^{E_0^*})^{\nu - 2}$, all the b_k–vectors belong to $\mathscr{R}$. In particular, $\cap_k (b_k + \mathfrak{c})$ intersects $\mathscr{R}$ along a semi-line $\mathscr{R}^{\geq c} = \mathbb{R}_{\geq const} \cdot E_0^*$ of $\mathscr{R}$. Since $Q_h(l')$ is quasipolynomial on $\cap_k (b_k + \mathfrak{c})$, cf. (10.2.12), and a certain restriction of it is determined by (9.6.13) whose right hand side is a quasipolynomial too, we obtain that the identity (9.6.13) is valid on $\mathscr{R}^{\geq c}$ as well. This reads as follows.

Recall that for any $h \in H$ and $l' \in L'$ we have a unique $q_{l',h}$ in the semi-open unit cube such that $l' + q_{l',h} \in r_h + L$. Hence we get

$$Q_h(l') = \mathfrak{sw}_{\sigma[K + 2l']}(M) - \frac{(K + 2l' + 2q_{l',h})^2 + |\mathscr{V}|}{8} \quad (l' \in \mathscr{R}^{\geq c}). \tag{10.3.19}$$

The term $q_{l',h}$ is responsible for the non-polynomial behavior. Nevertheless, if we assume that $l' = m \circ E_0^* \in \mathscr{R}^{\geq c} \cap L$, $m \in \mathbb{Z}$, then $q_{l',h} = r_h$, hence by (9.6.14)

$$Q_h(l') = -\frac{(l', l' + K + 2r_h)}{2} + \mathrm{pc}(Z_h). \tag{10.3.20}$$

By the Reduction theorem 10.3.10 $Q_h(l')$ from (10.3.20) depends only on the E_0-coefficient of $l' = m \circ E_0^*$, which is exactly $m\alpha$. One can see that in fact (10.3.20) agrees with (10.3.18) if we set $n = m\alpha$.

The non-equivariant version can be obtained by summation of (10.3.18). For this we need $\sum_h \mathfrak{a}(r_h)$. Note that $\mathfrak{a}(r_h)$ vanishes exactly $d/\mathfrak{o}$ times and in all other cases $\mathfrak{a}(r_h) \neq 0$, and $\mathfrak{a}(r_h) + \mathfrak{a}(r_{-h}) = 1$. In particular, $2\sum_h \mathfrak{a}(r_h) = d - d/\mathfrak{o}$. (Here

$d = |H|$.) Therefore, the summation of (10.3.18) provides

$$Q^{ne}(Z^{ne})(n) = -\frac{den^2}{2} + \frac{den}{2}(\mathfrak{r} + \frac{1}{\mathfrak{o}}) + \mathrm{pc}(Z^{ne}) \quad (\text{for } n \in \alpha\mathbb{Z}). \tag{10.3.21}$$

Next, we will identify the coefficients of (10.3.18) and (10.3.21) with the first three coefficients of the Ehrhart quasipolynomial $\mathscr{L}_h^{\mathfrak{c}}(\mathscr{T})$ via the identity (10.2.12).

For simplicity we will assume that $\mathfrak{o} = 1$, in particular all the b_k–vectors belong to L. If $l' \in \mathscr{R}$, then by 'Reduction Theorem' 10.3.10 the counting function $\mathscr{L}_h^{\mathfrak{c}}(\mathscr{T}, l')$ of the polytope $P^{(l')}$ depends only on the E_0-coefficient of l'; let us denote this coefficient by l'_0. Hence, this $\mathscr{L}_h^{\mathfrak{c}}(\mathscr{T}, l'_0)$ is the Ehrhart quasipolynomial of the ν-dimensional simplicial polytope, being its h–class counting function. Via (3.5.2) the definition (10.2.4) of this polytope becomes

$$P_0 = \{(x_1, \dots, x_\nu) \in (\mathbb{R}_{\geq 0})^\nu \ : \ \sum_i \frac{x_i}{|e|\alpha_i} < l'_0\}. \tag{10.3.22}$$

The representation is $\mathbf{x} \mapsto [\sum_{v \in \mathscr{V}^e} x_v E_v^*]$. Let

$$\mathscr{L}_h^{\mathfrak{c}}(\mathscr{T}, l'_0) = \sum_{j=0}^{\nu} \mathfrak{e}_{h,j}(l'_0) \cdot \frac{(l'_0)^j}{j!} \tag{10.3.23}$$

be the coefficients of the equivariant Ehrhart quasipolynomial: each $\mathfrak{e}_{h,j}(l'_0)$ is a periodic function in l'_0 and it is normalized by $1/j!$ by the convenience of the next expressions. Since the numerator of f is $(1 - t^{1/|e|})^{\nu-2}$, by (10.2.12) for $l' \in \mathscr{R}$

$$Q_h(l') = \sum_{j=0}^{\nu} \mathfrak{e}_{h,j}(l'_0) \cdot \frac{1}{j!} \sum_{k=0}^{\nu-2} (-1)^k \binom{\nu-2}{k} \Big(l'_0 - \frac{k}{|e|}\Big)^j. \tag{10.3.24}$$

This should equal the expression (10.3.19) above. The non-polynomial behavior of this expression indicates that $\mathfrak{e}_j(l'_0)$ is indeed nonconstant periodic.

Since we are interested primarily in the Seiberg–Witten invariant, namely in $\mathrm{pc}(Z_h)$, we perform this explicit identification only via the expressions (10.3.18) and (10.3.20). Hence, similarly as in these cases, we take $l' = m\mathfrak{o}E_0^* \in \mathscr{R}^{\geq c} \cap L$, and we identify (10.3.18) with (10.3.24) evaluated for l', whose E_0-coefficient is $l'_0 = m\alpha = n$. In this case $\mathfrak{e}_{h,j}(n)$ is a *constant*, denoted by $\mathfrak{e}_{h,j}$, and

$$-\frac{en^2}{2} + \frac{en}{2}(\mathfrak{r} + 1 - 2\mathfrak{a}) + \mathrm{pc}(Z_h) = \sum_{j=0}^{\nu} \mathfrak{e}_{h,j} \cdot \frac{1}{j!} \sum_{k=0}^{\nu-2} (-1)^k \binom{\nu-2}{k} \Big(n - \frac{k}{|e|}\Big)^j.$$

This implies

$$\begin{array}{l} \frac{\mathfrak{e}_{h,v}}{|e|^v} = \frac{1}{|e|} \\ \frac{\mathfrak{e}_{h,v-1}}{|e|^{v-1}} = \frac{v-2}{2|e|} - \frac{1}{2}(\mathfrak{r}+1-2\mathfrak{a}) \\ \frac{\mathfrak{e}_{h,v-2}}{|e|^{v-2}} = \mathrm{pc}(Z_h) + \frac{(v-2)(3v-7)}{24|e|} - \frac{v-2}{4}(\mathfrak{r}+1-2\mathfrak{a}). \end{array} \tag{10.3.25}$$

In particular, the $\mathfrak{e}_{h,v-2}$ can be identified (up to 'easy' extra terms) with $\mathrm{pc}(Z_h)$ (hence with the equivariant Seiberg–Witten invariant of the link). [The interpretation of the first two coefficients is the equivariant analogue of 10.1.6, though, in the second case, one has to remove the corresponding facets.] In this identification of the coefficients the next combinatorial identity is helpful (see [576, p. 7–8])

$$\frac{(-1)^v}{(v-2)!} \cdot \sum_{k=0}^{v-2} (-1)^k \binom{v-2}{k} k^j = \begin{cases} 0 & \text{if } j < v-2, \\ 1 & \text{if } j = v-2, \\ (v-2)(v-1)/2 & \text{if } j = v-1, \\ (v-2)(v-1)v(3v-5)/24 & \text{if } j = v. \end{cases}$$

In the non-equivariant case, if $\sum_{j=0}^{v} \mathfrak{e}_j \frac{n^j}{j!}$ is the Ehrhart polynomial of P_0, then

$$\begin{array}{l} \frac{\mathfrak{e}_v}{|e|^v} = \prod_i \alpha_i \\ \frac{\mathfrak{e}_{v-1}}{|e|^{v-1}} = \prod_i \alpha_i \cdot \left(-\frac{1}{\alpha} + \sum_i \frac{1}{\alpha_i} \right)/2 \\ \frac{\mathfrak{e}_{v-2}}{|e|^{v-2}} = \prod_i \alpha_i \left(\frac{\mathrm{pc}(Z^{ne})}{\prod_i \alpha_i} - \frac{(v-2)(3v-5)}{24} + \frac{v-2}{4}(-\frac{1}{\alpha} + \sum_i \frac{1}{\alpha_i}) \right). \end{array} \tag{10.3.26}$$

Remark 10.3.14 The article [349] contains the comparison of the coefficients of the equivariant Ehrhart quasipolynomial and the Seiberg–Witten invariants for any (rational homology sphere) plumbing graph (and a detailed analysis of the two-node case).

10.3.B Technical Lemmas Regarding the Intersection Form

One of the key properties of the periodic constant of the counting function Q_h is that it equals the correction term of a certain additivity formula. The most general additivity formula has the following form: Let us fix a subset $\mathscr{I} \subset \mathscr{V}$, $\mathscr{I} \neq \emptyset$, and denote by $\{\Gamma_k\}_k$ the collection of full connected subgraphs of Γ with vertices $\mathscr{V} \setminus \mathscr{I}$. Then one can compare different invariants of Γ and $\cup_k \Gamma_k$. In order to formulate and prove such surgery formulae we will need several combinatorial properties of the intersection form with respect to such decomposition. In this section we list several of them. ♣

10.3.15 We assume that Γ is as in 10.3.1, and we write $\mathscr{V}$ as a disjoint union $\mathscr{V}_1 \sqcup \mathscr{V}_2$. Recall that I denotes the intersection matrix $(E_u, E_v)_{u,v}$. For the E_u-coefficient of $l' \in L'$ the notation $l'|_u$ is also used. Write also $L_{\mathscr{V}_i} = \mathbb{Z}\langle E_v\rangle_{v\in\mathscr{V}_i}$, $i = 1, 2$.

In [354], based on the diagonalization procedure of [178, §21], the next lemma was proved for the case when the full subgraph with vertices $\mathscr{V}_2$ is connected.

Lemma 10.3.16 *According to the decomposition $\mathscr{V} = \mathscr{V}_1 \sqcup \mathscr{V}_2$ write the matrix $-I$ in the blocks $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ and its inverse $(-I)^{-1}$ as $\left(\begin{smallmatrix} A' & B' \\ C' & D' \end{smallmatrix}\right)$. Then one has:*

(a) D' is positive definite with positive definite inverse S, where, in fact, $S = D - CA^{-1}B$.
(b) For all entries $u, v \in \mathscr{V}_2$ one has $(-S)_{uv} \geq (-D)_{uv}$.
(c) Let $(\ ,\ )_{mod}$ be the symmetric negative definite integral bilinear form on $L_{\mathscr{V}_2}$ associated with $-S$. Then $(E_u^*|_{\mathscr{V}_2}, E_v|_{\mathscr{V}_2})_{mod} = -\delta_{uv}$ for any $u, v \in \mathscr{V}_2$.

In particular, if we embed the dual lattice $L'_{\mathscr{V}_2} = \mathrm{Hom}_{\mathbb{Z}}(L_{\mathscr{V}_2}, \mathbb{Z})$ into $L_{\mathscr{V}_2} \otimes \mathbb{Q}$ given by $(\ ,\ )_{mod}$ then the image is $\mathbb{Z}\langle E_u^*|_{\mathscr{V}_2}\rangle_{u\in\mathscr{V}_2}$.

Proof

(a) We use Jacobi's equality between complementary minors. Since $-I$ is definite and A is invertible (by Sylvester criterion), (†) $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) = \left(\begin{smallmatrix} Id & 0 \\ CA^{-1} & Id \end{smallmatrix}\right) \left(\begin{smallmatrix} A & 0 \\ 0 & S \end{smallmatrix}\right) \left(\begin{smallmatrix} Id & A^{-1}B \\ 0 & Id \end{smallmatrix}\right)$ follows, where $S = D - CA^{-1}B$ is the 'Schur complement' of A. Inverting (†) term–by–term gives $D' = S^{-1}$, hence $\det(-I) = \det(A)/\det(D')$ and $\det(D') > 0$. In order to get the positive definiteness of D' (and to apply Sylvester criterion for it) we can repeat the procedure for any block corresponding to the upper left minor of D'.
(b) All entries of B and C are non-positive, while all entries of A^{-1} are positive (cf. 2.1.19). Part *(c)* is the reformulation of $D' = S^{-1}$.

□

Lemma 10.3.17 *Let $\mathscr{V} = \mathscr{V}_1 \sqcup \mathscr{V}_2$ as above. Let $\mathscr{B}$ be the set of vertices from $\mathscr{V}_2$, which are adjacent to a vertex from $\mathscr{V}_1$. If $v_1 \in \mathscr{V}_1$ let $\mathscr{V}_{1,v_1}$ be the set of vertices of the connected component of $\Gamma(\mathscr{V}_1)$ (the full graph with vertices $\mathscr{V}_1$) which contains v_1, and let $\mathscr{B}_{v_1}$ be the set of vertices from $\mathscr{V}_2$, which are adjacent to $\mathscr{V}_{1,v_1}$.*

(a) Then for any $v_1 \in \mathscr{V}_1$ there exists a unique collection $\alpha_{v_2} \in \mathbb{Q}_{>0}$ indexed by $v_2 \in \mathscr{B}_{v_1}$ such that the $E_{v_2'}$-coordinate of

$$V^* := E_{v_1}^* - \sum_{v_2\in\mathscr{B}_{v_1}} \alpha_{v_2} E_{v_2}^*$$

for any $v_2' \in \mathscr{B}_{v_1}$ is zero.

(b) For any $v_1 \in \mathscr{V}_1$ with the coefficients $\{\alpha_{v_2} > 0\}_{v_2 \in \mathscr{B}_{v_1}}$ from (a) one has

$$E^*_{v_1}|_{\mathscr{V}_2} = \sum_{v_2 \in \mathscr{B}_{v_1}} \alpha_{v_2} E^*_{v_2}|_{\mathscr{V}_2}.$$

(c) The projection of the Lipman's cone $\pi_{\mathscr{V}_2}(\mathscr{S}_{\mathbb{R}})$ is top-dimensional in $L_{\mathscr{V}_2} \otimes \mathbb{R}$ and it equals $\mathbb{R}_{\geq 0}\langle E^*_{v_2}|_{\mathscr{V}_2}\rangle_{v_2 \in \mathscr{V}_2}$. Moreover, $\pi_{\mathscr{V}_2}(\mathscr{S}^{\circ}_{\mathbb{R}}) = (\pi_{\mathscr{V}_2}(\mathscr{S}_{\mathbb{R}}))^{\circ}$ and it equals $\mathbb{R}_{>0}\langle E^*_{v_2}|_{\mathscr{V}_2}\rangle_{v_2 \in \mathscr{V}_2}$.

(d) $\pi_{\mathscr{V}_2}(\mathscr{S}_{\mathbb{R}}) = \{l' \in L_{\mathscr{V}_2} \otimes \mathbb{R} : (l', E_v|_{\mathscr{V}_2})_{mod} \leq 0$ for any $v \in \mathscr{V}_2\}$.

Proof Consider the expression V^* with some coefficients $\{\alpha_{v_2}\}_{v_2}$ and the linear system $V^*|_{v'_2} = 0$ for every $v'_2 \in \mathscr{B}_{v_1}$. The matrix of the system is $(E^*_{v_2}, E^*_{v'_2})_{v_2, v'_2 \in \mathscr{B}_{v_1}}$, which is non-degenerate by 10.3.16. Hence the system has a unique rational solution. Next notice that for any $u \notin \mathscr{V}_{1,v_1}$ one has $V^*|_u = 0$. Indeed, write V^* as $V^*_1 + V^*_2$, where V^*_1 is supported in $\mathscr{V}_{1,v_1}$ and V^*_2 in $\mathscr{V} \setminus (\mathscr{V}_{1,v_1} \cup \mathscr{B}_{v_1})$. Then $(V^*_2, E_u) = (V^*, E_u) = 0$ for any $u \notin \mathscr{V}_{1,v_1} \cup \mathscr{B}_{v_1}$, hence $V^*_2 = 0$. This shows that V^* equals the cycle $E^*_{v_1}(\Gamma(\mathscr{V}_{1,v_1}))$ (computed in the graph $\Gamma(\mathscr{V}_{1,v_1})$). Furthermore, if for any $v_2 \in \mathscr{B}_{v_1}$ we set $v(v_2) \in \mathscr{V}_{1,v_1}$ adjacent to v_2, then $\alpha_{v_2} = (V^*, E_{v_2}) = (E^*_{v_1}(\Gamma(\mathscr{V}_{1,v_1})), E_{v_2})$, which equals the $v(v_2)$-coefficient of $E^*_{v_1}(\Gamma(\mathscr{V}_{1,v_1}))$. Since this coefficient is positive, $\alpha_{v_2} > 0$ follows. □

10.3.18 Multiplicity Systems Revisited Fix the integral cycle $l = \sum_v m_v E_v \in L$, which in the dual base is $l = \sum_v c_v E^*_v$. For each $v \in \mathscr{V}$ in $\widetilde{X}$ we consider 2-discs (cuts) $\{C_{v,i}\}_{i=1}^{k_v}$, each of them intersecting E_v transversally in generic points and with $\partial C_{v,i} \subset \partial \widetilde{X}$. Then, whenever $\sum_i c_{v,i} = c_v$ for every v, $C(l) := \sum_v (m_v E_v + \sum_i c_{v,i} C_{v,i})$ is a relative cycle in $(\widetilde{X}, \partial \widetilde{X})$ with $(C(l), E_v) = 0$ for every v (hence its class in $H_2(\widetilde{X}, \partial \widetilde{X}, \mathbb{Z})$ is zero). Each $\partial C_{v,i} \subset \partial \widetilde{X}$ is a link component in $\partial \widetilde{X}$, and their collection endowed with the multiplicities $\{c_{v,i}\}_{v,i}$ forms a multilink with '*multiplicity system*' $\{m_v, c_{v,i}\}_{v,i}$, cf. 3.3.6.

If $c_{v,i} \geq 0$ and at least one inequality is strict, then each $m_v > 0$ too, cf. 2.2.5(4). Moreover, under the same hypothesis, the multilink is fibered. [Indeed, by analytic realization 2.5, there exists an analytic structure in which the fibration is realized even by an analytic Milnor fibration.]

Fix $C(l)$ and a multiplicity system as above. Let Γ' be a full connected subgraph of Γ and $\widetilde{X}(\Gamma')$ a small tubular neighbourhood of $\cup_{v \in \mathscr{V}(\Gamma')} E_v$ in $\widetilde{X}$. Then $C(l)$ induces the homologically trivial relative cycle $\sum_v (m_v E_v \cap \widetilde{X}(\Gamma') + \sum_i c_{v,i} C_{v,i} \cap \widetilde{X}(\Gamma'))$ in $\widetilde{X}(\Gamma')$. This gives a multiplicity system of Γ'. In a different language, the E_v, respectively the E^*_v–multiplicities of l after restriction are the following. A cut C_v is preserved with its multiplicity c_v if $v \in \mathscr{V}(\Gamma')$, otherwise it becomes empty. The restriction of E_v becomes empty if $E_v \cap \widetilde{X}(\Gamma') = \emptyset$; it becomes the cut $E_v \cap \widetilde{X}(\Gamma')$ of E_w with multiplicity m_v if $E_v \cap E(\Gamma')$ is the point $E_v \cap E_w$; and it remains E_v with its multiplicity m_v if $v \in \mathscr{V}(\Gamma')$. (Homologically, this is the cohomology operator R associated with the inclusion $j : L(\Gamma') \hookrightarrow L(\Gamma)$, cf. (9.6.8).)

10.3.19 Write $\mathcal{V} = \mathcal{V}_1 \sqcup \mathcal{V}_2$ as above, and assume now that $\Gamma(\mathcal{V}_2)$ is *connected*. Let the boundary $\mathcal{B} := \{u \in \mathcal{V}_2 : \exists\ w \notin \mathcal{V}_2$ adjacent to u in $\Gamma\}$ be defined as in 10.3.17. For any fixed $u \in \mathcal{B}$, $\Gamma_{1,u}$ denotes that full connected subgraph of Γ, which contains u and all the connected components of $\Gamma(\mathcal{V}_1)$, which have vertices adjacent to u. Write $\mathcal{V}_{1,u} = \mathcal{V}(\Gamma_{1,u})$.

As above, $Z(\mathbf{t}_{\mathcal{V}_2})$ is the reduction of $Z(\mathbf{t})$ to the variables indexed by $\mathcal{V}_2$.

Lemma 10.3.20

(a) Any element from the support of $Z(\mathbf{t}_{\mathcal{V}_2})$ can be written in a unique way as $\sum_{v\in\mathcal{V}_2} r_v E_v^*|_{\mathcal{V}_2}$ for certain coefficients $r_v \in \mathbb{Q}_{\geq 0}$.

(b) Fix $u \in \mathcal{B}$. Let $\kappa_{2,u}$ be the number of edges from $\mathcal{V}_2$ adjacent to u. If $\kappa_{2,u} \geq 2$ and $\sum_{v\in\mathcal{V}_2} r_v E_v^*|_{\mathcal{V}_2}$ is in the support of $Z(\mathbf{t}_{\mathcal{V}_2})$, then $r_u \cdot E_u^*|_u \leq \sum_{v\in\mathcal{V}_{1,u}} (\kappa_v - 2) E_v^*|_u$.

Proof

(a) follows from 10.3.17*(a)–(b)*. In fact, for $u \in \mathcal{B}$ and $v \in \mathcal{V}_{1,u}$, the cycle $l := E_v^* \cdot E_u^*|_u - E_u^* \cdot E_v^*|_u \in L \otimes \mathbb{Q}$ is supported in $\mathcal{V}_{1,u} \setminus u$. In particular,

$$E_v^*|_{\mathcal{V}_2} = E_u^*|_{\mathcal{V}_2} \cdot (E_v^*|_u / E_u^*|_u). \tag{10.3.27}$$

Next, write $Z(\mathbf{t})$ as $Z_2(\mathbf{t}) \cdot \prod_{u\in\mathcal{B}} Z_{1,u}(\mathbf{t})$, where $Z_{1,u}(\mathbf{t}) := \prod_{v\in\mathcal{V}_{1,u}} (1 - \mathbf{t}^{E_v^*})^{\kappa_v - 2}$ and $Z_2(\mathbf{t}) := \prod_{v\in\mathcal{V}_2\setminus\mathcal{B}} (1 - \mathbf{t}^{E_v^*})^{\kappa_v - 2}$. Hence, in the support of $Z(\mathbf{t}_{\mathcal{V}_2})$, the term $Z_2(\mathbf{t}_{\mathcal{V}_2})$ chips in with $\{E_v^*|_{\mathcal{V}_2}\}_{v\in\mathcal{V}_2\setminus\mathcal{B}}$, while, $Z_{1,u}(\mathbf{t}_{\mathcal{V}_2})$ with $E_u^*|_{\mathcal{V}_2}$ for each $u \in \mathcal{B}$. Moreover, $\{E_v^*|_{\mathcal{V}_2}\}_{v\in\mathcal{V}_2}$ are linearly independent by 10.3.16.

(b) We construct the following graph $\Gamma'_{1,u}$ with arrowheads: $\Gamma'_{1,u}$ consists of all the vertices and edges of $\Gamma_{1,u}$, and we also add $\kappa_{2,u}$ arrowheads attached to u (more precisely, we replace the u–adjacent edges from $\mathcal{V}_2$ by arrowheads). Each arrowhead represents a cut of E_u in $\widetilde{X}(\Gamma_{1,u})$. We regard their collection as a multilink, that is, we endow the vertices and arrowheads with a multiplicity system (of $\Gamma'_{1,u}$) as in 10.3.18. We define the m_v–multiplicities as the multiplicities of dE_u^* restricted to $\mathcal{V}_{1,u}$, where $d = \det(-I)$. That is, the multiplicity of a vertex v is $dE_u^*|_v = dE_v^*|_u = -d(E_v^*, E_u^*) \in \mathbb{Z}_{>0}$. Then the sum of the multiplicities of the arrowheads (all of them at u) should be $c_u = d + d\sum_w E_u^*|_w$, where the sum runs over the adjacent vertices w of u in $\mathcal{V}_2$ (there are $\kappa_{2,u}$ of them). Since each $dE_u^*|_w \geq 1$, we get $c_u \geq d + \kappa_{2,u}$, hence we redistribute c_u into $\kappa_{2,u}$ positive integers, such that one of them is 1. These integers will be the multiplicities of the arrowheads (link components) of $\Gamma'_{1,u}$.

The constructed multiplicity system defines a fibred multilink (cf. 10.3.18). Since one of the multiplicities is 1, the corresponding Milnor fiber is connected, cf. 4.1.9. Furthermore, by A'Campo's theorem 4.1.22, the inverse of the monodromy zeta function of the Milnor fibration is $\zeta^{-1}(t) = \prod_{v\in\mathcal{V}_{1,u}} (1 - t^{dE_v^*|_u})^{\kappa_v - 2}$. Hence, comparing the definitions of ζ and $Z_{1,u}$, and using (10.3.27), we get that the reduced

series of $Z_{1,u}$ is obtained by the following substitution:

$$Z_{1,u}(\mathbf{t}_{\mathscr{V}_2}) = \zeta^{-1}(t)|_{t\mapsto \mathbf{t}_{\mathscr{V}_2}^{E_u^*|_{\mathscr{V}_2}/dE_u^*|_u}}. \tag{10.3.28}$$

We claim that if $\kappa_{2,u} \geq 2$ then $\zeta^{-1}(t)$ is a polynomial. Indeed, it has the form $\Delta(t)/(t-1)$, where $\Delta(t)$ is the characteristic polynomial of the monodromy of the first homology of the Milnor fiber, cf. 4.1.22. Hence, ζ^{-1} is a polynomial if and only if it has no pole at $t = 1$. But, since $\mathscr{T}_{1,u}$ is a tree, the vanishing order $\mathrm{ord}_{t-1}\zeta^{-1}(t) = \sum_{v\in\mathscr{V}_{1,u}}(\kappa_v - 2) = -2 + \kappa_{2,u} \geq 0$.

Furthermore, the degree of ζ^{-1} is $\sum_{v\in\mathscr{V}_{1,u}}(\kappa_v - 2)dE_v^*|_u$, and the (rational) $E_u^*|_{\mathscr{V}_2}$ degree of $Z_{1,u}(\mathbf{t}_{\mathscr{V}_2})$ is $\deg(\zeta^{-1})/(dE_u^*|_u)$. Finally, from *(a)* and its proof, all the contribution into the coefficient of $E_u^*|_{\mathscr{V}_2}$ in $Z(\mathbf{t}_{\mathscr{V}_2})$ comes from $Z_{1,u}(\mathbf{t}_{\mathscr{V}_2})$. □

10.4 The Modified Counting Function

We fix $\mathscr{I} \subset \mathscr{V}$, $\mathscr{I} \neq \emptyset$ and $h \in H$. By the inclusion–exclusion principle, the counting function $Q_{h,\mathscr{I}}$ of $Z_h(\mathbf{t}_{\mathscr{I}})$ can be written as $\sum_{\emptyset\neq\mathscr{J}\subset\mathscr{I}}(-1)^{|\mathscr{I}|+1}q_{h,\mathscr{J}}$, where the expressions $q_{h,\mathscr{J}}$ are the 'modified counting functions'.

A modified counting function has several advantages compared with the original one, namely: *(i)* from the point of view of Ehrhart theory it corresponds to a *convex* polytope (unlike $P^{l'}$ considered in (10.2.5)), see 10.4.5; *(ii)* its chamber decomposition is compatible with the Lipman's cone, that is, $\mathscr{S}_{\mathbb{R}}$ is not dismembered by the chambers of the new situation (in contrast with the chamber decomposition of Q_h, cf. 10.3.5), see the next 10.4.12 and 10.4.14 for this case; and *(iii)* it satisfies certain 'convexity' property, which generates a lot of combinatorial cancellations, see 10.4.A. Using these properties in 10.4.B we establish an additivity formula for $q_{h,\mathscr{J}}$, which will generate a general additivity formula for Q_h and for their periodic constants as well.

The advantage of the modified counting function, compared with the original one, was firstly realized in [354]. Here we follow [356]. ♣

10.4.1 In the next discussions the following notation is helpful.

Definition 10.4.2 We say that $a, b \in \mathbb{R}^k$, $a = \{a_v\}_v$, $b = \{b_v\}_v$, satisfy $a \prec b$ if $a_v < b_v$ for *all* the coordinates v.

10.4.3 We fix $\mathscr{I} \subset \mathscr{V}$, $\mathscr{I} \neq \emptyset$ and some $h \in H$. As we will see later, in the $\mathfrak{sw}$–surgery formula the counting functions of the reduced series $Z_h(\mathbf{t}_{\mathscr{I}})$ plays a role.

This can be defined using the coefficients $\mathfrak{z}(l')$ of $Z_h(\mathbf{t})$ too. Indeed, set

$$Q_{h,\mathscr{I}} : \{l' \in L' \,:\, [l'] = h\} \to \mathbb{Z}, \quad Q_{h,\mathscr{I}}(l') = \sum_{l''|_{\mathscr{I}} \not\succeq l'|_{\mathscr{I}},\ [l'']=h} \mathfrak{z}(l'').$$

This, indeed, equals the counting function of the reduced series $Z_h(\mathbf{t}_{\mathscr{I}})$.

If in a certain context we wish to emphasize that $Q_{h,\mathscr{I}}$ is associated with the pair $(Z, \mathscr{I})$ provided by the graph Γ, then we write $Q^{\Gamma}_{h,\mathscr{I}}$ instead of $Q_{h,\mathscr{I}}$.

By inclusion–exclusion principle, the sum which defines $Q_{h,\mathscr{I}}(l')$ is

$$\sum_{l''|_{\mathscr{I}} \not\succeq l'|_{\mathscr{I}}} \mathfrak{z}(l'') = \sum_{\exists w \in \mathscr{I} \,:\, l''_w \prec l'_w} \mathfrak{z}(l'') = \sum_{\emptyset \neq \mathscr{J} \subset \mathscr{I}} (-1)^{|\mathscr{I}|+1} \sum_{l''|_{\mathscr{J}} \prec l'|_{\mathscr{J}}} \mathfrak{z}(l''),$$

where everywhere $[l''] = [l'] = h$. This motivates the following definition.

Definition 10.4.4 For $l' \in L'$ with $[l'] = h$ and $\mathscr{I} \subset \mathscr{V}$ ($\mathscr{I} \neq \emptyset$), the 'modified counting function' associated with $Z_h(\mathbf{t})$ (or, with $Z_h(\mathbf{t}_{\mathscr{I}})$) is

$$q_{h,\mathscr{I}}(l') := \sum_{l''|_{\mathscr{I}} \prec l'|_{\mathscr{I}}} \mathfrak{z}(l'').$$

Again, sometimes we might write $q^{\Gamma}_{h,\mathscr{I}}(l')$ instead of $q_{h,\mathscr{I}}(l')$ as well.

10.4.5 Next, we list certain *Ehrhart theoretical properties of the modified counting function* $q_{h,\mathscr{I}}$. Since they are valid for the modified counting function of the Taylor expansion of any rational function, we present the statements in this *general setup*, as a parallel discussion of 10.2. Accordingly, we fix a rational function as in (10.2.1), we consider its expansion $Tf(\mathbf{t})$, its H-decompositions $Tf_h(\mathbf{t})$ as in (10.2.2), and the representation ρ as in 10.2.5. Then, for any $v \in \mathscr{V}$ and $l' \in L'$ the vectors $\{a_i\}_{i=1}^n$ determine a simplicial polytope (cf. (10.2.4) and (10.2.5))

$$P_v := \{\mathbf{x} \in (\mathbb{R}_{\geq 0})^n \,:\, \sum_i x_i a_{i,v} \leq l'_v\}. \tag{10.4.1}$$

In the original situation of 10.2, compatibly with $Q(l')$, we had to take the non-convex polytope $P^{(l')} := \cup_v P_v$ of $(\mathbb{R}_{\geq 0})^n$ for the summation region. In the present situation, for any $\mathscr{I} \subset \mathscr{V}$, $\mathscr{I} \neq \emptyset$, we define ${}^m P^{(l')}_{\mathscr{I}} := \cap_{v \in \mathscr{I}} P_v$. Clearly, ${}^m P^{(l')}_{\mathscr{I}}$ depends only on the projection $\pi_{\mathscr{I}}(L') = \{l'_{\mathscr{I}} := l'|_{\mathscr{I}} \,:\, l' \in L'\}$ on the $\mathscr{I}$-coordinates. Hence, ${}^m P^{(l')}_{\mathscr{I}}$ are closed dilated convex polytopes indexed (or, dilated) by $l'|_{\mathscr{I}} = \sum_{v \in \mathscr{I}} l'_v E_v \in \pi_{\mathscr{I}}(L')$.

We assume that $l'_{\mathscr{I}}$ moves in some connected region of $\pi_{\mathscr{I}}(L')$ in such a way that ${}^m P^{(l')}_{\mathscr{I}}$ stays combinatorially stable. In such a case we can associate the set (dilated

family) of facets $\mathscr{F}_{\mathscr{J}}^{(l')}$ to the stable combinatorial type of the dilated polytopes. Again, as in 10.2.5, we distinguish those facets which sit on coordinate hyperplanes $\cup_i\{x_i = 0\}$, and we denote the collection of other facets by $\mathscr{T}_{\mathscr{J}}^{(l')}$. In a simplified notation: $\mathscr{T}_{\mathscr{J}} \subset \mathscr{F}_{\mathscr{J}}$.

Furthermore, we consider the equivariant counting function

$$ {}^m\mathscr{L}^e(\mathscr{T}_{\mathscr{J}}, l'_{\mathscr{J}}) := \sum_{h\in H} {}^m\mathscr{L}_h(\mathscr{T}_{\mathscr{J}}, l'_{\mathscr{J}}) \cdot [h] := \sum 1 \cdot \left[\sum_i x_i a_i\right] \in \mathbb{Z}[H], \tag{10.4.2} $$

where the last sum runs over $\mathbf{x} \in {}^m P_{\mathscr{J}}^{(l'),\triangleleft}$, where ${}^m P_{\mathscr{J}}^{(l'),\triangleleft} := {}^m P_{\mathscr{J}}^{(l')} \setminus \cup_{F\in\mathscr{T}_{\mathscr{J}}} F$.

According to the equivariant Ehrhart theory, applied to the dilated polytopes ${}^m P_{\mathscr{J}}^{(l')}$, for any $h \in H$ the counting function ${}^m\mathscr{L}_h(\mathscr{T}_{\mathscr{J}}, l'_{\mathscr{J}})$ is a quasipolynomial, whenever $l'_{\mathscr{J}}$ moves in some connected region of $\pi_{\mathscr{J}}(L')$ in such a way that ${}^m P_{\mathscr{J}}^{(l')}$ stays combinatorially stable. The inequalities of (10.4.1) can be viewed as a vector partition $\sum_{i=1}^n x_i \cdot a_i|_{\mathscr{J}} + \sum_{v\in\mathscr{J}} y_v \cdot E_v|_{\mathscr{J}} = l'_{\mathscr{J}}$ with $x_i, y_v \geq 0$. Hence, by [354, 674], the stability of the combinatorial type of ${}^m P_{\mathscr{J}}^{(l')}$ is determined by the following chamber decomposition of $\pi_{\mathscr{J}}(L' \otimes \mathbb{R}) = \mathbb{R}^{|\mathscr{J}|}$: let ${}^m\mathfrak{B}_{\mathscr{J}}$ be the set of all bases $\mathfrak{b} \subset \{a_i|_{\mathscr{J}}, E_v|_{\mathscr{J}} : i \in \{1,\dots,n\}, v \in \mathscr{J}\}$ of $\mathbb{R}^{|\mathscr{J}|}$. Then a (big, open) chamber $\mathfrak{c}_{op}$ is a connected component of $\mathbb{R}^{|\mathscr{J}|} \setminus \cup_{\mathfrak{b}\in\mathfrak{B}_{\mathscr{J}}} \partial\mathbb{R}_{\geq 0}\mathfrak{b}$, where $\partial\mathbb{R}_{\geq 0}\mathfrak{b}$ is the boundary of the closed cone $\mathbb{R}_{\geq 0}\mathfrak{b}$. (See also 10.2.7 for another equivalent formulation how we can derive the set of chambers from the terms of the vector partition. Observe also the sign difference of E_v's compared with the non-convex polytopes from 10.2.7.) In particular, the chamber decomposition is independent of the choices of $h \in H$.

Hence, if $l'_{\mathscr{J}}$ moves in such an open chamber $\mathfrak{c}_{op}$, then the above counting function of lattice points ${}^m\mathscr{L}_h(\mathscr{T}_{\mathscr{J}}, l'_{\mathscr{J}})$ is a quasipolynomial, denoted by ${}^m\mathscr{L}_h^{\mathfrak{c}}(\mathscr{T}_{\mathscr{J}}, l'_{\mathscr{J}})$. One can extend ${}^m\mathscr{L}_h^{\mathfrak{c}}(\mathscr{T}_{\mathscr{J}}, l'_{\mathscr{J}})$ continuously to the closure $\mathfrak{c}$ of $\mathfrak{c}_{op}$ as a quasipolynomial. All these extensions glue together into a continuous piecewise quasipolynomial of $\mathbb{R}^{|\mathscr{J}|}$. This piecewise quasipolynomial is the expression ${}^m\mathscr{L}_h(\mathscr{T}_{\mathscr{J}}, l'_{\mathscr{J}})$.

By the *equivariant Ehrhart–Macdonald–Stanley reciprocity law*, for any fixed h and fixed chamber $\mathfrak{c}$ one has

$$ {}^m\mathscr{L}_h^{\mathfrak{c}}(\mathscr{T}_{\mathscr{J}}, l'_{\mathscr{J}}) = (-1)^n \cdot {}^m\mathscr{L}_{-h}^{\mathfrak{c}}(\mathscr{F}_{\mathscr{J}} \setminus \mathscr{T}_{\mathscr{J}}, -l'_{\mathscr{J}}). \tag{10.4.3} $$

(We warn again, cf. 10.1.11, that usually the parameter $-l'_{\mathscr{J}}$ is included in a different chamber than $\mathfrak{c}$, that is, ${}^m\mathscr{L}_{-h}^{\mathfrak{c}}(\mathscr{F}_{\mathscr{J}} \setminus \mathscr{T}_{\mathscr{J}}, -l'_{\mathscr{J}}) \neq {}^m\mathscr{L}_{-h}(\mathscr{F}_{\mathscr{J}} \setminus \mathscr{T}_{\mathscr{J}}, -l'_{\mathscr{J}})$.)

10.4.6 Next, let us consider the modified counting function associated with $Tf(\mathbf{t}) = \sum_{l'} z(l')\mathbf{t}^{l'}$, namely, for any $l' \in L'$ defined by the identity

$$\sum_{l''|_{\mathscr{I}} \prec l'|_{\mathscr{I}}} z(l'') \cdot [l''] = \sum_{h \in H} q_{h,\mathscr{I}}(l') \cdot [h].$$

Then the general theory above has the following consequences. (These are the analogues of Corollary 10.2.9 and Proposition 10.2.14.)

Corollary 10.4.7 *We fix h and $\mathscr{I}$.*

(a) $q_{h,\mathscr{I}}$ is a piecewise quasipolynomial, which can be written as

$$q_{h,\mathscr{I}}(l') = \sum_k \iota_k \cdot {}^m\mathscr{L}_{h-[b_k]}(\mathscr{T}_{\mathscr{I}}, l'_{\mathscr{I}} - b_k|_{\mathscr{I}}). \tag{10.4.4}$$

(b) For a fixed closed chamber $\mathfrak{c}$ of $\mathbb{R}^{|\mathscr{I}|}$ define the quasipolynomial

$$\overline{q}^{\mathfrak{c}}_{h,\mathscr{I}}(l') := \sum_k \iota_k \cdot {}^m\mathscr{L}^{\mathfrak{c}}_{h-[b_k]}(\mathscr{T}_{\mathscr{I}}, l'_{\mathscr{I}} - b_k|_{\mathscr{I}}). \tag{10.4.5}$$

Then $q_{h,\mathscr{I}}$ is a quasipolynomial on $\cap_k(b_k|_{\mathscr{I}} + \mathfrak{c})$, namely, for $l'_{\mathscr{I}} \in \cap_k(b_k|_{\mathscr{I}} + \mathfrak{c})$ one has $q_{h,\mathscr{I}}(l') = \overline{q}^{\mathfrak{c}}_{h,\mathscr{I}}(l')$.

(c) For any fixed $h \in H$, the quasipolynomial $\overline{q}^{\mathfrak{c}}_{h,\mathscr{I}}(l')$ satisfies the following property: for any $l' \in L'$ with $[l'] = h$, and any q from the semi-open unit cube one has

$$\overline{q}^{\mathfrak{c}}_{h,\mathscr{I}}(l' - q) = \overline{q}^{\mathfrak{c}}_{h,\mathscr{I}}(l'). \tag{10.4.6}$$

In particular, by taking $l' = q = r_h$:

$$\overline{q}^{\mathfrak{c}}_{h,\mathscr{I}}(0) = \overline{q}^{\mathfrak{c}}_{h,\mathscr{I}}(r_h). \tag{10.4.7}$$

(d) For any fixed chamber $\mathfrak{c}$ the modified counting function $q_{h,\mathscr{I}}$ admits a quasipolynomial $\mathfrak{q}^{\mathfrak{c}}_{h,\mathscr{I}}$ in the sense of 10.2.12, which satisfies the identity $\mathfrak{q}^{\mathfrak{c}}_{h,\mathscr{I}}(l) = \overline{q}^{\mathfrak{c}}_{h,\mathscr{I}}(l')$ for $l' = l + r_h$. Additionally, $q_{h,\mathscr{I}}$ admits a periodic constant $\mathrm{pc}^{\mathfrak{c}}(q_{h,\mathscr{I}}) = \mathfrak{q}^{\mathfrak{c}}_{h,\mathscr{I}}(0) = \overline{q}^{\mathfrak{c}}_{h,\mathscr{I}}(r_h)$ associated with the chamber $\mathfrak{c}$ as well.

10.4.8 The statements of (10.4.5) and (10.4.6), valid for any rational function f, will be applied for $Z(\mathbf{t})$ associated with a plumbing graph. That is, the modified counting function $q_{h,\mathscr{I}}$ is defined as in (10.4.4) using the coefficients of $Z_h(\mathbf{t})$, or, equivalently, of $Z_h(\mathbf{t}_{\mathscr{I}})$.

Definition 10.4.9 In this case $\mathrm{pc}^{\mathfrak{c}}(q_{h,\mathscr{I}})$ will be denoted by $\mathrm{mpc}^{\mathfrak{c}}(Z_h(\mathbf{t}_{\mathscr{I}}))$, and we call it the *modified periodic constant* of $Z_h(\mathbf{t}_{\mathscr{I}})$ associated with h, $\mathscr{I}$ and $\mathfrak{c}$.

10.4.10 The Chambers of the Modified Counting Function of $Z_h(\mathbf{t}_{\mathscr{I}})$ Recall from 10.4.5 that the chambers of the modified counting function associated with $Z(\mathbf{t}_{\mathscr{I}})$ are determined by the collection ${}^{m}\mathfrak{B}_{\mathscr{I}}$ of the bases $\mathfrak{b} \subset \{E^*_e|_{\mathscr{I}},\ E_v|_{\mathscr{I}} : e \in \mathscr{V}^e(\Gamma),\ v \in \mathscr{I}\}$, where $\mathscr{V}^e(\Gamma)$ denotes the end-vertices of Γ. In fact, since in the chamber-construction we use the real simplicial cones generated by $\mathfrak{b}$'s, any of the above vectors $\{E^*_e|_{\mathscr{I}}\}_e$ and $\{E_v|_{\mathscr{I}}\}_v$ can be replaced by any of its positive multiples.

These vectors are determined by a vector partition argument; the vectors $\{E^*_e|_{\mathscr{I}}\}$ are read from the denominator of $Z(\mathbf{t}_{\mathscr{I}})$. Indeed, $Z(\mathbf{t})$ has denominator $\prod_{e\in\mathscr{V}^e(\Gamma)}(1-\mathbf{t}^{E^*_e})$, hence by a 'rash argument', the denominator of $Z(\mathbf{t}_{\mathscr{I}})$ is $\prod_{e\in\mathscr{V}^e(\Gamma)}(1-\mathbf{t}_{\mathscr{I}}^{E^*_e|_{\mathscr{I}}})$. This is true, however, if by simplification we can reduce this denominator, then we can reduce the set ${}^{m}\mathfrak{B}_{\mathscr{I}}$ as well, and we get larger chambers. In fact, this is exactly the case. Though $\prod_{e\in\mathscr{V}^e(\Gamma)}(1-\mathbf{t}^{E^*_e})$ is the minimal denominator of $Z(\mathbf{t})$, usually $\prod_{e\in\mathscr{V}^e(\Gamma)}(1-\mathbf{t}_{\mathscr{I}}^{E^*_e|_{\mathscr{I}}})$ is *not* the minimal denominator of $Z(\mathbf{t}_{\mathscr{I}})$. More precisely, in the minimal denominator of $Z(\mathbf{t}_{\mathscr{I}})$ all the vectors which might appear have the form $\{\mathbb{R}_{>0}\langle E^*_e|_{\mathscr{I}}\rangle\}_{e\in\mathscr{V}^e(\Gamma)}$, *however some of them do not appear*.

Lemma 10.4.11 *Fix a tree Γ. Let $\mathscr{J}$ be a subset of $\mathscr{V}$ such that the full subgraph $\Gamma(\mathscr{J})$ of Γ with vertices $\mathscr{J}$ is connected. Let $\kappa_{\mathscr{J},v}$ be the valency of $v \in \mathscr{J}$ in $\Gamma(\mathscr{J})$. Define also $\mathscr{B}$, as the set of those vertices of $\mathscr{J}$, which are adjacent to some vertex from $\mathscr{V}\setminus\mathscr{J}$. Then*

$$(i)\quad \{\mathbb{R}_{>0}\langle E^*_v|_{\mathscr{J}}\rangle\}_{v\in\mathscr{V}} = \{\mathbb{R}_{>0}\langle E^*_v|_{\mathscr{J}}\rangle\}_{v\in\mathscr{J}}.$$

$$(ii)\quad \{\mathbb{R}_{>0}\langle E^*_v|_{\mathscr{J}}\rangle\}_{v\in\mathscr{V}^e(\Gamma)} = \{\mathbb{R}_{>0}\langle E^*_v|_{\mathscr{J}}\rangle\}_{v\in\mathscr{B}\cup(\mathscr{V}^e(\Gamma)\cap\mathscr{J})}.$$

Furthermore, for $v \in \mathscr{B}\cup(\mathscr{V}^e(\Gamma)\cap\mathscr{J})$ the ray $\mathbb{R}_{>0}\langle E^*_v|_{\mathscr{J}}\rangle$ appears in the denominator of $Z(\mathbf{t}_{\mathscr{J}})$ only if $v \in \mathscr{V}^e(\Gamma(\mathscr{J}))$.

Proof For *(i)–(ii)* we apply Lemmas 10.3.17*(a)* and 10.3.20 (and their proofs) for $\mathscr{J} = \mathscr{V}_2$, cf. (10.3.27). For the last sentence use again the proof of 10.3.20: if $\kappa_{2,u} \geq 2$ then the only term which might produce $\mathbb{R}_{>0}E^*_u|_{\mathscr{V}_2}$ comes from the contribution $Z_{1,u}(\mathbf{t}_{\mathscr{V}_2})$, which is a polynomial. □

In order to be able to apply 10.4.11, associated with an arbitrary $\mathscr{I} \subset \mathscr{V}$, we define $\overline{\mathscr{I}}$ as the smallest subset $\mathscr{J}$ of $\mathscr{V}$ with $\mathscr{I} \subset \mathscr{J}$, such that the full subgraph $\Gamma(\mathscr{J})$ of Γ with vertices $\mathscr{J}$ is connected. Then, using the argument of 10.4.5 (based on [354, 674]) and the structure of the denominator of $Z_h(\mathbf{t}_{\mathscr{I}})$ from Lemma 10.4.11*(ii)* we get that the collection ${}^{m}\mathfrak{B}_{\mathscr{I}}$ of bases can be reduced

to the collection

$$^m\mathfrak{B}'_{\mathscr{J}} \text{ consisting of bases } \mathfrak{b} \subset \{E_e^*|_{\mathscr{J}},\ E_v|_{\mathscr{J}} : e \in \mathscr{V}^e(\Gamma(\overline{\mathscr{J}})),\ v \in \mathscr{J}\}.$$

By Lemma 10.4.11 this is a subset of $^m\mathfrak{B}_{\mathscr{J}}$. Furthermore, $\mathscr{V}^e(\Gamma(\overline{\mathscr{J}})) \subset \mathscr{J}$ as well.

The point is that the chambers cut out by the collection $^m\mathfrak{B}'_{\mathscr{J}}$ are large compared with the projection of the real Lipman's cone. Recall that the projection of the Lipman's cone (for any subset) was characterized in 10.3.17.

Lemma 10.4.12 ([354]) *Let $\mathscr{J} \subset \mathscr{V}$ be an arbitrary subset, $\mathscr{J} \neq \emptyset$, and let $\overline{\mathscr{J}} \subset \mathscr{V}$ be as above. Consider the collection of vectors $\{E_e^*|_{\mathscr{J}},\ E_v|_{\mathscr{J}} : e \in \mathscr{V}^e(\Gamma(\overline{\mathscr{J}})),\ v \in \mathscr{J}\}$. Let $^m\mathfrak{B}'_{\mathscr{J}}$ be the collection of bases formed from these vectors, and let $^m\mathfrak{C}'_{\mathscr{J}}$ be the collection of chambers $\mathfrak{c}$ defined from $^m\mathfrak{B}'_{\mathscr{J}}$ as in 10.2.7 or 10.4.5. Then for any $\mathfrak{b} \in {}^m\mathfrak{B}'_{\mathscr{J}}$ either $\pi_{\mathscr{J}}(\mathscr{S}_{\mathbb{R}})^\circ \cap \mathbb{R}_{>0}\mathfrak{b} = \emptyset$, or $\pi_{\mathscr{J}}(\mathscr{S}_{\mathbb{R}})^\circ \subset \mathbb{R}_{>0}\mathfrak{b}$.*

In particular, there exists a top-dimensional open chamber $\mathfrak{c}_{op} \in {}^m\mathfrak{C}'_{\mathscr{J}}$, which entirely contains the interior of the projected real Lipman's cone $\pi_{\mathscr{J}}(\mathscr{S}_{\mathbb{R}})^\circ$.

Proof Assume that the statement is not true, hence a facet of $\mathbb{R}_{\geq 0}\mathfrak{b}$ intersects $\pi_{\mathscr{J}}(\mathscr{S}_{\mathbb{R}})^\circ$. This means that there exists $\mathscr{J}' := \{j_1, \dots, j_l\} \subset \mathscr{V}^e(\Gamma(\overline{\mathscr{J}}))$ and $\mathscr{J}'' := \{i_1, \dots, i_k\} \subset \mathscr{J}$ with $l + k = |\mathscr{J}| - 1$ and a vector of type

$$V = \sum_{t=1}^{l} \alpha_t \cdot E_{j_t}^*|_{\mathscr{J}} + \sum_{s=1}^{k} \beta_s \cdot E_{i_s}|_{\mathscr{J}},\quad \alpha_t \geq 0,\quad \beta_s \geq 0$$

with $V \in \pi_{\mathscr{J}}(\mathscr{S}_{\mathbb{R}})^\circ$. Since $\mathscr{V}^e(\Gamma(\overline{\mathscr{J}})) \subset \mathscr{J}$ and $l + k < |\mathscr{J}|$ we have $\mathscr{J}' \cup \mathscr{J}'' \subsetneq \mathscr{J}$. Fix $i \in \mathscr{J} \setminus (\mathscr{J}' \cup \mathscr{J}'')$. Next we apply Lemmas 10.3.16 and 10.3.17 with $\mathscr{V}_2 = \mathscr{J}$. Consider the form $(\ ,\ )_{mod}$ on $L_{\mathscr{J}} \otimes \mathbb{R}$. Then $(E_{j_t}^*|_{\mathscr{J}}, E_i|_{\mathscr{J}})_{mod} = 0$ by 10.3.16*(c)*, and $(E_{i_s}|_{\mathscr{J}}, E_i|_{\mathscr{J}})_{mod} \geq (E_{i_s}, E_i) \geq 0$ by 10.3.16*(b)*. In particular, $(V, E_i|_{\mathscr{J}})_{mod} \geq 0$. On the other hand, since $V \in \pi_{\mathscr{J}}(\mathscr{S}_{\mathbb{R}})^\circ$, one has $(V, E_i|_{\mathscr{J}})_{mod} < 0$ by 10.3.17*(c)–(d)*. This leads to a contradiction. □

10.4.13 Lemma 10.4.12 provides the h–independent chamber decomposition of $Z_h(\mathbf{t}_{\mathscr{J}})$ in $\mathbb{R}^{|\mathscr{J}|}$. Along such a chamber the corresponding polytopes have stable combinatorial types and the modified counting function $l'|_{\mathscr{J}} \mapsto q_{h,\mathscr{J}}(l'|_{\mathscr{J}})$ is a quasipolynomial. Here, for any $\mathscr{J}$ the ambient space is $\mathbb{R}^{|\mathscr{J}|}$; however, we can lift the parameter $l'|_{\mathscr{J}}$ naturally to $l' \in L' \otimes \mathbb{R}$, and consider the modified counting function $l' \mapsto q_{h,\mathscr{J}}(l') = q_{h,\mathscr{J}}(l'|_{\mathscr{J}})$ (as in 10.4.3). In this way the chambers of $l' \mapsto q_{h,\mathscr{J}}(l')$ are the pullbacks $\pi_{\mathscr{J}}^{-1}(\mathfrak{c})$ by $\pi_{\mathscr{J}} : L' \otimes \mathbb{R} \to \mathbb{R}^{|\mathscr{J}|}$ of the chambers $\mathfrak{c} \in {}^m\mathfrak{C}'_{\mathscr{J}}$ of $\mathbb{R}^{|\mathscr{J}|}$. They will have a cylinder structure over $\mathbb{R}^{|\mathscr{J}|}$.

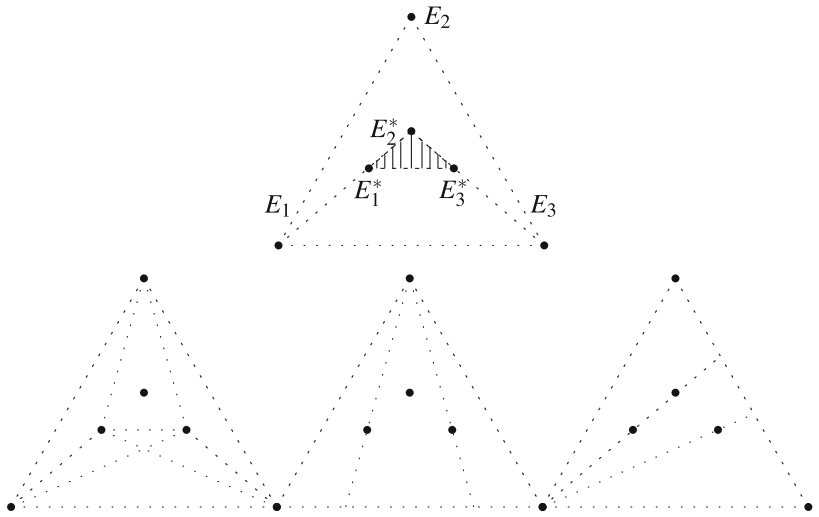
Let $^m\mathfrak{C}'$ be their minimal common refinement (when we take every possible $\mathscr{J}$).

Proposition 10.4.14 *Consider the chamber decomposition $^m\mathfrak{C}'$ defined above. Then there exists a (closed) chamber $\mathfrak{c} \in {}^m\mathfrak{C}'$ such that $\mathscr{S}_{\mathbb{R}} = \mathfrak{c}$.*

Proof The existence of $\mathfrak{c}$ with $\mathscr{S}_{\mathbb{R}} \subset \mathfrak{c}$ follows from Lemma 10.4.12. Next we prove that in fact the chamber which contains $\mathscr{S}_{\mathbb{R}}$ cannot be larger than $\mathscr{S}_{\mathbb{R}}$. For this we need to show that for any $V \notin \mathscr{S}_{\mathbb{R}}$ there exist $\mathscr{I} \subset \mathscr{V}$ and $\mathfrak{b} \in {}^{m}\mathfrak{B}'_{\mathscr{I}}$, such that the cone $\mathbb{R}_{\geq 0}\mathfrak{b}$ contains $\pi_{\mathscr{I}}(\mathscr{S}_{\mathbb{R}})$ but does not contain $\pi_{\mathscr{I}}(V)$.

Since $V \notin \mathscr{S}_{\mathbb{R}}$, there exists $v \in \mathscr{V}$ such that $(V, E_v) > 0$. Set $\mathscr{B} := \{w \in \mathscr{V} : (v, w) \text{ is an edge}\}$. Set $\mathscr{I} := \{v\} \cup \mathscr{B}$. Then $\overline{\mathscr{I}} = \mathscr{I}$, $\mathscr{V}^e(\overline{\mathscr{I}}) \supset \mathscr{B}$ and the chamber given by $\mathfrak{b} = \{E_v|_{\mathscr{I}}, \{E^*_w|_{\mathscr{I}}\}_{w \in \mathscr{B}}\}$ works. □

Example 10.4.15 Consider the minimal graph of A_3. The conical cambers from $\mathbb{R}^3$ are determined by their intersection with the affine plane $l_1 + l_2 + l_3 = 1$; this is what we will draw below. The first diagram shows the position of the vectors E_v and E^*_v. The second one is the chamber decomposition associated with $\mathscr{I} = \mathscr{V}$, the third one exemplifies $\mathscr{I} = \{1, 3\}$ while the last one $\mathscr{I} = \{2, 3\}$. (The reader is invited to check the statements of Example 10.3.6 on the picture as well, valid for E_2.)



10.4.16 Recall that the counting function Q_h of $Z_h(\mathbf{t})$ is associated with the piecewise quasipolynomial $\mathscr{L}_h(\mathscr{T}, \cdot)$, which is determined by the denominator of Z_h, or by the polytope $\cup_{v \in \mathscr{V}} P_v$, cf. 10.3.1. Similarly, for any $\mathscr{I} \subset \mathscr{V}$, one can consider the counting function $Q_{h,\mathscr{I}}$ of $Z_h(\mathbf{t}_{\mathscr{I}})$ (defined in 10.4.3). It is associated with a piecewise quasipolynomial $\mathscr{L}_h(\mathscr{T}_{\mathscr{I}}, \cdot)$, determined by the denominator of $Z_h(\mathbf{t}_{\mathscr{I}})$, or by the polytope $\cup_{v \in \mathscr{I}} P_v$ (whose detailed definition is left to the reader).

Hence, the analogue of the identity

$$Q_{h,\mathscr{I}} = \sum_{\emptyset \neq \mathscr{J} \subset \mathscr{I}} (-1)^{|\mathscr{J}|+1} \cdot q_{h,\mathscr{J}} \tag{10.4.8}$$

(see 10.4.3 and 10.4.4) is

$$\mathscr{L}_h(\mathscr{T}_{\mathscr{I}}, \cdot) = \sum_{\emptyset \neq \mathscr{J} \subset \mathscr{I}} (-1)^{|\mathscr{J}|+1} \cdot {}^m\mathscr{L}_h(\mathscr{T}_{\mathscr{J}}, \cdot). \tag{10.4.9}$$

Proposition 10.4.17 *Consider the chamber decomposition* ${}^m\mathfrak{C}'$ *defined in 10.4.13. Then* $\mathscr{L}_h(\mathscr{T}_{\mathscr{I}}, \cdot)$ *is a quasipolynomial on each chamber of* ${}^m\mathfrak{C}'$*. In particular, via Proposition 10.4.14, each* $\mathscr{L}_h(\mathscr{T}_{\mathscr{I}}, \cdot)$ *is a quasipolynomial on* $\mathscr{S}_{\mathbb{R}}$*.*

10.4.A Convexity Property of the Modified Counting Functions

10.4.18 Let Γ be as in 10.3, and let us fix a subset $\mathscr{I} \subset \mathscr{V}$, $\mathscr{I} \neq \emptyset$. Recall that $\overline{\mathscr{I}}$ is the set of vertices of the connected minimal full subgraph Γ_2 of Γ with $\mathscr{I} \subset \mathscr{V}(\Gamma_2)$. As above, $q_{h,\mathscr{I}} = q^{\Gamma}_{h,\mathscr{I}}$ denotes the modified counting function associated with $Z_h(\mathbf{t}_{\mathscr{I}})$.

Proposition 10.4.19 ([354, 356]) *Assume that* $l'_0 \in \sum_v (\kappa_v - 2)E_v^* + \mathscr{S}'$, $[l'_0] = h$. *Then* $q_{h,\mathscr{I}}(l'_0) = q_{h,\overline{\mathscr{I}}}(l'_0)$.

Proof We write $\mathscr{V}_2 := \mathscr{V}(\Gamma_2) = \overline{\mathscr{I}}$, and we adopt the notations of Sect. 10.3.B, especially of 10.3.19, associated with $\mathscr{V}_2$. Let $l'_0 = \sum_v a_v E_v^*$ be the fixed element of L' appearing in the statement. Let $l' = \sum_v b_v E_v^*$ denote an element from the support $\mathrm{Supp}(Z_h)$ of Z_h (i.e. $\mathfrak{z}(l') \neq 0$), such that $l := l' - l'_0$ is in L and it satisfies $l|_{\mathscr{I}} \prec 0$. Note that such elements l' parametrize the support of the sum in $q_{h,\mathscr{I}}(l'_0)$ (cf. Definition 10.4.4). Write $c_v = b_v - a_v$ and $l = \sum_v m_v E_v$ (hence $\{m_v, c_v\}_v$ is a multiplicity system in the sense of 10.3.18). Then $l|_{\mathscr{I}} \prec 0$ reads as $m_v < 0$ for all $v \in \mathscr{I}$.

We wish to compare the sets $\{l' \in \mathrm{Supp}(Z_h) \ : \ (l' - l'_0)|_{\mathscr{I}} \prec 0\}$ and $\{l' \in \mathrm{Supp}(Z_h) \ : \ (l' - l'_0)|_{\overline{\mathscr{I}}} \prec 0\}$ for a fixed l'_0. If they agree then definitely we get $q_{h,\mathscr{I}}(l'_0) = q_{h,\overline{\mathscr{I}}}(l'_0)$. The point is that these two sets can be different, however, we show that the sum of the coefficients over the support–difference is zero.

To start the proof, let us fix some $l' \in \mathrm{Supp}(Z_h)$ with $(l' - l'_0)|_{\mathscr{I}} \prec 0$.

First, we check an inequality regarding c_v. Let us take $v \in \mathscr{V}$ with $\kappa_v > 1$. Then, by the assumption of the proposition $a_v \geq \kappa_v - 2$. But, by the shape of the rational function $Z(\mathbf{t})$ (and from $\kappa_v > 1$), one has $b_v \leq \kappa_v - 2$. Hence

$$c_v \leq 0 \ \text{ whenever } \ \kappa_v > 1. \tag{10.4.10}$$

The proof in the 'easy case'. Assume that $m_u < 0$ for every $u \in \mathscr{B}$. We claim that $m_v < 0$ for every $v \in \overline{\mathscr{I}}$, hence $q_{h,\mathscr{I}}(l'_0) = q_{h,\overline{\mathscr{I}}}(l'_0)$ by the above discussion.

Assume that this is not the case, and choose a maximal connected full subgraph Γ' of Γ_2 with all m_v–multiplicities non-negative.

Let $C(l) = \sum_v (m_v E_v + c_v C_v)$ be the homologically trivial relative cycle in $\widetilde{X}$ associated with l (with some choices of cuts C_v) as in 10.3.18. Then $C(l)$ induces a relative cycle and a multiplicity system of Γ' via the multiplicity preserving intersection $C(l) \cap \widetilde{X}(\Gamma')$, as it is explained in 10.3.18.

By construction $\mathscr{V}(\Gamma') \subset \overline{\mathscr{I}} \setminus \mathscr{I}$ and for every $v \in \mathscr{V}(\Gamma')$ one has $\kappa_v > 1$. Therefore, those cut–multiplicities which come as restrictions of cuts of $C(l)$ are ≤ 0 by (10.4.10). The other cut–multiplicities, which come from the restriction of some neighboring E_v's have multiplicities $m_v < 0$ (by the maximality of Γ'). Therefore, the restriction of $C(l)$ to Γ' has all cut–multiplicities ≤ 0, with at least one < 0 (at the 'boundary' of Γ'). On the other hand, all E_v–multiplicities ≥ 0. These facts contradict the second paragraph of 10.3.18 (with signs reversed). □

The proof in the General Case Assume that $m_u \geq 0$ for at least one $u \in \mathscr{B}$.

Let us define the rational coefficients $\{c_{1,u}\}_{u \in \mathscr{V}_2}$ as follows:

$$c_{1,u} := \begin{cases} \sum_{v \in \mathscr{V}_{1,u}} c_v E_v^*|_u / E_u^*|_u & \text{if } u \in \mathscr{B} \\ c_u & \text{if } u \notin \mathscr{B}. \end{cases} \tag{10.4.11}$$

Then, by (10.3.27), for each $u \in \mathscr{B}$ one has $\sum_{v \in \mathscr{V}_{1,u}} c_v E_v^*|_{\mathscr{V}_2} = c_{1,u} E_u^*|_{\mathscr{V}_2}$, hence

$$\sum_{v \in \mathscr{V}_2} c_{1,v} E_v^*|_{\mathscr{V}_2} = \sum_{v \in \mathscr{V}} c_v E_v^*|_{\mathscr{V}_2} = l|_{\mathscr{V}_2} = \sum_{v \in \mathscr{V}_2} m_v E_v|_{\mathscr{V}_2}. \tag{10.4.12}$$

Lemma 10.4.20 *Fix $l' \in \mathrm{Supp}(Z_h)$ with $(l' - l'_0)|_{\mathscr{I}} \prec 0$. If there exists $u \in \mathscr{B}$ with $m_u \geq 0$ then there exists $u \in \mathscr{B}$ with $m_u \geq 0$, $c_{1,u} > 0$ and $\kappa_{2,u} \geq 2$ as well. (For the definition of $\kappa_{2,u}$ see 10.3.20(b).)*

Proof Set $\mathscr{V}_2^{<0} := \{v \in \mathscr{V}_2 : m_v < 0\}$ and $\mathscr{V}_2^{\geq 0} := \{v \in \mathscr{V}_2 : m_v \geq 0\}$. By assumptions $\mathscr{I} \subset \mathscr{V}_2^{<0}$ and $\mathscr{V}_2^{\geq 0} \neq \emptyset$ too. Write $l|_{\mathscr{V}_2}$ as $l_1 - l_2$, l_1 supported in $\mathscr{V}_2^{\geq 0}$, while l_2 supported in $\mathscr{V}_2^{<0}$, both effective. Consider also the negative definite 'modified' form $(\ ,\ )_{mod}$ defined in 10.3.16. If $l_1 \neq 0$ then $(l|_{\mathscr{V}_2}, l_1)_{mod} \leq (l_1, l_1)_{mod} < 0$, hence there exists $u \in \mathscr{V}_2^{\geq 0}$ (in the support of l_1) such that $(l|_{\mathscr{V}_2}, E_u)_{mod} < 0$. If $l_1 = 0$, since Γ_2 is connected, one can find $u \in \mathscr{V}_2^{\geq 0}$ such that E_u intersects the support of $l|_{\mathscr{V}_2}$, hence $(l|_{\mathscr{V}_2}, E_u)_{mod} < 0$ again. But, via Lemma 10.3.16 and the identity (10.4.12) $(l|_{\mathscr{V}_2}, E_u)_{mod} = -c_{1,u}$. Hence, there exists $u \in \mathscr{V}_2^{\geq 0}$ such that $c_{1,u} > 0$. Note also that

$$\text{if } u \in \mathscr{V}_2 = \overline{\mathscr{I}} \text{ and either } \kappa_u = 1 \text{ or } \kappa_{2,u} = 1 \text{ then } u \in \mathscr{I} \subset \mathscr{V}_2^{<0}. \tag{10.4.13}$$

Therefore, using (10.4.10) and definition (10.4.11) we get that $u \in \mathscr{B}$ necessarily. On the other hand, $\kappa_{2,u} \geq 2$ too by (10.4.13). □

Now we continue the proof of Proposition 10.4.19 (general case). Let l' as in the assumption of 10.4.20, and u as in the conclusion of of 10.4.20. Let us introduce the coefficients $\{b_{1,v}\}_{v\in\mathscr{V}_2}$ starting from the integers $\{b_v\}_{v\in\mathscr{V}}$ in a similar way as (10.4.11) defines $\{c_{1,v}\}_{v\in\mathscr{V}_2}$. By the assumption regarding a_v's, and $c_{1,u} > 0$, we obtain that the $E_u^*|_{\mathscr{V}_2}$-coefficient $b_{1,u}$ of $l'|_{\mathscr{V}_2}$ satisfies

$$b_{1,u} = \sum_{v\in\mathscr{V}_{1,u}} b_v E_v^*|_u/E_u^*|_u > \sum_{v\in\mathscr{V}_{1,u}} a_v E_v^*|_u/E_u^*|_u \geq d_{1,u} := \sum_{v\in\mathscr{V}_{1,u}} (\kappa_v - 2)E_v^*|_u/E_u^*|_u.$$

In particular, by Lemma 10.3.20, $l'|_{\mathscr{V}_2}$ is not in the support of $Z_h(\mathbf{t}_{\mathscr{V}_2})$.

This fact reads as follows. For any element u of $\mathscr{B}_2 := \{u \in \mathscr{B} : \kappa_{2,u} \geq 2\}$ set

$$\mathrm{Supp}_u := \{l' \in \mathrm{Supp}(Z) : [l'] = [l'_0],\ l'|_{\mathscr{I}} \prec l'_0|_{\mathscr{I}},\ \ b_{1,u} > d_{1,u}\}.$$

Then we claim that the sub-summation $\sum \mathfrak{z}(l')$ over $l' \in \cup_{u\in\mathscr{B}_2}\mathrm{Supp}_u$ is zero.

Indeed, write $Z(\mathbf{t}_{\mathscr{V}_2})$ as $Z_2(\mathbf{t}_{\mathscr{V}_2}) \cdot \prod_{u\in\mathscr{B}} Z_{1,u}(\mathbf{t}_{\mathscr{V}_2})$, as in the proof of Lemma 10.3.20. Then $Z_{1,u}(\mathbf{t}_{\mathscr{V}_2})$ collects the contribution from $\Gamma_{1,u}$ (as in the proof of 10.3.20). Then, if $u \in \mathscr{B}_2$ then the coefficient $b_{1,u}$ is larger than the $E_u^*|_{\mathscr{V}_2}$-degree of $Z_{1,u}(\mathbf{t}_{\mathscr{V}_2})$, and by Lemma 10.3.20*(b)* $l'|_{\mathscr{V}_2}$ is not in the support of $Z_h(\mathbf{t}_{\mathscr{V}_2})$. This means that the sum over Supp_u of the corresponding coefficients is zero. Furthermore, in the product $Z_2(\mathbf{t}_{\mathscr{V}_2}) \cdot \prod_{u\in\mathscr{B}} Z_{1,u}(\mathbf{t}_{\mathscr{V}_2})$, due to Lemma 10.3.20*(a)*, there is no additional cancellation of the monomials of the factors. Hence the summation over the whole $\cup_{u\in\mathscr{B}_2}\mathrm{Supp}_u$ is zero as well.

Hence, up to these zero sums in the 'modified counting function', we can consider only the cycles l' from $\{\mathrm{Supp}(Z_h) : (l' - l'_0)|_{\mathscr{I}} \prec 0\} \setminus \cup_{u\in\mathscr{B}_2}\mathrm{Supp}_u$. But by the above discussion such a cycle satisfies $m_u < 0$ for every $u \in \mathscr{B}$, hence $m_u < 0$ for every $u \in \overline{\mathscr{I}}$ (by the 'easy case'). Hence $q_{h,\mathscr{I}}(l'_0) = q_{h,\overline{\mathscr{I}}}(l'_0)$. □

10.4.21 For any $h \in H$ and $\mathscr{I} \subset \mathscr{V}$ let $\overline{Q}_{h,\mathscr{I}}$ and $\overline{q}_{h,\mathscr{I}}$ denote the quasipolynomials of $Q_{h,\mathscr{I}}$ and $q_{h,\mathscr{I}}$ in the Lipman's cone $\mathscr{S}_{\mathbb{R}}$ (or, equivalently, the counting functions $Q_{h,\mathscr{I}}$ and $q_{h,\mathscr{I}}$ restricted on $l'_* + \mathscr{S}'$ for a convenient l'_*).

Note that $\mathscr{V}^e(\Gamma(\overline{\mathscr{J}})) \subset \mathscr{J}$ for any nonempty subset $\mathscr{J} \subset \mathscr{V}$. The subset $\mathscr{J}$ is called *boundary* if either $|\mathscr{J}| = 1$ or $\mathscr{V}^e(\Gamma(\overline{\mathscr{J}})) = \mathscr{J}$. Let Θ be the set of boundary subsets. Note that if $\mathscr{B} \in \Theta$ and $\emptyset \neq \mathscr{B}_0 \subset \mathscr{B}$ then $\mathscr{B}_0 \in \Theta$ too. Furthermore, $\mathscr{V}^e(\Gamma(\overline{\mathscr{J}})) \in \Theta$ for any $\mathscr{J} \subset \mathscr{V}$, $\mathscr{J} \neq \emptyset$.

Next let us fix some $h \in H$ and $\mathscr{I} \subset \mathscr{V}$, $\mathscr{I} \neq \emptyset$. We also write $\Theta_{\mathscr{I}} := \{\mathscr{B} \in \Theta : \overline{\mathscr{B}} \cap \mathscr{I} = \mathscr{B}\}$. Again,

$$\text{if } \mathscr{B} \in \Theta_{\mathscr{I}} \text{ and } \emptyset \neq \mathscr{B}_0 \subset \mathscr{B} \text{ then } \mathscr{B}_0 \in \Theta_{\mathscr{I}} \text{ too.} \tag{10.4.14}$$

By (10.4.8) we have $\overline{Q}_{h,\mathscr{I}} = \sum_{\mathscr{J}\subset\mathscr{I},\ \mathscr{J}\neq\emptyset} (-1)^{|\mathscr{J}|+1} \cdot \overline{q}_{h,\mathscr{J}}$.

Next, in this sum we group the subsets $\mathscr{J}$ with the same $\mathscr{B} = \mathscr{V}^e(\Gamma(\overline{\mathscr{J}})) \in \Theta$, and then we apply Proposition 10.4.19:

$$\sum_{\substack{\mathscr{J}\subset\mathscr{I}\\ \mathscr{J}\neq\emptyset}} \overline{q}_{h,\mathscr{J}}(-1)^{|\mathscr{J}|+1} = \sum_{\substack{\mathscr{B}\in\Theta\\ \mathscr{B}\subset\mathscr{I}}} \overline{q}_{h,\overline{\mathscr{B}}} \sum_{\substack{\mathscr{J}\subset\mathscr{I}\\ \overline{\mathscr{J}}=\overline{\mathscr{B}}}} (-1)^{|\mathscr{J}|+1} = \sum_{\substack{\mathscr{B}\in\Theta\\ \mathscr{B}\subset\mathscr{I}}} \overline{q}_{h,\mathscr{B}} \sum_{\substack{\mathscr{J}\subset\mathscr{I}\\ \overline{\mathscr{J}}=\overline{\mathscr{B}}}} (-1)^{|\mathscr{J}|+1}.$$

Note that for $\mathscr{B} \in \Theta$ fixed $\{\mathscr{J} : \mathscr{J} \subset \mathscr{I},\ \overline{\mathscr{J}} = \overline{\mathscr{B}}\} = \{\mathscr{J} : \mathscr{B} \subset \mathscr{J} \subset \mathscr{I} \cap \overline{\mathscr{B}}\}$. Hence $\sum_{\mathscr{J}\subset\mathscr{I},\ \overline{\mathscr{J}}=\overline{\mathscr{B}}}(-1)^{|\mathscr{J}|+1} = 0$ if $\mathscr{B} \notin \Theta_{\mathscr{I}}$ and $= (-1)^{|\mathscr{B}|+1}$ otherwise. Thus

$$\overline{Q}_{h,\mathscr{I}} = \sum_{\mathscr{B}\in\Theta_{\mathscr{I}}} \overline{q}_{h,\mathscr{B}}(-1)^{|\mathscr{B}|+1}. \tag{10.4.15}$$

This identity applied for some $\mathscr{B}' \in \Theta_{\mathscr{I}}$ reads as

$$\overline{Q}_{h,\mathscr{B}'} = \sum_{\mathscr{B}\in\Theta_{\mathscr{B}'}} \overline{q}_{h,\mathscr{B}}(-1)^{|\mathscr{B}|+1} = \sum_{\emptyset\neq\mathscr{B}\subset\mathscr{B}'} \overline{q}_{h,\mathscr{B}}(-1)^{|\mathscr{B}|+1}. \tag{10.4.16}$$

Therefore, by combinatorial inversion, for $\mathscr{B}' \in \Theta_{\mathscr{I}}$ one also has

$$\overline{q}_{h,\mathscr{B}'} = \sum_{\emptyset\neq\mathscr{B}\subset\mathscr{B}'} \overline{Q}_{h,\mathscr{B}}(-1)^{|\mathscr{B}|+1}. \tag{10.4.17}$$

All these combined give

$$\overline{Q}_{h,\mathscr{I}} = \sum_{\mathscr{B}\in\Theta_{\mathscr{I}}} \overline{Q}_{h,\mathscr{B}} \sum_{\mathscr{B}\subset\mathscr{B}'\in\Theta_{\mathscr{I}}} (-1)^{|\mathscr{B}'\setminus\mathscr{B}|}. \tag{10.4.18}$$

Hence any $\overline{Q}_{h,\mathscr{I}}$ is a linear combination of the quasipolynomials $\{\overline{Q}_{h,\mathscr{B}}\}_{\mathscr{B}\in\Theta_{\mathscr{I}}}$.

Example 10.4.22 ([354]) If $\mathscr{I} = \mathscr{V}$ then $\Theta_{\mathscr{V}} = \{\{v\}_{v\in\mathscr{V}}, \{u,v\}_{(u,v)\in\mathscr{E}}\}$, and

$$\overline{Q}_{h,\mathscr{V}} = \sum_{v\in\mathscr{V}} \overline{q}_{h,\{v\}} - \sum_{(u,v)\in\mathscr{E}} \overline{q}_{h,\{u,v\}} = \sum_{v\in\mathscr{V}} (1-\kappa_v)\cdot \overline{Q}_{h,\{v\}} + \sum_{(u,v)\in\mathscr{E}} \overline{Q}_{h,\{u,v\}}.$$

10.4.B Additivity Formula for the Modified Counting Functions

10.4.23 Let Γ be as in 10.3. We consider the 'additivity setup' as in 9.3.A. In particular, we choose some $v \in \mathscr{V}$, and write $\Gamma\setminus v = \cup_k \Gamma_k$ as the union of connected

components. Let $R : L'(\Gamma) \to L'(\Gamma \setminus v)$ and $R_k : L'(\Gamma) \to L'(\Gamma_k)$ be the dual cohomological restriction operators defined in 9.3.8.

Since we need to discuss invariants associated with different graphs simultaneously, we use superscripts to identify the corresponding graph.

Proposition 10.4.24 ([356]) *Fix one of the components, say $\Gamma_{k'}$, and let $\mathscr{I} \subset \mathscr{V}(\Gamma_{k'})$, $\mathscr{I} \neq \emptyset$. Then for any $l'_0 \in \sum_w (\kappa_w - 2)E^*_w + \mathscr{S}'$ with $[l'_0] = h$ one has*

$$q^{\Gamma}_{h,\mathscr{I}}(l'_0) - q^{\Gamma}_{h,\mathscr{I}\cup v}(l'_0) = q^{\Gamma_{k'}}_{[R_{k'}(l'_0)],\mathscr{I}}(R_{k'}(l'_0)). \tag{10.4.19}$$

In fact, if we write $Z^{\Gamma\setminus v}(\mathbf{t}_{\mathscr{V}\setminus v}) = \prod_{k''} Z^{\Gamma_{k''}}(\mathbf{t}_{\mathscr{V}(\Gamma_{k''})})$ and we note that $Z^{\Gamma_{k''}}(\mathbf{t}_{\mathscr{I}}) \equiv 1$ for $k'' \neq k'$, then the right hand side of (10.4.19) equals $q^{\Gamma\setminus v}_{[R(l'_0)],\mathscr{I}}(R(l'_0))$ too.

Proof The proof is divided into three steps.

Step 1. First we assume that v is an end-vertex of Γ, and the adjacent vertex w has $\kappa_w = 2$. Partly, we will follow the strategy of the proof of Proposition 9.6.15. We will use the notations of the proof of Proposition 10.4.19: $l' = \sum_u b_u E^*_u$, $l'_0 = \sum_u a_u E^*_u$, $l = l' - l'_0 \in L$, $l = \sum_u c_u E^*_u = \sum_u m_u E_u$. Define also

$$\begin{aligned}\mathrm{Supp}(\Gamma) &:= \{l' \,:\, l'|_{\mathscr{I}} \prec 0,\ m_v \geq 0\} \cap \mathrm{Supp}(Z^{\Gamma}(\mathbf{t})),\\ \mathrm{Supp}(\Gamma\setminus v) &:= \{\tilde{l}' \,:\, \tilde{l}'|_{\mathscr{I}} \prec R(l'_0)|_{\mathscr{I}},\ [\tilde{l}'] = [R(l'_0)]\} \cap \mathrm{Supp}(Z^{\Gamma\setminus v}(\mathbf{t}_{\mathscr{V}\setminus v})).\end{aligned}$$

In the left (respectively right) hand side of (10.4.19) we summation is over $\mathrm{Supp}(\Gamma)$ (respectively $\mathrm{Supp}(\Gamma\setminus v)$). It is convenient to make the change of variables $x_u := \mathbf{t}^{E^*_u}$ ($u \in \mathscr{V}$), hence $Z^{\Gamma}(\mathbf{t})$ becomes $Z^{\Gamma}(\mathbf{x}) = \prod_u (1-x_u)^{\kappa_u - 2}$. In particular,

$$Z^{\Gamma}(\mathbf{x}) = Z_0(\mathbf{x}_0)\cdot(1-x_v)^{-1}, \quad \text{and} \quad Z^{\Gamma\setminus v}(\mathbf{x}) = Z_0(\mathbf{x}_0)\cdot(1-x_w)^{-1}, \tag{10.4.20}$$

where $Z_0(\mathbf{x}_0)$ is a series in variable $\{x_u\}_{u\in\mathscr{V}\setminus\{v,w\}}$ only. Therefore, if $M_0 \in \mathrm{Supp}(Z_0)$ appears in the monomial $a_0\mathbf{x}_0^{M_0}$ then for any $k \geq 0$ the monomial $a_0\mathbf{x}_0^{M_0}x_v^k$ appears in Z^{Γ}, and $a_0\mathbf{x}_0^{M_0}x_w^k$ appears in $Z^{\Gamma\setminus v}$. Note that if $l' \in \mathrm{Supp}(\Gamma)$ then $m_v \geq 0$ and $b_v \geq 0$. Moreover, since $\kappa_w = 2$, then $b_w = 0$ whenever $l' \in \mathrm{Supp}(Z^{\Gamma}(\mathbf{t}))$.

If we apply R to the identity $l' - l'_0 = l$ we get

$$R(l') - R(l'_0) = -m_v E^{\Gamma\setminus v,*}_w + \sum_{u\in\mathscr{V}\setminus v} m_u E^{\Gamma\setminus v}_u.$$

If we write $l' = \bar{l}' + b_v E^*_v$ with $\bar{l}' := \sum_{u\neq w,v} b_u E^*_u$ then $R(l') = \tilde{l}' := \sum_{u\neq w,v} b_u E^{\Gamma\setminus v,*}_u$. Moreover, $l' \in \mathrm{Supp}(Z^{\Gamma})$ if and only if $\prod_{u\neq v,w} x_u^{b_u} \in$

$\mathrm{Supp}(Z_0)$, hence $R(l') + m_v E_w^{\Gamma\setminus v,*} \in \mathrm{Supp}(Z^{\Gamma\setminus v})$. In particular, $\Phi : \mathrm{Supp}(\Gamma) \to \mathrm{Supp}(\Gamma \setminus v)$, $l' \mapsto R(l') + m_v E_w^{\Gamma\setminus v,*}$ is well-defined.

Write also $\bar{l}'_0 = \sum_{u\neq w,v} a_u E_u^*$. Since $m_v = -(l, E_v^*)$ one has

$$m_v = -(\bar{l}' - \bar{l}'_0, E_v^*) + a_w(E_w^*, E_v^*) - (b_v - a_v)(E_v^*, E_v^*). \tag{10.4.21}$$

Hence Φ (that is, $(\bar{l}', b_v) \mapsto (\tilde{l}', m_v)$) is injective. Furthermore, for any $(\bar{l}', m_v) \in \mathrm{Supp}(\Gamma \setminus v)$ the Eq. (10.4.21) provides a unique well-defined candidate for b_v, such that $(\bar{l}', b_v)$ satisfies all the requirements of the elements of $\mathrm{Supp}(\Gamma)$ except maybe $b_v \geq 0$ (cf. (10.4.20)). Define $\mathrm{Supp}(\Gamma \setminus v)^{\geq 0}$ as a subset of $\mathrm{Supp}(\Gamma \setminus v)$ consisting of those elements $(\bar{l}', m_v)$ for which b_v computed via (10.4.21) is ≥ 0; that is, $\mathrm{Supp}(\Gamma \setminus v)^{\geq 0} = \mathrm{im}(\Phi)$. Then, using the bijection Φ onto its image and (10.4.20)

$$\sum_{l'\in \mathrm{Supp}(\Gamma)} \mathfrak{z}^{\Gamma}(l') \;=\; \sum_{(\tilde{l}', m_v)\in \mathrm{Supp}(\Gamma\setminus v)^{\geq 0}} \mathfrak{z}^{\Gamma\setminus v}(\tilde{l}' + m_v E^{\Gamma\setminus v,*}).$$

Set $\mathrm{Supp}(\Gamma \setminus v)^{<0} := \mathrm{Supp}(\Gamma \setminus v) \setminus \mathrm{Supp}(\Gamma \setminus v)^{\geq 0}$. In order to finish the proof of (10.4.19), we need

$$\sum_{(\tilde{l}', m_v)\in \mathrm{Supp}(\Gamma\setminus v)^{<0}} \mathfrak{z}^{\Gamma\setminus v}(\tilde{l}' + m_v E^{\Gamma\setminus v,*}) = 0. \tag{10.4.22}$$

For this we follow a similar argument as in the last part of the proof of Proposition 10.4.19. First note that using 10.4.19 we can assume that $\mathscr{I} = \overline{\mathscr{I}}$. Let u_0 be the vertex of $\mathscr{I}$, which is closest to v, and let $[u_0, w]$ be the geodesic string connecting u_0 and w in Γ. Then we write $\mathscr{V}_2 := \overline{\mathscr{I} \cup w} \subset \mathscr{V}(\Gamma \setminus v)$. Consider the homological trivial cycle $C(l) = \sum_{v\in\mathscr{V}}(m_v E_v + c_v E_v^*)$ (of Γ) for some conveniently chosen cuts of E_v's. Let $C(l)|_{\mathscr{V}_2}$ be its restriction on $\mathscr{V}_2$. Since $\mathscr{V}_2 = \mathscr{I} \cup [u_0, w]$, we can concentrate on the $c_{1,u}$-coefficients, defined similarly as in the proof of Proposition 10.4.19, for any u along the path $[u_0, w]$, $u \neq w$. Let Supp_u be the collection of parameter values $\{(\tilde{l}', m_v) : c_{1,u} > 0\}$ and $\mathrm{Supp} = \cup_u \mathrm{Supp}_u$, where the union is over such vertices of $[u_0, w]$, $u \neq w$. Then similarly as in that proof, based on Lemma 10.3.20, the summation of coefficients $\mathfrak{z}^{\Gamma\setminus v}(\tilde{l}' + m_v E^{\Gamma\setminus v,*})$ over Supp is zero.

So, let us take some $(\tilde{l}', m_v) \in \mathrm{Supp}(\Gamma \setminus v)^{<0} \setminus \mathrm{Supp}$. Recall that it was obtained by a certain restriction of $C(l)$. In fact, it is much more convenient to look at the restriction of $C(l)$ to $[u'_0, v]$, where u'_0 is adjacent to u_0 in the path $[u_0, v]$. Indeed, $C(l)|_{[u'_0,v]}$ has the following properties: it has a cut at v with multiplicity $b_v - a_v$, no cut at w, the other interior cut–multipicities $c_{1,u}$ are ≤ 0, and the cut–multiplicity at u'_0 is $c_{1,u'_0} + m_{u_0} < 0$ (since $c_{1,u'_0} \leq 0$ and $m_{u_0} < 0$ due to the fact that $u_0 \in \mathscr{I}$). On the other hand, the multiplicity m_v of E_v is non-negative. This can happen (by the second paragraph of 10.3.18) only if $b_v - a_v > 0$. But by the assumption, $a_v \geq \kappa_v - 2 = -1$, hence $b_v \geq 0$, a contradiction. Hence $\mathrm{Supp}(\Gamma \setminus v)^{<0} \setminus \mathrm{Supp} = \emptyset$.

Step 2. Next, we assume that v is an end-vertex of Γ, and the adjacent vertex w has $\kappa_w \geq 3$. Then we reduce this case to the previous *Step 1*: we blow up the edge connecting v and w, then we apply *Step 1*. For this we have to verify that the modified counting functions are stable with respect to this operation.

When we blow up the edge (v, w), then we create a new graph, denoted by $\widetilde{\Gamma}$ with a newly created base element $\widetilde{E}_{new} \in L(\widetilde{\Gamma})$. There is a natural projection $\rho : L(\widetilde{\Gamma}) \to L(\Gamma)$, and its dual $\rho^* : L'(\Gamma) \to L'(\widetilde{\Gamma})$, which satisfy the projection formula $(\rho^*(l'), \widetilde{l}) = (l', \rho(\widetilde{l}))$. In particular, $\rho^*(E_u^*) = \widetilde{E}_u^*$ (with natural notations). Hence if we denote by $\widetilde{\mathscr{I}} \subset \mathscr{V}(\widetilde{\Gamma})$ the same index set as $\mathscr{I} \subset \mathscr{V}(\Gamma)$, then

$$\mathfrak{z}^{\Gamma}_{[l'_0]}(l') = \mathfrak{z}^{\widetilde{\Gamma}}_{[\rho^*(l'_0)]}(\rho^*(l')) \quad \text{and} \quad q^{\Gamma}_{[l'_0],\mathscr{I}}(l'_0) = q^{\widetilde{\Gamma}}_{[\rho^*(l'_0)],\widetilde{\mathscr{I}}}(\rho^*(l'_0)). \tag{10.4.23}$$

Next, $\widetilde{\Gamma} \setminus v$ is the graph obtained from $\Gamma \setminus v$ by blowing up the vertex w. By definition, $Z^{\widetilde{\Gamma}\setminus v}$ has the shape $\rho^*(Z^{\Gamma\setminus v}(\mathbf{t})) \cdot (1 - \mathbf{t}^{\widetilde{E}^*_w})/(1 - \mathbf{t}^{\widetilde{E}^*_{new}})$, where $\rho^*(\sum z(l')\mathbf{t}^{l'}) = \sum z(l')\mathbf{t}^{\rho^*(l')}$. Note that $\widetilde{E}^*_{new} = \widetilde{E}^*_w + \widetilde{E}_{new}$, hence $\mathbf{t}^{\widetilde{E}^*_{new}} = \mathbf{t}^{\widetilde{E}^*_w} \cdot t_{new}$. Therefore, when we restrict it to the $\widetilde{\mathscr{I}}$ variables and we substitute $t_{new} = 1$, the term $(1 - \mathbf{t}^{\widetilde{E}^*_w})/(1 - \mathbf{t}^{\widetilde{E}^*_{new}})$ becomes 1. Hence, the coefficients of the reduced series associated with $\Gamma \setminus v$ and $\widetilde{\Gamma} \setminus v$ can be compared as in the previous case (10.4.23).

Step 3. Now we consider the general situation. We prove (10.4.19) by induction over $\sum_{k\neq k'} |\mathscr{V}(\Gamma_k)|$. If this sum is zero, then we apply *Step 2*. Consider the general situation, and let $e \in \cup_{k\neq k'}\mathscr{V}(\Gamma_k)$ be an end vertex of Γ. Then, by *Step 1–Step 2*

$$q^{\Gamma}_{[l'_0],\mathscr{I}}(l'_0) - q^{\Gamma}_{[l'_0],\mathscr{I}\cup e}(l'_0) = q^{\Gamma\setminus e}_{[R(l'_0)],\mathscr{I}}(R(l'_0)),$$

$$q^{\Gamma}_{[l'_0],\mathscr{I}\cup v}(l'_0) - q^{\Gamma}_{[l'_0],\mathscr{I}\cup v\cup e}(l'_0) = q^{\Gamma\setminus e}_{[R(l'_0)],\mathscr{I}\cup v}(R(l'_0)).$$

Since $q^{\Gamma}_{[l'_0],\mathscr{I}\cup e}(l'_0) = q^{\Gamma}_{[l'_0],\mathscr{I}\cup v\cup e}(l'_0)$ by Proposition 10.4.19, and

$$q^{\Gamma\setminus e}_{[R(l'_0)],\mathscr{I}}(R(l'_0)) = q^{\Gamma\setminus e}_{[R(l'_0)],\mathscr{I}\cup v}(R(l'_0)) + q^{\Gamma_{k'}}_{[R_{k'}(l'_0)],\mathscr{I}}(R_{k'}(l'_0))$$

by the inductive step, the identity (10.4.19) follows. □

10.5 General Additivity Formulae

In this section we prove a general additivity formula for quasipolynomials associated with $Q_{h,\mathscr{I}}$. At the level of their periodic constants we obtain a general additivity formula for the (normalized) Seiberg–Witten invariants as well. In this section we follow [356]. ♣

10.5.1 Recall from 10.4.3 that if $\mathscr{I} \subset \mathscr{V}$, $\mathscr{I} \neq \emptyset$ and $h \in H$ one defines

$$Q^{\Gamma}_{h,\mathscr{I}} : \{l' \in L' \,:\, [l'] = h\} \to \mathbb{Z}, \quad Q^{\Gamma}_{h,\mathscr{I}}(l') = \sum_{l''|_{\mathscr{I}} \not\geq l'|_{\mathscr{I}},\ [l'']=h} \mathfrak{z}(l'').$$

In fact, $Q_{h,\mathscr{I}}$ equals the counting function of the reduced series $Z_h(\mathbf{t}_{\mathscr{I}})$ too.

The setup of the additivity formula is the following. We fix $h \in H$, and we choose $l' = \sum_{v\in\mathscr{V}} a_v E^*_v \in L'$ with $[l'] = h$. We also fix $\mathscr{I} \subset \mathscr{V}$, $\mathscr{I} \neq \emptyset$, and $\mathscr{V} \setminus \mathscr{I}$ is the vertex set of $\cup_k \Gamma_k$, of disjoint connected subgraphs Γ_k. The restriction operator is denoted by $R_k : L'(\Gamma) \to L'(\Gamma_k)$.

Theorem 10.5.2 ([356]) *For any l' with every a_v sufficiently large one has*

$$Q^{\Gamma}_{[l']}(l') = Q^{\Gamma}_{[l'],\mathscr{I}}(l') + \sum_k Q^{\Gamma_k}_{[R_k(l')]}(R_k(l')). \tag{10.5.1}$$

Proof We prove (10.5.1) by induction on the cardinality of $\mathscr{I}$.
Step 1. Assume that $\mathscr{I} = \{v\}$.

We rewrite the identity (10.5.1) in terms of modified counting functions (as in (10.4.8)), where we use the notation $\mathscr{J}$ for the corresponding subsets. For the last sum we have to consider nonempty subsets $\mathscr{J} \subset \mathscr{V} \setminus v$. Hence, it is natural to organize the nonempty subsets of $\mathscr{V}$ as $\{v\} \cup \{\mathscr{J}, \mathscr{J} \cup v\}_{\mathscr{J}\subset\mathscr{V}\setminus v,\ \mathscr{J}\neq\emptyset}$. The modified counting function $q^{\Gamma}_{[l'],v}$ associated with the left hand side cancels the term $Q^{\Gamma}_{[l'],v}(l')$ from the right hand side of (10.5.1). Hence the left hand side of (10.5.1) becomes an alternating sum of expressions of type $q^{\Gamma}_{[l'],\mathscr{J}}(l') - q^{\Gamma}_{[l'],\mathscr{J}\cup v}(l')$. If $\mathscr{J}$ is not contained totally in only one $\mathscr{V}(\Gamma_k)$ then $v \in \overline{\mathscr{J}}$, hence this expression is zero by Proposition 10.4.19. Therefore, we can assume that there exists k such that $\mathscr{J} \subset \mathscr{V}(\Gamma_k)$. Hence the expression (10.5.1) decomposes as a sum over k according to such inclusions. Then the needed identity is the subject of Proposition 10.4.24.
Step 2. Next, we take $\mathscr{I}$ with $|\mathscr{I}| \geq 2$, and we assume, by the inductive step, that the identity (10.5.1) is true for any graph Γ', any $h' \in H(\Gamma')$, and any subset $\mathscr{I}' \subset \mathscr{V}(\Gamma')$ with $|\mathscr{I}'| < |\mathscr{I}|$.

Recall that $\mathscr{V} \setminus \mathscr{I} = \cup_k \mathscr{V}(\Gamma_k)$, and we wish to prove (10.5.1).

We choose some $v \in \mathscr{I}$, and we apply the inductive step for $\Gamma \setminus v$ and $\mathscr{I} \setminus v$. Let $\Gamma \setminus v = \cup_j \Gamma_{v,j}$ be the connected components of $\Gamma \setminus v$, and let $R_{v,j}$ be the corresponding dual operators. Note that if Γ_k is contained in $\Gamma_{v,j}$ then $R_{\Gamma_k\subset\Gamma_{v,j}} \circ R_{v,j} = R_k$. In particular, we get the following identity

$$\sum_j Q^{\Gamma_{v,j}}_{[R_{v,j}(l')]}(R_{v,j}(l')) = \sum_j Q^{\Gamma_{v,j}}_{[R_{v,j}(l')],\mathscr{I}\setminus v}(R_{v,j}(l')) + \sum_k Q^{\Gamma_k}_{[R_k(l')]}(R_k(l')). \tag{10.5.2}$$

By induction this identity is valid for any $\tilde{l}'$ (instead of $R_{\Gamma\setminus v\subset\Gamma}(l') = \oplus_j R_{v,j}(l')$) from the lattice of $\Gamma\setminus v$ (satisfying the required assumptions that its E^*-coefficients are sufficiently high). Hence it is true also for $\tilde{l}' = \oplus_j R_{v,j}(l')$, and this identity, in this way, will be considered as a quasipolynomial identity in variable $l' \in L'$ (cf. discussion from 10.5.3). The difference between the needed identity (10.5.1) and (10.5.2) is

$$Q^{\Gamma}_{[l']}(l') - \sum_j Q^{\Gamma_{v,j}}_{[R_{v,j}(l')]}(R_{v,j}(l')) = Q^{\Gamma}_{[l'],\mathscr{I}}(l') - \sum_j Q^{\Gamma_{v,j}}_{[R_{v,j}(l')],\mathscr{I}\setminus v}(R_{v,j}(l')).$$

This identity (via induction) is equivalent to (10.5.1). But, for its left hand side one can apply the statement of *Step 1*. In particular, (10.5.1) is equivalent to

$$Q^{\Gamma}_{[l'],\mathscr{I}}(l') - Q^{\Gamma}_{[l'],v}(l') = \sum_j Q^{\Gamma_{v,j}}_{[R_{v,j}(l')],\mathscr{I}\setminus v}(R_{v,j}(l')). \tag{10.5.3}$$

Next, we rewrite the identity (10.5.3) in terms of modified counting functions by the same principle as in *Step 1*. For the last sum we have to consider nonempty subsets $\mathscr{J}\subset\mathscr{I}\setminus v$, hence, again as above, we organize the nonempty subsets of $\mathscr{I}$ as $\{v\}\cup\{\mathscr{J},\ \mathscr{J}\cup v\}_{\mathscr{J}\subset\mathscr{I}\setminus v,\ \mathscr{J}\neq\emptyset}$. The modified counting function associated with $\{v\}$ cancels the second term of (10.5.3). Hence the left hand side of (10.5.3) is again a combination of expressions of type $q^{\Gamma}_{h,\mathscr{J}}(l') - q^{\Gamma}_{h,\mathscr{J}\cup v}(l')$. If $\mathscr{J}$ is not contained totally in only one $\mathscr{V}(\Gamma_{v,j})$ then $v\in\overline{\mathscr{J}}$, hence this expression is zero by Proposition 10.4.19. Hence we can assume that there exists j such that $\mathscr{J}\subset\mathscr{V}(\Gamma_{v,j})$ and the expression (10.5.3) also decomposes as a sum over j according to this inclusion, and it becomes the statement of Proposition 10.4.24. □

10.5.3 Let us deduce some consequences and corollaries. Write $l' = r_h + l$. By Corollary 10.3.2, if every a_v is large, then $Q^{\Gamma}_{[l']}(l')$ equals the quasipolynomial $\mathfrak{Q}^{\Gamma}_h(l)$, and the same is true for each term in the last sum (by the same statement applied for Γ_k). Therefore, the identity (10.5.1) guarantees that for every a_v large $Q^{\Gamma}_{[l'],\mathscr{I}}(l')$ is a quasipolynomial as well. Let us define the quasipolynomial

$$\begin{aligned}\overline{Q}^{\Gamma}_{h,\mathscr{I}}(l') := {} & \mathfrak{sw}_{\sigma[K+2l']}(M(\Gamma)) - \frac{(K+2l')^2+|\mathscr{V}|}{8} \\ & - \sum_k \left(\mathfrak{sw}_{\sigma[R_k(K+2l')]}(M(\Gamma_k)) - \frac{(R_k(K+2l'))^2+|\mathscr{V}(\Gamma_k)|}{8}\right),\end{aligned} \tag{10.5.4}$$

and set also, as usual, the quasipolynomial $\mathfrak{Q}^{\Gamma}_{h,\mathscr{I}}(l) := \overline{Q}^{\Gamma}_{h,\mathscr{I}}(l')$ defined on L.

Corollary 10.5.4 *For $h \in H$ fixed and $l'_0 = r_h + l$, $l \in L$, if all a_v are sufficiently large, the counting function $Q^{\Gamma}_{h,\mathscr{J}}(l')$ equals the quasipolynomial $\mathfrak{Q}^{\Gamma}_{h,\mathscr{J}}(l)$.*

Since $R_k(K(\Gamma)) = K(\Gamma_k)$, the above formula in (10.5.4) transforms accordingly.

Except for the free term, $\mathfrak{Q}^{\Gamma}_{h,\mathscr{J}}(l)$ is a multivariable quadratic polynomial. However, the free term is a *periodic* function $L \to \mathbb{Q}$. Indeed, $l \mapsto \{[R_k(r_h + l)]\}_k$, in general, is not constant. However, if we define $\widetilde{L} \subset L$ as the kernel of $L \to \oplus_k H(\Gamma_k)$, $l \mapsto \oplus_k [R_k(l)]$, then the free term of $\mathfrak{Q}^{\Gamma}_{h,\mathscr{J}}(l)|_{l\in\widetilde{L}}$ is constant as well.

Corollary 10.5.5 *For any $h \in H$ fixed, the counting function $Q^{\Gamma}_{h,\mathscr{J}}(l')$ ($l' \in r_h + L$) of $Z_h(\mathbf{t}_{\mathscr{J}})$ admits a quasipolynomial in the Lipman's cone $\mathscr{S}_{\mathbb{R}}$, namely $\mathfrak{Q}^{\Gamma}_{h,\mathscr{J}}(l)$, and its periodic constant is*

$$\mathrm{pc}^{\mathscr{S}_{\mathbb{R}}}(Z_h(\mathbf{t}_{\mathscr{J}})) = \mathfrak{Q}^{\Gamma}_{h,\mathscr{J}}(0) = \overline{Q}^{\Gamma}_{h,\mathscr{J}}(r_h) = \mathfrak{sw}_{\sigma_{[K+2r_h]}}(M(\Gamma)) - \frac{(K+2r_h)^2 + |\mathscr{V}|}{8}$$
$$- \sum_k \left(\mathfrak{sw}_{\sigma_{[K(\Gamma_k)+R_k(2r_h)]}}(M(\Gamma_k)) - \frac{(K(\Gamma_k)+2R_k(r_h))^2 + |\mathscr{V}(\Gamma_k)|}{8} \right).$$

10.5.6 Here we considered $Q^{\Gamma}_{[l'],\mathscr{J}}(l')$ (and $\mathfrak{Q}^{\Gamma}_{h,\mathscr{J}}(l)$) as functions in $l' \in r_h + L$ (or in $l \in L$). This is the right point of view: when we take periodic constants of a sum of different quasipolynomials, one has to consider this operation in the same lattice. In this way the periodic constant will behave as an additive operator.

However, $Q^{\Gamma}_{[l'],\mathscr{J}}(l')$ depends only on the restriction $l'|_{\mathscr{J}}$, and basically counts the coefficients of the reduced series $Z_h(\mathbf{t}_{\mathscr{J}})$, hence it can be considered also as a counting function defined on the lattice $L|_{\mathscr{J}}$, via $l'|_{\mathscr{J}} = l|_{\mathscr{J}} + r_h|_{\mathscr{J}}$. In this way, its periodic constant in this lattice should be computed via a substitution of $l'|_{\mathscr{J}}$ by its representative in the semi-open cube associated with variables $\mathscr{J}$. But, the point is that this is exactly $r_h|_{\mathscr{J}}$ (since all the entries of $r_h|_{\mathscr{J}}$ are automatically in $[0, 1)$). Hence, the two periodic constants (computed in L or $L|_{\mathscr{J}}$) agree.

Corollary 10.5.7 *The series $Z_h(\mathbf{t}_{\mathscr{J}})$ admits a periodic constant in the real cone $\pi_{\mathscr{J}}(\mathscr{S}_{\mathbb{R}})$, and, in fact, $\mathrm{pc}^{\pi_{\mathscr{J}}(\mathscr{S}_{\mathbb{R}})}(Z_h(\mathbf{t}_{\mathscr{J}})) = \mathrm{pc}^{\mathscr{S}_{\mathbb{R}}}(Z_h(\mathbf{t}_{\mathscr{J}}))$. In particular, the formula from Corollary 10.5.5 applies for $\mathrm{pc}^{\pi_{\mathscr{J}}(\mathscr{S}_{\mathbb{R}})}(Z_h(\mathbf{t}_{\mathscr{J}}))$ too.*

Example 10.5.8 In the next examples we list some particular cases of Corollaries 10.5.5–10.5.7.

(i) If $\mathscr{J} = \mathscr{V}$ then each Γ_k is empty, and we recover Corollary 10.3.2.
(ii) If $\mathscr{J}$ consists of one vertex then we recover Theorem 9.6.5.

(iii) Assume next that $\mathscr{I} = \mathscr{N}$. Then we recover Theorem 10.3.10*(c)*, once we verify for each k the vanishing

$$\mathfrak{sw}_{\sigma_{[K(\Gamma_k)+R_k(2r_h)]}}(M(\Gamma_k)) - \frac{(K(\Gamma_k)+2R_k(r_h))^2+|\mathscr{V}(\Gamma_k)|}{8} = 0. \qquad (10.5.5)$$

In this case the graph Γ_k is a string, hence $M(\Gamma_k)$ is a lens space. This vanishing was already proved for $h = 0$ (see e.g. (9.7.2)), but it is not so evident for arbitrary h. The difficulty is that r_h is a global object induced from Γ, and for any fixed Γ_k it is not clear what classes of $L'(\Gamma_k)$ might appear as $R_k(r_h)$ for certain 'extension graphs' Γ. However, the vanishing (10.5.5) can be verified even directly (see [356, §7]).

(iv) Assume that $\mathscr{I}$ is an WR-set, that is, all the subgraphs Γ_k are *rational*.

Let $s_{h_k} \in L'(\Gamma_k) \cap \mathscr{S}'(\Gamma_k)$ be the minimal element associated with a certain $h_k \in H(\Gamma_k)$, as usual. Then, by Theorem 7.1.11*(a)* one has $h^1(\mathscr{O}(-s_{h_k})) = 0$. Hence, from the SWIC, valid for rational singularities cf. 9.7.15, we obtain

$$\mathfrak{sw}_{\sigma_{[K(\Gamma_k)+2s_{h_k}]}}(M(\Gamma_k)) - \frac{(K(\Gamma_k)+2s_{h_k})^2+|\mathscr{V}(\Gamma_k)|}{8} = 0. \qquad (10.5.6)$$

In particular, for $h_k = R_k(r_h)$, from (10.5.5) and (10.5.6) we obtain

$$\mathfrak{sw}_{\sigma_{[K(\Gamma_k)+R_k(2r_h)]}}(M(\Gamma_k)) - \frac{(K(\Gamma_k)+2R_k(r_h))^2+|\mathscr{V}(\Gamma_k)|}{8}$$
$$= \chi(R_k(r_h)) - \chi(s_{R_k(r_h)}).$$

Hence, Corollary 10.5.5 reads as

$$\mathfrak{sw}_{\sigma_{[K+2r_h]}}(M(\Gamma)) - \frac{(K+2r_h)^2+|\mathscr{V}|}{8} =$$
$$\mathrm{pc}^{\pi_{\mathscr{I}}(\mathscr{S}_{\mathbb{R}})}(Z_h(\mathbf{t}_{\mathscr{I}})) + \sum_k \big(\chi(R_k(r_h)) - \chi(s_{R_k(r_h)})\big).$$

Usually $s_{R_k(r_h)} \neq R_k(r_h)$, see the example from part (v).

An alternative formula can be obtained as follows. Consider the identity of quasipolynomials from (10.5.4), and substitute $l' = s_h = r_h + \Delta_h$ in it. The point is that in the last sum $R_k(s_h) = s_{R_k(s_h)}$ (see e.g. [354, Lemma 8.4.2]),

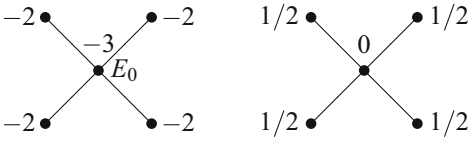
hence by the vanishing (10.5.6) applied for $s_{R_k(s_h)}$ the sum $\sum_k$ is zero, hence

$$\mathfrak{sw}_{\sigma[K+2s_h]}(M(\Gamma)) - \frac{(K+2s_h)^2+|\mathcal{V}|}{8} = \overline{\mathcal{Q}}^{\Gamma}_{h,\mathcal{I}}(s_h) = \mathfrak{Q}^{\Gamma}_{h,\mathcal{I}}(\Delta_h).$$

(v) Assume that each Γ_k is rational and $h = 0$. Then clearly $\sum_k \big(\chi(R_k(r_h)) - \chi(s_{R_k(r_h)})\big) = 0$, hence $\mathrm{pc}^{\mathcal{S}_{\mathbb{R}}}(Z_0(\mathbf{t})) = \mathrm{pc}^{\pi_{\mathcal{I}}(\mathcal{S}_{\mathbb{R}})}(Z_0(\mathbf{t}_{\mathcal{I}}))$.

Note that in case (iii) we have this identity for any h. However, in general, if $h \neq 0$ then even if we assume that $\mathcal{I}$ is an SR–set (in fact, even if we assume that the graph Γ itself is rational) the correction term $\chi(R_k(r_h)) - \chi(s_{R_k(r_h)})$ from (iv) is not necessarily zero. In such cases $\mathrm{pc}^{\mathcal{S}_{\mathbb{R}}}(Z_h(\mathbf{t})) \neq \mathrm{pc}^{\pi_{\mathcal{I}}(\mathcal{S}_{\mathbb{R}})}(Z_h(\mathbf{t}_{\mathcal{I}}))$. (This fact should be compared with the 'Reduction Theorems' 11.4.3 and 11.4.8.)

Indeed, consider the next graph and $r_h \in L'$ with E–multiplicities on the right.



Let $\mathcal{I}$ be the union of the four (-2)–vertices, hence Γ_1 consists of the (-3)–vertex v_0. Then $h_1 := R_1(r_h) = -2E_0^* = -2E_0/3$. Its representative $r_{h_1} = s_{h_1}$ is $E_0/3$, hence, $R_1(r_h) \neq s_{R_1(r_h)}$. Then, by a computation $\chi(R_1(r_h)) = 1$ and $\chi(s_{h_1}) = 0$.

(vi) The difference between the case (iii), when each Γ_k is a string in Γ, and the more general cases from (iv)–(v) can be seen in the structure of the exceptional curves of $\widetilde{c}^{-1}(\widetilde{X}(\Gamma_k))$ induced by the UAC (cf. 8.2.3). (Here $\widetilde{X}(\Gamma_k)$ is a small neighbourhood of $\cup_{v\in\mathcal{V}(\Gamma_k)} E_v$.) In the case (iii) this exceptional set is a union of strings (this is the main point in the proof of (iii) in [356]), while in the cases (iv)–(v) it is not necessarily a union of rational configurations. E.g., in the case of the example from (v), in the double covering associated with r_h above E_0 we get an elliptic curve.

Example 10.5.9 Let us take $h = 0$. We wish to compare the expressions $\mathfrak{Q}^{\Gamma}_{0,\mathcal{I}}(0) = \mathrm{pc}^{\mathcal{I}}(Z_0(\mathbf{t}_{\mathcal{I}}))$, respectively $\mathfrak{q}^{\Gamma}_{0,\mathcal{I}}(0) = \mathrm{mpc}^{\mathcal{I}}(Z_0(\mathbf{t}_{\mathcal{I}}))$, with the (Euler-type) summation of certain weighted cubes in some region R as in Lemma 9.6.9 *(d)*.

First take $l_0 \in L$, $l_0 \in Z_K + \mathcal{S}'$, effective. Then, for any nonempty $\mathcal{I} \subset \mathcal{V}$ we have an analogous $l_0|_{\mathcal{I}} \in L(\Gamma(\mathcal{I}))$, $l_0 \in Z_K(\Gamma(\mathcal{I})) + \mathcal{S}'(\Gamma(\mathcal{I}))$. Then, also define

$$\overline{\mathfrak{sw}}_{\sigma[K]}(M(\Gamma)) := \mathfrak{sw}_{\sigma[K]}(M(\Gamma)) - \frac{K^2+|\mathcal{V}|}{8}.$$

Then, via the notation $R := R(0, l_0)$, Lemma 9.6.9 and Corollary 9.6.20 imply

$$\mathrm{pc}^{\mathscr{S}}(Z_0(\mathbf{t})) = \mathfrak{Q}^{\Gamma}_{0,\mathscr{V}}(0) = \overline{\mathfrak{sw}}_{\sigma[K]}(M(\Gamma)) = Eu_{\chi}(R).$$

This applied for $\mathscr{V} \setminus \mathscr{I}$ we get

$$\overline{\mathfrak{sw}}_{\sigma[K]}(M(\Gamma \setminus \mathscr{I})) = Eu_{\chi}(R(0, l_0|_{\mathscr{V}\setminus\mathscr{I}})) = Eu_{\chi}\big(\cap_{v\in\mathscr{I}}\ (R \cap \{x_v = 0\})\big).$$

Therefore, the surgery formula from Corollary 10.5.5 reads as

$$\mathrm{pc}^{\mathscr{S}}(Z_0(\mathbf{t}_{\mathscr{I}})) = \mathfrak{Q}^{\Gamma}_{0,\mathscr{I}}(0) = Eu_{\chi}(R) - Eu_{\chi}\big(\cap_{v\in\mathscr{I}}\ (R \cap \{x_v = 0\})\big).$$

Since $\mathfrak{Q}^{\Gamma}_{0,\mathscr{I}}(0) = \sum_{\emptyset\neq\mathscr{J}\subset\mathscr{I}}(-1)^{|\mathscr{J}|+1}\mathfrak{q}^{\Gamma}_{0,\mathscr{J}}(0)$, by inclusion-exclusion principle

$$\mathrm{mpc}^{\mathscr{S}}(Z_0(\mathbf{t}_{\mathscr{I}})) = \mathfrak{q}^{\Gamma}_{0,\mathscr{I}}(0) = Eu_{\chi}(R) - Eu_{\chi}\big(\cup_{v\in\mathscr{I}}\ (R \cap \{x_v = 0\})\big).$$

10.6 Duality for the Topological Series

Based on the symmetry $l' \mapsto Z_K - l'$ of $Z(\mathbf{t})$ and on the equivariant Ehrhart–Macdonald–Stanley reciprocity, for any $\mathscr{I} \subset \mathscr{V}$ and $h \in H$ we show that both periodic constants $\mathrm{mpc}^{\pi_{\mathscr{I}}(\mathscr{S}_{\mathbb{R}})}(Z_h(\mathbf{t}_{\mathscr{I}}))$ and $\mathrm{pc}^{\pi_{\mathscr{I}}(\mathscr{S}_{\mathbb{R}})}(Z_h(\mathbf{t}_{\mathscr{I}}))$ can be represented as a certain finite sum of coefficients of the 'dual' counting functions $q_{[Z_k]-h,\mathscr{I}}$ and $Q_{[Z_k]-h,\mathscr{I}}$. This has two immediate important applications.

The first one determines explicitly the polynomial part $Z^{+}_{h,\mathscr{I}}$ of $Z_{h,\mathscr{I}}$ (this is a *multivariable polynomial extension of the Seiberg–Witten invariant* $Z^{+}_{h,\mathscr{I}}(\mathbf{1})$). The second application expresses the Seiberg–Witten invariant of $M(\Gamma)$ in terms of 'inclusion-exclusion lattice point counting'. We follow [355]. ♣

10.6.1 Consider the setup of 10.2.1 of a multivariable rational function $f(\mathbf{t})$; let $Tf(\mathbf{t}) = \sum_{l'} z(l')\mathbf{t}^{l'} \in \mathbb{Z}[[\mathbf{t}^{1/d}]][\mathbf{t}^{-1/d}]$ be its Laurent expansion at the origin, and write $\sum_h Tf_h(\mathbf{t})[h]$ its equivariant decomposition.

We also define the Laurent expansion of $f(\mathbf{t})$ at infinity

$$T^{\infty}f(\mathbf{t}) = \sum_{\tilde{l}} r(\tilde{l})\mathbf{t}^{\tilde{l}} \in \mathbb{Z}[[\mathbf{t}^{-1/d}]][\mathbf{t}^{1/d}].$$

$T^{\infty}f$ is obtained by the substitution $\mathbf{s} = 1/\mathbf{t}$ into the Laurent expansion at $\mathbf{s} = 0$ of the function $f(1/\mathbf{s})$. Note that $\sum_{\tilde{l}\geq 0} r(\tilde{l})$ is a finite sum.

E.g., if $f(t) = t^2/(1-t)$, then $T^{\infty}f(t) = -t(1 + t^{-1} + t^{-2} + \cdots)$.

The series $T^{\infty}f(\mathbf{t})$ also has an equivariant decomposition $\sum_h T^{\infty}f_h(\mathbf{t})[h]$, defined in a very similar way as the decomposition of Tf. In fact, f can be rewritten in the form $\sum_{b'} \iota_{b'}\mathbf{t}^{b'}/\prod_{i=1}^{n}(1-\mathbf{t}^{da_i})$, hence f itself decomposes into $\sum_h f_h$, where $z_h(\mathbf{t})$ is the rational function of the form $\sum_{b'\in h+L} \iota_{b'}\mathbf{t}^{b'}/\prod_{i=1}^{n}(1-\mathbf{t}^{da_i})$. Its expansions at the origin is Tf_h and at infinity is $T^{\infty}f_h$. (In other words, the operation $Tf \mapsto T^{\infty}f$, defined via the original f, preserves all the h-components.)

Recall that for any $\mathscr{I} \subset \mathscr{V}$ and h we defined in 10.4.5 the modified counting function $q_{h,\mathscr{I}}$. Its periodic constant associated with a chamber $\mathfrak{c}$ will be denoted by $\mathrm{mpc}^{\mathfrak{c}}(Tf_h(\mathbf{t}_{\mathscr{I}}))$, cf. 10.4.7 and 10.4.9.

10.6.2 In general, the computation of quasipolynomials of counting functions and of their periodic constants (via their original definitions) is hard: the periodic constant measures the asymptotic behaviour of the coefficients in a certain cone. The next result based on the equivariant Ehrhart–Macdonald–Stanley reciprocity (10.4.3) shows that (under some conditions) the periodic constant of the modified counting function equals a *finite sum of certain coefficients of the Laurent expansion at infinity*.

Theorem 10.6.3 *Fix h and $\mathscr{I}$ as above. Write the h-component of the Laurent expansion at infinity as $(T^{\infty}f)_h(\mathbf{t}_{\mathscr{I}}) = \sum_{\tilde{l}} r_{h,\mathscr{I}}(\tilde{l})\mathbf{t}^{\tilde{l}}_{\mathscr{I}}$. Assume that a (closed) chamber $\mathfrak{c}$ has the property that $b_k|_{\mathscr{I}} \in \mathfrak{c}$ for every k. Then*

$$\mathrm{mpc}^{\mathfrak{c}}(Tf_h(\mathbf{t}_{\mathscr{I}})) = \sum_{\tilde{l}\geq 0} r_{h,\mathscr{I}}(\tilde{l}). \tag{10.6.1}$$

Proof By Corollary 10.4.7 the function $q_{h,\mathscr{I}}$ on the set $\cap_k(b_k|_{\mathscr{I}} + \mathfrak{c})$ is

$$\sum_k \iota_k \cdot {}^{m}\mathscr{L}^{\mathfrak{c}}_{h-[b_k]}(\mathscr{T}_{\mathscr{I}}, l'_{\mathscr{I}} - b_k|_{\mathscr{I}}).$$

Hence, by definitions (see again 10.4.7, especially (10.4.7)) $\mathrm{mpc}^{\mathfrak{c}}(Tf_h(\mathbf{t}_{\mathscr{I}})) = \mathrm{pc}^{\mathfrak{c}}(q_{h,\mathscr{I}})$ exists and equals

$$\sum_k \iota_k \cdot {}^{m}\mathscr{L}^{\mathfrak{c}}_{h-[b_k]}(\mathscr{T}_{\mathscr{I}}, -b_k|_{\mathscr{I}}).$$

On the other hand, since $b_k|_{\mathscr{I}} \in \mathfrak{c}$, from reciprocity (10.4.3) one has

$$(-1)^n \cdot {}^{m}\mathscr{L}^{\mathfrak{c}}_{-h+[b_k]}(\mathscr{F}_{\mathscr{I}} \setminus \mathscr{T}_{\mathscr{I}}, b_k|_{\mathscr{I}}) = {}^{m}\mathscr{L}^{\mathfrak{c}}_{h-[b_k]}(\mathscr{T}_{\mathscr{I}}, -b_k|_{\mathscr{I}}).$$

The expression ${}^{m}\mathscr{L}^{\mathfrak{c}}_{-h+[b_k]}(\mathscr{F}_{\mathscr{I}} \setminus \mathscr{T}_{\mathscr{I}}, b_k|_{\mathscr{I}})$ counts solutions of $\sum_i x_i a_{i,v} \leq b_{k,v}$ for every $v \in \mathscr{I}$ under the restrictions $[b_k - \sum_i x_i a_i] = h$ and $x_i > 0$ for every i.

On the other hand, in the expansion at infinity,

$$(-1)^n \cdot \frac{\mathbf{t}^{b_k}}{\prod_i (1-\mathbf{t}^{a_i})} \xrightarrow{T^\infty} \mathbf{t}^{b_k} \cdot \sum_{x_i>0} \mathbf{t}^{-\sum_i x_i a_i}.$$

Hence

$$\sum_k \iota_k \cdot (-1)^n \cdot {}^m\mathscr{L}^{\mathfrak{c}}_{-h+[b_k]}(\mathscr{F}_{\mathscr{I}} \setminus \mathscr{T}_{\mathscr{I}}, b_k|_{\mathscr{I}}) = \sum_{\tilde{l}\geq 0} r_{h,\mathscr{I}}(\tilde{l}).$$

□

10.6.4 In the sequel $f(\mathbf{t})$ will be the topological series $Z(\mathbf{t})$ associated with the graph Γ. Furthermore, thanks to Proposition 10.4.14, $\mathscr{S}_{\mathbb{R}}$ is a (closed) chamber $\mathfrak{c} \in {}^m\mathfrak{C}'$. It also satisfies the assumption of Theorem 10.6.3, since each $E_v^* \in \mathscr{S}_{\mathbb{R}}$.

10.6.5 Duality for Counting Functions and Periodic Constants Now we prove a duality/pairing between special evaluations of the counting functions associated with $Z(\mathbf{t})$ and the periodic constants. The duality is the upshot of *two 'symmetries'*, manifested at two different levels. The first one is the *equivariant Ehrhart–Macdonald–Stanley reciprocity of the polytopes*, while the second is a topological imprint of the Gorenstein duality present at the level of the topological Poincaré series: a $\{x \leftrightarrow Z_K - x\}$ symmetry.

Theorem 10.6.6 *Fix any $\mathscr{I} \subset \mathscr{V}$, $\mathscr{I} \neq \emptyset$ and $h \in H$. Then*

$$(a)\ \mathrm{mpc}^{\pi_{\mathscr{I}}(\mathscr{S}_{\mathbb{R}})}(Z_h(\mathbf{t}_{\mathscr{I}})) = q_{[Z_K]-h,\mathscr{I}}(Z_K - r_h);$$

$$(b)\ \mathrm{pc}^{\pi_{\mathscr{I}}(\mathscr{S}_{\mathbb{R}})}(Z_h(\mathbf{t}_{\mathscr{I}})) = Q_{[Z_K]-h,\mathscr{I}}(Z_K - r_h).$$

That is, (in principle, the hardly computable) periodic constant of the series $Z_h(\mathbf{t})$ equals a precise finite sum of coefficients of the 'dual series' $Z_{[Z_K]-h}(\mathbf{t})$.

Proof By (10.4.8) (b) follows from (a). Next we prove (a).

The substitution $x \mapsto Z_K - x$ in $Z(\mathbf{t})$, together with the identities $Z_K - E = \sum_{v\in\mathscr{V}}(\kappa_v - 2)E_v^*$ from (6.3.1) and $-2 = \sum_{n\in\mathscr{V}}(\kappa_n - 2)$ (since Γ is a tree) gives

$$Z(\mathbf{t}_{\mathscr{I}}) = \mathbf{t}_{\mathscr{I}}^{Z_K-E} \cdot Z(\mathbf{t}_{\mathscr{I}}^{-1}). \tag{10.6.2}$$

This on the Taylor expansion level transforms into the symmetry $(T^\infty Z)(\mathbf{t}_{\mathscr{I}}) = \mathbf{t}_{\mathscr{I}}^{Z_K-E} Z(\mathbf{t}_{\mathscr{I}}^{-1}) = \sum_{l'\in\mathscr{S}'} \mathfrak{z}(l')\mathbf{t}_{\mathscr{I}}^{Z_K-E-l'}$. The corresponding h-equivariant parts are

$$(T^\infty Z)_h(\mathbf{t}_{\mathscr{I}}) = \sum_{l'\in\mathscr{S}',[l']=[Z_K]-h} \mathfrak{z}(l')\mathbf{t}_{\mathscr{I}}^{Z_K-E-l'}. \tag{10.6.3}$$

This combined with Theorem 10.6.3 and 10.6.4 give

$$\mathrm{mpc}^{\pi_{\mathscr{I}}(\mathscr{S}_{\mathbb{R}})}(Z_h(\mathbf{t}_{\mathscr{I}})) = \sum_{l'|_{\mathscr{I}} \le (Z_K - E)|_{\mathscr{I}}, [l'] = [Z_K] - h} \mathfrak{z}(l').$$

But the right-hand side is exactly the counting function $q_{[Z_K]-h,\mathscr{I}}(Z_K - r_h)$, since $l'|_{\mathscr{I}} \le (Z_K - E)|_{\mathscr{I}}$ is equivalent to $l'|_{\mathscr{I}} \prec (Z_K - r_h)|_{\mathscr{I}}$ if $[l'] = [Z_K] - h$. □

Corollary 10.6.7

$$Q_{[Z_K]-h,\mathscr{V}}(Z_K - r_h) = \mathrm{pc}^{\mathscr{S}_{\mathbb{R}}}(Z_h(\mathbf{t})) = \mathfrak{sw}_{\sigma[K+2r_h]}(M) - \frac{(K+2r_h)^2 + |\mathscr{V}|}{8}.$$

In particular, the Seiberg–Witten invariant can be expressed via the counting function as a finite sum of $Z(\mathbf{t})$-coefficients.

Proof Combine (10.3.1), (10.3.9) and Theorem 10.6.6. □

Remark 10.6.8 (Duality/Symmetry of the Quasipolynomials $\overline{Q}^{\Gamma}_{h,\mathscr{I}}(l')$) Let us analyse the quasipolynomial $\overline{Q}_{h,\mathscr{I}}(l') := \overline{Q}^{\Gamma}_{h,\mathscr{I}}(l')$ defined in (10.5.4) from the point of view of the symmetry $l' \leftrightarrow \bar{l}'$ given by $l' + \bar{l}' = -K = Z_K$. Then $\mathfrak{sw}_{\sigma[K+2l']} = \mathfrak{sw}_{\sigma[-K-2l']} = \mathfrak{sw}_{\sigma[K+2\bar{l}']}$ and $(K+2l')^2 = (K+2\bar{l}')^2$, and similar identities hold for the R_k-restrictions. Therefore,

$$\overline{Q}_{h,\mathscr{I}}(l') = \overline{Q}_{[Z_k]-h,\mathscr{I}}(\bar{l}') = \overline{Q}_{[Z_K]-h,\mathscr{I}}(Z_K - l').$$

In particular, $\overline{Q}_{h,\mathscr{I}}(r_h) = \overline{Q}_{[Z_K]-h,\mathscr{I}}(Z_K - r_h)$. This together with the definition of the periodic constant, namely with $\mathrm{pc}^{\pi_{\mathscr{I}}(\mathscr{S}_{\mathbb{R}})}(Z_h(\mathbf{t}_{\mathscr{I}})) = \mathfrak{Q}^{\Gamma}_{h,\mathscr{I}}(0) = \overline{Q}_{h,\mathscr{I}}(r_h)$, provide $\mathrm{pc}^{\pi_{\mathscr{I}}(\mathscr{S}_{\mathbb{R}})}(Z_h(\mathbf{t}_{\mathscr{I}})) = \overline{Q}_{[Z_K]-h,\mathscr{I}}(Z_K - r_h)$. Hence, Theorem 10.6.6*(b)* is equivalent to $\overline{Q}_{[Z_K]-h,\mathscr{I}}(Z_K - r_h) = Q_{[Z_K]-h,\mathscr{I}}(Z_K - r_h)$. This means that the value of the counting function at $Z_K - r_h$ is computable already from its quasipolynomial, that is, $Z_K - r_h$ is already so large that it sits in the 'asymptotic region'.

Example 10.6.9 (Continuation of Example 10.5.9) Consider the situation and notations of 10.5.9, and additionally assume that $Z_K \in L$. For technical reasons we also assume that the graph is minimal good, or at least that Z_K is effective (cf. 6.3.4). Then Z_K is a good candidate for the cycle l_0 of 10.5.9. In particular, $R = R(0, Z_K)$. Also, by Theorem 10.6.6 we have (for $r_h = 0$) $\mathrm{mpc}^{\mathscr{S}}(Z_0(\mathbf{t}_{\mathscr{I}})) = \mathfrak{q}^{\Gamma}_{0,\mathscr{I}}(0) = q_{0,\mathscr{I}}(Z_K)$ and $\mathrm{pc}^{\mathscr{S}}(Z_0(\mathbf{t}_{\mathscr{I}})) = \mathfrak{Q}^{\Gamma}_{0,\mathscr{I}}(0) = Q_{0,\mathscr{I}}(Z_K)$. Hence,

$$Q_{0,\mathscr{I}}(Z_K) = Eu_{\chi}(R) - Eu_{\chi}\big(\cap_{v\in\mathscr{I}}\ (R \cap \{x_v = 0\})\big),$$

(which for $\mathscr{I} = \mathscr{V}$ is compatible with (9.6.6), or with Theorem 9.6.12), and

$$q_{0,\mathscr{I}}(Z_K) = Eu_\chi(R) - Eu_\chi\big(\cup_{v\in\mathscr{I}}\ (R \cap \{x_v = 0\})\big).$$

These provide identifications between evaluations of the counting and modified counting functions at Z_K and the Euler summation of certain weighted cubes in $R(0, Z_K)$. Since $R(0, Z_K)$ and χ are stable with respect to the symmetry $l \leftrightarrow Z_K - l$, the right hand sides of the above identities can be rewritten as another summations (when we eliminate certain 'exterior' facets of the cube of type $\{x_v = Z_K|_v\}$).

10.6.A The 'Polynomial Part' of the Series $Z(\mathbf{t})$

10.6.10 The 'Polynomial–Negative Degree Part' Decomposition; Motivation Consider a one-variable rational function $f(t) = B(t)/A(t)$ with $A(t) = \prod_{i=1}^{n}(1-t^{a_i})$ and $a_i > 0$. In 8.7.2 it is observed that any such function has a unique decomposition of the form $f(t) = f^+(t) + f^-(t)$, where $f^+(t)$ is a polynomial (with non-negative exponents) and $f^-(t)$ is a rational function of negative degree. Furthermore, $f(t)$ admits a periodic constant (associated with the Taylor expansion of f and the cone $\mathbb{R}_{\geq 0}$), which equals $f^+(1)$. The decomposition can be established using the Euclidean division algorithm. f^+ is called the *polynomial part* while the rational function f^- is the *negative degree part* of the decomposition.

The formulation of the multivariable generalization is not immediate; main guiding questions/principles are: (a) what are the universal properties of the parts f^+ and f^-, which guarantee that a decomposition $f = f^+ + f^-$ exists and it is unique; and, (b) what algorithm provides this decomposition. Additionally, the wished decomposition must satisfy (at least) the next basic property: (c) in the geometric/singularity context $f(\mathbf{t}) = Z(\mathbf{t})$, $\mathrm{pc}^{\mathscr{I}}(Z(\mathbf{t})) = Z^+(1)$.

Hence, in this geometric situation, Z^+ *is a polynomial generalization of the Seiberg–Witten invariant*.

Below we show that such a decomposition is possible, with all the three requirements satisfied.

10.6.11 The Polynomial Part by Division We consider again the setup of 10.2.1, with a pair of free $\mathbb{Z}$-modules $L \subset L'$, and a general multivariable rational function

$$f(\mathbf{t}) = \frac{\sum_{k=1}^{r} \iota_k \mathbf{t}^{b_k}}{\prod_{i=1}^{n}(1-\mathbf{t}^{a_i})},$$

where $\{\iota_k\}_{k=1}^r \in \mathbb{Z}$, $\{b_k\}_{k=1}^r$, $\{a_i\}_{i=1}^n \in L'$ such that $b_k \not\succ 0$ for every k and $0 \prec a_i$ for every i.

Proposition 10.6.12 ([353, 355])

(a) $f(\mathbf{t})$ *can be written in the following form by a 'multivariable Euclidean division algorithm':*

$$f(\mathbf{t}) = \sum_{S\subset\{1,\dots,n\}} \frac{\sum_j \iota_{S,j}\mathbf{t}^{b_{S,j}}}{\prod_{i\in S}(1-\mathbf{t}^{a_i})}, \tag{10.6.4}$$

where $\iota_{S,j} \in \mathbb{Z}$, $b_{S,j} \not\prec 0$ *for all* (S, j), *and* $b_{S,j} \prec a_i$ *for all* (S, j) *and all* $i \in S$ *whenever* $S \neq \emptyset$.

(b) $f(\mathbf{t})$ *has a decomposition of type* $f^+(\mathbf{t}) + f^-(\mathbf{t})$ *with the next properties:*

(i) $f^+(\mathbf{t})$ *is a finite sum (polynomial)* $\sum_j n_j\mathbf{t}^{c_j}$ *with* $c_j \not\prec 0$ *for all* j;

(ii) $f^-(\mathbf{t})$ *is a rational function with negative degree in all variables* t_v.

Furthermore, a decomposition of $z(\mathbf{t})$ *with properties (i)-(ii) is unique.*

(c) In fact, the terms f^+ *and* f^- *of the decomposition from (b) are given via (10.6.4) as follows:* f^+ *(resp.* f^-*) is the sum of terms from (10.6.4) over* $S = \emptyset$ *(resp.* $S \neq \emptyset$*).*

Proof

(a) Assume that we have an expression $g(\mathbf{t}) = \mathbf{t}^b/\prod_{i\in S}(1-\mathbf{t}^{a_i})$ such that $b \not\prec 0$ and there exists some i_0 such that $b \not\prec a_{i_0}$. Then we replace g by $-\mathbf{t}^{b-a_{i_0}}/\prod_{i\in S\setminus\{i_0\}}(1-\mathbf{t}^{a_i}) + \mathbf{t}^{b-a_{i_0}}/\prod_{i\in S}(1-\mathbf{t}^{a_i})$. Note that $b-a_{i_0} \not\prec 0$, hence the new fractions have similar form. Starting from the original expression of f, by repeating the above step whenever it is applicable, after finitely many steps we obtain (10.6.4).

(b) Define f^+ and f^- as indicated in (c). Then properties (i)–(ii) are automatically satisfied. Next, we prove the uniqueness of the decomposition. We need to show that if $f^+(\mathbf{t}) + f^-(\mathbf{t}) = 0$ (†) and if f^+ and f^- satisfy (i)–(ii), then both are zero. But, from (†), both f^+ and f^- should satisfy the assumptions of both (i)–(ii), a fact which happens only if they are zero.

□

10.6.13 The Polynomial Part of $f_h(\mathbf{t}_{\mathscr{J}})$ by Duality Assume again that we are in the situation of a plumbing graph and its rational function $f(\mathbf{t}) = Z(\mathbf{t})$.

We fix $h \in H$. Then, by the proof of Theorem 10.6.6 we have $Z_h(\mathbf{t}_{\mathscr{J}}) = \mathbf{t}_{\mathscr{J}}^{Z_K-E} \cdot Z_{[Z_K]-h}(\mathbf{t}_{\mathscr{J}}^{-1})$ and $(T^\infty Z)_h(\mathbf{t}_{\mathscr{J}}) = \mathbf{t}_{\mathscr{J}}^{Z_K-E} Z_{[Z_K]-h}(\mathbf{t}_{\mathscr{J}}^{-1})$. Write this $\mathbf{t}_{\mathscr{J}}^{Z_K-E} Z_{[Z_K]-h}(\mathbf{t}_{\mathscr{J}}^{-1})$ as $\sum_{l'\in Z_K-E-\mathscr{S}'} w(l')\mathbf{t}_{\mathscr{J}}^{l'}$, where $[l'] = h$ automatically whenever $w(l') \neq 0$. Define

$$Z^+_{h,\mathscr{J}}(\mathbf{t}_{\mathscr{J}}) := \sum_{l'|_{\mathscr{J}}\not\prec 0|_{\mathscr{J}}} w(l')\mathbf{t}_{\mathscr{J}}^{l'}, \quad Z^-_{h,\mathscr{J}}(\mathbf{t}_{\mathscr{J}}) := \sum_{l'|_{\mathscr{J}}\prec 0|_{\mathscr{J}}} w(l')\mathbf{t}_{\mathscr{J}}^{l'}. \tag{10.6.5}$$

Write also $Z^+_{h,\mathscr{J}}(\mathbf{1}) := Z^+_{h,\mathscr{J}}(\mathbf{t}_{\mathscr{J}})|_{t_i=1,\forall i}$.

Theorem 10.6.14 *Consider the decomposition $Z_h(\mathbf{t}_{\mathscr{J}}) = Z^+_{h,\mathscr{J}}(\mathbf{t}_{\mathscr{J}}) + Z^-_{h,\mathscr{J}}(\mathbf{t}_{\mathscr{J}})$ from (10.6.5).*

(a) $Z^+_{h,\mathscr{J}}(\mathbf{t}_{\mathscr{J}})$ and $Z^-_{h,\mathscr{J}}(\mathbf{t}_{\mathscr{J}})$ satisfy the requirements (i)–(ii) from Proposition 10.6.12(b). In particular, by the uniqueness of the decomposition, this decomposition agrees with the decomposition from 10.6.12(a)–(c) given by Euclidean division.

(b) $Z^+_{h,\mathscr{J}}(\mathbf{1}) = Q_{[Z_K]-h,\mathscr{J}}(Z_K - r_h)$. In particular, $Z^+_{h,\mathscr{J}}(\mathbf{1}) = \mathrm{pc}^{\pi_{\mathscr{J}}(\mathscr{S}_{\mathbb{R}})}(Z_h(\mathbf{t}_{\mathscr{J}}))$.

Proof

(a) By its definition, $Z^+_{h,\mathscr{J}}(\mathbf{t}_{\mathscr{J}})$ is a finite sum, hence $Z^-_{h,\mathscr{J}}(\mathbf{t}_{\mathscr{J}}) = Z_h(\mathbf{t}_{\mathscr{J}}) - Z^+_{h,\mathscr{J}}(\mathbf{t}_{\mathscr{J}})$ is a rational function. Since in its expansion all monomials $w(l')\mathbf{t}^{l'}_{\mathscr{J}}$ satisfy $l'|_{\mathscr{J}} \prec 0|_{\mathscr{J}}$, it has negative degree in all the variables.

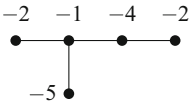
(b) By the above transformations $w(l') = z(Z_K - E - l')$, $[l'] = h$. Hence $l'|_{\mathscr{J}} \not\prec 0|_{\mathscr{J}}$ transforms into $(Z_K - E)|_{\mathscr{J}} \not\prec (Z_K - E - l')|_{\mathscr{J}}$, or $(Z_K - E - l')|_{\mathscr{J}} \not\succeq (Z_K - r_h)|_{\mathscr{J}}$. This proves the first identity. For the second one use Theorem 10.6.6 *(b)*.

□

Corollary 10.6.7 and Theorem 10.6.14 combined gives

Corollary 10.6.15 $Z^+_{h,\mathscr{V}}(\mathbf{1}) = Z^+_{h,\mathscr{N}}(\mathbf{1}) = \mathfrak{sw}_{\sigma_{[K+2r_h]}}(M) - \frac{(K+2r_h)^2+|\mathscr{V}|}{8}$ *for any h.*

Example 10.6.16 Let us consider the Brieskorn sphere $M = \Sigma(2,5,7)$.



$(H = 0)$

$Z(\mathbf{t})$ reduced to the node $\mathscr{N} = \{v_0\}$ and its 'polynomial-negative degree part' decomposition (obtained by simple division of polynomials) is

$$Z(\mathbf{t}_{\mathscr{N}}) = Z(t) = \frac{1-t^{70}}{(1-t^{35})(1-t^{14})(1-t^{10})} = t + t^{11} + \frac{1-t+t^{15}+t^{21}}{(1-t^{14})(1-t^{10})}.$$

The same result follows by duality as well. Indeed, the Taylor expansion is $Z(t) = 1 + t^{10} + t^{14} + \dots$. The E_{v_0}-coefficient of Z_K is 12, hence $t^{(Z_K)_{v_0}-1}Z(t^{-1}) = t^{11} + t^1 + t^{-3} + \cdots$, which gives $Z^+(t) = t^{11} + t$ again.

Example 10.6.17 (The Seiberg–Witten Invariant of Elliptic Singularity Links [454]) We fix an elliptic graph Γ associated with the minimal resolution, and we assume

that this resolution in fact is good. We also assume that the link is a rational homology sphere. In particular, results of the Sect. 7.2.E and the present section can be applied. The reader is invited to review the notations of Sect. 7.2.E.

As an application of the surgery formula from Corollary 10.5.5 and the duality identity 10.6.6*(b)* (applied for $h = 0$) we prove the following identity:

$$\overline{\mathfrak{sw}}_{\sigma[K]}(M(\Gamma)) = \mathfrak{sw}_{\sigma[K]}(M(\Gamma)) - \frac{K^2 + |\mathcal{V}|}{8} = m + 1.$$

(This identity can be proved using the techniques of the lattice cohomology and graded roots as well; see 7.3.22(g), 7.2.66 and 11.3.9.)

In the present proof we will use an inductice procedure based on the structure of the elliptic sequence $\{B_j\}_{j=-1}^{m}$. (We also write $B_{m+1} := \emptyset$.)

Case 1. First assume that $Z_K \in L$. Since SWIC is true for minimally elliptic singularities (see e.g. 9.7.15) the $m = 0$ case follows. (It can also be proved similarly as the inductive step below.) Next, we run induction. Assume that the statement is already proved for any graph with length m and we fix some elliptic Γ with length $m + 1$. We fix $\mathscr{I} := B_0 \setminus B_1$. Since $M(\Gamma(B_1))$ is elliptic with length m, $\overline{\mathfrak{sw}}_{\sigma[K]}(M(\Gamma(B_1))) = m$ from the inductive step. On the other hand, from 10.5.5 and 10.6.6*(b)* we have

$$\overline{\mathfrak{sw}}_{\sigma[K]}(M(\Gamma(B_0))) = \overline{\mathfrak{sw}}_{\sigma[K]}(M(\Gamma(B_1))) + Q_{[Z_K],\mathscr{I}}(Z_K).$$

Hence, we need to show that $Q_{[Z_K],\mathscr{I}}(Z_K) = 1$. But $Z_K \in L$, hence $[Z_K] = 0 \in H$, and $Q_{[Z_K],\mathscr{I}}(Z_K)$ becomes $\sum \mathfrak{z}(l)$, summed over $l \in L$ with $l|_{\mathscr{I}} \not\geq Z_K|_{\mathscr{I}}$.

Since $Z_0(\mathbf{t})$ is supported in $\mathscr{S}$, any $l \neq 0$ in the support of $Z_0(\mathbf{t})$ has the property that $l \geq Z_{min}$. On the other hand, along $\mathscr{I} = B_0 \setminus B_1$ we have $Z_K|_{\mathscr{I}} = Z_{min}|_{\mathscr{I}}$. This means that any $l \neq 0$ form the support of Z_0 satisfies $l|_{\mathscr{I}} \geq Z_K|_{\mathscr{I}}$. In particular, in the sum only one term is relevant, namely the one corresponding to $l = 0$ with $\mathfrak{z}(0) = 1$.

Case 2. Assume that Γ is an elliptic graph of length $m + 1$ with $Z_K \notin L$. Now we set $\mathscr{I} := B_{-1} \setminus B_0 = E \setminus B_0$. Since B_0 supports a numerically Gorentein graph, from Step 1 we already know that $\overline{\mathfrak{sw}}_0(M(\Gamma(B_0))) = m + 1$. We wish to show that $\overline{\mathfrak{sw}}_{\sigma[K]}(M(\Gamma(B_{-1}))) = m + 1$ too. Hence from the surgery formula we need to verify that $\sum \mathfrak{z}(l') = 0$, where the sum is over $\{l'|_{\mathscr{I}} \not\geq Z_K|_{\mathscr{I}},\ [l'] = [Z_K]\}$.

Recall that if $l' \in \mathscr{S}'$ and $[l'] = [Z_K]$, then $l' \geq s_{[Z_K]}$. But $(s_{[Z_K]})|_{\mathscr{I}} = Z_K|_{\mathscr{I}}$. This reads as $l'|_{\mathscr{I}} \geq Z_K|_{\mathscr{I}}$ for any relevant l'.

10.6.18 Analogy with Invariants of Plane Curve Singularities Consider a plane curve singularity (C, o), let $\mathscr{S} := \mathscr{S}_{C,o}$ be its semigroup and μ its Milnor number. For details see 5.2.2. If $\zeta(t)$ is the inverse of the monodromy zeta function, then by [94] (see (5.2.2) here) its Taylor expansion at the origin is $T\zeta(t) = \sum_{s \in \mathscr{S}} t^s$. We determine the polynomial part of $T\zeta(t)$ and we verify the analogue of the construction from 10.6.13 and of Theorem 10.6.14. Here the needed Gorenstein

duality reads as follows: (†) $s \notin \mathscr{S}$ if and only if $\mu - 1 - s \in \mathscr{S}$. This also shows that $\mathbb{Z} \setminus \mathscr{S}$ is finite with cardinality $\mu/2$, and the largest element of $\mathbb{Z} \setminus \mathscr{S}$ is $\mu - 1$.

Since $T\zeta(t) = \sum_{s\in\mathscr{S}} t^s = \sum_{s\geq 0} t^s - \sum_{s\notin\mathscr{S}, s\geq 0} t^s$ we get that $\zeta^-(t) = \sum_{s\geq 0} t^s = 1/(1-t)$, and $\zeta^+(t) = -\sum_{s\notin\mathscr{S}, s\geq 0} t^s$. In particular, $\mathrm{pc}(\zeta) = \zeta^+(1) = -\mu/2$. On the other hand, by duality (†) we have the identities $(T^\infty\zeta)(t) = -t^{\mu-1}\cdot T\zeta(t^{-1}) = -t^{\mu-1}(\sum_{s\in\mathscr{S}, s<\mu} t^{-s} + \sum_{s\geq\mu} t^{-s}) \overset{(\dagger)}{=} -\sum_{s\notin\mathscr{S}, s\geq 0} t^s - (t^{-1} + t^{-2} + \cdots) = \zeta^+(t) - (t^{-1} + t^{-2} + \cdots)$, whose part with positive exponents is exactly $\zeta^+(t)$.

Example 10.6.19 For the decompositions of the reduced series $P(t) = P^+ + P^-$ associated with the central vertex of the resolution of a weighted homogeneous singularity, or $Z(t) = Z^+ + Z^-$ (its topological counterpart), see 8.1.B, 9.7.A, 10.3.12.

10.6.B Polytopes, Lattice Points and the Seiberg–Witten Invariant

10.6.20 Before we make our statement regarding the Seiberg–Witten invariant, let us comment some parallel results regarding certain analytic invariants. Here we also wish to focus on the relationship of the algebraic geometry (singularity theory) with the combinatorics of polytopes and enumeration of lattice points in certain polytopes. Such connections culminate in toric geometry, however, in singularity theory we witness an even more direct bridge in the theory of Newton non-degenerate hypersurface singularities (see e.g. Sect. 5.5). In this case one defines the Newton boundary $N\Gamma$ associated with the defining equation and one proves that several invariants of the singularity can be recovered from $N\Gamma$, see e.g. the list from 5.5.4. In the present discussion we wish to highlight only the geometric genus correspondence, which in the hypersurface case equals the number of lattice points with strictly positive coordinates in $N\Gamma_-$, cf. Merle–Teissier formula from 5.5.4(c). More generally, consider an isolated complete intersection singularity $\{f_1 = \ldots = f_{n-1} = 0\}$, where the system is non-degenerate with respect to Newton boundaries $\{N\Gamma_i\}_i$ associated with the equations $\{f_i\}_i$, $f_i : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, o)$. Then, by work of Khovanskii and Morales [313, 443], p_g can be expressed as an alternating sum of lattice points in certain polytopes obtained from the $N\Gamma_i$'s. More precisely,

$$p_g = \#(N\Gamma_{1,+} + \cdots + N\Gamma_{n-1,+}) - \sum_{i=1}^{n-1} \#(N\Gamma_{1,+} + \cdots + \widehat{N\Gamma_{i,+}} + \cdots + N\Gamma_{n-1,+}) + \cdots + (-1)^n \sum_{i=1}^{n-1} \#(N\Gamma_{i,+}),$$

where $\#(N\Gamma_+)$ is the number of lattice points with positive coordinates below or on the boundary $N\Gamma$ of $N\Gamma_+$, and $N\Gamma_{1,+} + N\Gamma_{2,+}$ is the Minkowski sum, and $\widehat{\cdot}$ means that the corresponding term is omitted.

Having in mind the Seiberg–Witten Invariant Conjecture, it is natural to ask whether there is an analogous interpretation of the Seiberg–Witten invariant in terms of an alternating sum of lattice points in certain polytopes associated with the topological type (valid for any negative definite rational homology sphere plumbed 3-manifold). We will call such an expression *inclusion-exclusion lattice point counting*, or IELP counting. In this section we provide a positive answer to this task. However, there will be certain differences. The first one is dictated by the geometric situation: in the case of p_g and Merle–Teissier–Khovanskii–Morales type formulae the polytopes were associated with the equation of the germ, in the next topological version they are associated with the resolution graph. Also, in the next topological context we have to replace the Minkowski sum by a different construction (for Minkowski-sum versions, valid in certain situations, see e.g. [355]).

10.6.21 Preparation for an IELP Expression We consider the notations and constructions from 10.2.5 associated with a plumbing graph Γ as in Sect. 10.3.

In particular, $P^{(l')} = \cup_{v\in\mathcal{V}} P_v^{(l')}$, where $P_v^{(l')} = \{\mathbf{x} \in \mathbb{R}_{\geq 0}^{|\mathcal{V}^e|} : \alpha(\mathbf{x})|_v \leq l'_v\}$, where $\alpha : \mathbb{Z}^{|\mathcal{V}^e|} \to L'$ (or its real extension) is given by $\alpha(\mathbf{x}) := \sum_{v\in\mathcal{V}^e} x_v E_v^*$.

Let us consider for any $\mathcal{J} \subset \mathcal{V}$, $\mathcal{J} \neq \emptyset$, the polytope $P_{\mathcal{J}}^{(l')} = \cup_{v\in\mathcal{J}} P_v^{(l')}$ as well. From the definition it is clear that $P_{\mathcal{J}}^{(l')}$ depends only on the restriction $l'|_{\mathcal{J}}$. Furthermore, it can happen that in the union the contribution of some $P_v^{(l')}$ is superfluous (see e.g 10.3.A). This is what we establish in the next lemma.

If $l' = \sum_{v\in\mathcal{J}} a_v E_v^*$ with every $a_v \neq 0$ ($v \in \mathcal{J}$), then we say that the E^*-support of l' is $\mathcal{J}$, and we write $|l'|^* = \mathcal{J}$.

Lemma 10.6.22 ([355]) *Assume that $|l'|^* = \mathcal{J} \neq \emptyset$. Then $P_v^{(l')} \subset P_{\mathcal{J}}^{(l')}$ for any $v \in \mathcal{V}$.*

Proof Proceed as in the proof of Theorem 10.3.10. □

The duality from (10.6.2) motivates the introduction of the next special element

$$l'_{top} := Z_K - E + \sum_{v\in\mathcal{V}^e} E_v^* \in L'. \tag{10.6.6}$$

It defines the special polytope $P_{\mathcal{V}}^{(l'_{top})}$. By (6.3.1) $l'_{top} = \sum_{v\in\mathcal{N}}(\kappa_v - 2)E_v^*$ too, hence $|l'_{top}|^* = \mathcal{N}$ and by Lemma 10.6.22 we obtain $P_{\mathcal{V}}^{(l'_{top})} = P_{\mathcal{N}}^{(l'_{top})}$. We call it the *topological polytope* associated with the graph Γ and we denote it by $P_{\mathcal{N}}^{top}$.

A family of sub-polytopes of $P_{\mathcal{N}}^{(l'_{top})}$ are defined as follows. Let us write the set of nodes as $\mathcal{N} = \{v_1, \ldots, v_s\}$. Then consider the multiset $\mathcal{N}^m :=$

$\{v_1^{\kappa_{v_1}-2},\dots,v_s^{\kappa_{v_s}-2}\}$, where in $\mathscr{N}^m$ each node v has multiplicity κ_v-2. Since $\sum_{v\in\mathscr{N}}(\kappa_v-2)=|\mathscr{V}^e|-2$, $\mathscr{N}^m$ contains $|\mathscr{V}^e|-2$ symbols. Let $\mathscr{I}^m\subset\mathscr{N}^m$ be a non-trivial sub-multiset, $\mathscr{I}^m=\{v_{i_1}^{k_{i_1}},\dots,v_{i_r}^{k_{i_r}}\}$, where $\{v_{i_1},\dots,v_{i_r}\}=\mathscr{I}\subset\mathscr{N}$ and $0<k_{i_j}\le\kappa_{v_{i_j}}-2$ for every j, and $0<r\le s$. To such $\mathscr{I}^m$ we define $|\mathscr{I}^m|:=\sum_j k_{i_j}$, $\binom{\kappa-2}{k_{\mathscr{I}^m}}:=\prod_{j=1}^r\binom{\kappa_{v_{i_j}}-2}{k_{i_j}}$, the cycle $l'(\mathscr{I}^m)=\sum_j k_{i_j}E^*_{v_{i_j}}$ with E^*-support $\mathscr{I}$, and the polytope $P_{\mathscr{I}^m}:=P_{\mathscr{I}}^{(l'(\mathscr{I}^m))}$.

Note that $P_{\mathscr{N}^m}=P_{\mathscr{N}}^{(l'(\mathscr{N}^m))}=P_{\mathscr{N}}^{(l'_{top})}=P_{\mathscr{N}}^{top}$.

Recall that all these polytopes $P_{\mathscr{I}^m}$ sit in $\mathbb{R}^{|\mathscr{V}^e|}_{\ge 0}$, hence they are $|\mathscr{V}^e|$-dimensional. Their number is bounded from above by the cardinality of the index set $\mathscr{I}^m$, namely by $-1+\prod_v(\kappa_v-1)$, however some of the polytopes might contain no lattice points.

Theorem 10.6.23 ([355]) *For any concave dilated polytope of type $P_{\mathscr{I}}^{(l')}$ we denote by $\#_h(P_{\mathscr{I}}^{(l')})$ the number of integral points in $P_{\mathscr{I}}^{(l')}\cap\rho^{-1}([l']-h)$ with all coordinates strictly positive. Then*

$$\mathfrak{sw}_{\sigma_{[K+2r_h]}}(M)-\frac{(K+2r_h)^2+|\mathscr{V}|}{8}=\sum_{\emptyset\neq\mathscr{I}^m\subset\mathscr{N}^m}(-1)^{|\mathscr{V}^e|-|\mathscr{I}^m|}\cdot\binom{\kappa-2}{k_{\mathscr{I}^m}}\cdot\#_h(P_{\mathscr{I}^m}).$$

Proof Let us fix some $\mathscr{I}\subset\mathscr{V}$, and compute $Z^+_{h,\mathscr{I}}(\mathbf{t}_{\mathscr{I}})$ via the truncation of the series $Z_{[Z_K]-h}(\mathbf{t}_{\mathscr{I}})$, the *'dual' polynomial part*,

$$\check{Z}^+_{h,\mathscr{I}}(\mathbf{t}_{\mathscr{I}}):=\sum_{\substack{[l']=[Z_K]-h\\ l'|_{\mathscr{I}}\not\succ(Z_K-E)|_{\mathscr{I}}}}z(l')\mathbf{t}^{l'}_{\mathscr{I}}\tag{10.6.7}$$

using the defining identity (10.6.5)

$$Z^+_{h,\mathscr{I}}(\mathbf{t}_{\mathscr{I}})=\mathbf{t}_{\mathscr{I}}^{Z_K-E}\check{Z}^+_{h,\mathscr{I}}(\mathbf{t}^{-1}_{\mathscr{I}}).\tag{10.6.8}$$

Then (10.6.7) gives for the dual polynomial part

$$\check{Z}^+_{h,\mathscr{I}}(\mathbf{t}_{\mathscr{I}})=\sum(-1)^{\sum_{v\in\mathscr{N}}k_v}\cdot\binom{\kappa-2}{k}\mathbf{t}_{\mathscr{I}}^{\sum_{v\in\mathscr{N}}k_vE^*_v+\alpha(\mathbf{x})}\tag{10.6.9}$$

where $\binom{\kappa-2}{k}:=\prod_{v\in\mathscr{N}}\binom{\kappa_v-2}{k_v}$ and the sum runs over all $0\le k_v\le\kappa_v-2$ for every $v\in\mathscr{N}$ and $\mathbf{x}\in\mathbb{Z}^{|\mathscr{V}^e|}_{\ge 0}\cap\rho^{-1}([Z_K-\sum_{v\in\mathscr{N}}k_vE^*_v]-h)$ such that $\alpha(\mathbf{x})|_{\mathscr{I}}\not\succ(Z_K-E-\sum_{v\in\mathscr{N}}k_vE^*_v)|_{\mathscr{I}}$. Now apply (10.6.8), (6.3.1) and replace k_v by κ_v-2-k_v in order to get for the polynomial part

$$Z^+_{h,\mathscr{I}}(\mathbf{t}_{\mathscr{I}})=(-1)^{|\mathscr{V}^e|}\sum(-1)^{\sum_{v\in\mathscr{N}}k_v}\binom{\kappa-2}{k}\cdot\mathbf{t}_{\mathscr{I}}^{\sum_{\mathscr{N}}k_vE^*_v-\alpha(\mathbf{x})},\tag{10.6.10}$$

where the sum runs over all $0 \le k_v \le \kappa_v - 2$ and $\mathbf{x} \in P_{\mathscr{V}}^{(\sum_{v\in\mathscr{N}} k_v E_v^*)} \cap \rho^{-1}([\sum_{v\in\mathscr{N}} k_v E_v^*] - h) \cap \mathbb{Z}_{>0}^{|\mathscr{V}^e|}$.

If we specify $\mathscr{I} = \mathscr{N}$, 10.6.22 implies that the sum in (10.6.10) runs over all $\mathscr{I}^m \subset \mathscr{N}^m$, $\mathscr{I}^m \neq \emptyset$, and lattice points of $P_{\mathscr{I}^m} \cap \rho^{-1}([\sum_I k_v E_v^*] - h)$ with all coordinates being strictly positive. Now, by Corollary 10.6.15 the formula 10.6.23 follows. □

Corollary 10.6.24 *If for every node v the condition $Z_K|_{\mathscr{N}} \le E_v^*|_{\mathscr{N}}$ is satisfied then*

$$\mathfrak{sw}_{\sigma_{[K+2r_h]}}(M) - \frac{(K+2r_h)^2 + |\mathscr{V}|}{8} = \#_h(P_{\mathscr{N}^m}) = \#_h(P_{\mathscr{N}}^{top}).$$

Moreover, the polynomial part is

$$Z_{h,\mathscr{N}}^+(\mathbf{t}_{\mathscr{N}}) = \sum \mathbf{t}_{\mathscr{N}}^{Z_K - E - \alpha(\mathbf{x})},$$

where the sum is over $\mathbf{x} \in \rho^{-1}([Z_K] - h)$ and $\mathbf{x} + (1, \dots, 1) \in P_{\mathscr{N}}^{top} \setminus \cup_i \{x_i = 0\}$.

Proof In 10.6.23 only the submultiset $\mathscr{I}^m = \mathscr{N}^m$ has non-trivial contribution. Indeed, in the 'symmetric expression' (10.6.9) each k_v should be zero, since $\alpha(\mathbf{x})|_{\mathscr{I}} \not\succ (Z_K - E - \sum_{v\in\mathscr{N}} k_v E_v^*)|_{\mathscr{I}}$. □

Example 10.6.25 Let M be a negative definite Seifert fibered rational homology sphere, which means that Γ is a star-shaped tree. Assume that the central node v_0 has $\nu \ge 3$ legs, each with determinant $\alpha_j \ge 2$ ($j = 1, \dots, \nu$). Let e be the orbifold Euler number of M. Then by (3.5.2) and (6.3.1) $(Z_K - E)|_{v_0} = (\nu - 2 - \sum_j 1/\alpha_j)/|e|$ and $E_{v_0}^*|_{v_0} = 1/|e|$. Also, the polytope of v_0 is $P_{v_0} = \{\mathbf{x} \in \mathbb{R}_{\ge 0}^{\nu} : \sum_{j=1}^{\nu} x_j/\alpha_j \le 1\}$. Hence, the assumption of Corollary 10.6.24 reads as

$$\nu - 3 < \sum_{j=1}^{\nu} \frac{1}{\alpha_j}. \tag{10.6.11}$$

This inequality cannot hold if $\nu \ge 6$ and it is always satisfied for $\nu = 3$. If (10.6.11) holds then one needs exactly one polytope in 10.6.23, otherwise one needs more.

Assume that $\{\alpha_j\}_j$ are pairwise relative prime, or, equivalently, that M is an integral homology sphere, the link of a Brieskorn complete intersection. Then for $\nu = 3$ by Corollary 10.6.24 we recover the well-known result that the normalized Casson invariant of $M(\alpha_1, \alpha_2, \alpha_3)$ is $\#(P_0)$. (Combine 9.2, 6.9.7 and 5.5.4(c).) For arbitrary $\nu \ge 3$ the right hand side of the formula from Theorem 10.6.23 is

$$\sum_{0<k\le\nu-2} (-1)^{\nu-2-k} \binom{\nu-2}{k} \cdot \#(kP_{v_0}).$$

The common Newton polytope of the $(\nu - 2)$-Brieskorn equations is P_0, hence the result fits perfectly with the geometric genus formula of [443], see here in 10.6.20.

Example 10.6.26 For an explicit computation of the involved polytoles and lattice points in the case of a graph with two nodes and $H \neq 0$ see [355].

Chapter 11
Lattice Cohomology

11.1 The Lattice Cohomology Associated with a System of Weights

We provide two equivalent definitions for the lattice cohomology $\{\mathbb{H}^q\}_{q\geq 0}$ associated with a free $\mathbb{Z}$-module endowed with a fixed basis and with a set of 'compatible weight functions'. The first definition is based on the construction of a cochain complex. The second one involves the spaces $\{S_n\}_n$ introduced in 7.3.2. In 11.1.A we apply the definition for a negative definite plumbing graph Γ (where $M(\Gamma)$ is a rational homology sphere). Once Γ is fixed, any characteristic element $k \in \mathrm{Char}$ determines a set of weights (via the RR expression χ_k), hence the lattice cohomology $\mathbb{H}^*(\Gamma, k)$. It turns out that they depend only on $M(\Gamma)$ and $[k] \in \mathrm{Spin}^c(M(\Gamma))$. In 11.1.B we show that the Euler characteristic of $\mathbb{H}^*(\Gamma, k)$ is the normalized Seiberg–Witten invariant of $M(\Gamma)$.
The first subsections follow [478, 480]. ♣

11.1.1 We consider a free $\mathbb{Z}$-module, with a fixed basis $\{E_v\}_{v\in\mathcal{V}}$, denoted by $\mathbb{Z}^s$. It is also convenient to fix a total ordering of the index set $\mathcal{V}$, which in the sequel will be denoted by $\{1, \dots, s\}$.

Our goal is to define a graded $\mathbb{Z}[U]$-module associated with the pair $(\mathbb{Z}^s, \{E_v\}_v)$ and a set of weights. First we set some notations regarding $\mathbb{Z}[U]$-modules.

11.1.2 $\mathbb{Z}[U]$**-Modules** Consider the graded $\mathbb{Z}[U]$-module $\mathbb{Z}[U, U^{-1}]$, and (following [547]) denote by $\mathscr{T}_0^+$ its quotient by the submodule $U \cdot \mathbb{Z}[U]$. This has a grading in such a way that $\deg(U^{-d}) = 2d$ $(d \geq 0)$. Similarly, for any $n \geq 1$, the quotient of $U^{-(n-1)} \cdot \mathbb{Z}[U]$ by $U \cdot \mathbb{Z}[U]$ (with the same grading) defines the graded module $\mathscr{T}_0(n)$. Hence, $\mathscr{T}_0(n)$, as a $\mathbb{Z}$-module, is freely generated by $1, U^{-1}, \dots, U^{-(n-1)}$, and has finite $\mathbb{Z}$-rank n.

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More generally, for any graded $\mathbb{Z}[U]$-module P with d-homogeneous elements P_d, and for any $r \in \mathbb{Q}$, we denote by $P[r]$ the same module graded (by $\mathbb{Q}$) in such a way that $P[r]_{d+r} = P_d$. Then set $\mathscr{T}_r^+ := \mathscr{T}_0^+[r]$ and $\mathscr{T}_r(n) := \mathscr{T}_0(n)[r]$. Hence, for $m \in \mathbb{Z}$, $\mathscr{T}_{2m}^+ = \mathbb{Z}\langle U^{-m}, U^{-m-1}, \dots\rangle$ as a $\mathbb{Z}$-module.

11.1.3 The Cochain Complex The space $\mathbb{Z}^s \otimes \mathbb{R}$ has a natural decomposition into cubes (see also 7.3.1). The set of zero-dimensional cubes consists of the lattice points $\mathbb{Z}^s$. Any $l \in \mathbb{Z}^s$ and subset $I \subset \mathscr{V}$ of cardinality q defines a q-dimensional cube, which has its vertices in the lattice points $(l + \sum_{v\in I'} E_v)_{I'\subset I}$. On each such cube we fix an orientation. This can be determined, e.g., by the order $(E_{v_1}, \dots, E_{v_q})$, where $v_1 < \cdots < v_q$, of the involved base elements $\{E_v\}_{v\in I}$. The set of oriented q-dimensional cubes defined in this way is denoted by $\mathscr{Q}_q$ $(0 \le q \le s)$.

Let $\mathscr{C}_q$ be the free $\mathbb{Z}$-module generated by oriented cubes $\square_q \in \mathscr{Q}_q$. Clearly, for each $\square_q \in \mathscr{Q}_q$, the oriented boundary $\partial\square_q$ (of 'classical' cubical homology) has the form $\sum_k \varepsilon_k \square_{q-1}^k$ for some $\varepsilon_k \in \{-1, +1\}$. These are the *faces* of $\square_q$.

It is clear that $\partial \circ \partial = 0$. But, obviously, the homology of the chain complex $(\mathscr{C}_*, \partial)$ (or, of the dual cochain complex $(\mathrm{Hom}_{\mathbb{Z}}(\mathscr{C}_*, \mathbb{Z}), \delta)$) is not very interesting: it is the (co)homology of $\mathbb{R}^s$. A more interesting (co)homology can be constructed as follows. For this, we consider a set of compatible *weight functions* $\{w_q\}_q$.

Definition 11.1.4 (Compare with 7.3.6) A set of functions $w_q : \mathscr{Q}_q \to \mathbb{Z}$ $(0 \le q \le s)$ is called a *set of compatible weight functions* if the following hold:

(a) For any integer $k \in \mathbb{Z}$, the set $w_0^{-1}(\,(-\infty, k]\,)$ is finite;
(b) $w_q(\square_q) \ge w_{q-1}(\square_{q-1})$ for any $\square_q \in \mathscr{Q}_q$ and any of its faces $\square_{q-1} \in \mathscr{Q}_{q-1}$.

In the sequel sometimes we will omit the index q of w_q.

11.1.5 In the presence of any fixed set of compatible weight functions $\{w_q\}_q$ we define $\mathscr{F}^q$ as the set of morphisms $\mathrm{Hom}_{\mathbb{Z}}(\mathscr{C}_q, \mathscr{T}_0^+)$ with finite support on $\mathscr{Q}_q$.

Notice that $\mathscr{F}^q$ is a $\mathbb{Z}[U]$-module by $(p * \phi)(\square_q) := p(\phi(\square_q))$ $(p \in \mathbb{Z}[U])$. Moreover, $\mathscr{F}^q$ has a $\mathbb{Z}$-grading: $\phi \in \mathscr{F}^q$ is homogeneous of degree $\deg(\phi) = d \in \mathbb{Z}$ if for each $\square_q \in \mathscr{Q}_q$ with $\phi(\square_q) \ne 0$, $\phi(\square_q)$ is a homogeneous element of $\mathscr{T}_0^+$ of degree $d - 2 \cdot w(\square_q)$. (In fact, the grading is $2\mathbb{Z}$-valued; hence, the reader interested only in the present construction may divide all the degrees by two. Nevertheless, we prefer to keep the present form in our presentation because of its resonance with the Heegaard Floer homology of the link.)

Next, we define $\delta_w : \mathscr{F}^q \to \mathscr{F}^{q+1}$. For this, fix $\phi \in \mathscr{F}^q$ and we indicate how $\delta_w\phi$ acts on a cube $\square_{q+1} \in \mathscr{Q}_{q+1}$. First write $\partial\square_{q+1} = \sum_k \varepsilon_k \square_q^k$, then set

$$(\delta_w\phi)(\square_{q+1}) := \sum_k \varepsilon_k\, U^{w(\square_{q+1}) - w(\square_q^k)}\, \phi(\square_q^k).$$

Lemma 11.1.6 $\delta_w \circ \delta_w = 0$, *i.e.* $(\mathscr{F}^*, \delta_w)$ *is a cochain complex.*

Proof With the obvious notations, $(\delta_w^2\phi)(\square_{q+2}^j)$ equals

$$\sum_k \varepsilon_k^j U^{w(\square_{q+2}^j)-w(\square_{q+1}^k)} \sum_l \varepsilon_l^k U^{w(\square_{q+1}^k)-w(\square_q^l)} \phi(\square_q^l) = \sum_l U^{w(\square_{q+2}^j)-w(\square_q^l)} \Big(\sum_k \varepsilon_k^j \varepsilon_l^k \Big) \phi(\square_q^l).$$

But, for any j and l, $\sum_k \varepsilon_k^j \varepsilon_l^k = 0$ since $\partial^2 = 0$. □

11.1.7 In fact, $(\mathscr{F}^*, \delta_w)$ has a natural **augmentation**. Indeed, set $m_w := \min_{l\in\mathbb{Z}^s} w_0(l)$ and choose $l_w \in \mathbb{Z}^s$ such that $w_0(l_w) = m_w$. Then define the $\mathbb{Z}[U]$-linear map

$$\epsilon_w : \mathscr{T}^+_{2m_w} \longrightarrow \mathscr{F}^0$$

such that $\epsilon_w(U^{-m_w-s})(l)$ is the class of $U^{-m_w+w_0(l)-s}$ in $\mathscr{T}_0^+$ for any $l \in L$ and $s \geq 0$.

Lemma 11.1.8 *ϵ_w is injective, and $\delta_w \circ \epsilon_w = 0$.*

Proof Since $\epsilon_w(U^{-m_w-s})(l_w) = U^{-s}$, the injectivity follows. Take $\square \in \mathscr{Q}_1$ with $\partial\square = a - b$. Then $(\delta_w\epsilon_w)(t)(\square) = U^{w(\square)-w(a)}\epsilon_w(t)(a) - U^{w(\square)-w(b)}\epsilon_w(t)(b) = U^{w(\square)}t - U^{w(\square)}t = 0$. □

We invite the reader to verify that ϵ_w and δ_w are morphisms of $\mathbb{Z}[U]$-modules, and are homogeneous of degree zero.

Definition 11.1.9 The homology of the cochain complex $(\mathscr{F}^*, \delta_w)$ is called the *lattice cohomology* of the pair $(\mathbb{R}^s, w)$, and it is denoted by $\mathbb{H}^*(\mathbb{R}^s, w)$. The homology of the augmented cochain complex

$$0 \longrightarrow \mathscr{T}^+_{2m_w} \xrightarrow{\epsilon_w} \mathscr{F}^0 \xrightarrow{\delta_w} \mathscr{F}^1 \xrightarrow{\delta_w} \dots$$

is called the *reduced lattice cohomology* of the pair $(\mathbb{R}^s, w)$, and it is denoted by $\mathbb{H}^*_{red}(\mathbb{R}^s, w)$.

If the pair $(\mathbb{R}^s, w)$ is clear from the context, we omit it from the notation.

For any $q \geq 0$ fixed, the $\mathbb{Z}$-grading of $\mathscr{F}^q$ induces a $\mathbb{Z}$-grading on $\mathbb{H}^q$ and $\mathbb{H}^q_{red}$; the homogeneous part of degree d is denoted by $\mathbb{H}^q_d$, or $\mathbb{H}^q_{red,d}$. Moreover, both $\mathbb{H}^q$ and $\mathbb{H}^q_{red}$ admit an induced graded $\mathbb{Z}[U]$-module structure and $\mathbb{H}^q = \mathbb{H}^q_{red}$ for $q > 0$.

It is easy to see that $\mathbb{H}^*(\mathbb{R}^s, w)$ depends essentially on the choice of w.

Lemma 11.1.10 *One has a graded $\mathbb{Z}[U]$-module isomorphism $\mathbb{H}^0 \simeq \mathscr{T}^+_{2m_w} \oplus \mathbb{H}^0_{red}$.*

Proof Consider the isomorphism $U^{-m_w} : \mathscr{T}_0^+ \to \mathscr{T}^+_{2m_w}$. Then define $r_w : \mathbb{H}^0 \to \mathscr{T}^+_{2m_w}$ by $r_w(\phi) := U^{-m_w}\phi(l_w)$. Since $r_w \circ \epsilon_w = 1$, the exact sequence $0 \to \mathscr{T}^+_{2m_w} \xrightarrow{\epsilon_w} \mathbb{H}^0 \to \mathbb{H}^0_{red} \to 0$ splits. □

11.1.11 Next, we present another realization of the module $\mathbb{H}^*$. In 7.3.2 for each $n \in \mathbb{Z}$ we defined $S_n = S_n(w) \subset \mathbb{R}^s$ as the union of all the cubes $\square_q$ (of any dimension) with $w(\square_q) \leq n$. Clearly, $S_n = \emptyset$, whenever $n < m_w$. For any $q \geq 0$, set

$$\mathbb{S}^q(\mathbb{R}^s, w) := \oplus_{n \geq m_w} H^q(S_n, \mathbb{Z}).$$

Then $\mathbb{S}^q$ is $\mathbb{Z}$ (in fact, $2\mathbb{Z}$)-graded, the $d = 2n$-homogeneous elements $\mathbb{S}^q_d$ consist of $H^q(S_n, \mathbb{Z})$. Also, $\mathbb{S}^q$ is a $\mathbb{Z}[U]$-module; the U-action is given by the restriction map $r_{n+1} : H^q(S_{n+1}, \mathbb{Z}) \to H^q(S_n, \mathbb{Z})$. Moreover, for $q = 0$, the fixed base-point $l_w \in S_n$ provides an augmentation (splitting) $H^0(S_n, \mathbb{Z}) = \mathbb{Z} \oplus \widetilde{H}^0(S_n, \mathbb{Z})$, hence an augmentation of the graded $\mathbb{Z}[U]$-modules

$$\mathbb{S}^0 = \mathscr{T}^+_{2m_w} \oplus \mathbb{S}^0_{red} = (\oplus_{n \geq m_w} \mathbb{Z}) \oplus (\oplus_{n \geq m_w} \widetilde{H}^0(S_n, \mathbb{Z})).$$

Theorem 11.1.12

(a) There exists a graded $\mathbb{Z}[U]$-module isomorphism, compatible with the augmentations:

$$\mathbb{H}^*(\mathbb{R}^s, w) = \mathbb{S}^*(\mathbb{R}^s, w).$$

(b) For any degree d, there exists an integer $N(d) \geq 0$, such that

$$\mathbb{H}^*_{red,d} \cap im(U^{N(d)}) = 0.$$

(c) For any n one has $U^{n-m_w+1}\mathbb{H}^*_{2n} = 0$. If there exists N such that S_n is contractible for any $n \geq N$, then $U^{N-m_w}\mathbb{H}^*_{red} = 0$.

Proof *(a)* Let $\mathscr{F}^q_d$ be the set of $d = 2n$-homogeneous elements $\phi \in \mathscr{F}^q$. Since δ_w is homogeneous of degree zero, $(\mathscr{F}^*_d, \delta_w)$ is a complex. Let $(\mathscr{C}^*(S_n), \delta)$ be the usual cochain complex of S_n. Then the two complexes can be naturally identified. Indeed, take $\phi \in \mathscr{F}^q_d$. Then, for any $\square_q$, $\phi(\square_q)$ has the form $a_\phi(\square_q)U^{w(\square_q)-n}$. Hence $a_\phi(\square_q) \in \mathbb{Z}$ is well-defined for any q-cube $\square_q$ of S_n, and the correspondence $\phi \mapsto a_\phi$ realizes the bijection $\mathscr{F}^*_d \to \mathscr{C}^*(S_n)$.

Since $\widetilde{H}_q(\mathbb{R}^s, \mathbb{Z}) = 0$, for any n there exists N such that $\widetilde{H}_q(S_n) \to \widetilde{H}_q(S_{n+N})$ is trivial. [Indeed, take a finite set of generators $\{c_\alpha\}_\alpha$ of $\widetilde{H}_q(S_n)$ and choose cycles $\{d_\alpha\}_\alpha$ with $\partial(d_\alpha) = c_\alpha$ in $\mathbb{R}^s$. Then the restriction of w_q on $\cup_\alpha d_\alpha$ is bounded.] Part *(b)* is the dual statement of this fact. *(c)* follows from *(a)–(b)*. □

Remark 11.1.13 The definition $\mathbb{S}^*(\mathbb{R}^s, w) := \oplus_{n \geq m_w} H^*(S_n, \mathbb{Z})$, based on the tower of space $\{S_n\}_{n \geq m_w}$ suggests the possibility to introduce several other invariants of $(\mathbb{R}^s, w)$; e.g., H^* can be replaces by other generalized (extraordinary) cohomology groups, or by the homotpy groups, or, one can also consider different

cohomology operations as additional structures. Such potential possibilities are not exploited in this book, but definitely should be considered in the future.

Remark 11.1.14 Though $\mathbb{H}^*_{red}(\mathbb{R}^s, w)$ has finite $\mathbb{Z}$-rank in any fixed homogeneous degree, in general, it is not finitely generated over $\mathbb{Z}$, in fact, not even over $\mathbb{Z}[U]$. E.g., set $s = 1$, and define w_0 by $w_0(-n) = w_0(n) = [n/2] + 4\{n/2\}$ for any $n \in \mathbb{Z}_{\geq 0}$, where $[\,]$ and $\{\,\}$ are the integral, respectively the fractional parts; and let w_1 on the segment $[n, n+1]$ take the value $\max\{w_0(n), w_0(n+1)\}$. Then $\mathbb{H}^0_{red} = \oplus_{k\geq 1}\mathscr{T}_k(1)^2$.

11.1.15 Restrictions Assume that $T \subset \mathbb{R}^s$ is a subspace of $\mathbb{R}^s$ consisting of a union of some cubes (from $\mathscr{Q}_*$). Let $\mathscr{C}_q(T)$ be the free $\mathbb{Z}$-module generated by q-cubes of T, $\mathscr{F}^q(T)$ be the restriction of $\mathscr{F}^q$ to $\mathscr{C}_q(T)$. Then $(\mathscr{F}^*(T), \delta_w)$ is a complex, whose homology will be denoted by $\mathbb{H}^*(T, w)$. It has a natural graded $\mathbb{Z}[U]$-module structure. Again, $\mathbb{H}^0(T, w) \simeq \mathscr{T}^+_{2\min\{w|_T\}} \oplus \mathbb{H}^0_{red}(T, w)$. The restriction map induces a natural graded $\mathbb{Z}[U]$-module homogeneous homomorphism (of degree zero)

$$r^* : \mathbb{H}^*(\mathbb{R}^s, w) \to \mathbb{H}^*(T, w).$$

Definition 11.1.16 Fix $T \subset \mathbb{R}^s$ as above, and assume that $\mathbb{H}^*_{red}(T, w)$ has finite $\mathbb{Z}$–rank. Then we define the *Euler characteristic* of $\mathbb{H}^*(T, w)$ as

$$eu(\mathbb{H}^*(T, w)) := -\min\{w_0|_T\} + \sum_q (-1)^q \operatorname{rank}_{\mathbb{Z}} \mathbb{H}^q_{red}(T, w).$$

For motivation of the $-\min\{w_0|_T\}$ term see 11.2.10 and the computations from 11.1.32.

11.1.A The Lattice Cohomology Associated with a Plumbing Graph

11.1.17 We consider a connected negative definite plumbing graph Γ and we assume that Γ is a tree and $g_v = 0$ for every v (that is, $M(\Gamma)$ is a rational homology sphere). We also fix a characteristic element $k \in \mathrm{Char}$. We also write $s := |\mathscr{V}|$.

Note that Γ automatically and naturally provides a free $\mathbb{Z}$-module $L = \mathbb{Z}^s$ with a fixed bases $\{E_v\}_v$, cf. 2.1.15 and 7.3.1. Using Γ and k, we define a set of compatible weight functions w as in 7.3.1: $w_k(\square_q) = \max\{\chi_k(v) \,:\, v \text{ is a vertex of } \square_q\}$.

Definition 11.1.18 The $\mathbb{Z}[U]$-modules $\mathbb{H}^*(\mathbb{R}^s, w)$ and $\mathbb{H}^*_{red}(\mathbb{R}^s, w)$ obtained by these weight functions are called the *lattice cohomologies* associated with the pair (Γ, k) and are denoted by $\mathbb{H}^*(\Gamma, k)$, respectively $\mathbb{H}^*_{red}(\Gamma, k)$.

(Recall that we also have the notation $m_k := m_w = \min_{l\in L} \chi_k(l)$, cf. 7.3.2).

Proposition 11.1.19

(a) $\mathbb{H}^*_{red}(\Gamma, k)$ *is finitely generated over* $\mathbb{Z}$.
(b) $\mathbb{H}^0_{red,d}(\Gamma, K) = 0$ *for the canonical characteristic element* K *and* $d > 0$.

Proof (a) Use either 7.3.9 or 7.3.10. (b) Use 7.3.13. □

Remark 11.1.20 There is a symmetry present in the picture. Indeed, the involution $x \mapsto -x$ ($x \in L'$) induces identities $\chi_{-k}(-l) = \chi_k(l)$, hence isomorphisms

$$\mathbb{H}^*(\Gamma, k) = \mathbb{H}^*(\Gamma, -k) \text{ and } \mathbb{H}^*_{red}(\Gamma, k) = \mathbb{H}^*_{red}(\Gamma, -k).$$

The involution $[k] \mapsto [-k]$ corresponds to the natural involution of $\mathrm{Spin}^c(M)$, cf. 6.10.19.

11.1.21 Assume that $[k] = [k']$, hence $k' = k + 2l$ for some $l \in L$. Then $\chi_{k'}(x - l) = \chi_k(x) - \chi_k(l)$ for any $x \in L$. Therefore, the transformation $x \mapsto x' := x - l$ realizes the following identification:

Lemma 11.1.22 *If* $k' = k + 2l$ *for some* $l \in L$, *then:* $\mathbb{H}^*(\Gamma, k') = \mathbb{H}^*(\Gamma, k)[-2\chi_k(l)]$.

11.1.23 In fact, there is an easy way to choose one module from the multitude $\{\mathbb{H}^*(\Gamma, k)\}_{k \in [k]}$. Indeed, set $m_k = \min_{l \in L} \chi_k(l)$ as above. Since $(k + 2l)^2 = k^2 - 8\chi_k(l)$, we get

$$8m_k = k^2 - \max_{k' \in [k]} (k')^2 \leq 0. \tag{11.1.1}$$

Set $M_{[k]} := \{k \in [k] : m_k = 0\}$. Hence, if k_0 and $k_0 + 2l \in M_{[k]}$, then $-\chi_{k_0}(l) = 0$. In particular, for any fixed orbit $[k]$, any choice of $k_0 \in M_{[k]}$ provides the same module $\mathbb{H}^*(\Gamma, k_0)$, in the sequel denoted by $\mathbb{H}^*(\Gamma, [k])$. Hence, for any $k \in [k]$

$$\mathbb{H}^*(\Gamma, k) = \mathbb{H}^*(\Gamma, [k])[2m_k]. \tag{11.1.2}$$

Proposition 11.1.24 *For each fixed* $[k] \in \mathit{Spin}^c(M(\Gamma))$, $\mathbb{H}^*(\Gamma, [k])$ *depends only on* $M(\Gamma)$ *and is independent of the choice of the graph* Γ, *which provides* $M(\Gamma)$.

Proof Use Proposition 7.3.5. □

Remark 11.1.25

(a) The well–defined homotopy type of the tower $\{S_n\}_{n \geq m_w}$ is a much stronger invariant than the lattice cohomology, we can call it the *lattice homotopy type* (following the model of 'Khovanov homotopy type' of [391]). Usually to find the 'homotopy type' of a cohomology theory (or to realize it as the singular homology of some natural spectrum) is not automatical at all. However, in our

lattice cohomology case we have such a realization from one of the definitions and from 7.3.5, which guarantees that it is well–defined up to homotopy. See also 11.1.13.

(b) The author knows no example when the $\mathbb{Z}$–module $\mathbb{H}^*(\Gamma, k)$ has a non-zero $\mathbb{Z}$–torsion. *It is a challenge to prove that this cannot occur indeed.*

Consider the distinguished characteristic element k_r, cf. 7.3.11. The following statement follows from 7.3.12.

Proposition 11.1.26 *The restriction $\mathbb{H}^*(\Gamma, k_r) \to \mathbb{H}^*((\mathbb{R}_{\geq 0})^s, k_r)$ induced by the inclusion $(\mathbb{R}_{\geq 0})^s \hookrightarrow \mathbb{R}^s$ is an isomorphism of graded $\mathbb{Z}[U]$ modules.*

Remark 11.1.27 Assume that Γ is either rational or elliptic, in particular, $\min(\chi) = 0$. Then by 7.3.12 $\min(\chi_{k_r}) \geq 0$. Hence, by (11.1.1), in fact, $\min(\chi_{k_r}) = 0$.

Example 11.1.28 (Rational Graphs) Theorem 7.3.14 transforms into the following statement. The following facts are equivalent: (a) Γ is rational; (b) $\mathbb{H}^*_{red}(\Gamma, K) = 0$; (b') $\mathbb{H}^0_{red}(\Gamma, K) = 0$; (c) $\mathbb{H}^*_{red}(\Gamma, k) = 0$ for every $k \in \mathrm{Char}$.
Additionally, by Remark 11.1.27, if Γ is rational then $\mathbb{H}^0(\Gamma, k_r) = \mathscr{T}_0^+$ for any k_r.

Example 11.1.29 (Elliptic Graphs) Theorem 7.3.15 and Remark 11.1.27 transform into the following statement. The following facts are equivalent:

(a) Γ is elliptic;
(b) $\mathbb{H}^0(\Gamma, K) = \mathscr{T}_0^+ \oplus \mathbb{H}^0_{red}(\Gamma, K)$ with $\mathbb{H}^0_{red}(\Gamma, K) \neq 0$.

Furthermore, if Γ is elliptic then $\mathbb{H}^0_{red}(\Gamma, K) = \mathscr{T}_0(1)^\ell$ (that is, it is $\mathbb{Z}^\ell$ of homogeneous degree 0), where $\ell = \ell_{seq} > 0$ is the length of the elliptic sequence.

Example 11.1.30 (Minimally Elliptic Graphs) Assume that Γ is minimally elliptic and the minimal resolution is good. Then by 7.3.18 and 11.1.27 $\mathbb{H}^*(\Gamma, k_r) = \mathscr{T}_0^+$ for any $[k_r] \neq [K]$. (By 11.1.29 $\mathbb{H}^*(\Gamma, K) = \mathscr{T}_0^+ \oplus \mathscr{T}_0(1)$.)

Example 11.1.31 (Almost Rational Graphs) By 7.3.35 $\mathbb{H}^q(\Gamma, k) = 0$ for any $q \geq 1$ and $k \in \mathrm{Char}$. (For $\mathbb{H}^0(\Gamma, k)$ see 11.3.)

11.1.B The Lattice Cohomology and the Seiberg–Witten Invariant

Fix Γ and k as in Sect. 11.1.A. Our goal is to identify the 'Euler characteristic' of the lattice cohomology $\mathbb{H}^*(\Gamma, k)$. Recall that by 11.1.19 $\mathrm{rank}_{\mathbb{Z}}(\mathbb{H}^*_{red}(\Gamma, k)) < \infty$. Then, cf. 11.1.16, the Euler characteristic of $\mathbb{H}^*(\Gamma, k)$ is defined as

$$eu(\mathbb{H}^*(\Gamma, k)) := -m_k + \sum_q (-1)^q \mathrm{rank}_{\mathbb{Z}}(\mathbb{H}^q_{red}(\Gamma, k)). \tag{11.1.3}$$

11.1.32 Fix l_- and l_+ and the rectangle $R = R(-l_-, l_+)$ as in 7.3.9. We set

$$Eu_{\chi_k}(R) := \sum_{\square_q \subset R} (-1)^{q+1} w_k(\square_q) \quad \text{and} \quad Eu^{pol}_{\chi_k}(\mathfrak{q}) := \sum_{\square_q \subset R} (-1)^q \mathfrak{q}^{w_k(\square_q)} \in \mathbb{Z}[\mathfrak{q}, \mathfrak{q}^{-1}].$$

In particular, if we write $Eu^{pol}_{\chi_k}(\mathfrak{q})/(1-\mathfrak{q})$ as $\sum_{n \geq m_k} a_n \mathfrak{q}^n$ then

$$a_n = \sum_{\square_q \subset R,\ w_k(\square_q) \leq n} (-1)^q = \chi_{top}(S_n \cap R),$$

where χ_{top} is the topological Euler characteristic. But, by 7.3.9, $S_n \cap R \hookrightarrow S_n$ is a homotopy equivalence, hence $a_n = \chi_{top}(S_n)$. This by 11.1.12 reads as

$$\frac{Eu^{pol}_{\chi_k}(\mathfrak{q}) - \mathfrak{q}^{m_k}}{1-\mathfrak{q}} = \sum_{n \geq m_k} (a_n - 1)\mathfrak{q}^n = \sum_{n \geq m_k} \Big(\sum_{q \geq 0} (-1)^q \mathrm{rank}_{\mathbb{Z}}(\mathbb{H}^q_{red,2n}(\Gamma, k)) \Big) \mathfrak{q}^n.$$

In particular, this expression is independent of the choice of R.

Since all the sums are finite one might take the limit $\lim_{\mathfrak{q} \to 1}$:

$$Eu_{\chi_k}(R) + m_k = \sum_{q \geq 0} (-1)^q \mathrm{rank}_{\mathbb{Z}}(\mathbb{H}^q_{red}(\Gamma, k)) \Big),$$

or, cf. [482, Theorem 2.3.7],

$$Eu_{\chi_k}(R) = eu(\mathbb{H}^*(\Gamma, k)). \tag{11.1.4}$$

The above identity is a generalization to the level of weighted cubes of the classical fact that the Euler characteristic computed at the level of cubes (à la Euler) equals the homological Euler characteristic (à la Poincaré or Betti).

11.1.33 Recall that if $k' = k + 2l$, $l \in L$, then $\mathbb{H}^*(\Gamma, k') = \mathbb{H}^*(\Gamma, k)[-2\chi_k(l)]$, cf. 11.1.A. Hence, the lattice cohomologies associated with different k's with the same class $[k]$ are equal up to a shift. This has no effect on $\sum_q (-1)^q \mathrm{rank}_{\mathbb{Z}}(\mathbb{H}^q_{red}(\Gamma, k))$, however it has on m_k. This can be remedied either by choosing k from $M_{[k]}$ (cf. 11.1.23), or by taking k_r (cf. 11.1.21). Here is another way to eliminate this shift.

Let us replace the weight function $w_k(\square_q) := \{\chi_k(v) \ : \ v \text{ is a vertex of } \square_q\}$ by

$$\overline{w}_k(\square_q) := w_k(\square_q) + \mathfrak{d}_k, \ \text{ where } \ \mathfrak{d}_k := -\frac{k^2 + |\mathscr{V}|}{8} + \frac{K^2 + |\mathscr{V}|}{8} = \chi\Big(\frac{k-K}{2}\Big),$$

and denote the corresponding lattice cohomologies by $\overline{\mathbb{H}}^*(\Gamma, k)$. Then

Lemma 11.1.34 $\overline{\mathbb{H}}^*(\Gamma, k) = \mathbb{H}^*(\Gamma, k)[\mathfrak{d}_k]$ *is independent of the choice of* k *from* $[k]$.

Remark 11.1.35 In the spirit of 9.6.7, and with the notation $k = K + 2l'_h$, $\overline{\mathbb{H}}^*(\Gamma, k)$ is the lattice cohomology of the cubes of $l'_h + L$, where the weight function is generated by the restriction of χ on this shifted lattice $l'_h + L$. (Indeed, for $l \in L$, $\chi(l + l'_h) = \chi_k(l) + \mathfrak{d}_k$.) Hence, $eu(\mathbb{H}^*(\Gamma, k)) = eu(\overline{\mathbb{H}}^*(\Gamma, k)) + \mathfrak{d}_k$.

On the other hand, from (9.6.6) $Eu_{\overline{\chi}_k}(R) = \overline{\mathfrak{s}}_h$, and by 9.6.20

$$\mathfrak{sw}_{\sigma[K+2l'_h]}(M(\Gamma)) = \overline{\mathfrak{s}}_h + (K^2 + |\mathscr{V}|)/8.$$

All these combined together with (11.1.4) give

Theorem 11.1.36 ([482])

$$eu(\mathbb{H}^*(\Gamma, k)) = \mathfrak{sw}_{\sigma[k]}(M(\Gamma)) - \frac{k^2 + |\mathscr{V}|}{8}.$$

11.1.37 The SWIC Revisited For any $h \in H$ assume that the representative l'_h is either r_h or s_h. Then via the first extension 9.7.7 of the SWIC combined with 11.1.36 from above, the SWIC(h) is equivalent to

$$\text{(SWIC}(h)) \qquad h^1(\widetilde{X}, \mathscr{O}(-l'_h)) = eu(\mathbb{H}^*(\Gamma, K + 2l'_h)). \tag{11.1.5}$$

We wish to emphasize that to some extent this conjectured identity (SWIC) (and the efforts to prove it for different cases) lead to the definition of graded roots and lattice cohomology (at least, of $\mathbb{H}^0$), cf. [478]. Indeed, for several singularities with AR graphs (e.g. for the weighted homogeneous germs) the left hand side was computed by a concatenated Laufer computations sequence, and its χ-fluctuation was reformulated as the key topological object at the right hand side (cf. 7.3.D and 11.3).

Example 11.1.38 (Verification of SWIC for Rational Singularities) We wish to verify that $h^1(\widetilde{X}, \mathscr{O}(-s_h)) = eu(\mathbb{H}^*(\Gamma, K + 2s_h))$ for any h. But, in fact, both sides vanish; the left hand side by 7.1.11, the right hand side by 11.1.28. In particular, by 11.1.36, $\mathfrak{sw}_{\sigma[k]}(M(\Gamma)) = (k_r^2 + |\mathscr{V}|)/8$ for any rational singularity and $[k] = [k_r]$.

Example 11.1.39 (Verification of SWIC for Minimally Elliptic Singularities) Again we wish to verify (11.1.5) for $l'_h = s_h$. If $[k_r] \neq [K]$ then again both sides vanish, the left hand side by 7.2.31(c), the right hand side by 11.1.30. Otherwise, the left hand side is $p_g = 1$, while the right hand side is 1 by the same 11.1.30.

Example 11.1.40 (Discussion of SWIC for Elliptic Singularities, $k = K$) In this case the left hand side of (11.1.5) (with $l'_h = 0$) is p_g, while the right hand side is the length of the elliptic sequence ℓ_{seq}, cf. 11.1.29. In general, for an arbitrary analytic structure $p_g \leq \ell_{seq}$, however, for (X_0, o_0) Gorenstein, they equal, cf. 7.2.65.

11.2 Graded Roots and Their Cohomologies

We introduce the abstract graded root (R, χ) and we define its cohomology $\mathbb{Z}[U]$-module $\mathbb{H}(R, \chi)$. We provide several constructions, which provide graded roots. One of them associates a graded root (R_w, χ_w) with a system of weights $\{w_q\}_q$. In particular, any resolution graph Γ and characteristic element k provides a graded root. If $\{w_q\}_q$ is given by w_0 via max-principle, then $\mathbb{H}(R_w, \chi_w) = \mathbb{H}^0(\mathbb{R}^s, w)$. In particular, (R_w, χ_w) is an enhancement of $\mathbb{H}^0$. ♣

11.2.1 In this subsection we give the definition and the first properties of abstract graded roots. We follow [478, 479]. Section 11.2.A contains the construction of the graded roots R_k from the plumbing graphs Γ. [Although both graphs, namely Γ, the plumbing graph, and the constructed graded root R_k are 'connected graphs', they serve rather different roles, and emphasize rather different topological aspects.]

Definition 11.2.2

(1) Let R be an infinite tree with vertices $\mathcal{V}$ and edges $\mathcal{E}$. We denote by $[u, v]$ the edge with end-vertices u and v. We say that R is a *graded root* with grading $\chi : \mathcal{V} \to \mathbb{Z}$ if

   (a) $\chi(u) - \chi(v) = \pm 1$ for any $[u, v] \in \mathcal{E}$;
   (b) $\chi(u) > \min\{\chi(v), \chi(w)\}$ for any $[u, v],\ [u, w] \in \mathcal{E}$, $v \neq w$;
   (c) χ is bounded below, $\chi^{-1}(k)$ is finite for any $k \in \mathbb{Z}$, and $|\chi^{-1}(k)| = 1$ if $k \gg 0$.

(2) $v \in \mathcal{V}$ is a *local minimum point* of the graded root (R, χ) if $\chi(v) < \chi(w)$ for any edge $[v, w]$. Their set is denoted by $\mathcal{V}_{lm}$.
(3) A geodesic path connecting two vertices is *monotone* if χ restricted to the set of vertices on the path is strictly monotone. If a vertex v can be connected by another vertex w by a monotone geodesic and $\chi(v) > \chi(w)$, then we write $v \succ w$. $\succ$ is an ordering of $\mathcal{V}$.
(4) If (R, χ) is a graded root, and $r \in \mathbb{Z}$, then $(R, \chi)[r]$ denotes the same R with the new grading $\chi[r](v) := \chi(v) + r$.
(5) There is a natural definition of isomorphism of graded roots: it is a graph isomorphism, which preserves the gradings.

One verifies the following facts as well:

(6) By Part (1)(b), $\mathcal{V}_{lm}$ coincides with the set of vertices with adjacent degree one.

(7) For any pair $v, w \in \mathscr{V}$ there is a unique $\succ$*-minimal vertex* $\sup(v, w)$*, which dominates both (that is,* $\sup(v, w) \succ v$ *and* $\sup(v, w) \succ w$*).*

Example 11.2.3

(1) For any integer $n \in \mathbb{Z}$, let $R_{(n)}$ be the tree with $\mathscr{V} = \{v^k\}_{k \geq n}$ and $\mathscr{E} = \{[v^k, v^{k+1}]\}_{k \geq n}$. The grading is $\chi(v^k) = k$.

(2) Let I be a finite index set. For each $i \in I$ fix an integer $n_i \in \mathbb{Z}$; and for each pair $i, j \in I$ fix $n_{ij} = n_{ji} \in \mathbb{Z}$ with the next properties:

(i) $n_{ii} = n_i$;
(ii) $n_{ij} \geq \max\{n_i, n_j\}$; and
(iii) $n_{jk} \leq \max\{n_{ij}, n_{ik}\}$ for any $i, j, k \in I$.

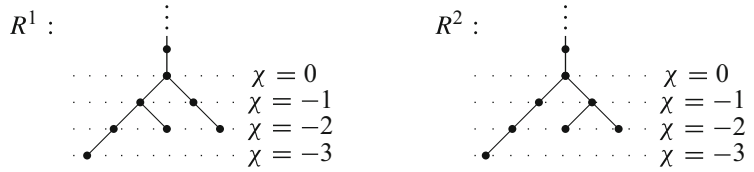
For any $i \in I$ consider $R_i := R_{(n_i)}$ with vertices $\{v_i^k\}$ and edges $\{[v_i^k, v_i^{k+1}]\}$, $(k \geq n_i)$. In the disjoint union $\sqcup_i R_i$, for any pair (i, j), identify v_i^k and v_j^k, resp. $[v_i^k, v_i^{k+1}]$ and $[v_j^k, v_j^{k+1}]$, whenever $k \geq n_{ij}$. Write $\bar{v}_i^k$ for the class of v_i^k. Then $\sqcup_i R_i/\sim$ is a graded root with $\chi(\bar{v}_i^k) = k$. It will be denoted by $R = R(\{n_i\}, \{n_{ij}\})$.

Clearly $\mathscr{V}_{lm}(R)$ is a subset of $\{\bar{v}_i^{n_i}\}_{i \in I}$, and this last set can be identified with I. $\mathscr{V}_{lm}(R) = I$ if in (ii) all the inequalities are strict. Otherwise all the indices $I \setminus \mathscr{V}_{lm}(R)$ are superfluous, i.e. the corresponding R_i's produce no additional vertices.

Any graded root (R', χ') is isomorphic with some $R(\{n_i\}, \{n_{ij}\})$. Indeed, set $I := \mathscr{V}_{lm}(R')$, $n_v := \chi'(v)$ and $n_{uv} := \chi'(\sup(u, v))$ for $u, v \in I$.

(3) Any map $\tau : \{0, 1, \ldots, T_0\} \to \mathbb{Z}$ produces a starting data for construction (2). Indeed, set $I = \{0, \ldots, T_0\}$, $n_i := \tau(i)$ $(i \in I)$, and $n_{ij} := \max\{n_k : i \leq k \leq j\}$ for $i \leq j$. Then $\sqcup_i R_i/\sim$ constructed in (2) using this data will be denoted by (R_τ, χ_τ).

For example, for $T_0 = 4$, take for the values of τ: $-3, -1, -2, 0$ and -2 (respectively $-3, 0, -2, -1$ and -2). Then the two graded roots are:



This construction can be extended to the case of a map $\tau : \mathbb{N} \to \mathbb{Z}$, whenever τ has the property that there exists $k_0 \geq 0$ such that $\tau(k+1) \geq \tau(k)$ for any $k \geq k_0$. In this case one can take any $T_0 \geq k_0$ and construct the root associated with the restriction of τ to $\{0, \ldots, T_0\}$. It is independent of the choice of T_0 (since all contributions for $k \geq k_0$ are superfluous). By definition, this is the root associated with τ.

Definition 11.2.4 (The (Cohomology) $\mathbb{Z}[U]$-Modules Associated with a Graded Root) For any graded root (R, χ), let $\mathbb{H}(R, \chi)$ (briefly $\mathbb{H}(R)$) be the set of functions $\phi : \mathscr{V} \to \mathscr{T}_0^+$ with the following property: whenever $[v, u] \in \mathscr{E}$ with $\chi(v) < \chi(u)$, then $U \cdot \phi(v) = \phi(u)$. Or, equivalently, one requires

$$U^{\chi(u)-\chi(v)} \cdot \phi(v) = \phi(u) \text{ for any } w \succ v. \tag{11.2.1}$$

Clearly $\mathbb{H}(R)$ is a $\mathbb{Z}[U]$-module via $(U\phi)(v) = U \cdot \phi(v)$. Moreover, $\mathbb{H}(R)$ has a $\mathbb{Z}$-grading: the element $\phi \in \mathbb{H}(R)$ is homogeneous of degree $d \in \mathbb{Z}$ if for each $v \in \mathscr{V}$ with $\phi(v) \neq 0$, $\phi(v) \in \mathscr{T}_0^+$ is homogeneous of degree $d - 2\chi(v)$. Since $2\chi(v) + \deg \phi(v) = 2\chi(u) + \deg \phi(u)$ in (11.2.1), d is well-defined.

Note also that any ϕ as above is automatically finitely supported.

Remark 11.2.5 By the definitions $\mathbb{H}((R, \chi)[r]) = \mathbb{H}(R, \chi)[2r]$ for any $r \in \mathbb{Z}$.

Proposition 11.2.6 *Let (R, χ) be a graded root. We order $\mathscr{V}_{lm}$ as follows. The first element v_1 is an arbitrary vertex with $\chi(v_1) = \min_v \chi(v)$. If $v_1, \dots, v_k$ is already determined, and $J_k := \{v_1, \dots, v_k\} \subsetneq \mathscr{V}_{lm}$, then let v_{k+1} be an arbitrary vertex in $\mathscr{V}_{lm} \setminus J_k$ with $\chi(v_{k+1}) = \min_{v \in \mathscr{V}_{lm} \setminus J_k} \chi(v)$. We continue this procedure till we order all the elements of $\mathscr{V}_{lm}$. Finally, for any $k \geq 2$, let $w_k \in \mathscr{V}$ be the unique $\succ$-minimal vertex of R which dominates v_k and at least one vertex from J_{k-1}. Then one has the following isomorphism of $\mathbb{Z}[U]$-modules*

$$\mathbb{H}(R, \chi) = \mathscr{T}^+_{2\chi(v_1)} \oplus \oplus_{k \geq 2} \mathscr{T}_{2\chi(v_k)}\big(\chi(w_k) - \chi(v_k)\big).$$

In particular, with the notations $m := \min_v \chi(v)$ and

$$\mathbb{H}_{red}(R, \chi) := \oplus_{k \geq 2} \mathscr{T}_{2\chi(v_k)}\big(\chi(w_k) - \chi(v_k)\big),$$

one has a (non-canonical) direct sum decomposition of graded $\mathbb{Z}[U]$-modules:

$$\mathbb{H}(R, \chi) = \mathscr{T}^+_{2m} \oplus \mathbb{H}_{red}(R, \chi),$$

such that the $\mathbb{Z}[U]$-module $\mathbb{H}_{red}(R)$ has finite $\mathbb{Z}$-rank.

Proof We use induction over the cardinality of $\mathscr{V}_{lm}$. If $|\mathscr{V}_{lm}| = 1$ then $(R, \chi) = R_n$ for a certain n, and the verification is elementary. Assume that we already know the validity of the statement for $|\mathscr{V}_{lm}| = r - 1$. Take any (R, χ) with $\mathscr{V}_{lm} = \{v_1, \dots, v_r\}$, where this set is ordered as in the statement of the proposition. Set $\mathscr{V}'_{lm} := \{v_1, \dots, v_{r-1}\}$, and define R' by $\{w : w \succ v_i \text{ for some } v_i \in \mathscr{V}'_{lm}\}$. Then the restriction χ' of χ defines a root (R', χ') with $\mathscr{V}_{lm}(R', \chi') = \mathscr{V}'_{lm}$. Consider the natural restriction map $\rho : \mathbb{H}(R, \chi) \to \mathbb{H}(R', \chi')$. We show that ρ is onto and admits a splitting. For this, we fix $\phi' \in \mathbb{H}(R', \chi')$ and we will extend it to $\phi : \mathscr{V}(R) \to \mathscr{T}_0^+$. Take any vertex w such that $w_r \succ w \succ v_r$. We already know $\phi'(w_r)$, but we need to define each $\phi(w)$ for the sequence of vertices $w_r \succ w \succ v_r$

such that $\phi(w_r) = \phi'(w_r)$ and (11.2.1) is satisfied too. First we choose some $v_j \in \mathscr{V}'_{lm}$ such that $w_r \succ v_j$. Next, for any w as above we define w' as the unique vertex w' with $w_r \succ w' \succ v_j$ and $\chi(w) = \chi(w')$. (Here we use the fact that $\chi(v_j) \leq \chi(v_r)$.). Then we define ϕ as an extension of ϕ' by $\phi(w) := \phi'(w')$.

On the other hand, if $\phi \in \ker(\rho)$, then with the notation $\delta := \chi(w_r) - \chi(v_r)$, one has $U^\delta \phi(v_r) = \rho(\phi)(w_r) = 0$, hence $\phi(v_r) = \sum_{k=0}^{\delta-1} n_k U^{-k}$ (and all the other values $\phi(w)$ are determined from this by (11.2.1)). □

Example 11.2.7

(a) $\mathbb{H}(R_n) = \mathscr{T}^+_{2n}$.

(b) The graded roots R^1 and R^2 constructed in 11.2.3(3) are not isomorphic but their $\mathbb{Z}[U]$-modules are isomorphic: $\mathbb{H}(R^1) = \mathbb{H}(R^2) = \mathscr{T}^+_{-6} \oplus \mathscr{T}_{-4}(1) \oplus \mathscr{T}_{-4}(2)$. Hence, in general, a graded root carries more information than its $\mathbb{Z}[U]$-module.

Definition 11.2.8 We define the **Euler characteristic of a graded root** (R, χ) as

$$eu(R, \chi) := -\min \chi + \operatorname{rank}_{\mathbb{Z}} \mathbb{H}_{red}(R, \chi).$$

Proposition 11.2.9 *Fix some n such that $|\chi^{-1}(n')| = 1$ for any $n' \geq n$. Let $\mathscr{V}_{\leq n} := \{v \in \mathscr{V} : \chi(v) \leq n\}$ and $\mathscr{E}_{\leq n} := \{[v, u] \in \mathscr{E} : v, u \in \mathscr{V}_{\leq n}\}$. Then*

$$eu(R, \chi) = -\sum_{v \in \mathscr{V}_{\leq n}} \chi(v) + \sum_{[v,u] \in \mathscr{E}_{\leq n}} \max\{\chi(v), \chi(u)\}.$$

Proof Use 11.2.6 (or repeat the proof from 11.1.32). □

Corollary 11.2.10 *Let (R_τ, χ_τ) be a graded root associated with a function $\tau : \mathbb{N} \to \mathbb{Z}$, cf. 11.2.3(3). Then*

$$rank\, \mathbb{Z}\mathbb{H}_{red}(R_\tau, \chi_\tau) = -\tau(0) + \min_{i \geq 0} \tau(i) + \sum_{i \geq 0} \max\{\tau(i) - \tau(i+1), 0\}.$$

The summand $\mathscr{T}^+_{2m}$ of $\mathbb{H}(R_\tau, \chi_\tau)$ has index $m = \min_{i \geq 0} \tau(i) = \min_v \chi_\tau(v)$.

Proof Use 11.2.9 (or induction over T_0, where $\tau : \{0, \ldots, T_0\} \to \mathbb{Z}$). □

Corollary 11.2.11 *Assume that $(R_1, \chi_1) \hookrightarrow (R_2, \chi_2)$ is a sub–graded root (graded injection). Then $eu(R_1, \chi_1) \leq eu(R_2, \chi_2)$.*

11.2.12 The Graded Root Associated with a System of Weigh Functions Fix a free $\mathbb{Z}$-module and a system of weights $\{w_q\}_q$ of cubes of $\mathbb{R}^s$, cf. 11.1.4. Consider the sequence of topological spaces (finite cubical complexes) $\{S_n\}_{n \geq m_w}$ with $S_n \subset S_{n+1}$. Let $\pi_0(S_n) = \{\mathscr{C}^1_n, \ldots, \mathscr{C}^{p_n}_n\}$ be the set of connected components of S_n.

Then we define the graded graph (R_w, χ_w) as follows. The vertex set $\mathcal{V}(R_w)$ is $\sqcup_{n\in\mathbb{Z}}\pi_0(S_n)$. The grading $\chi_w : \mathcal{V}(R_w) \to \mathbb{Z}$ is $\chi_w(\mathscr{C}_n^j) = n$, that is, $\chi_w|_{\pi_0(S_n)} = n$.

Furthermore, if $\mathscr{C}_n^i \subset \mathscr{C}_{n+1}^j$ for some n, i and j, then we introduce an edge $[\mathscr{C}_n^i, \mathscr{C}_{n+1}^j]$. All the edges of R_w are obtained in this way.

Lemma 11.2.13 *(R_w, χ_w) satisfies all the required properties of the definition of a graded root, except maybe the last one: $|\chi_w^{-1}(n)| = 1$ whenever $n \gg 0$.*

Proof Part 11.1.4*(a)* implies that w_0 is bounded below. If $x, y \in \mathbb{Z}^s$ are lattice points representing the connected components $\mathscr{C}_n^i$ and $\mathscr{C}_m^j$ respectively, then there is a finite path (for definition see 6.8.B) in $\mathbb{Z}^s$ connecting x and y in $\mathbb{Z}^s$, which is totally included in some S_k for some $k \gg 0$. Hence R_w is connected. Finally, 11.2.2*(b)* must be also satisfied, otherwise there would exist certain connected components $\mathscr{C}_n^i$, $\mathscr{C}_{n+1}^j$ and $\mathscr{C}_{n+1}^k$, $\mathscr{C}_{n+1}^j \neq \mathscr{C}_{n+1}^k$, such that $\mathscr{C}_n^i \subset \mathscr{C}_{n+1}^j \cap \mathscr{C}_{n+1}^k$. But this is not possible. □

The property $|\chi_w^{-1}(n)| = 1$ (i.e. S_n connected) for $n \gg 0$ is not satisfied for any $\{w_q\}_q$. This is e.g. the case for $\{w_q\}_q$ induced by w_0 from 11.1.14.

11.2.14 Next, assume that the system $\{w_q\}_q$ is determined by w_0 by the 'max–principle': $w_q(\square_q) = \max\{w_0(v) : v \text{ is a vertex of} \square_q\}$. Then the definition from 11.2.4 is compatible with the definition of $\mathbb{H}^0(\mathbb{R}^s, w)$ from 11.1.9. Indeed, let $\phi : L \to \mathscr{T}_0^+$ be a function which satisfies the analogue of (11.2.1): for any $\square_1 = [v, u] \in \mathscr{Q}_1$ with $w_0(v) \leq w_0(u)$ we have $U^{w_0(u)-w_0(v)} \cdot \phi(v) = \phi(u)$. But this means $\delta_w(\phi)[v, u] = 0$, hence $\phi \in \ker\left(\delta_w : \mathscr{F}^0 \to \mathscr{F}^1\right) = \mathbb{H}^0(\mathbb{R}^s, w)$.

Theorem 11.2.15 *Let $\{w_q\}_q$ be a system of weights of cubes in $\mathbb{R}^s$ as in 11.2.14. Let (R_w, χ_w) be the graded root associated with $\{w_q\}_q$. Then $\mathbb{H}(R_w, \chi_w) = \mathbb{H}^0(\mathbb{R}^s, w)$.*

Proof Fix $d = 2n$. The homogeneous elements of $\mathbb{H}^0(\mathbb{R}^s, w)$ are identified with $H^0(S_n, \mathbb{Z})$, cf. 11.1.12. They associate an integer $a_n^i \in \mathbb{Z}$ to each connected component $\mathscr{C}_n^i$ of S_n. We construct a $2n$-homogeneous element $\phi = \phi_{(a_n^i)_i}$ of $\mathbb{H}(R_w, \chi_w)$ by $\phi_{(a_j^i)_i}(\mathscr{C}_m^j) = 0$ for $m > n$ and $\phi_{(a_j^i)_i}(\mathscr{C}_m^j) = a_n^{i_0} U^{m-n}$ for $m \leq n$, where i_0 is determined uniquely by $\mathscr{C}_n^{i_0} \succ \mathscr{C}_m^j$. Then ϕ satisfies Definition 11.2.4, and it is homogeneous of degree $2n$. This provides a $\mathbb{Z}$-linear morphism $\Phi : \mathbb{H}^0(\mathbb{R}^s, w)_{2n} \to \mathbb{H}(R_w, \chi_w)_{2n}$. Clearly, it is injective. To see the surjectivity, notice that for any $\phi \in \mathbb{H}(R_w, \chi_w)_{2n}$, $\phi(\mathscr{C}_n^i)$ necessarily has the form $a_n^i \in \mathbb{Z} \cdot 1$ (by homogeneity), hence $\phi = \Phi((a_n^i)_i)$.

Finally, this identification is compatible with the U-action. □

In the above discussions $\mathbb{R}^s$ can be replaced by some $T \subset \mathbb{R}^s$ and its cubes.

11.2.A The Graded Root Associated with a Plumbing Graph

11.2.16 Fix a graph Γ and $k \in \mathrm{Char}$, their compatible weight functions and the graded cubes as in 11.1.17. The graded root associated with this system of weight functions (cf. 11.2.12 and 11.2.13) is denoted by (R_k, χ_k).

For the system of weight functions induced by χ_k the sequence of spaces $\{S_n\}_n$ have a finiteness property: only finitely many S_n is not contractible, cf. 7.3.10.

Corollary 11.2.17 *(R_k, χ_k) is a graded root. (In particular, $\mathbb{H}(R_k, \chi_k)$ is a finitely generated $\mathbb{Z}[U]$-module, and $\mathbb{H}_{red}(R_k, \chi_k)$ is a finitely generated $\mathbb{Z}$-module.)*

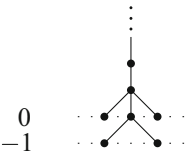
Remark 11.2.18 There are several natural **symmetries** in the picture.

(a) **The** Spin^c**-involution.** The involution $l' \mapsto -l'$ ($l' \in L'$) induces the identity $\chi_{-k}(-l) = \chi_k(l)$, hence an isomorphism of the graded roots $(R_k, \chi_k) \simeq (R_{-k}, \chi_{-k})$. ($[k] \mapsto [-k]$ is the natural involution of Spin$^c(M(\Gamma))$, cf. 6.10.19.)

(b) **The Gorenstein symmetry.** If Γ is numerically Gorenstein then χ_K is stable with respect to the transformation $L \to L$, $x \mapsto Z_K - x$. This shows that (R_K, χ_K) has a $\mathbb{Z}_2$-symmetry.

More generally, if $k \in L$ (that is, k is spin) then $x \mapsto -k - x$ induces a $\mathbb{Z}_2$-symmetry of (R_k, χ_k).

Example 11.2.19 Consider the example from 7.3.3. Those computations show that the graded root (R_K, χ_K) is



Then using 11.2.6 we get $\mathbb{H}^0\Gamma, K) = \mathscr{T}^+_{-2} \oplus \mathscr{T}_{-2}(1) \oplus \mathscr{T}_0(1) \oplus \mathscr{T}_0(1)$. The computation from 7.3.3 also shows that $\mathbb{H}^1(\Gamma, K) = \mathscr{T}_0(1)$ and $\mathbb{H}^q(\Gamma, K) = 0$ for $q \geq 2$.

Let us check the SWIC for singularities with this resolution graph for $k = K$. On one side $eu(\mathbb{H}^*(\Gamma, K)) = 1 + 3 - 1 = 3$. On the other hand, we know from 6.8.44 that $p_g \leq 3$ for any analytic structure supported on Γ. Equality holds if (X, o) is Kulikov or Gorenstein, cf. 6.8.44 or [495].

Here one can notice that the very same graded root (R_K, χ_K) is realized by the star-shaped graph, say Γ_{Br}, of the Brieskorn singularity $x^3 + y^5 + z^7 = 0$ too. However, since in this realization Γ_{Br} is AR, $\mathbb{H}^q(\Gamma_{Br}, K) = 0$ for $q \geq 1$ by 11.1.31. (For more on the general case of AR graph see 11.3.) Thus $eu(\mathbb{H}^*(\Gamma, K)) = 1+3 = 4$. Note that this Brieskorn singularity has $p_g = 4$, and $p_g \leq 4$ for any analytic structure supported on Γ_{Br}.

11.2.20 Next, with the notations from 11.1.21, by the very same argument as in that paragraph, we have the analogues of 11.1.22, (11.1.2), 11.1.24:

Proposition 11.2.21

(a) If $k' = k + 2l$ for some $l \in L$, then: $(R_{k'}, \chi_{k'}) = (R_k, \chi_k)[-2\chi_k(l)]$.
(b) $(R_k, \chi_k) = (R_{[k]}, \chi_{[k]})[2m_k]$
(c) The set $(R_{[k]}, \chi_{[k]})$, indexed by $[k] \in Spin^c(M(\Gamma))$, depends only on $M = M(\Gamma)$ and is independent of the choice of the plumbing graph Γ which provides M.

These identities induce similar identities at the level of their $\mathbb{H}$-homologies, compatible with the identities from Sect. 11.1.A via Theorem 11.2.15.

Example 11.2.22 (Rational Graphs) The following facts are equivalent: (a) Γ is rational; (b) $R_K = R_{(0)}$; (c) $R_K = R_{(m)}$ for some $m \in \mathbb{Z}$; (d) For all characteristic elements $k \in \mathrm{Char}$, $R_k = R_{(m_k)}$ for some $m_k \in \mathbb{Z}$.

Recall from 11.1.27 that $\min \chi_{k_r} = 0$ for rational Γ.

Example 11.2.23 (Elliptic Graphs) The following facts are equivalent:

(a) Γ is elliptic;
(b) $(R_K, \chi_K) = R(\{n_i\}, \{n_{ij}\})$ for some index set I, $|I| = \ell + 1 \geq 2$, such that $n_i = 0$ for any $i \in I$ and $n_{ij} = 1$ for any pair $i \neq j$;

Above ℓ can be identified with the length of the elliptic sequence (cf. 7.2.C).

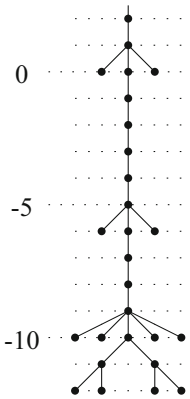
Example 11.2.24 If Γ is **minimally elliptic**, and Γ is the graph of the minimal resolution which is good (cf. 7.3.18), then $(R_{k_r}, \chi_{k_r}) = R_{(0)}$ for any $[k_r] \neq [K]$.

11.2.25 The 'age' of a Graded Root

A possible way to measure the complexity of a graded root is by its 'age', defined as follows. By definition, a root of type $R_{(n)}$ has age $\mathrm{AGE}(R_{(n)}) = 0$. Next, assume that R is not of type $R_{(n)}$. Then for each $v \in \mathscr{V}_{lm}$ let v' be the $\succ$-minimal vertex so that there exists $w \in \mathscr{V}_{lm}$, $w \neq v$, with $v' = \mathrm{supp}(v, w)$ (i.e. v' is the closest 'node' above v). Let R^{old} be the root (with induced χ), obtained from R by deleting the strings of type $[v, v') = \{v\} \cup \{w : v \succ w \succ v'\}$ (and the corresponding edges) for all $v \in \mathscr{V}_{lm}$.

Then one sets inductively $\mathrm{AGE}(R) := \mathrm{AGE}(R^{old}) + 1$.

E.g., elliptic roots (R_K, χ_K), or those from 11.2.3(3), or 11.2.19 have $\mathrm{AGE} = 1$. Below is a root with $\mathrm{AGE} = 2$. It is the root of the star-shaped graph with $g = 0$, $b_0 = 2$, $\nu = 6$, and Seifert invariants $(\alpha_j, \omega_j) \in \{(2, 1), (2, 1), (3, 1), (3, 1), (7, 1), (7, 1)\}$.



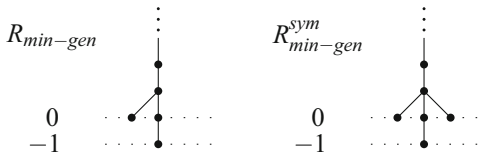
It would be nice to understand the meaning of AGE(R) associated with the root of (Γ, K) in terms of the topology of $M(\Gamma)$.

11.2.26 Continuation of the Artin–Laufer Program In a simplified sense, by the 'Artin-Laufer program' we understand results in the spirit of sections 7.1 and 7.2. They provide different analytic invariants (e.g. the Hilbert–Samuel function and the geometric genus) for a topologically identified family (maybe under certain additional analytic assumptions).

Recall that the family of rational and elliptic singularities were identified by $\min \chi$. However, in the continuation, using only $\min \chi$ (or merely numerical invariants), the program cannot be continued. Indeed, the class of singularities fixed in this way is too large to be uniform. In order to continue the program, we need to identify subfamilies for which one can show that they share 'common properties'.

One possibility is to identify these subfamilies by graded roots: For each fixed possible graded root, we consider all the singularities which have their canonical graded root (R_K, χ_K) identical with the fixed one.

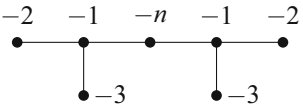
Let us be more precise regarding the first new steps. Let us say that a singularity is of 'general type' if it is not rational or elliptic. This happens if and only if $\min \chi < 0$. Now, let us define the family of *'minimally general type' singularities*, by the property that their canonical graded root (R_K, χ_K) is the root $R_{min-gen}$ from the left:



The minimal vertex should be realized by the component of the minimal cycle, hence in this case $\min \chi = \chi(Z_{min}) = -1$. Furthermore, the root has no $\mathbb{Z}_2$-Gorenstein symmetry, hence all the resolution graphs having $R_{min-gen}$ as the canonical root, will be non-numerically Gorenstein.

This root can indeed be realized, e.g. all the graphs below with $n \geq 15$ work:

$\Gamma(n)$:



These graphs admit even splice quotient analytic realizations, cf. 5.4. For $n = 15$ one has $\lfloor Z_K \rfloor = Z_{min}$, moreover, $\lfloor Z_K \rfloor$ is decreasing as n increases, and $\lim_{n\to\infty} \lfloor Z_K \rfloor = C$, the characteristic cycle, the unique minimal cycle Z with $\chi(Z) = -1$, cf. 6.6.11–6.6.12. Note also that for all singularities having this canonical root one has $p_g = h^1(\mathscr{O}_C) = h^1(\mathscr{O}_{Z_{min}}) = 2$ necessarily.

On the other hand, if we wish to consider the family of topological '*minimally numerically Gorenstein general type*' singularities then we have to fix the root $R^{sym}_{min-gen}$ from the right hand side. In this case $\min \chi = \chi(Z_{min}) = -1$. At zero level the local minimums are symmetrical, they correspond to the connected components of the cycles 0 and Z_K. Cycles at level -1 form a connected component, the maximal element (which necessarily stays in $\mathscr{S}$) is Z_{min}, while the minimal element is the characteristic cycle C. In fact, $Z_K = Z_{min} + C$. Furthermore, for this class $2 \leq p_g \leq 3$, and $p_g = 3$ if and only if the singularity is Gorenstein (use Proposition 6.4.20). (In this case the identity $Z_K = Z_{min} + C$ is the analogue of the package of combinatorial identities realized by the elliptic sequence in the case of elliptic graphs.)

This root is realized e.g. by $\Gamma(14)$. On the other hand, $\Gamma(14)$ is realized by the hypersurface $w^2 = (u^3+v^4)(u^4+v^3)$ with $\mathrm{mult}(X, o) = -Z^2_{min} = 2$. $\Gamma(14)$ has a Kulikov singularity realization as well, cf. 6.7.A. Another realization of $R^{sym}_{min-gen}$ is by the Brieskorn singularity $x^5+y^5+z^3$ with $p_g = 3$ and $\mathrm{mult}(X, o) = -Z^2_{min} = 3$. (This shows that the possible analytic structures still have some variations.)

We expect that the family of singularities with root $R_{min-gen}$ (or, similarly, germs with root $R^{sym}_{min-gen}$) can be organised and studied via a common skeleton. (See the article of Tomaru [694] for singularities with the property $Z_K = Z_{min} + C$, and articles of Konno with similar combinatorial restrictions [337–339].)

There is a second (stronger, but technically harder) possibility to identify topological families with common behaviours when we fix the lattice cohomology (or both the graded root and the lattice cohomology).

11.2.27 The following tasks appear very naturally.

Problem Determine all the possible canonical (R_K, χ_K) (and non-canonical (R_k, χ_k)) graded roots.

The possible resolution graphs are characterized by Grauert Theorem, namely they are connected and negative definite. For each negative definite graph (tree) we construct a canonical graded root in a direct combinatorial way. The problem is to find a combinatorial characterization of all of them.

Problem Determine all the possible graded $\mathbb{Z}[U]$-modules, which might appear as $\mathbb{H}^*(\Gamma, k)$ for some pair (Γ, k).

11.3 Graded Roots of Almost Rational Graphs

For almost rational graphs the lattice cohomology computation reduces basically to a 1-dimensional lattice. In particular, $\mathbb{H}^{>0}(\Gamma, k) = 0$, and $\mathbb{H}^0(\Gamma, k_r)$ and the graded roots are determined by a τ-function associated with the concatenated computation sequence connecting the universal cycles $\{x(\ell)\}_\ell$. Hence, in principle, all the graded roots, $\{\mathbb{H}^0(\Gamma, k)\}_k$, and all the Seiberg–Witten invariants are computable algorithmically.

First, in 11.3.A we perform this computation for star-shaped graphs, and we compare the output with the Reidemeister–Turaev torsion expressions (or with the Dolgachev–Pinkham–Demazure expression of the geometric genus of weighted homogeneous singularities).

The second application in 11.3.B targets the surgery manifold $S^3_{-d}(K)$ with arbitrary $d > 0$. In this case we determine the τ-function in terms of d and the Alexander polynomial (or, equivalently, the semigroup) of the knot K. This in 11.3.C applies for superisolated singularities associated with rational unicuspidal curves. Reinterpretation of the formulae shows that the SWIC for such germs is equivalent with the (valid) Semigroup Distribution Property, and also with the coincidence of the canonical lattice cohomologies (or of the canonical graded roots) of the 3-manifolds $S^3_{-d}(K)$ and $\Sigma(d, d, d+1)$. ♣

11.3.1 Let us fix $\mathbb{R}^s$ and a set of weight functions as in 11.1. If T is a subspace of $\mathbb{R}^s$ consisting of a union of certain cubes (as in 11.1.15) then we can define the graded root $(R_w(T), \chi_w)$, similarly as we defined $\mathbb{H}^*(T, w)$ in 11.1.15. Whilst the vertices of (R_w, χ_w) at level n correspond to the connected components of $S_n = \cup_{w(\square)\le n}\square$, the vertices at level n of $(R_w(T), \chi_w)$ correspond to the connected components of $S_n \cap T$, and the edges are defined by the similar principle as the edges of (R_w, χ_w).

More generally, in the definition of the graded root associated with the cubical decomposition of $\mathbb{R}^s$ (endowed compatibly with weights) we can replace $\mathbb{R}^s$ with any cubical CW complex $X = \cup_i \square_i$ (or even with a simplicial one), endowed with a set of compatible weights. Then the spaces S_n, the lattice cohomology and the graded root are naturally well-defined.

Several examples appear naturally. E.g., T as in the first paragraph works. Another example is the topological realization of a given graded root (R, χ). If we regard R as a union of 0 and 1 dimensional cells, and we take $w_0(v) := \chi(v)$ while $w_1([v, w]) = \max\{\chi(v), \chi(w)\}$, then (R, χ) might serve as a weighted 1-dimensional CW complex. It is an instructive exercise to check that the graded root associated with this weighted CW complex is the original graded root itself. For this, let us fix $n \in \mathbb{Z}$, and $\chi^{-1}(n) = \{v_1, \dots, v_{c_n}\}$. Then it turns out that $S_n = S_n(R, \chi)$ (that is, the topological realization of the full subgraph identified by $\chi \le n$) has c_n connected components and each of them contains exactly one vertex of type v_i.

In fact, in several situations it is more convenient to think about a graded root as a weighted 1-dimensional CW complex.

11.3.2 Let us assume that X is a 1-dimensional CW complex consisting of 0-cells $\{v_i\}_{i\geq 0}$ and 1-cells $\{[v_i, v_{i+1}]\}_{i\geq 0}$ (hence it is homeomorphic to $[0, \infty)$). We endow X with a set of compatible weight functions, namely, we define $w_0(v_i) \in \mathbb{Z}$ and $w_1([v_i, v_{i+1}]) \in \mathbb{Z}$ in such a way that $w_1([v_i, v_{i+1}]) = \max\{w_0(v_i), w_0(v_{i+1})\}$. Let $(R_w(X), \chi_w)$ be the graded root associated with this setup. The verification of the next combinatorial/topological statement is left to the reader.

Lemma 11.3.3 *If we define τ as w_0, and (R_τ, χ_τ) is associated with τ as in 11.2.3(3) then $(R_w(X), \chi_w) = (R_\tau, \chi_\tau)$.*

11.3.4 In the sequel we assume that Γ is an AR graph, with SR-set consisting the vertex $\{v_0\}$. We fix a distinguished characteristic element $k_r = K + 2s_h$. The reader is invited to review the definitions and the statements from 7.3.D. In particular, we consider the universal cycles $\{x_{s_h}(\ell)\}_{\ell\geq 0}$ associated with (Γ, k_r), and their τ-function $\tau : \mathbb{Z}_{\geq 0} \to \mathbb{Z}$ defined as $\tau(\ell) := \chi_{k_r}(x_{s_h}(\ell))$. Associated with this τ-function we consider its graded root (R_τ, χ_τ) as well, cf. 11.2.3(3).

Theorem 11.3.5 *[478] Assume that Γ is AR, and set $k_r = K + 2s_h$ for some $h \in H$. Then*

(a) $\mathbb{H}^q(\Gamma, k_r) = 0$ *for* $q \geq 1$;
(b) $\mathbb{H}^0(\Gamma, k_r) = \mathbb{H}(R_{k_r}, \chi_{k_r})$;
(c) $(R_{k_r}, \chi_{k_r}) = (R_\tau, \chi_\tau)$;
(d) $x_{s_h}(0) = 0$, $\tau(0) = 0$, $\tau(1) = 1 - (s_h, E_{v_0}) \geq 1$, $m_{k_r} = \min_{\ell\geq 0}\{\tau(\ell)\}$ *and*

$$eu(\mathbb{H}^*(\Gamma, k_r)) = -\min_{\ell}\{\tau(\ell)\} + rank_{\mathbb{Z}}(\mathbb{H}^0_{red}(\Gamma, k_r))$$
$$= \sum_{\ell\geq 0} \max\{\tau(\ell) - \tau(\ell+1), 0\}.$$

(e) $\tau(\ell) - \tau(\ell+1) = -1 + (x_{s_h}(\ell) + s_h, E_{v_0})$.
(f) *If* $h = 0$ *then* $Path^i \leq eu(\mathbb{H}^*(\Gamma, K))$.

Proof The main technical statement behind the theorem is Theorem 7.3.37. Part *(a)* was already mentioned in 11.1.31. Part *(b)* was proved in 11.2.15. We prove *(c)* in two steps. First, from Theorem 7.3.37 we obtain that $\mathbb{H}^*(\Gamma, k_r)$ is isomorphic with the lattice cohomology $\mathbb{H}^*(C_\gamma, \chi_{k_r})$ associated with the embedded concatenated path C_γ constructed in 7.3.D, and the same fact is true for the corresponding graded roots. On the other hand, by Lemma 11.3.3, the graded root associated with (C_γ, k_r) and (R_τ, χ_τ) coincide. The first part of *(d)* follows from Lemma 7.3.33 and the last identity from 11.2.10. Part *(e)* follows from 7.3.36*(e)*.

Finally we prove *(f)*. Consider the path γ_{AR}, the concatenated computation sequence constructed in 7.3.D. Then comparing the definition of $S(\gamma_{AR})$ from 6.8.37 with the formula from part *(d)* (and Lemma 7.3.36) we get that $eu(\mathbb{H}^*(\Gamma, K)) = S(\gamma_{AR})$ (this realizability of $eu(\mathbb{H}^*(\Gamma, K))$ as $S(\gamma)$ for some γ is

exactly the main point of 6.8.37 and part *(d)* above). Then $\mathrm{Path}^i = \min_{\gamma\in\mathscr{P}^i} S(\gamma) \le S(\gamma_{AR})$. □

Remark 11.3.6

(a) The above theorem shows that for almost rational graphs, any graded tree (R_k, χ_k) is completely determined by the values of χ_k along a very natural, universal, infinite (increasing) computation sequence (depending on k), which contains the elements $\{x(\ell)\}_{\ell\ge 0}$. (For the construction of the sequence see 7.3.D.) In particular, all the important vertices of R_k can be represented by some special cycles in L, which can be arranged in an increasing linear order (with respect to $\le$). [This property was extremely useful in the study of the elliptic singularities, where this role was played by the elliptic sequence, but here we realize that it is a more general phenomenon. Compare with 7.3.38(b).]

(b) The set $\{x(\ell)\}_\ell$ usually is not very economical: only some of the $x(\ell)$'s carry substantial information, which will survive in (R_τ, χ_τ). The others are intermediate steps in some monotone paths. E.g., for rational singularities, $\chi(x(\ell+1)) \ge \chi(x(\ell))$, hence only the information $\chi(x(0)) = 0$ is preserved in R_τ.

11.3.7 Next we connect the topological invariant $eu(\mathbb{H}^*(\Gamma, k_r)) = \sum_{\ell\ge 0} \max\{\tau(\ell) - \tau(\ell+1), 0\}$ with analytic invariants.

Proposition 11.3.8 ([478]) *Consider a normal surface singularity (X, o) with a fixed resolution $\widetilde{X}$. Assume that the resolution graph is AR. Next, consider any holomorphic line bundle $\mathscr{L} \in Pic(\widetilde{X})$ with $c_1(\mathscr{L}) = -s_h$. Then*

$$h^1(\widetilde{X}, \mathscr{L}) \le eu(\mathbb{H}^*(\Gamma, k_r)). \tag{11.3.1}$$

This combined with Theorem 11.1.36 gives

$$h^1(\widetilde{X}, \mathscr{L}) \le \mathfrak{sw}_{\sigma[k_r]}(M(\Gamma)) - \frac{k_r^2 + |\mathscr{V}|}{8}. \tag{11.3.2}$$

Furthermore, if $h = 0$ and $k_r = K$ then $p_g \le \mathrm{Path}^i \le eu(\mathbb{H}^*(\Gamma, K))$.

Proof Use the infinite sequence $\gamma = \{l_i\}_{\ge 0}$ from 7.3.D and the cohomology long exact sequences associated with $0 \to \mathscr{L} \otimes \mathscr{O}_{E_{v(i)}}(-l_i) \to \mathscr{L}|_{l_{i+1}} \to \mathscr{L}|_{l_i} \to 0$ similarly as in 6.8.36. For the $h = 0$ case use 6.8.B and 11.3.5*(f)*. □

Corollary 11.3.9 *If an AR topological type supports an analytic structure, which satisfies the SWIC (i.e. $p_g = eu(\mathbb{H}^*(\Gamma, K))$) then $\mathrm{Path}^i = eu(\mathbb{H}^*(\Gamma, K))$.*

Definition 11.3.10 In order to comment the above inequalities and for applications in the next subsections it is helpful to introduce

$$eu(\mathbb{H}^0(\Gamma,k)) := -m_k + \operatorname{rank}_{\mathbb{Z}}(\mathbb{H}^0_{red}(\Gamma,k))$$

for any (not necessarily AR) pair (Γ,k).

Though it is not really an 'Euler characteristic'—since, in general, $\mathbb{H}^{\geq 1}$ might be non-zero—, its applicability emphasizes that certain geometrical properties are guided only by $\mathbb{H}^0$ instead of $\mathbb{H}^*$. In the AR case $eu(\mathbb{H}^0(\Gamma,k)) = eu(\mathbb{H}^*(\Gamma,k))$.

Remark 11.3.11 Historically (in the process in which the author tried to understand the analytic/topological connections, like the SWIC) the inequality (11.3.2) was a crucial momentum, but at some point it became a misleading factor as well.

At the early stage, in the effort to bound/recover $h^1(\mathscr{L})$ from topology, for several examples (e.g. for rational, elliptic, or weighted homogeneous germs) we realized (11.3.2), even with equality for 'nice' analytic structures. By semicontinuity, via deformation of these 'nice' analytic structures, $h^1(\mathscr{L})$ usually drops.

Based on these facts it was natural to formulate the conjecture that (11.3.2) is valid for any singularity with equality for 'nice' (specific) analytic structures (see e.g.[489]). Well, later it turned out that this is not the case: The prediction works for AR graphs, but not in general. The point is that basically in the proof of (11.3.2) $h^1(\mathscr{L}) \leq eu(\mathbb{H}^0(\Gamma,k_r))$ is proved via computation sequences. Furthermore, at that moment, it was not clear that the Seiberg–Witten invariant from the right hand side of (11.3.2) has a cohomological categorification, namely the lattice cohomology (and in the inequality only the term $\mathbb{H}^0$ appears naturally). So, the correct generalization for arbitrary germs is $h^1(\mathscr{L}) \leq eu(\mathbb{H}^0(\Gamma,k_r))$ (see 11.6), while $h^1(\mathscr{L}) \leq eu(\mathbb{H}^*(\Gamma,k_r))$ (which is equivalent to (11.3.2)) for non-AR graphs might not hold . E.g., when $h^1(\mathscr{L}) = eu(\mathbb{H}^0(\Gamma,k_r))$ but $\mathbb{H}^1 \neq 0$ (and $\mathbb{H}^{\geq 2} = 0$) then $h^1(\mathscr{L}) > eu(\mathbb{H}^*(\Gamma,k_r))$. (That is, for some analytic structures, the inequality (11.3.2) even turned into the unexpected direction, a fact which before the construction of the lattice cohomology was hard to explain.) For such a key example see 9.7.32. In that case $\mathbb{H}^0(\Gamma,K) = \mathscr{T}^+_{-10} \oplus \mathscr{T}_{-10}(3) \oplus \mathscr{T}_0(1)^2$, hence $eu(\mathbb{H}^0) = 10$, but $\mathbb{H}^1 = \mathscr{T}_{-4}(1)^2$ and $\mathbb{H}^{\geq 2} = 0$, hence $eu(\mathbb{H}^*) = 8$. The superisolated analytic realization has $p_g = 10$ while the splice quotient realization has $p_g = 8$, cf. 9.7.32.

Now, the phenomenon is rather complex, even for p_g (when the analytic variation along Pic^0 is not even considered). Indeed, $eu(\mathbb{H}^0)$ stays as a topological upper bound, which even is improved by a 'path lattice cohomology bound' in 11.6 (equivalent with the bound Path^i from 6.8.B). For several graphs this bound Path^i is even realized by very concrete 'nice' analytic structures (e.g. this happens for certain surgery 3-manifolds, where this topological bound is realized by the superisolated singularities, see 6.8.43, 11.6.8 and 11.6.11). It turns out that some analytic structures 'choose' $eu(\mathbb{H}^0)$ for their p_g (in such cases $p_g = \mathrm{Path}^i$ too, cf. 11.6.8), and also, some other ones choose Path^i (with $\mathrm{Path}^i < eu(\mathbb{H}^0)$, see e.g. 11.4.11) and 11.6.10. [However, for some graphs, the optimal topological bound

is neither $eu(\mathbb{H}^0)$, nor Path^i; see e.g 11.6.10 and the comments form 6.8.44.] On the other hand, some other analytic structures prefer $eu(\mathbb{H}^*)$ for their p_g (e.g. the splice quotients).

Here is a wonderful project: list all the possible topological candidates from which the analytic types might choose for their p_g. Are they all related with (parts/versions of) the lattice cohomology?

Example 11.3.12 In general, Path^i and $eu(\mathbb{H}^*(\Gamma, K))$ cannot be compared.

For the (non AR) graph from 9.7.32 $eu(\mathbb{H}^*(\Gamma, K)) = 8 < 10 = \mathrm{Path}^i = eu(\mathbb{H}^0(\Gamma, K))$ (cf. 11.4.12, see also 11.3.11). On the other hand, for the surgery 3–manifold $S^3_{-5}(T_{2,3}\#T_{2,3}\#T_{3,5})$ is proved in 11.5.38 that $eu(\mathbb{H}^*(\Gamma, K)) > eu(\mathbb{H}^0(\Gamma, K))$; and from 11.6.7 follows that (in any case) $eu(\mathbb{H}^0(\Gamma, K)) \geq \mathrm{Path}^i$.

11.3.A Example. Star-Shaped Graphs

Assume that Γ is star-shaped with ν legs. In the sequel we will use the notations from 5.1. We also fix $l'_h = a_0E^*_0 + \sum_{j=1}^{\nu}\sum_{t=1}^{s_j} a_{jt}E^*_{jt}$. The coefficients of l'_h also determine the integers $\tilde{a}_{jk} := \sum_{t\geq k} n^j_{t+1,s_j}a_{jt}$ for $1 \leq k \leq s_j$. We also write $a_j = \tilde{a}_{j1}$ (compare with the formulae (6.6.6), (6.6.7), (6.6.16), and (SI) in 6.6.20) when $l'_h = s_h$).

Γ is AR, where its SR-set consists of the central vertex, cf. 7.3.22(f). Hence, for any $\bar{l} = \ell E_0$ (and for the fixed l'_h and $k := K + 2l'_h$) we have a cycle $x(\bar{l})$, which will be denoted simply by $x(\ell)$ ($\ell \in \mathbb{Z}$). For details see 7.3.D. In the sequel we describe the cycles $x(\ell)$ in terms of the Seifert invariants and the coefficients of l'_h.

Proposition 11.3.13 ([478, Proposition 11.11])

(1) Define the integers $\{v_{jk}\}$ $(1 \leq j \leq \nu,\ 1 \leq k \leq s_j)$ inductively by

$$v_{j1} := \left\lceil \frac{\ell\omega_j - a_j}{\alpha_j} \right\rceil = \left\lceil \frac{\ell n^j_{2s_j} - \tilde{a}_{j1}}{n^j_{1s_j}} \right\rceil; \quad v_{jk} := \left\lceil \frac{v_{j,k-1}n^j_{k+1,s_j} - \tilde{a}_{jk}}{n^j_{ks_j}} \right\rceil (1 < k \leq s_j).$$

Set $z(\ell) := \ell E_0 + \sum_{j,k} v_{jk}E_{jk}$. Then $(z(\ell) + l'_h, E_{jk}) \leq 0$ for any j and k.

(2) Assume that an effective integral cycle $\bar{z}(\ell) := \ell E_0 + \sum_{j,k} \bar{v}_{jk}E_{jk}$ satisfies $(\bar{z}(\ell) + l'_h, E_{jk}) \leq 0$ for any j and k. Then

$$\bar{v}_{j1} \geq \left\lceil \frac{\ell\omega_j - a_j}{\alpha_j} \right\rceil \quad \text{and} \quad \bar{v}_{jk} \geq \left\lceil \frac{\bar{v}_{j,k-1}n^j_{k+1,s_j} - \tilde{a}_{jk}}{n^j_{ks_j}} \right\rceil (1 < k \leq s_j).$$

(3) With the notations of (1)–(2) one has $\bar{v}_{jk} \geq v_{jk}$ for every j, k.

(4) In particular, $z(\ell)$ defined in (1) equals $x_{l'_h}(\ell)$.

Proof

(1) By the definition of v_{jk} one gets $n^j_{ks_j}v_{jk} \geq v_{j,k-1}n^j_{k+1,s_j} - \tilde{a}_{jk}$, which (via identities (2.2.4)) is equivalent to

$$b_{jk}v_{jk} - v_{j,k-1} + a_{jk} \geq \big(v_{jk}n^j_{k+2,s_j} - \tilde{a}_{j,k+1}\big)/n^j_{k+1,s_j}. \tag{11.3.3}$$

(Here, if $k = 1$ then $v_{j,k-1} := \ell$.) Now, using the definition of $v_{j,k+1}$ (compared with the right hand side of (11.3.3), and the fact that the left hand side of (11.3.3) is an integer) one gets $b_{jk}v_{jk} - v_{j,k-1} + a_{jk} \geq v_{j,k+1}$ which shows $(z(\ell) + s_h, E_{jk}) \leq 0$.

(2) $(\bar{z}(\ell) + l'_h, E_{js_j}) \leq 0$ is equivalent to the wanted inequality for $k = s_j$. In general, for arbitrary k, the inequality follows from $(\bar{z}(\ell) + l'_h, y_k) \leq 0$, where $y_k = \sum_{t\geq k} n^j_{t+1,s_j}E_{jt}$.

(3) Follows from (1)–(2) and increasing induction on k.

□

Example 11.3.14 Assume that $g = 0$ and $l'_h = s_h$, and set $\tau(\ell) := \chi_{k_r}(x_{s_h}(\ell))$ $(\ell \geq 0)$. If $\ell = 0$ then $x_{s_h}(0) = 0$, hence $\tau(0) = 0$ too. For $\ell \geq 0$, from 7.3.29 one gets

$$\tau(\ell+1) - \tau(\ell) = 1 - (x_{s_h}(\ell) + s_h, E_0) = 1 + a_0 + \ell b_0 - \sum_j \left\lceil \frac{\ell\omega_j - a_j}{\alpha_j} \right\rceil. \tag{11.3.4}$$

In particular, $\tau(1) = 1 + a_0 \geq 1$, and in general,

$$\tau(\ell) = \sum_{k=0}^{\ell-1} \Big(1 + a_0 + kb_0 - \sum_j \left\lceil \frac{k\omega_j - a_j}{\alpha_j} \right\rceil\Big). \tag{11.3.5}$$

This expression, obtained here combinatorially via the cycles $\{x(\ell)\}_\ell$, appears as the output of other type of computations in several different places: (i) in the Dolgachev–Pinkham–Demazure theory, cf. 5.1.26; (ii) in the expressions of s_h, cf. 6.6.20; (iii) in the cohomological computations from 6.6.20 and 6.8.19; (iv) in the Poincaré series of a weighted homogeneous singularity, cf. 8.1.B; (v) in the expression of the topological series Z, cf. 8.6.13; (vi) or in the Reidemeister–Turaev torsion computations, see 9.7.A.

These coincidences have the following consequence for the weighted homogeneous analytic structure.

If $h = 0$, $k_r = K$, then $\tau(\ell+1) - \tau(\ell) = 1 + N(\ell)$. Then Lemma 7.3.36 and the above computation for the concatenated path of the subsection 7.3.D show that (via (5.1.5), (5.1.8) and 11.3.5)

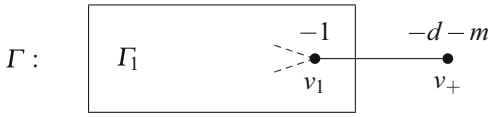
$$\mathrm{Path}^i \leq \sum_{\ell\geq 0} \max\{0, \tau(\ell) - \tau(\ell+1)\} = \sum_{\ell\geq 0} \max\{0, -1 - N(\ell\} = p_g. \tag{11.3.6}$$

Since $p_g \le \mathrm{Path}^i$ (cf. (6.8.14)) we get that necessarily $p_g = \mathrm{Path}^i$, and the pure topological equality $\mathrm{Path}^i = \sum_{\ell \ge 0} \max\{0, \tau(\ell) - \tau(\ell+1)\}$ also holds. That is, Path^i is realized by the universal concatenated computation sequence constructed in 7.3.D.

Note that the right hand side of (11.3.6), via 11.3.5 and 11.1.36, reproves the SWIC for weighted homogeneous singularities. (For the first proof see 9.7.A.)

11.3.B Example. The Surgery Manifold $S^3_{-d}(K)$

11.3.15 The Surgery Manifold $M(\Gamma) = S^3_{-d}(K)$ Fix $d \in \mathbb{Z}_{>0}$ and an *irreducible* plane curve singularity (C, o) with local algebraic knot $(K_1 \subset S^3)$. Several invariants of (C, o) are listed in 5.2.2. For the shape and structure of the surgery 3-manifold $S^3_{-d}(K_1)$ see 3.5.12. If it appears as the link of a superisolated surface singularity associated with a rational unicuspidal curve of degree d (cf. 5.2.3) then necessarily $(d-1)(d-2) = \mu(C, o)$. However, in the next discussion below we will not assume this additional restriction (in particular, d can be any $d \in \mathbb{Z}_{>0}$). We use the following schematic diagram (cf. 3.5.C and 9.1.17):



The basis elements in $L = L(\Gamma)$ corresponding to v_1 and v_+ are denoted by E_1 and E_+. The lattice associated with Γ_1 is L_1, its dual is L_1'. The elements $\{E_v\}_{v \ne v_+}$ of L are identified with the basis elements of L_1.

Recall that Γ is an AR graph with $\overline{\mathcal{V}} = \{v_1\}$, cf. 7.3.22(h). In the next presentation we follow [477, 479, 497].

Assume that (C, o) is determined by the function f; denote its compact divisor by $Z \in L_1$ and the E_1-multiplicity of Z by m. Then, $Z = E_1^*(\Gamma_1)$, hence $-(Z, Z)_{L_1} = m$. This fact together with 2.2.5(5), and 2.2.5(6) (applied for the edge (v_1, v_+) as G), gives that $\det(\Gamma) = d$. Since $\det(\Gamma_1) = 1$ (by Cramer's rule) the coefficient of E_+ in E_+^* is $1/d$. Hence $[E_+^*]$ has order d in H, and $H = \mathbb{Z}_d$.

We abridge $s_{a[E_+^*]}$ by s_a for $a = 0, 1, \ldots, d-1$.

Lemma 11.3.16 *$s_a = aE_+^*$ for any $a = 0, 1, \ldots, d-1$.*

Proof By the minimality of s_a one has $aE_+^* - s_a = l \in L_{\ge 0}$. We have to show that $l = 0$. Let l_+ be the coefficient of E_+ in l; hence $l_+ \ge 0$. Since the coefficient of E_+ in dE_+^* is 1, l can be written as $l_+ dE_+^* + l_1$ for some $l_1 \in L_1$. But $(l, E_v) = (-s_a, E_v) \ge 0$ for $v \ne v_+$, hence $(l_1, E_v) \ge 0$, which shows that $l_1 \le 0$ by 2.1.19.

Next, $0 \le (-s_a, E_+) = -l_+ d + a + (l_1, E_+) \le -l_+ d + a$. Since $l_+ \ge 0$ and $a < d$ this shows that $l_+ = 0$. But then $l_1 = l$, and since $l \ge 0$ and $l_1 \le 0$ we get $l = 0$. □

11.3.17 Our goal is to determine $\{x_{k_r}(\ell)\}_{\ell\geq 0}$ for Γ and for any spin^c structure. If $k_r = K + 2aE_+^*$ for a certain a then we abridge $x_{k_r}(\ell)$ as $x_a(\ell)$, where $0 \leq a < d$.

Let us write $x_a(\ell)$ as $y_a(\ell) + n_a E_+$, where $n_a \in \mathbb{Z}_{\geq 0}$ and $y_a(\ell) \in L_1$. The inequality $(x_a(\ell) + aE_+^*, E_+) \leq 0$ reads as $n_a(m+d) \geq \ell - a$. Hence $n_a = \lceil (l-a)/(m+d) \rceil$.

On the other hand, for all other vertices $v \in \mathcal{V} \setminus \{v_+, v_1\}$ we have $(x_a(\ell) + aE_+^*, E_v) = (y_a(\ell), E_v)$, hence $y_a(\ell)$ is independent of a; let us denote it by $y(\ell)$. It satisfies the universal property *(a)-(b)-(c)* from 7.3.24 for the graph Γ_1, vertex v_1 and $l'_h = 0$. Namely, $y(\ell)$ is minimal with *(a)* $m_{v_1}(y(\ell)) = \ell$ and *(b)* $(y(\ell), E_v) \leq 0$ for any $v \neq v_1$. For example, $y(0) = 0$.

Proposition 11.3.18 *Let $Z = div_{E(\Gamma_1)}(f) = E_1^*(\Gamma_1)$ be the cycle as above. Then*

(a) if $\ell = tm + \ell_0$ with $t \geq 0$ and $0 \leq \ell_0 < m$, then $y(\ell) = tZ + y(\ell_0)$;
(b) for any $\ell < m$ one has

$$(y(\ell), E_1) = \begin{cases} 1 & \text{if } \ell \notin \mathcal{S}_{C,o}; \\ 0 & \text{if } \ell \in \mathcal{S}_{C,o}. \end{cases}$$

Proof For *(a)* use 7.3.25. Part *(b)* follows from the next sequence of Claims.

Claim 1 $(y(\ell), E_1) \leq 1$ *for any* $\ell \geq 0$.

Indeed, since Γ_1 is rational, $\ell \mapsto \chi(y(\ell))$ is non-decreasing, cf. 7.3.38. Furthermore, by 7.3.29 $\chi(y(\ell+1)) = \chi(y(\ell)) + 1 - (y(\ell), E_1)$.

Claim 2 *Fix $\ell \geq 0$. Then $(y(\ell), E_1) \leq 0$ if and only if $\ell \in \mathcal{S}_{C,o}$.*

'$\Rightarrow$' If $(y(\ell), E_1) \leq 0$ then $y(\ell) \in \mathcal{S}(\Gamma_1)$. Since Γ_1 is the dual graph of a modification of $(\mathbb{C}^2, 0)$, the cycle $y(\ell)$ is the divisorial part $Z_g = \mathrm{div}_{E(\Gamma_1)}(g)$ of a holomorphic germ $g \in \mathcal{O}_{(\mathbb{C}^2,0)}$. Since the intersection multiplicity $i_0(f,g)$ of the germs f and g at $0 \in \mathbb{C}^2$ is the multiplicity of Z_g along the exceptional divisor supporting the arrow of f, one gets $i_0(f,g) = (E_1^*(\Gamma_1), y(\ell)) = \ell$. Thus, from the definition, $i_0(f,g) \in \mathcal{S}_{C,o}$.

'$\Leftarrow$' If $\ell \in \mathcal{S}_{C,o}$, then there exists $g \in \mathcal{O}_{(\mathbb{C}^2,0)}$ with $i_0(f,g) = \ell$; that is, $m_{v_1}(Z_g) = \ell$ and $Z_g \in \mathcal{S}(\Gamma_1)$. Then the minimality of $y(\ell)$ implies $y(\ell) \leq Z_g$. Since $E_1 \notin |Z_g - y(\ell)|$ one gets $(E_1, y(\ell)) \leq (E_1, Z_g) \leq 0$.

Claim 3 *Assume that $\ell < m$. Then $(y(\ell), E_1) = 0$ if and only if $\ell \in \mathcal{S}_{C,o}$.*

Via Claim 2, we need to show that $(y(\ell), E_1) < 0$ cannot hold for $\ell < m$. Indeed, otherwise, $y(\ell) - Z \in \mathcal{S}(\Gamma_1)$, hence $y(\ell) - Z \geq 0$, and $\ell - m = m_{v_1}(y(\ell) - Z) \geq 0$.

□

Corollary 11.3.19 *Fix* $0 \le a < d$ *and write* $\ell = tm + \ell_0$ *for some* $t \ge 0$ *and* $0 \le \ell_0 < m$. *Then*

$$x_a(\ell) = t \cdot Z + y(\ell_0) + \left\lceil \frac{\ell - a}{m + d} \right\rceil E_+.$$

In particular,

$$(x_a(\ell), E_1) = -t + \left\lceil \frac{\ell - a}{m + d} \right\rceil + (y(\ell_0), E_1).$$

Furthermore, $\chi_{k_r}(x_a(0)) = 0$ *and for any* $\ell \ge 0$ *one has*

$$\chi_{k_r}(x_a(\ell+1)) - \chi_{k_r}(x_a(\ell)) = t + 1 - \left\lceil \frac{\ell - a}{m + d} \right\rceil - \begin{cases} 1 & \text{if } \ell_0 \notin \mathscr{S}_{C,o} \\ 0 & \text{if } \ell_0 \in \mathscr{S}_{C,o}. \end{cases} \tag{11.3.7}$$

11.3.20 The τ-Function τ_a Corresponding to the main strategy from 7.3.D we write $\tau_a(\ell) := \chi_{k_r}(x_a(\ell))$. Then in (11.3.7) one has

$$\frac{\ell - a}{m + d} \le t + 1,$$

hence $\tau_a(\ell + 1) - \tau_a(\ell) \ge -1$ for any ℓ, and $= -1$ only in very special situations, namely if

$$\frac{tm + \ell_0 - a}{m + d} > t \quad \text{and} \quad \ell_0 \notin \mathscr{S}_{C,o}. \tag{11.3.8}$$

In order to analyze the cases when this holds, we will consider sequences $Seq(t) := \{tm + \ell_0 : 0 \le \ell_0 < m\}$ for fixed $t \ge 0$. In such a sequence, notice that the very last element of $\mathbb{N} \setminus \mathscr{S}_{C,o}$, namely $\mu - 1 = 2\delta - 1$, is strictly smaller than $m - 1$ (a fact, which can be proved by induction over the number of Newton pairs), hence the complete set $\mathbb{N} \setminus \mathscr{S}_{C,o}$ sits in $\{0, \ldots, m-1\}$. Therefore, in $Seq(t)$ there exists an ℓ_0 satisfying (11.3.8) if and only if

$$\frac{tm + 2\delta - 1 - a}{m + d} > t.$$

This is equivalent to $t \le t_a$, for $t_a := \lfloor (2\delta - 2 - a)/d \rfloor$. In other words, if $\ell \ge T_0 := (t_a + 1)m$, then $\tau_a(\ell + 1) \ge \tau_a(\ell)$, hence those values of τ_a provide no contribution in the graded root. Moreover, for $t \in \{0, \ldots, t_a\}$, in $Seq(t)$ one has:

$$\Delta(\ell_0) := \tau_a(tm + \ell_0 + 1) - \tau_a(tm + \ell_0) = \begin{cases} 0 & \text{if } \ell_0 \le td + a, \text{ and } \ell_0 \notin \mathscr{S}_{C,o}; \\ +1 & \text{if } \ell_0 \le td + a, \text{ and } \ell_0 \in \mathscr{S}_{C,o}; \\ -1 & \text{if } \ell_0 > td + a, \text{ and } \ell_0 \notin \mathscr{S}_{C,o}; \\ 0 & \text{if } \ell_0 > td + a, \text{ and } \ell_0 \in \mathscr{S}_{C,o}. \end{cases}$$

In particular, $\Delta(\ell_0) \geq 0$ for any ℓ_0 with $0 \leq \ell_0 \leq td + a$, and $\Delta(\ell_0) \geq 0$ takes the value $+1$ exactly

$$A_t := \#\{s \in \mathscr{S}_{C,o} : s \leq td + a\}$$

times, otherwise it is zero. Furthermore, $\Delta(\ell_0) \leq 0$ for any $\ell_0 > td + a$ and it takes value -1 exactly

$$B_t := \#\{s \notin \mathscr{S}_{C,o} : s > td + a\}$$

times, otherwise it is zero. Recall that in 5.2.2 we rewrote the characteristic polynomial (Alexander polynomial) as $\Delta(t) = 1 + \delta(t-1) + (t-1)^2 Q(t)$, where $Q(t) = \sum_{i=0}^{\mu-2} \alpha_i t^i$. The above B_t compared with (5.2.3) reads as $B_t = \alpha_{td+a}$.

Notice that both A_t and B_t are strictly positive (since $0 \in \mathscr{S}_{C,o}$, respectively $2\delta - 1 \notin \mathscr{S}_{C,o}$ and $2\delta - 1 > td + a$). This shows that

$$M_t := \max_{0 \leq \ell_0 < m} \tau_a(tm + \ell_0) = \tau_a(tm) + A_t = \tau_a((t+1)m) + B_t \tag{11.3.9}$$

and

$$M_t > \max\{\, \tau_a(tm), \tau_a(tm + m) \,\}.$$

Therefore, the graded root associated with the values $\{\tau_a(\ell)\}_{0 \leq \ell \leq (t_a+1)m}$ is the same as the graded root associated with the values

$$\tau_a(0), M_0, \tau_a(m), M_1, \tau_a(2m), M_2, \ldots, \tau_a(t_a m), M_{t_a}, \tau_a(t_a m + m).$$

Finally, since $\#\{s \notin \mathscr{S}_{C,o}\} = \delta$, one has $\delta - B_t = \#\{s \notin \mathscr{S}_{C,o} : s \leq td + a\}$, hence $\delta - B_t + A_t = td + a + 1$. Thus, by (11.3.9),

$$\tau_a((t+1)m) - \tau_a(tm) = td + a + 1 - \delta.$$

Since $\tau_a(0) = 0$, this gives $\tau_a(tm)$ inductively.

Clearly, the graded root associated with τ_a is the same as the graded root associated with $\tilde{\tau}_a : \{0, 1, 2, \ldots, 2t_a + 2\} \to \mathbb{Z}$, where $\tilde{\tau}_a(2t) := \tau_a(tm)$ and $\tilde{\tau}_a(2t+1) := M_t$.

The above discussion gives the following statement.

Theorem 11.3.21 *For each fixed $a = 0, 1, \ldots, d-1$,—corresponding to the d different spin^c-structures of M—one defines the following objects:*

- $t_a := \left\lfloor \frac{2\delta - 2 - a}{d} \right\rfloor$, *($t_a \geq -1$ automatically);*

- *a function* $\tau_a : \{0, 1, \dots, 2t_a + 2\} \to \mathbb{Z}$ *by*

$$\begin{cases} \tau_a(2t) = d \cdot \frac{t(t-1)}{2} - t(\delta - 1 - a), & (t = 0, \dots, t_a + 1); \\ \\ \tau_a(2t+1) = \tau_a(2t+2) + \alpha_{td+a}, & (t = 0, \dots, t_a). \end{cases}$$

- *and the graded root* $(R_{\tau_a}, \chi_{\tau_a})$ *associated with* τ_a.

Then $(R_{\tau_a}, \chi_{\tau_a})$ *is the graded root of* M *associated with* (Γ, k_r).
Note also that $\min \tau_a = \tau_a(\, 2\lceil t_a/2 \rceil\,)$.

Remark 11.3.22

(a) Since for any $t \in \{0, \dots, t_a\}$, $\tau_a(2t+1) > \max\{\tau_a(2t), \tau_a(2t+2)\}$, the above representation of the graded root is the most 'economical': all the values are essential. This also shows that $(R_{\tau_a}, \chi_{\tau_a})$ has exactly $t_a + 2$ local minimum points, and they correspond to the values $\tau_a(2t)$, $t = 0, 1, \dots, t_a + 1$.
(b) The values $\tau_a(2t)$, $t = 0, 1, \dots, t_a + 1$ depend only on t, d and δ, that is, for these values no other information is needed from the semigroup $\mathscr{S}_{C,o}$.

Lemma 11.3.23 ***The Value of*** $k_r^2 + |\mathscr{V}|$ *For any* $0 \le a < d$, $k_r := K + 2s_h = K + 2aE_+^*$ *satisfies* $k_r^2 + |\mathscr{V}| = 1 - (d + 2\delta - 2 - 2a)^2/d$.

Proof $K^2 + |\mathscr{V}|$ is independent of the resolution, hence we can consider the minimal resolution: it has only one exceptional curve, say F^+, with self-intersection $-d$ and a singularity with delta-invariant δ. Hence $-K = ((d + 2\delta - 2)/d)F^+$ and $K^2 + |\mathscr{V}| = 1 - (d + 2\delta - 2)^2/d$. Now, we consider the minimal good resolution, where $k_r^2 + |\mathscr{V}| = K^2 + |\mathscr{V}| + 4a(K, E_+^*) + 4a^2(E_+^*)^2$, where $(K, E_+^*) = (d + 2\delta - 2)/d$ and $(E_+^*)^2 = -1/d$. □

Corollary 11.3.24

(a) $eu(\mathbb{H}^*(\Gamma, k_r)) = \sum_{t=0}^{t_a} \alpha_{td+a}$
(b) $\mathfrak{sw}_{\sigma[k_r]}(M(\Gamma)) = \sum_{t=0}^{t_a} \alpha_{td+a} + \frac{1}{8}(1 - \frac{(d+2\delta-2-2a)^2}{d})$.

Proof Use 11.2.10 for (a) and 11.1.36 for (b). □

Example 11.3.25 Without proof we provide the graded root and the Seiberg–Witten invariant of the (rational) surgery 3-manifold $M = S^3_{-p/q}(K)$ too ($p/q \in \mathbb{Q}_{>0}$). For details and more information see [479]. For Γ see 3.5.13.

In this case $H := H_1(M, \mathbb{Z}) = \mathbb{Z}_p$. There is a natural identification of the spin^c-structures of M with the classes of $\mathbb{Z}_p$—or, with the integers $a = 0, 1, \dots, p-1$, $a = 0$ corresponding to the canonical spin^c-structure. The symbol $\mathbf{s}(q, p)$ denotes the Dedekind sum (see 2.2.A). The integer q' is uniquely determined by $1 \le q' \le p$ and $qq' \equiv 1 \pmod{p}$.

For each fixed $a = 0, 1, \dots, p-1$,—corresponding to the p different spin^c-structures of M—one defines the following objects:

- $t_a := \left\lfloor \frac{(2\delta-1)q-a-1}{p} \right\rfloor$;
- a function $\tau_a : \{0, 1, \dots, 2t_a + 2\} \to \mathbb{Z}$ by

$$\begin{cases} \tau_a(2t) = t(1-\delta) + \sum_{i=0}^{t-1} \left\lfloor \frac{ip+a}{q} \right\rfloor, & (t = 0, \dots, t_a + 1); \\ \\ \tau_a(2t+1) = \tau_a(2t+2) + \alpha_{\lfloor (tp+a)/q \rfloor}, & (t = 0, \dots, t_a). \end{cases}$$

Then $(R_{\tau_a}, \chi_{\tau_a})$ is the graded root of (Γ, k_r). Furthermore $\min \tau_a = \tau_a(2\lceil t_a/2 \rceil)$.

Furthermore, $-(k_r^2 + |\mathcal{V}|)/4$ equals

$$3\mathbf{s}(q, p) + 2\sum_{j=1}^{a} \left\{ \frac{jq'}{p} \right\} - \frac{(1+2a)(p-1)}{2p} + \delta\Big(1 - \frac{q+1}{p}\Big) + \frac{\delta^2 q}{p} - \frac{2\delta a}{p}.$$

For example, when $p = 1$, i.e. $M = S^3_{-1/q}(K)$, then $q' = 1$ and there is only one spin^c-structure corresponding to $a = 0$. In this case $-(K^2 + |\mathcal{V}|)/4 = q\delta(\delta-1)$.

In particular,

$$eu(\mathbb{H}^*(\Gamma, k_r)) = \sum_{t \geq 0} \alpha_{[(tp+a/q]}. \tag{11.3.10}$$

Example 11.3.26 If $a \geq (2\delta-1)q$, then $t_a = -1$, hence $R_{\tau_a} = R_{(0)}$ (cf. 11.2.3(1)). In particular, for such a, $\mathbb{H}^0_{red}(R_{\tau_a}) = 0$. For all the other a's, $\mathbb{H}^0_{red}(R_{\tau_a})$ is not trivial. E.g., for the canonical spin^c-structure corresponding to $a = 0$, $\mathbb{H}^0_{red}(R_{\tau_0})$ is never trivial. In particular, Γ is never rational.

Example 11.3.27 Assume $p = q = 1$. Then M is an integral homology sphere; $a = 0$ and $t_0 = 2\delta - 2 = \mu - 2$. Moreover, $-(K^2 + |\mathcal{V}|)/4 = \delta(\delta-1)$ and $\tau_0(2t) = t(t - 2\delta + 1)/2$. The reader is invited to draw the graded root and verify that

$$\mathbb{H}^0(\Gamma, K) = \Big(\mathcal{T}_0^+ \oplus \mathcal{T}_0(\alpha_{\delta-1}) \oplus \bigoplus_{i=1}^{\delta-1} \mathcal{T}_{i(i+1)}(\alpha_{i-1+\delta})^{\oplus 2} \Big)[-\delta(\delta-1)].$$

Example 11.3.28 More generally, assume only that $p = 1$. Similarly as in the above case, only $a = 0$ is possible, and in this case $\min \chi = (K^2 + |\mathcal{V}|)/4$ as well. Moreover, $\mathbb{H}^0(\Gamma, K)$ equals

$$\Big(\mathcal{T}_0^+ \oplus \mathcal{T}_0(\alpha_{\delta-1})^{\oplus q} \oplus \bigoplus_{i=1}^{(\delta-1)q} \mathcal{T}_{([i/q]+1)(\{i/q\}q+l)}(\alpha_{\delta-1+\lceil i/q \rceil})^{\oplus 2} \Big)[-q\delta(\delta-1)].$$

From this expression (since it involves only the coefficients α_i for $i \geq \delta - 1$), seemingly $\mathbb{H}^0(\Gamma, K)$ contains less information than the polynomial $Q(t)$. However, this is not the case. Indeed, the symmetry of $\Delta(t)$ implies that $\alpha_{\mu-2-i} - \alpha_i$ is a Δ-independent universal number (see e.g. (11.5.32)). In particular, Q can be recovered from $\{\alpha_\ell\}_{\ell \geq \delta-1}$. In fact, from $\mathbb{H}^0(\Gamma, K)$ both the integer q and the isotopy type of $K \subset S^3$ can be recovered.

Remark 11.3.29 In the above example the graded root and $\mathbb{H}^0(\Gamma, K)$ have a striking $\mathbb{Z}_2$-*symmetry*. We explain this for $a = 0$. The point is that if the graph is numerically Gorenstein, then the whole theory associated with the integral cycles has a duality. This happens for example if $p = 1$; or if $q = 1$ and $2\delta - 2$ is divisible by p. In our case, in these situations, the function τ_0 is stable with respect to the $\mathbb{Z}_2$ action $i \mapsto 2t_0 + 2 - i$, i.e. $\tau_0(i) = \tau_0(2t_0 + 2 - i)$. This induces a symmetry of the root and of the lattice cohomology.

But, for general p/q, the graphs are not numerically Gorenstein. In fact, even if they are numerically Gorenstein, for general a, the symmetry may fail.

For more examples see e.g. [479, 5.6.5].

Remark 11.3.30 In 11.3.8 we proved that for any analytic structure supported on an AR topological type we have

$$p_g \leq eu(\mathbb{H}^*(\Gamma, K)) = -\min(\chi_K) + \operatorname{rank}_{\mathbb{Z}} \mathbb{H}^0(\Gamma, K).$$

In the case of $M = S^3_{-p/q}(K)$ this via (11.3.10) reads as $p_g \leq \sum_{t \geq 0} \alpha_{[tp/q]}$.

When $p/q = d$ satisfies $(d-1)(d-2) = \mu(K)$ then in the next subsection we will see (cf. 11.6.11 and 11.6.12) that this inequality is optimal, and equality is realized by the superisolated singularities: in fact, the equality is equivalent to the validity of the SWIC. However, for other surgery coefficients it is hard to find/guess the right analytic structures, which might have $p_g = \sum_{t \geq 0} \alpha_{[tp/q]}$. Nevertheless, the graph $S^3_{-p/q}(K)$ is Kulikov (cf. 6.7.29), hence it admits a Kulikov analytic structure (and in several examples the Kulikov analytic type maximizes p_g, see e.g. 6.8.44).

11.3.31 Question

(a) Is it true that $p_g = \sum_{t \geq 0} \alpha_{[tp/q]}$ for a Kulikov singularity with link $S^3_{-p/q}(K)$?
(b) Does a Gorenstein Kulikov analytic structure with link $S^3_{-p/q}(K)$ exist?

Note that the two-cuspidal graph from 6.8.44 is also of type $S^3_{-1}(K_1\#K_2)$, it is Kulikov, but it does not support a Gorenstein Kulikov analytic type.

11.3.C Superisolated Singularities with One Cusp

11.3.32 In the sequel we will consider a superisolated singularity as in 5.2.3. For different invariants see 5.2, whose notations we will adopt. We will assume that C is an irreducible rational unicuspidal curve.

We invite the reader to review the 'Semigroup Distribution Inequality' from 5.2.9(III)) and the 'Semigroup Distribution Property' from 5.2.9(IV)). The reinterpretations in terms of reduced Poincaré series can be found in 8.6.B, and the connection with the Seiberg–Witten Invariant Conjecture (as the basic motivation and source of the Semigroup Distribution Property) is presented in 9.7.B. Here we present further connections with the graded roots. We will follow [199].

11.3.33 In this part we will compare the invariants of the link $M = S^3_{-d}(K)$ of the superisolated singularity with the corresponding invariants of the Seifert 3-manifold $\Sigma(d,d,d+1)$, the link of the hypersurface Brieskorn singularity $x^d+y^d+z^{d+1}=0$, cf. 5.1.17. Before we state the next theorem, we recall that the plumbing graph of (or the 3-manifold itself) $S^3_{-d}(K)$ contains complete information about the embedded link $K \subset S^3$. Moreover, by the statements of 11.3.21, the graded root or lattice cohomology still preserves essential data about the Alexander polynomial. However, the Seifert 3-manifold $\Sigma(d,d,d+1)$ has information only about the degree μ of Δ via $(d-1)(d-2)=\mu$. The point is that the algebraic realizability of C (that is, the existence of an analytic superisolated singularity with link $S^3_{-d}(K)$) imposes the following very surprising necessary topological obstructions.

Theorem 11.3.34 ([199]) *The following facts are equivalent:*

(a) The Seiberg–Witten Invariant Conjecture is true for the superisolated germ.
(b) The Semigroup Distribution Property is true.
(c) The canonical graded roots of $S^3_{-d}(K)$ and $\Sigma(d,d,d+1)$ are the same.
(d) The canonical lattice homologies of $S^3_{-d}(K)$ and $\Sigma(d,d,d+1)$ are the same.
(e)

$$\left(\mathfrak{sw}_{\sigma[K]}(M) - \frac{K^2+\#\mathscr{V}}{8}\right)\Big|_{M=S^3_{-d}(K)} = \left(\mathfrak{sw}_{\sigma[K]}(M) - \frac{K^2+\#\mathscr{V}}{8}\right)\Big|_{M=\Sigma(d,d,d+1)}.$$

Recall that, in fact, the Semigroup Distribution Property *is true* by [68] (cf. 5.2.9(III-IV)), hence all the statements of 11.3.34 *are true* as well. However, we formulated above a weaker statement, only the *equivalence* of the above statements, whose proof is independent of the Heegaard Floer theory based proof of [68].

The proof of 11.3.34 is given in several steps. The starting point is that both 3-manifolds $S^3_{-d}(K)$ and $\Sigma(d,d,d+1)$ are almost rational. In particular, in both cases, the canonical graded root can be determined via the τ-function, cf. 11.3. In the first case this is done explicitly in 11.3.21, while for the second case see 11.3.A.

In this second case the minimal good resolution graph of $\Sigma(d,d,d+1)$ is star-shaped with a (-1) central vertex, and it has d identical legs, each consisting of one vertex with self-intersection number $-(d+1)$. The algorithm from 11.3.A gives a 'non-economical' τ–function, below we replace this one by the most economical one.

Fact 1 Let us rewrite 11.3.21 for $S^3_{-d}(K)$ and for the canonical spin^c structure $a=0$. Set $c_l := \alpha_{(d-3-l)d}$ and define $\tau : \{0,1,\dots,2d-4\} \to \mathbb{Z}$ by

$$\tau(2l) = \frac{l(l-1)}{2}d - l(\delta-1), \quad \tau(2l+1) = \tau(2l+2) + c_{d-3-l}. \tag{11.3.11}$$

Then $(R_{can}, \chi_{can}) = (R_\tau, \chi_\tau)$.

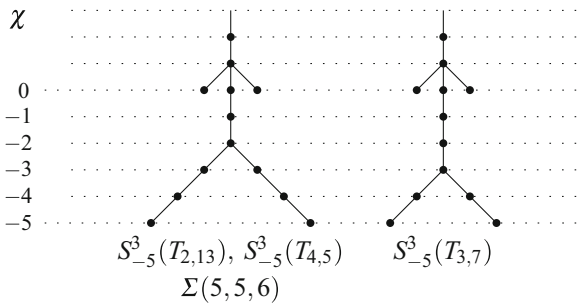
Fact 2 Consider next the Seifert manifold $\Sigma(d,d,d+1)$. Its canonical graded root is the following. For any $0 \le l \le d-3$ write $c^u_l := (l+1)(l+2)/2$, and $2\delta := (d-1)(d-2)$. Then define $\tau^u : \{0,1,\dots,2d-4\} \to \mathbb{Z}$ by

$$\tau^u(2l) = \frac{l(l-1)}{2}d - l(\delta-1), \quad \tau^u(2l+1) = \tau^u(2l+2) + c^u_{d-3-l}. \tag{11.3.12}$$

Then $(R_{can}, \chi_{can}) = (R_{\tau^u}, \chi_{\tau^u})$.

Notice the shocking similarities of (11.3.11) and (11.3.12): the graded roots associated with $S^3_{-d}(K)$ and $\Sigma(d,d,d+1)$ coincide exactly when $c_l = c^u_l$ for every l. However, by the Semigroup Distribution Inequality (a consequence of the Bézout's Theorem, cf. 5.2.9(III)) $c_l \ge c^u_l$ for every l. Hence $c_l = c^u_l$ for every l if and only if $\sum_l c_l = \sum_l c^u_l$. But this is exactly the vanishing of $N(1)$, cf. 8.6.21*(b)*, hence 9.7.30 applies.

Example 11.3.35 Assume that $d=5$ and C is unicuspidal and its singular point has only one Puiseux pair (a,b) with $a<b$. Then by the genus formula the possible values of (a,b) are $(4,5)$, $(3,7)$ and $(2,13)$. It turns out that the first and the third cases can be realized, while the second case not (see e.g. 5.2.8). This fact is compatible with the above Theorem 11.3.34. Indeed, the corresponding canonical graded roots (together with the root of $\Sigma(5,5,6)$) are shown in the next picture.



In the case of $(3, 7)$ the semigroup distribution in the intervals $((l-1)d, ld]$ $(l \geq 0)$ is 1,1,4,4,5,... instead of 1,2,3,4,5,... required by (SDP). That is, a semigroup element 'moves wrongly to the right'.

Wrong move to the left is also possible: Consider $d = 6$ and the semigroup $(2, 21)$ (i.e., $S^3_{-6}(T_{2,21})$), then the distribution is 1,3,3,3,5,6,... instead of 1,2,3,4,5,6,.... The fact that this data is not realized algebraically can be verified by the Matsuoka–Sakai inequality too (cf. 5.2.9.I), see also the classification from 5.2.8.II.

Remark 11.3.36 As we already mentioned in 5.2.9(III)–(IV), the Semigroup Distribution Property (in the unicuspidal case) was partially verified in [198] and proved in [68]. The first approach is based on a case-by-case verification of the families of cuspidal rational projective curves which appear in the classification theorems. The second approach is based on the Heegaard Floer theory. The discussion from 11.3.38 traces a possible third approach, which would lead to a different proof, and would open a new chapter in the deformation theory of surface singularities.

Corollary 11.3.37 *The Seiberg–Witten Invariant Conjecture is true for superisolated germs associated with rational unicuspidal curves.*

11.3.38 Why $\Sigma(d, d, d+1)$? At the first glance the pairing of $\Sigma(d, d, d+1)$ with $S^3_{-d}(K)$ in Theorem 11.3.34 looks very unmotivated, even incidental or accidental. In the next paragraphs we wish to convince the reader that this is not the case, and conjecturally a very deep structure might exist behind the scene.

Assume that the rational unicuspidal curve is given by $f_d(x, y, z) = 0$ in $\mathbb{P}^2$ (for notations see 5.2.3). We can fix the homogeneous coordinates in $\mathbb{P}^2$ in such a way that $z = 0$ intersects C generically. A possible choice for the superisolated singularity $f : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ is $f = f_d + z^{d+1}$. Write f_d as $\sum_{i=0}^{d} g_{d-i}(x, y)z^i$. Then g_d is a product of d linear factors corresponding to the points $C \cap \{z = 0\}$, hence the germ $g_d : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ is equisingular with $(x, y) \mapsto x^d + y^d$.

Next, consider the following deformation $f_t : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ of isolated hypersurface germs, given by $f_t(x, y, z) = f_d(x, y, tz) + z^{d+1} = \sum_i g_{d-i}(x, y)z^i t^i + z^{d+1}$. For $t \neq 0$ the deformation is μ-constant, the embedded topological type stays also constant, and it is equivalent (up to such equivalences)

to the type of f. However, for $t = 0$ it is equivalent (in similar sense) to the germ $x^d + y^d + z^{d+1}$.

Along this deformation not only the embedded topological type jumps (e.g. the Milnor number), but even the (non-embedded abstract) link as well: for $t \neq 0$ it is $S^3_{-d}(K)$, while for $t = 0$ it is $\Sigma(d, d, d + 1)$. Their graphs are rather different (cf. 11.3.15 and the proof of 11.3.34). Their first homology groups also differ: it is $\mathbb{Z}_d$ for $t \neq 0$ and $\mathbb{Z}_{d+1}^{d-1}$ for $t = 0$. (However, both graphs are AR.)

The point is that several key invariants still stay stable. For example, using the corresponding sections above we obtain that in both cases $p_g = d(d-1)(d-2)/6$. If in both cases we compute the (resolution independent) invariant $K^2 + |\mathscr{V}|$ we realizes that they are different. However, if we denote by K^2_{min} the self-intersection of K in the *minimal resolution*, then it turns out that in both cases it is $-d(d-2)^2$. Hence we are dealing with a Gorenstein K^2_{min}-constant deformation. By a result of Laufer [365] such deformations admit a *very weak simultaneous resolution* (possible after a finite base change). This give possibility to compare the lattices associated with their minimal resolutions. Indeed, $S^3_{-d}(K)$ and $\Sigma(d, d, d + 1)$ admit certain non-minimal resolution graphs with lattices $L_{t\neq 0}$ and $L_{t=0}$ and a homological map $\iota : L_{t\neq 0} \to L_{t=0}$, which preserves the intersection matrices, the canonical classes, the χ-expression. [This created the possibility to prove the validity of the statements of Theorem 11.3.34—not just their equivalences –, but the plan was obstructed by the stability of the crucial relations (11.2.1), at least in the author's try.]

We formulate the next conjecture, whose positive answer would produce an extremely strong test for the existence of certain analytic deformations.

Conjecture 11.3.39 Along a flat K^2_{min}–constant deformation $\{X_t\}_t$ of Gorenstein surface singularities, such that the links of $X_{t=0}$ and $X_{t\neq 0}$ are both rational homology spheres, the graded roots of the links of $X_{t=0}$ and $X_{t\neq 0}$ associated with $\chi = \chi_K$ are the same.

Note that along a deformation as in 11.3.39 in general we *cannot* expect the stability of the whole module $\{\mathbb{H}^q\}_q$. Indeed, for the deformation described in 11.3.38 valid for superisolated germs, for $t = 0$ we have an AR case with $\mathbb{H}^{\geq 1} = 0$. However, for $t \neq 0$, for certain superisolated germs with $\nu \geq 2$ we might have $\mathbb{H}^{\geq 1} \neq 0$, see e.g. 11.4.12. In fact, for any superisolated germ which produced a counterexample for the SWIC, along the above deformation the canonical Seiberg–Witten invariant is non-constant too (cf. 11.3.34).

11.4 The Reduction Theorem

Fix Γ and $k_r = K + 2s_h$ ($h \in H$). If $\overline{\mathscr{V}}$ is an SR-set of $\mathscr{V}$, then for each $\overline{l} \in L(\overline{\mathscr{V}})$ in 7.3.C we defined the universal cycle $x_{s_h}(\overline{l}) \in L$. Furthermore, $\overline{l} \mapsto \chi_{k_r}(x_{s_h}(\overline{l}))$ generates a system of weights $\overline{w}$ of the cubes of $(\mathbb{R}_{\geq 0})^{\overline{s}}$, $\overline{s} := |\overline{\mathscr{V}}|$. Then the Reduction Theorem states that $\mathbb{H}^*(\Gamma, k_r) \simeq \mathbb{H}^*((\mathbb{R}_{\geq 0})^{\overline{s}}, \overline{w})$.

Furthermore, if π denotes the projection $L\otimes\mathbb{Q} \to L(\overline{\mathscr{V}})\otimes\mathbb{Q}$, then $\mathrm{pc}^{\mathscr{S}}(Z_h(\mathbf{t})) = \mathrm{pc}^{\pi(\mathscr{S})}(Z_{h,\overline{\mathscr{V}}}(\mathbf{t}_{\overline{\mathscr{V}}}))$. Hence we obtain a reduction to an $\bar{s}$-dimensional cubical decomposition. ♣

11.4.1 We consider a graph Γ as in 11.1.A. We also fix a distinguished class $k_r \in \mathrm{Char}$ and the corresponding lattice cohomology $\mathbb{H}^*(\Gamma, k_r)$. Recall that there is an isomorphism of graded $\mathbb{Z}[U]$-modules $\mathbb{H}^*(\Gamma, k_r) \simeq \mathbb{H}^*((\mathbb{R}_{\geq 0})^s, k_r)$, where the second module is generated by weighted cubes in $(\mathbb{R}_{\geq 0})^s$, cf. 11.1.26. Here $s := |\mathscr{V}|$.

This $\mathbb{Z}[U]$-module was drastically simplified in the case of AR graphs, basically the cubes from $(\mathbb{R}_{\geq 0})^s$ were replaced by 0 and 1 dimensional cubes along an infinite increasing path (starting with $0 \in L$), cf. Theorem 11.3.5. Here the AR-assumption is really necessary: such a reduction to a 1-dimensional path cannot be done for any graph (e.g. when $\mathbb{H}^1 \neq 0$). In this subsection we discuss the analogue of this statement for an arbitrary graph.

The reader is invited to review the definition of an SR-set of vertices from 7.3.B. Recall that this definition does not involve any $k \in \mathrm{Char}$, hence such a set can be uniformly used for any k_r. In this section we fix such an SR-set $\overline{\mathscr{V}} \subset \mathscr{V}$ as in 7.3.21, and any $k_r \in \mathrm{Char}$. Then, for each $\bar{l} = \sum_{v\in\overline{\mathscr{V}}} \ell_v E_v \in L(\overline{\mathscr{V}})$, with every $\ell_v \geq 0$, we define the universal cycle $x_{s_h}(\bar{l})$ associated with $\bar{l}$ and s_h (where $k_r = K + 2s_h$) as in 7.3.24. For several properties of the cycles $x(\bar{l})$ and of the values $\chi_{k_r}(x(\bar{l}))$ see 7.3.C. Let $\bar{s}$ be the cardinality of $\overline{\mathscr{V}}$. In the next paragraphs we follow [348, 350].

11.4.2 Preparation for the Lattice Reduction Our goal is to replace the cubes of the lattice $\mathbb{R}^s$ (or from $(\mathbb{R}_{\geq 0})^s$) with cubes from $(\mathbb{R}_{\geq 0})^{\bar{s}}$. In order to run the theory we need to define the new weights. Define the function $\overline{w}_0 : (\mathbb{Z}_{\geq 0})^{\bar{s}} \to \mathbb{Z}$ as

$$\overline{w}_0(\bar{l}) := \chi_{k_r}(x_{s_h}(\bar{l})). \tag{11.4.1}$$

Then $\overline{w}_0$ defines a set $\{\overline{w}_q\}_{q=0}^{\bar{s}}$ of compatible weight functions as in 11.1.17, $\overline{w}_q(\square) = \max\{\overline{w}_0(v) : v \text{ is a vertex of } \square\}$. This system is denoted by $\overline{w}[k_r]$.

Here some comments are appropriate. We wish to emphasize that in the definition of the lattice cohomology the *lattice* (that is, the *linear*) *structure* in not used, it is unessential. The important structure consists of the weight-levels of the *lattice points* in some regions (e.g. in some quadrants, rectangles) and their neighboring properties. Note that in the new situation we do not use the linear structure of $\mathbb{Z}^{\bar{s}}$ either, and we do not even define the weights of the lattice points outside the first quadrant. Furthermore, $\bar{l} \mapsto \chi_{k_r}(x_{s_h}(\bar{l}))$ is a complicated arithmetical function (usually not polynomial). However, it is a quasipolynomial, at least for points in the real cone generated by the projection of vectors of type $\{E_v^*\}_{v\in\overline{\mathscr{V}}}$. [Indeed, let $\mathfrak{o}_v$ be the order of $[E_v^*]$ in H, and let us consider $z(n) := \sum_{v\in\overline{\mathscr{V}}} n_v\mathfrak{o}_v E_v^*$ for $n_v \in \mathbb{Z}_{\geq 0}$. Then, via 7.3.25*(b)*, $x(\bar{l} + \overline{z(n)}) = x(\bar{l}) + z(n)$, hence $\overline{w}_0(\bar{l} + \overline{z(n)}) = \chi_{k_r}(x(\bar{l}) + z(n))$.]

Let us denote the associated lattice cohomology by $\mathbb{H}^*((\mathbb{R}_{\geq 0})^{\bar{s}}, \overline{w}[k_r])$.

Theorem 11.4.3 (Reduction Theorem [348, 350]) *Assume that $\overline{\mathscr{V}}$ is an SR–set. Then, with the above notations, there exists a graded $\mathbb{Z}[U]$-module isomorphism*

$$\mathbb{H}^*((\mathbb{R}_{\geq 0})^s, k_r) \cong \mathbb{H}^*((\mathbb{R}_{\geq 0})^{\bar{s}}, \overline{w}[k_r]). \tag{11.4.2}$$

The original proof is long and technical, and since the author knows no other/shorter proof, it will not be reproduced here: we send the reader to the original source [350].

Corollary 11.4.4 *Fix an arbitrary graph Γ. If it admits an SR-set of cardinality $\bar{s}$ then $\mathbb{H}^q(\Gamma, k) = 0$ for any $q \geq \bar{s}$ and $k \in Char$.*

Proof An isomorphism $\mathbb{H}^*(\Gamma, k) = \oplus_n H^*(\overline{S}_n, \mathbb{Z})$ follows from 11.4.3 and 11.1.12. But $\overline{S}_n$ is a compact cubical subcomplex of $\mathbb{R}^{\bar{s}}$, hence the vanishing follows. □

This vanishing can be proved by surgery exact sequences of lattice cohomology as well, see [483].

11.4.5 The Reduction Theorem has its effect on the relation of the lattice cohomology with the counting function of the coefficients of topological Poincaré series $Z(\mathbf{t})$ as well. Let us rewrite (use e.g. 9.6.7) the statements of Theorems 8.4.13, 9.6.21 and 11.1.36 in a form, which will resonate better with their reduced versions.

Theorem 11.4.6 *Fix h, s_h and $k_r = K + 2s_h$ as above. Let $w = w[k_r]$ be the system of weights associated with k_r. Then the following facts hold.*

(1)

$$Z_h(\mathbf{t}) = \sum_{l \in L} \Big(\sum_{I \subseteq \mathscr{V}} (-1)^{|I|+1} w((l, I)) \Big) \mathbf{t}^{l+s_h}.$$

(2) Fix some $l \in L$ with $l + s_h \in -K + \mathscr{S}'$. Then, for $Z_h(\mathbf{t}) = \sum_{[l']=h} \mathfrak{z}(l')\mathbf{t}^{l'}$

$$\sum_{x \in L,\, x \not\geq l} \mathfrak{z}(x + s_h) = \chi_{k_r}(l) + eu(\mathbb{H}^*(\Gamma, k_r)).$$

11.4.7 The Reduced Series Let us return to the SR-set $\overline{\mathscr{V}}$, write $\mathscr{V}$ as $\overline{\mathscr{V}} \sqcup \mathscr{V}^*$, and let $\pi : L' \to L(\overline{\mathscr{V}}) \otimes \mathbb{Q}$ be the projection to the $\overline{\mathscr{V}}$-coordinates. As usual, we also write $\mathbf{t}_{\overline{\mathscr{V}}} = \{t_v\}_{v \in \overline{\mathscr{V}}}$ for the variables of $L(\overline{\mathscr{V}})$, and $\mathbf{t}_{\overline{\mathscr{V}}}^{\bar{l}} = \prod_{v \in \overline{\mathscr{V}}} t_v^{\ell_v}$ for $\bar{l} = \sum_{v \in \overline{\mathscr{V}}} \ell_v E_v \in L(\overline{\mathscr{V}}) \otimes \mathbb{Q}$. For any $h \in H$ set $Z_{h,\overline{\mathscr{V}}}(\mathbf{t}_{\overline{\mathscr{V}}}) = Z_h(\mathbf{t})|_{t_v=1 \text{ for all } v \in \mathscr{V}^*}$. It is supported on the projection of $\mathscr{S}' \cap (s_h + L)$. Write

$$Z_{h,\overline{\mathscr{V}}}(\mathbf{t}_{\overline{\mathscr{V}}}) = \sum_{\bar{l} \in L(\overline{\mathscr{V}})} \overline{\mathfrak{z}}(\bar{l} + \pi(s_h)) \cdot \mathbf{t}_{\overline{\mathscr{V}}}^{\bar{l}+\pi(s_h)}.$$

Theorem 11.4.8 ([348, 350]) *With the above notations (and $\overline{w} = \overline{w}[k_r]$)*

(1)

$$Z_{h,\overline{\mathscr{V}}}(\mathbf{t}_{\overline{\mathscr{V}}}) = \sum_{\bar{l}\in L(\overline{\mathscr{V}})} \Big(\sum_{\overline{I}\subseteq\overline{\mathscr{V}}} (-1)^{|\overline{I}|+1}\overline{w}((\bar{l},\overline{I}))\Big)\mathbf{t}_{\overline{\mathscr{V}}}^{\bar{l}+\pi(s_h)}.$$

(2) There exists $\bar{l}_0 \in \pi(\mathscr{S})$ such that for any $\bar{l} \in \bar{l}_0 + \pi(\mathscr{S})$

$$\sum_{\bar{x}\not\geq\bar{l},\ \bar{x}\in L(\overline{\mathscr{V}})} \overline{\mathfrak{z}}(\bar{x}+\pi(s_h)) = \overline{w}(\bar{l}) + eu(\mathbb{H}^*((\mathbb{R}_{\geq 0})^{\bar{s}}, \overline{w}[k_r])).$$

Note that

$$\sum_{\bar{x}\not\geq\bar{l}} \overline{\mathfrak{z}}(\bar{x}+\pi(s_h)) = \sum_{\bar{x}\not\geq\bar{l}} \sum_{\bar{y}=\bar{x}} \mathfrak{z}(y+s_h) = Q^{\Gamma}_{h,\mathscr{J}}(\bar{l}+s_h).$$

The quasipolynomial of this counting function is $\mathfrak{Q}^{\Gamma}_{h,\mathscr{J}}(\bar{l}+\Delta_h)$, cf. 10.5.4. Therefore, as a corollary, we have the quasipolynomial equality as well:

$$\mathfrak{Q}^{\Gamma}_{h,\mathscr{J}}(\bar{l}+\Delta_h) = \overline{w}(\bar{l}) + eu(\mathbb{H}^*((\mathbb{R}_{\geq 0})^{\bar{s}}, \overline{w}[k_r])) \quad (\bar{l}\in L(\overline{\mathscr{V}})). \tag{11.4.3}$$

For another expression of $\mathfrak{Q}^{\Gamma}_{h,\mathscr{J}}(\bar{l})$ see 10.5.3 (that formula depends only on $\bar{l}$ as it is mentioned in 10.5.6).

Example 11.4.9 In the AR case we recover 11.3.5. For star-shaped graph the statements (applied for $\overline{\mathscr{V}}$ as the central node) are compatible with 11.3.A. More generally, for any graph, the results 11.4.3 and 11.4.8 can be applied for $\overline{\mathscr{V}} = \mathscr{N}$.

See 10.5.8(v) for a discussion regarding the periodic constants $\mathrm{pc}^{\mathscr{S}_{\mathbb{R}}}(Z_h(\mathbf{t}))$ and $\mathrm{pc}^{\pi(\mathscr{S}_{\mathbb{R}})}(Z_{h,\overline{\mathscr{V}}}(\mathbf{t}_{\overline{\mathscr{V}}}))$, and 10.5.9 for another expression which involves $\mathfrak{Q}^{\Gamma}_{h,\mathscr{J}}(0)$.

Example 11.4.10 Let us collect (and show compatibility of) some formulae valid in the AR case. Set $\{v\}$ for the SR–set. By 11.4.8*(1)*

$$Z_{h,\{v\}}(t) = \sum_{\ell\in\mathbb{Z}} \big(\max\{\overline{w}(\ell), \overline{w}(\ell+1)\} - \overline{w}(\ell)\big)\, t^{\ell+\pi(s_h)}. \tag{11.4.4}$$

This by 7.3.36 reads as

$$Z_{h,\{v\}}(t) = \sum_{\ell\in\mathbb{Z}} \max\{0,\ 1-(x_{s_h}(\ell)+s_h, E_v)\}\, t^{\ell+\pi(s_h)}. \tag{11.4.5}$$

Recall the notation $\Delta_h := s_h - r_h$ and $\Delta_{h,v} = \pi(\Delta_h) \in \mathbb{Z}_{\geq 0}$. In general, $\Delta_{h,v}$ can be positive (hence $\lfloor \ell + \pi(s_h) \rfloor = \ell + \Delta_{h,v} > \ell$ in the exponent of t). However, we claim that $(x_{s_h}(\ell) + s_h, E_v) \geq 1$ for any $\ell < 0$. (This fact and the next proof is compatible with the proof of the (SI) property from 6.6.20 valid for star–shaped graphs.)

Indeed, if $\ell < 0$ then $(\ell E_v + s_h, E_u) \leq 0$ for any $u \neq v$, hence by the universal property of x_{s_h} we obtain $x_{s_h} \leq \ell E_v$, hence $x_{s_h}(\ell) < 0$. On the other hand, $(x_{s_h}(\ell) + s_h, E_u) \leq 0$ for any $u \neq v$, hence $(x_{s_h}(\ell) + s_h, E_v) \leq 0$ cannot happen since in that case $x_{s_h}(\ell) + s_h$ would contradict the universal property of s_h.

Hence, in (11.4.5) the summation is for $\ell \geq 0$ (hence the 'gap' $[0, \Delta_{h,v})$ in the exponent of t survives).

Since $x_{s_h}(\ell) + s_h = x_{r_h}(\ell + \Delta_{h,v}) + r_h$ (cf. 7.3.25*(a)*)

$$Z_{h,\{v\}}(t) = \sum_{\ell \geq 0} \max\{0,\ 1 - (x_{r_h}(\ell) + r_h, E_v)\}\, t^{\ell + \pi(r_h)}. \tag{11.4.6}$$

Though the sum for $\Delta_{h,v} > \ell \geq 0$ has zero entries we prefer the above summation $\ell \geq 0$ (because of the split below). The advantage of this second sum is that $\pi(r_h) \in [0, 1)$ essential from the point of view of the periodic constant. The point is that if we use $\max\{0, x\} = x + \max\{0, -x\}$ and

$$Z^-_{h,\{v\}}(t) = \sum_{\ell \geq 0} \big(1 - (x_{r_h}(\ell) + r_h, E_v)\big)\}\, t^{\ell + \pi(r_h)}, \tag{11.4.7}$$

$$Z^+_{h,\{v\}}(t) = \sum_{\ell \geq 0} \max\{0,\ -1 + (x_{r_h}(\ell) + r_h, E_v)\}\, t^{\ell + \pi(r_h)}, \tag{11.4.8}$$

then $Z_{h,\{v\}}(t) = Z^-_{h,\{v\}}(t) + Z^+_{h,\{v\}}(t)$, $Z^+_{h,\{v\}}(t)$ is a finite sum and $Z^-_{h,\{v\}}(t)$ has negative degree as a rational function. In this last statement it is important to make the summation from 0 and that $\pi(r_h) \in [0, 1)$. [The proof runs over the following steps. Choose $\mathfrak{o}$ such that $\mathfrak{o} E_v^* \in L$, and let m_v be its E_v–coefficient. Then $x(\ell + pm_v) = x(\ell) + p\mathfrak{o}_v E_v^*$, cf. 7.3.25*(b)*. Then replace ℓ by $\ell_0 + pm_v$ with $p \geq 0$ and $0 \leq \ell_0 < m_v$ and compute the sum. See also 8.1.9.]

In particular,

$$\begin{aligned} \mathrm{pc}\, Z_{h,\{v\}}(t) = Z^+_{h,\{v\}}(1) &= \sum_{\ell \geq 0} \max\{0,\ -1 + (x_{r_h}(\ell) + r_h, E_v)\} \\ &= \sum_{\ell \geq -\Delta_{h,v}} \max\{0,\ -1 + (x_{s_h}(\ell) + s_h, E_v)\}. \end{aligned} \tag{11.4.9}$$

Now, from 11.3.5

$$\begin{aligned} eu\,\mathbb{H}^*(\Gamma, k_r) &= \sum_{\ell\geq 0} \max\{0,\ -1 + (x_{s_h}(\ell) + s_h, E_v)\} \\ &= \sum_{\ell\geq \Delta_{h,v}} \max\{0,\ -1 + (x_{r_h}(\ell) + r_h, E_v)\}. \end{aligned} \tag{11.4.10}$$

Recall also that $-\chi_{K+2s_h}(x - \Delta_h) + \chi_{K+2r_h}(x) = \chi_{K+2r_h}(\Delta_h) = \chi(s_h) - \chi(r_h)$, thus

$$\begin{aligned} &eu\,\mathbb{H}^*(\Gamma, K + 2r_h) - \mathrm{pc}\, Z_{h,\{v\}}(t) = \\ &\quad - \sum_{\Delta_{h,v}>\ell\geq 0} \max\{0,\ -1 + (x_{r_h}(\ell) + r_h, E_v)\} + \chi(r_h) - \chi(s_h). \end{aligned} \tag{11.4.11}$$

But, from 10.5.8 and 11.1.36 we also have

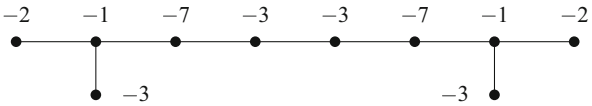
$$eu\,\mathbb{H}^*(\Gamma, K + 2r_h) - \mathrm{pc}\, Z_{h,\{v\}}(t) = \chi(R(r_h)) - \chi(s_{R(r_h)}), \tag{11.4.12}$$

where R is the restriction operator to the lattice of $\mathscr{V} \setminus \{v\}$.

The equality of the right hand sides of (11.4.11) and (11.4.12) follows from the steps of the Laufer algorithm connecting r_h with s_h (cf. 6.6.3). Note also that the right hand side of (11.4.12) equals also $\chi(r_h) - \chi(x_{r_h}(0) + r_h)$ (proved in 8.7.27) used in analytic expressions in 8.7.12 and 8.7.27. These analytic expressions can be compared with the above 'parallel' topological expressions (in the spirit of SWIC).

For more about $\chi(r_h) - \chi(x_{r_h}(0) + r_h)$ see 8.7.13 and 8.7.28.

Example 11.4.11 [350] Consider the following graph Γ



It is the minimal good resolution graph of the Newton non-degenerate hypersurface singularity $x^{13} + y^{13} + x^2y^2 + z^3 = 0$. In particular, Z_K is integral, and the weight function $\chi(l) = -(l, l - Z_K)/2$ is fixed by the symmetry $l \mapsto Z_K - l$.

In the sequel we exemplify the reduction theorem and we describe the lattice cohomology of $M(\Gamma)$ associated with $k_r = K$. We choose the two nodes as an SR-set. Then Reduction Theorem 11.4.3 implies that $\mathbb{H}^*(\Gamma, K) \cong \mathbb{H}^*((\mathbb{R}_{\geq 0})^2, \overline{w})$, where $\overline{w}(i, j) := \chi(x(i, j))$ for any $(i, j) \in (\mathbb{Z}_{\geq 0})^2$.

Since $\pi(Z_K) = (14, 14)$, the projection of the rectangle $R(0, Z_K)$ is $\pi(R(0, Z_K)) = R((0, 0), (14, 14))$. Hence by 7.3.12*(b)* the rectangle $R((0, 0), (14, 14)) = \{(i, j) \in (\mathbb{R}_{\geq 0})^2 : (i, j) \leq (14, 14)\}$ contains all the lattice

cohomological information. The values $\overline{w}(i, j)$ are given in the next diagram. ((0, 0) is at the lower left corner.)

1	1	0	0	0	0	0	1	0	0	0	0	0	1	0
1	1	0	0	0	0	0	1	0	0	0	0	0	1	1
0	0	−1	−1	−1	−1	−1	0	−1	−1	−1	−1	−1	0	0
0	0	−1	−1	−1	−1	−1	0	−1	−1	−1	−1	−1	0	0
0	0	−1	−1	−1	−1	−1	0	−1	−1	−1	−1	−1	0	0
0	0	−1	−1	−1	−1	−1	0	−1	−1	−1	−1	−1	0	0
0	0	−1	−1	−1	−1	−1	0	−1	−1	−1	−1	−1	0	0
1	1	0	0	0	0	0	1	0	0	0	0	0	1	1
0	0	−1	−1	−1	−1	−1	0	−1	−1	−1	−1	−1	0	0
0	0	−1	−1	−1	−1	−1	0	−1	−1	−1	−1	−1	0	0
0	0	−1	−1	−1	−1	−1	0	−1	−1	−1	−1	−1	0	0
0	0	−1	−1	−1	−1	−1	0	−1	−1	−1	−1	−1	0	0
0	0	−1	−1	−1	−1	−1	0	−1	−1	−1	−1	−1	0	0
1	1	0	0	0	0	0	1	0	0	0	0	0	1	1
0	1	0	0	0	0	0	1	0	0	0	0	0	1	1

Using the $\mathbb{S}^*$-description from 11.1.11, one can read off the lattice cohomology: the large frames illustrate the generators of $H^0(S_{-1}, \mathbb{Z})$, the small ones the generators of $\tilde{H}^0(S_0, \mathbb{Z})$ and the circle shows the generator of $H^1(S_0, \mathbb{Z})$. Hence,

$$\mathbb{H}^0(\Gamma, K) = \mathscr{T}^+_{-2} \oplus \mathscr{T}^3_{-2}(1) \oplus \mathscr{T}^2_0(1) \quad \text{and} \quad \mathbb{H}^1(\Gamma, K) = \mathscr{T}_0(1).$$

(Then, $eu(\mathbb{H}^*(\Gamma, K)) = 5$, which equals the geometric genus of $x^{13} + y^{13} + x^2y^2 + z^3 = 0$, hence the SWIC holds for this Newton non-degenerate hypersurface.)

Example 11.4.12 Consider the numerically Gorenstein graph from 9.7.32, and chose $k = K$ and $\overline{\mathscr{V}} = \mathscr{N}$ (the set of nodes). Then $\pi(R(0, Z_K)) = R((0, 0), (30, 34))$, and the lattice cohomology computation is reduced to this 2-dimensional rectangle. For the complete list of weights see [348, page 179]. The symmetric local minima $(0, 0)$ and $(30, 34)$ have similar neighborhoods as in the previous example. They will generate a summand $\mathscr{T}_0(1)^2$ in $\mathbb{H}^0(\Gamma, K)$. The other generators (and relations) can be read from the rectangle $R((12, 14), (18, 20))$:

−2	**−2**	**−3**	−4	−4	−4	**−5**
−2	−1	**−2**	−3	−3	−3	−4
−3	**−2**	**−2**	−3	−3	−3	−4
−3	−2	−2	−2	−2	−2	−3
−4	−3	−3	−3	**−2**	**−2**	**−3**
−4	−3	−3	−3	**−2**	−1	**−2**
−5	−4	−4	−4	**−3**	**−2**	**−2**

Hence $\mathbb{H}^0(\Gamma, K) = \mathscr{T}^+_{-10} \oplus \mathscr{T}_{-10}(3) \oplus \mathscr{T}_0(1)^2$ and $\mathbb{H}^1 = \mathscr{T}_{-2}(1)^2$.

Example 11.4.13 For several concrete formulae and other explicit examples when Γ has two nodes, and $\overline{\mathscr{V}} = \mathscr{N}$, see [348].

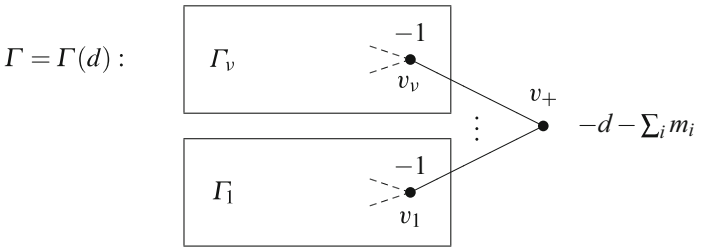
11.5 Application. $\mathbb{H}^*$ of the Surgery Manifold $S^3_{-d}(\#_i K_i)$

We consider the surgery 3-manifold $M = S^3_{-d}(\#_{i=1}^{\nu} K_i)$, where K_i are algebraic knots in S^3 and $d > 0$. It turns out that $H = \mathbb{Z}_d$ and it is generated by $E^*_{v_+}$. Using the Reduction Theorem and several deformation retractions of the spaces $\{S_n\}_n$, we compute $\mathbb{H}^*(M, k_r)$ ($k_r = K + 2aE^*_{v_+}$, $0 \le a < d$) for every $d > 0$, every spin^c structure $\sigma[k_r]$ of M and $\{K_i \subset S^3\}_i$. The lattice cohomology $\mathbb{H}^*(M, k_r)$ can be expressed in terms of d, a and the semigroups $\{\mathscr{S}_i\}_i$ of the local plane curve singularities. Then, in 11.5.A, for $\mathbb{H}^0$ we rewrite the semigroup information in terms of the Hilbert functions $\{H_i\}_i$ of the local singularities. It turns out that $\mathbb{H}^0$ depends only on d, a and the 'minimum convolution' $H = H_1 \diamond \cdots \diamond H_\nu$ of the functions $\{H_i\}_i$. Surprisingly, this minimum convolution H depend only on the multiset of the collection of multiplicity sequences associated with the local topological types.

In 11.5.B we apply these results to superisolated singularities with arbitrary number of cusps. We reinterpret Conjectures 8.6.23 and 8.6.25 and the Semigroup Distribution Property in terms of lattice cohomology. ♣

11.5.1 Consider the notations of 3.5.12, or of 9.3.12 with $d > 0$ (compare also with 11.3.B when $\#_i K_i = K_1$). Here we do not assume that $\mu = (d-1)(d-2)$ (as in the superisolated link case 5.2 or 8.6.B). In this section we follow [497].

11.5.2 Invariants of $M(\Gamma) = S^3_{-d}(\#_i K_i)$ for All Spin^c Structures By 3.5.12



Similarly as in 11.3.B, $H = \mathbb{Z}_d$, it is generated by the class of the dual of $E_+ := E_{v_+}$. Furthermore, as in Lemma 11.3.16 (with the same proof) one has $s_{[aE^*_+]} = aE^*_+$ for any $a = 0, 1, \ldots, d-1$. We will use the notations $h := [aE^*_+] \in H$ and $k_r := K + 2aE^*_+ \in \mathrm{Char}$. With $\mathscr{I} = \{v_+\}$ one has (cf. 8.4.2 and 9.7.B)

$$Z_{\mathscr{I}}(t) = \frac{\Delta(t^{1/d})}{(1-t^{1/d})^2} \quad \text{and} \quad Z_{h,\mathscr{I}}(t) = \frac{1}{d} \cdot \sum_{\xi^d=1} \xi^{-a} \frac{\Delta(\xi t^{1/d})}{(1-\xi t^{1/d})^2}. \tag{11.5.1}$$

Using $\Delta(t) = 1+(t-1)\delta+(t-1)^2 Q(t)$ and $Q(t) = \sum_{n=0}^{\mu-2} \alpha_n t^n$, by a computation

$$Z_{h,\mathscr{J}}(t) = \frac{t^{a/d}(a+1) + t^{1+a/d}(d-a-1)}{(t-1)^2} + \frac{\delta \cdot t^{a/d}}{t-1} + \sum_{n\equiv a \pmod d} \alpha_n t^{n/d}. \tag{11.5.2}$$

Since the polynomial part $Z^+_{h,\mathscr{J}}(t)$ of this expression is $\sum_{n\equiv a \pmod d} \alpha_n t^{n/d}$, we get

$$\mathrm{pc}(Z_{h,\mathscr{J}}(t)) = \mathrm{pc}(Z_{h,\mathscr{J}}(t^d)) = \sum_{n\equiv a \pmod d} \alpha_n. \tag{11.5.3}$$

Next we apply the surgery formula from Theorem 9.6.5 for $v = v_+$ and $l' = aE^*_+$. Then $l'_{v_+} = a/d \in [0,1)$. Furthermore, $R_i(aE^*_+) = 0$, hence all the contributions $\mathfrak{sw}_{\sigma[K_i]}(M(\Gamma_i)) - (K_i^2 + |\mathscr{V}_i|)/8$ vanish by 11.1.38. Therefore, from 9.6.5,

$$\mathfrak{sw}_{\sigma[k_r]}(M(\Gamma)) - \frac{k_r^2 + |\mathscr{V}|}{8} = \sum_{n\equiv a \pmod d} \alpha_n. \tag{11.5.4}$$

This combined with Theorem 11.1.36 give

$$eu(\mathbb{H}^*(\Gamma, k_r)) = \sum_{n\equiv a \pmod d} \alpha_n. \tag{11.5.5}$$

It is instructive to recover this formula via an explicit expression of the torsion $\mathcal{T}_{\sigma[k_r]}(1)$ as well. First, by 9.5.9

$$\mathcal{T}_{\sigma[k_r]}(1) = \lim_{t\to 1} \big(Z_{h,\mathscr{J}}(t) - \frac{1}{d} Z_{\mathscr{J}}(t)\big). \tag{11.5.6}$$

This via (11.5.1) gives

$$\mathcal{T}_{\sigma[k_r]}(1) = \lim_{t\to 1} \frac{1}{d} \cdot \sum_{\xi^d=1\neq\xi} \xi^{-a} \frac{\Delta(\xi t^{1/d})}{(1-\xi t^{1/d})^2} = \frac{1}{d} \cdot \sum_{\xi^d=1\neq\xi} \xi^{-a} \frac{\Delta(\xi)}{(1-\xi)^2}. \tag{11.5.7}$$

Now, a computation based either on (11.5.2), or on (11.5.7) together with (9.5.11) and (9.5.12) gives:

$$\mathcal{T}_{\sigma[k_r]}(1) = -\frac{(d-1)(d-5)}{12d} - \frac{a^2+2a-ad}{2d} - \delta\cdot\frac{d-1-2a}{2d} - \frac{Q(1)}{d} + \sum_{j\equiv a \pmod d} \alpha_j.$$

Note that $2Q(1) = \Delta''(1)$. Also, by similar formulae and proof as in 11.3.23

$$k_r^2 + |\mathcal{V}| = 1 - (\mu - 2 + d - 2a)^2/d. \tag{11.5.8}$$

Then the definition $\mathfrak{sw}_{\sigma[k_r]}(M(\Gamma)) = \mathcal{T}_{\sigma[k_r]}(1) - \lambda/d$, the λ-formula from 9.3.12 and the formula (11.5.8) reprove (11.5.4).

11.5.3 The Lattice Reduction The set $\overline{\mathcal{V}} := \{v_1, \dots, v_\nu\}$ of the (-1)-vertices form an SR-set, cf. 7.3.22(i). Set $E_1, \dots, E_\nu$ for the corresponding elements of L. Next we apply the Reduction Theorem from 11.4, whose notations we will adopt.

Write $\bar{l} = \sum_{i=1}^{\nu} \ell_i E_i \in L(\overline{\mathcal{V}}) = \overline{L}$, and let $x_{k_r}(\bar{l})$ be the universal cycle associated with k_r and $\bar{l}$ as in 7.3.24 and 11.4. Set $\overline{w}(\bar{l}) := \chi_{k_r}(x(\bar{l}))$ as in (11.4.1). Then, by the Reduction Theorem 11.4.3 one has a graded $\mathbb{Z}[U]$-module isomorphism:

$$\mathbb{H}^*(\Gamma, k_r) \cong \mathbb{H}^*((\mathbb{R}_{\geq 0})^\nu, \overline{w}). \tag{11.5.9}$$

Based on the arguments and formulae from 11.3.B we deduce the following facts. For each $\ell_i \geq 0$ consider the cycle $y_i(\ell_i)$ determined in the graph Γ_i as in 11.3.17 and 11.3.18. Set $\Sigma m := \sum_i m_i$ and $\Sigma\ell := \sum_i \ell_i$ (and, in general, $\Sigma x := \sum_i x_i$ for $x \in \mathbb{R}^\nu$). Then the E_+-coefficient of $x_{k_r}(\bar{l})$ is $m_+(\bar{l}) = \lceil (\Sigma\ell - a)/(\Sigma m + d)\rceil$ and

$$x_{k_r}(\bar{l}) = \sum_i y_i(\ell_i) + \left\lceil \frac{\Sigma\ell - a}{\Sigma m + d} \right\rceil \cdot E_+. \tag{11.5.10}$$

Write $\ell_i = t_i m_i + \ell_{i,0}$ with $t_i \in \mathbb{Z}_{\geq 0}$ and $0 \leq \ell_{i,0} < m_i$. Let Z_i be the cycle $\mathrm{div}_{E(\Gamma_i)}(f_i) = E_i^*(\Gamma_i)$. Then $y_i(\ell_i) = t_i Z_i + y_i(\ell_{i,0})$ (cf. 11.3.18). Furthermore, if for any $i = 1, \dots, \nu$ we takes $1_i = (0, \dots, 0, 1, 0, \dots, 0)$ (1 at entry i) then $\overline{w}(0) = 0$ (cf. 7.3.33), and

$$\overline{w}(\bar{l} + 1_i) - \overline{w}(\bar{l}) = t_i + 1 - \left\lceil \frac{\Sigma\ell - a}{\Sigma m + d} \right\rceil - \begin{cases} 1 & \text{if } \ell_{i,0} \notin \mathcal{S}_i \\ 0 & \text{if } \ell_{i,0} \in \mathcal{S}_i. \end{cases} \tag{11.5.11}$$

Here $\mathcal{S}_i$ is the abbreviation for the semigroup $\mathcal{S}_{C,p_i}$.

Next, we reduce $(\mathbb{R}_{\geq 0})^\nu$ to a finite multi-rectangle. We write $\mathbf{m}$ for the vector $(m_1, \dots, m_\nu)$, and $R(\bar{l}_1, \bar{l}_2)$ denotes the rectangle $\{x \in \mathbb{R}^\nu : \bar{l}_1 \leq x \leq \bar{l}_2\}$, as usual.

Set $\min := \min_i\{m_i - \mu_i\}$ and $\mathfrak{t}_0 := \lceil (\Sigma m - \min - a - \nu)/d\rceil$.

Lemma 11.5.4

(a) Take any integer $\mathfrak{t} \geq \mathfrak{t}_0$ *and* $\bar{l} \in R\big(\mathfrak{t}\,\mathbf{m}, (\mathfrak{t}+1)\,\mathbf{m}\big)$. *Then* $\overline{w}(\bar{l}+1_i) \geq \overline{w}(\bar{l})$ *for any* $i \in \{1, \dots, \nu\}$.

(b)

$$\mathbb{H}^*\big((\mathbb{R}_{\geq 0})^\nu, \overline{w}\big) \cong \mathbb{H}^*\big(R(0, \mathfrak{t}_0\,\mathbf{m}), \overline{w}\big).$$

Proof

(a) Assume that $\Sigma\ell \leq (\mathfrak{t}+1)\Sigma m - \min - \nu$ (†). Since $\mathfrak{t} \geq \mathfrak{t}_0$ we get $d\mathfrak{t} \geq \Sigma m - \min - a - \nu$. This together with (†) implies $(\Sigma\ell - a)/(\Sigma m + d) \leq \mathfrak{t}$, hence *(a)* follows from (11.5.11). Similarly, if $\Sigma\ell > (\mathfrak{t}+1)\Sigma m - \min - \nu$, that is, if $\sum_i \ell_{i,0} > \Sigma m - \min - \nu$, then necessarily $\ell_{i,0} \in \mathscr{S}_i$ for all i (since $\ell_{i,0} \leq m_i - 1$ for all i, then necessarily $\ell_{i,0} \geq \mu_i$ for all i, but μ_i is the conductor of $\mathscr{S}_i$). Hence, the last contribution in (11.5.11) is zero. On the other hand, $\Sigma\ell \leq (\mathfrak{t}+1)\Sigma m$ (from the choice of $\bar{l}$), hence $(\Sigma\ell - a)/(\Sigma m + d) \leq \mathfrak{t}+1$.

(b) We show that if $\bar{l} \in R\big(\mathfrak{t}\,\mathbf{m}, (\mathfrak{t}+1)\,\mathbf{m}\big)$ as in *(a)* then for any i we have:

$$\mathbb{H}^*\big(R(0, \bar{l}), \overline{w}\big) = \mathbb{H}^*\big(R(0, \bar{l}+1_i), \overline{w}\big).$$

For this we use deformation retractions as in the proof of 7.3.9*(II)*. Fix i and any $\bar{l}^\circ \in R(0, \bar{l}) \cap \overline{L}$ with the same i-entry: $\bar{l}^\circ_i = \bar{l}_i$. Then from 7.3.29 one has $\overline{w}(\bar{l}+1_i) - \overline{w}(\bar{l}) = 1 - (x_{k_r}(\bar{l}) + aE^*_+, E_i)$ and the same identity for $\bar{l}^\circ$. Hence $\overline{w}(\bar{l}^\circ + 1_i) - \overline{w}(\bar{l}^\circ) = \overline{w}(\bar{l}+1_i) - \overline{w}(\bar{l}) + (x_{k_r}(\bar{l}) - x_{k_r}(\bar{l}^\circ), E_i)$. Now $x_{k_r}(\bar{l}) - x_{k_r}(\bar{l}^\circ)$ is effective and its support does not contain E_i. Therefore $(x_{k_r}(\bar{l}) - x_{k_r}(\bar{l}^\circ), E_i) \geq 0$ and $\overline{w}(\bar{l}+1_i) \geq \overline{w}(\bar{l})$ by *(a)*. Hence $\overline{w}(\bar{l}^\circ + 1_i) \geq \overline{w}(\bar{l}^\circ)$ for such cycles and the contraction holds.

Since this step can be applied for any of the situations $\bar{l} \in R\big((\mathfrak{t}_0+n)\,\mathbf{m}, (\mathfrak{t}_0+n+1)\,\mathbf{m}\big)$ and for any $n \in \mathbb{Z}_{\geq 0}$, by induction part *(b)* follows as well. □

The rectangle $R(0, \mathfrak{t}_0\,\mathbf{m})$ can be divided further into smaller parts: 'small rectangles' and 'stripes'. A 'small rectangle' has the form

$$R_{\mathfrak{t}_1, \dots, \mathfrak{t}_\nu} := \big\{\, \mathbf{x} \in \mathbb{R}^\nu \;:\; \mathfrak{t}_i m_i \leq x_i \leq (\mathfrak{t}_i + 1)m_i \;\text{ for any } i \,\big\}.$$

For simplicity, we also write $R_{\mathfrak{t}} := R_{\mathfrak{t}, \dots, \mathfrak{t}} = R(\mathfrak{t}\,\mathbf{m}, (\mathfrak{t}+1)\,\mathbf{m})$.

Clearly, $R(0, \mathfrak{t}_0\,\mathbf{m}) = \cup R_{\mathfrak{t}_1, \dots, \mathfrak{t}_\nu}$, where the union is over $0 \leq \mathfrak{t}_i < \mathfrak{t}_0$ for every i.

On the other hand, we also consider the following subsets, indexed by $\mathfrak{t} \in \mathbb{Z}_{\geq 0}$:

$$B_{\mathfrak{t}} := \Big\{\, x \in (\mathbb{R}_{\geq 0})^\nu \;:\; \left\lceil \frac{\Sigma x - a}{\Sigma m + d} \right\rceil = \mathfrak{t} \Big\},$$

$$T_{\mathfrak{t}}^{+} := \Big\{ x \in (\mathbb{R}_{\geq 0})^{\nu} \ : \ \frac{\Sigma x - a}{\Sigma m + d} = \mathfrak{t} \Big\},$$

$$T_{\mathfrak{t}}^{-} := \Big\{ x \in (\mathbb{R}_{\geq 0})^{\nu} \ : \ \frac{\Sigma x - a - 1}{\Sigma m + d} = \mathfrak{t} - 1 \Big\}.$$

Obviously, $T_{\mathfrak{t}}^{-} \cup T_{\mathfrak{t}}^{+} \subset B_{\mathfrak{t}}$ are the two limiting planes of the 'latticial' $\mathfrak{t}$-stripe $B_{\mathfrak{t}} \cap \overline{L}$ (with $T_0^{-} = \emptyset$). For any $\bar{l} \in B_{\mathfrak{t}} \cap \overline{L}$ the coefficient $m_+(\bar{l})$ takes the constant value $\mathfrak{t}$.

The lattice points $\mathfrak{t}\mathbf{m}$ depend only on the local data $\{\mathscr{S}_i\}_i$, but the width of each $B_{\mathfrak{t}}$ depends on d as well: it increases when d increases.

A direct computation shows that for any $\mathfrak{t} \geq 0$ one has:

$$\begin{aligned} &(a) \ \ \mathfrak{t}\mathbf{m} \in \cup_{\mathfrak{t}' \leq \mathfrak{t}} B_{\mathfrak{t}'}; \\ &(b) \ \ \text{if } \ (\mathfrak{t}+1)\mathbf{m} \in \cup_{\mathfrak{t}' \leq \mathfrak{t}} B_{\mathfrak{t}'}, \ \text{ then } \mathfrak{t} \geq \mathfrak{t}_0; \\ &(c) \ \ \text{if } \mathfrak{t}\mathbf{m} \in T_{\mathfrak{t}}^{-} \text{ then } \mathfrak{t} > \mathfrak{t}_0. \end{aligned} \tag{11.5.12}$$

In particular, if $0 \leq \mathfrak{t} \leq \mathfrak{t}_0$, then $\mathfrak{t}\mathbf{m} \in B_{\mathfrak{t}} \setminus T_{\mathfrak{t}}^{-}$; and $\mathfrak{t}\mathbf{m} \in T_{\mathfrak{t}}^{+}$ if and only if $a = \mathfrak{t} = 0$. Moreover, for $0 \leq \mathfrak{t}, \mathfrak{t}' \leq \mathfrak{t}_0$ one has $T_{\mathfrak{t}}^{+} \cap R_{\mathfrak{t}'} \neq \emptyset$ if and only if $\mathfrak{t} = \mathfrak{t}'$.

Lemma 11.5.5 *Consider a small rectangle $R = R_{\mathfrak{t}_1,\dots,\mathfrak{t}_\nu}$ such that $\mathfrak{t} := \max_i\{\mathfrak{t}_i\} < \mathfrak{t}_0$. If $\bar{l} \in R \cap \overline{L}$ then $\bar{l} \leq (\mathfrak{t}+1)\mathbf{m}$, hence $R \cap \overline{L} \subset \cup_{\mathfrak{t}' \leq \mathfrak{t}+1} B_{\mathfrak{t}'}$. Furthermore,*

(i) if $R \cap B_{\mathfrak{t}+1} \cap \overline{L} \neq \emptyset$, then $R \cap \overline{L} \subset B_{\mathfrak{t}} \cup B_{\mathfrak{t}+1}$;
(ii) if $R \cap B_{\mathfrak{t}+1} \cap \overline{L} = \emptyset$, then $R \cap \overline{L} \subset \cup_{\mathfrak{t}' \leq \mathfrak{t}} B_{\mathfrak{t}'}$.

Proof Use (11.5.12) and note that the difference between the two $\Sigma\ell$-values of the two opposite corners of R is Σm, which is less than $\Sigma m + d$, showing *(i)*. □

The next lemma shows that in the right hand side of 11.5.4*(b)* all the small rectangles can be omitted except those of type $R_{\mathfrak{t}}$.

Lemma 11.5.6 $\mathbb{H}^*\big(R(0, \mathfrak{t}_0\,\mathbf{m}), \overline{w}\big) \cong \mathbb{H}^*\big(\cup_{0 \leq \mathfrak{t} < \mathfrak{t}_0} R_{\mathfrak{t}}, \overline{w}\big)$.

Proof We proceed by induction. Assume that we already reduced the rectangle $R(0, \mathfrak{t}_0\mathbf{m})$ by suitable deformation retractions to $R(0, (\mathfrak{t}+1)\,\mathbf{m}) \cup (\cup_{\mathfrak{t} < \mathfrak{t}' < \mathfrak{t}_0} R_{\mathfrak{t}'})$. Next, we wish to eliminate all the small rectangles not contained in $R(0, \mathfrak{t}\,\mathbf{m})$, and not equal to $R_{\mathfrak{t}}$. Consider such a small rectangle R. Then $\max_i\{\mathfrak{t}_i\} = \mathfrak{t}$, and $\min_i\{\mathfrak{t}_i\} < \mathfrak{t}$. We consider the two cases of 11.5.5.

In the case of *(i)*, $\lceil (\Sigma\ell - a)/(\Sigma m + d) \rceil \geq \mathfrak{t}$ for all $\bar{l} \in R$, hence for any $\bar{l} \in R \cap \overline{L}$ and any index j with $\mathfrak{t}_j < \mathfrak{t}$ one has $\overline{w}(\bar{l} + 1_j) \leq \overline{w}(\bar{l})$. Hence R can be contracted in the increasing direction of the j-coordinate similarly as in the proof of 11.5.4*(b)*.

In the case of *(ii)*, $\lceil (\Sigma\ell - a)/(\Sigma m + d) \rceil \leq \mathfrak{t}$ for all $\bar{l} \in R$, hence for any $\bar{l} \in R \cap \overline{L}$ and any index j with $\mathfrak{t}_j = \mathfrak{t}$ one has $\overline{w}(\bar{l} + 1_j) \geq \overline{w}(\bar{l})$. Hence R can be contracted in the decreasing direction of the j-coordinate similarly as above.

The order how we eliminate the small rectangles is the following. In the first series we eliminate those which satisfy *(ii)* (i.e. $R \cap B_{\mathfrak{t}+1} \cap \overline{L} = \emptyset$) in the following

order: first those with $\mathfrak{t}_1 = \mathfrak{t}$ (in decreasing direction of the first coordinate), then those from the remaining ones with $\mathfrak{t}_2 = \mathfrak{t}$, and continuing in this way, finally all those remaining rectangles satisfying $R \cap B_{\mathfrak{t}+1} \cap \overline{L} = \emptyset$ with $\mathfrak{t}_\nu = \mathfrak{t}$.

After this series of contractions, all rectangles which still have to be contracted satisfy *(i)*, hence any coordinate j with $\mathfrak{t}_j < \mathfrak{t}$ is a good candidate for a contraction in increasing direction. Then we contract in increasing direction of 1_1 all the rectangles with $\mathfrak{t}_1 < \mathfrak{t}$, then all the remaining ones with $\mathfrak{t}_2 < \mathfrak{t}$ in direction 1_2, and we continue in this way, at the end we contract all those remaining with $\mathfrak{t}_\nu < \mathfrak{t}$ in direction 1_ν. □

11.5.7 The Lattice Cohomology of $S^3_{-d}(\#_i K_i)$ via the Semigroups $\{\mathscr{S}_i\}_i$

Next we determine the weights of lattice points sitting in $\cup_{0\le\mathfrak{t}<\mathfrak{t}_0} R_\mathfrak{t}$ in terms of the semigroups $\{\mathscr{S}_i\}_i$. As a consequence, we get a description of the lattice cohomology of $S^3_{-d}(\#_i K_i)$ in terms of these semigroups.

Lemma 11.5.8 $\overline{w}(\mathfrak{t}\,\mathbf{m}) = \mathfrak{t}(1 + a - \delta) + d\mathfrak{t}(\mathfrak{t}-1)/2$ *for any* $0 \le \mathfrak{t} \le \mathfrak{t}_0$. *Hence, these values memorize only the integer* δ *from the collection of semigroups* $\{\mathscr{S}_i\}_i$.

Proof Since $y_i(t_i m_i) = t_i Z_i$ (cf. 11.5.3), (11.5.10) gives $\mathfrak{t}\,\mathbf{m} = \mathfrak{t}\sum_i Z_i + m_+ E_+$. By (11.5.12) $m_+ = \mathfrak{t}$. Then use $\chi(Z_i) = m_i - \delta_i$ (cf. 6.3.9) and $Z_i^2 = -m_i$. □

(11.5.12), Lemma 11.5.5 and the identity (11.5.11) give the following.

Lemma 11.5.9 *Fix* $0 \le \mathfrak{t} < \mathfrak{t}_0$.

(a) Assume that $\bar{l} \in R_\mathfrak{t} \cap B_\mathfrak{t} \cap \overline{L}$. *Then for every* i *one has*

$$\overline{w}(\bar{l} + 1_i) - \overline{w}(\bar{l}) = \begin{cases} 0 & \text{if } \ell_{i,0} \notin \mathscr{S}_i \\ 1 & \text{if } \ell_{i,0} \in \mathscr{S}_i. \end{cases}$$

Therefore, for any $\bar{l} \in R_\mathfrak{t} \cap \overline{L}$, $\ell_i = \mathfrak{t}m_i + \ell_{i,0}$, *with* $\Sigma\ell \le \mathfrak{t}(\Sigma m + d) + a + 1$ *(in particular, for any* $\bar{l} \in T^-_{\mathfrak{t}+1} \cap R_\mathfrak{t} \cap \overline{L}$ *too), one has:*

$$\overline{w}(\bar{l}) - \overline{w}(\mathfrak{t}\,\mathbf{m}) = \sum_i \#\{s \in \mathscr{S}_i \,:\, s \le \ell_{i,0} - 1\}. \tag{11.5.13}$$

(b) Assume that $\bar{l} \in R_\mathfrak{t} \cap B_{\mathfrak{t}+1} \cap \overline{L}$. *Then for every* i *with* $\bar{l} + 1_i \in B_{\mathfrak{t}+1}$ *one has*

$$\overline{w}(\bar{l} + 1_i) - \overline{w}(\bar{l}) = -\begin{cases} 1 & \text{if } \ell_{i,0} \notin \mathscr{S}_i \\ 0 & \text{if } \ell_{i,0} \in \mathscr{S}_i. \end{cases}$$

Therefore, for any $\bar{l} \in R_\mathfrak{t} \cap \overline{L}$, $\ell_i = \mathfrak{t}m_i + \ell_{i,0}$, *with* $\Sigma\ell \ge \mathfrak{t}(\Sigma m + d) + a + 1$ *(in particular, for any* $\bar{l} \in T^-_{\mathfrak{t}+1} \cap R_\mathfrak{t} \cap \overline{L}$ *too), one has:*

$$\overline{w}(\bar{l}) - \overline{w}((\mathfrak{t}+1)\,\mathbf{m}) = \sum_i \#\{s \notin \mathscr{S}_i \,:\, s \ge \ell_{i,0}\}. \tag{11.5.14}$$

All the above statements summarized provide the following facts. In this final version, we decrease even the bound $\mathfrak{t}_0$ (notice that $\Sigma m - \min -\nu \geq \mu - 1$).

Theorem 11.5.10 (Final Reduction Theorem) *Set* $\tilde{\mathfrak{t}}_0 := \lceil (\mu - a - 1)/d \rceil$, *and for any* $0 \leq \mathfrak{t} < \tilde{\mathfrak{t}}_0$ *introduce* $\min T^-_{\mathfrak{t}+1} := \min\{\, \overline{w}(\bar{l}) \,:\, \bar{l} \in T^-_{\mathfrak{t}+1} \cap R_{\mathfrak{t}} \cap \overline{L}\}$. *Then the following facts hold:*

(a) $\overline{w}(\mathfrak{t}\,\mathbf{m}) \leq \min T^-_{\mathfrak{t}+1}$, $\overline{w}((\mathfrak{t}+1)\,\mathbf{m}) \leq \min T^-_{\mathfrak{t}+1}$.
(b) $m_{k_r} := \min \chi_{k_r} = \min_{0 \leq \mathfrak{t} \leq \tilde{\mathfrak{t}}_0}\{\, \overline{w}(\mathfrak{t}\,\mathbf{m}) \,\}$.
(c) Let $\mathfrak{t}_{\min}$ *be the smallest integer satisfying* $\overline{w}(\mathfrak{t}_{\min}\mathbf{m}) = m_{k_r}$. *Then*

$$\mathbb{H}^0_{red}(\Gamma, k_r) = \bigoplus_{0 \leq \mathfrak{t} < \mathfrak{t}_{\min}} \mathscr{T}_{2\overline{w}(\mathfrak{t}\,\mathbf{m})}\big(\min T^-_{\mathfrak{t}+1} - \overline{w}(\mathfrak{t}\,\mathbf{m})\big) \oplus \bigoplus_{\mathfrak{t}_{\min} \leq \mathfrak{t} < \tilde{\mathfrak{t}}_0} \mathscr{T}_{2\overline{w}((\mathfrak{t}+1)\,\mathbf{m})}\big(\min T^-_{\mathfrak{t}+1} - \overline{w}((\mathfrak{t}+1)\,\mathbf{m})\big).$$

(d) $\mathit{rank}_{\mathbb{Z}}\mathbb{H}^0_{red}(\Gamma, k_r)$ *equals*

$$\sum_{0 \leq \mathfrak{t} < \mathfrak{t}_{\min}} \big(\min T^-_{\mathfrak{t}+1} - \overline{w}(\mathfrak{t}\,\mathbf{m})\big) + \sum_{\mathfrak{t}_{\min} \leq \mathfrak{t} < \tilde{\mathfrak{t}}_0} \big(\min T^-_{\mathfrak{t}+1} - \overline{w}((\mathfrak{t}+1)\,\mathbf{m})\big),$$

or

$$-m_{k_r} + \mathit{rank}_{\mathbb{Z}}\mathbb{H}^0_{red}(\Gamma, k_r) = \sum_{0 \leq \mathfrak{t} < \tilde{\mathfrak{t}}_0} \big(\min T^-_{\mathfrak{t}+1} - \overline{w}((\mathfrak{t}+1)\,\mathbf{m})\big).$$

(e) For any $q > 0$ *one has*

$$\mathbb{H}^q(\Gamma, k_r) = \bigoplus_{0 \leq \mathfrak{t} < \tilde{\mathfrak{t}}_0} \mathbb{H}^q(R_{\mathfrak{t}}, \overline{w}).$$

Proof First we prove all the statements for $\mathfrak{t}_0$ instead of $\tilde{\mathfrak{t}}_0$. Parts *(a)–(b)* follow from 11.5.9, *(c)* from 11.2.6, *(d)* is based on *(c)*, while *(e)* from a version of 11.1.12 applied for the union of cubes and a Mayer–Vietoris argument.

Next, we prove that all the small rectangles $R_{\mathfrak{t}}$ for $\mathfrak{t} \geq \tilde{\mathfrak{t}}_0$ can also be eliminated. We prove this by increasing induction: we verify that for any $\mathfrak{t}'_0 > \tilde{\mathfrak{t}}_0$ one has

$$\bigoplus_{0 \leq \mathfrak{t} < \mathfrak{t}'_0} \mathbb{H}^q(R_{\mathfrak{t}}, \overline{w}) = \bigoplus_{0 \leq \mathfrak{t} < \mathfrak{t}'_0 - 1} \mathbb{H}^q(R_{\mathfrak{t}}, \overline{w}). \tag{11.5.15}$$

that is, the last small rectangle can be contracted into $(\mathfrak{t}'_0 - 1)\mathbf{m}$ compatibly with the weights. The first step is the verification of the following equivalences: $\mathfrak{t} \geq \tilde{\mathfrak{t}}_0$ if and only if $\sum_i \ell_{i,0} \geq \mu$ for all $\bar{l} \in T^-_{\mathfrak{t}+1} \cap R_{\mathfrak{t}} \cap \overline{L}$, if and only if there is an $\bar{l} \in T^-_{\mathfrak{t}+1} \cap R_{\mathfrak{t}} \cap \overline{L}$ with $\ell_{i,0} \geq \mu_i$ for all i. Since the largest element of $\mathbb{Z} \setminus \mathscr{S}_i$ is $\mu_i - 1$, the last statement (via (11.5.14)) is equivalent to $\min T^-_{\mathfrak{t}+1} = \overline{w}((\mathfrak{t}+1)\mathbf{m})$. Hence, in (c), proved for the bound $\mathfrak{t}_0$, all the contributions in $\mathbb{H}^0_{red}(\Gamma, k_r)$ corresponding to $\mathfrak{t} \geq \tilde{\mathfrak{t}}_0$ are zero.

For $q > 0$ we need a better argument, a sequence of contraction (which reproves the $q = 0$ case too). For any $0 \leq \ell_{i,0} \leq m_i$ set the rectangle

$$R_{\mathfrak{t}}(\ell_{1,0}, \dots, \ell_{\nu,0}) := \{x \in \mathbb{R}^\nu \,:\, \mathfrak{t}m_i \leq x_i \leq \mathfrak{t}m_i + \ell_{i,0} \text{ for all } i\,\}.$$

Fix some $\bar{l}^\circ \in T^-_{\mathfrak{t}+1} \cap R_{\mathfrak{t}} \cap \overline{L}$ with $\ell^\circ_{i,0} \geq \mu_i$ for all i. Then we show that $R_{\mathfrak{t}}$ contracts onto $R_{\mathfrak{t}}(\ell^\circ_{1,0}, \dots, \ell^\circ_{\nu,0})$. Assume that $R_{\mathfrak{t}}$ is already contracted onto $R_{\mathfrak{t}}(\bar{\ell}_{1,0}, \dots, \bar{\ell}_{\nu,0})$ with $\bar{\ell}_{i,0} \geq \ell^\circ_{i,0}$ for all i and $\bar{\ell}_{j,0} > \ell^\circ_{j,0}$ for some j. Fix such an index j. We claim that for any $\bar{l} \in R_{\mathfrak{t}}(\bar{\ell}_{1,0}, \dots, \bar{\ell}_{\nu,0}) \setminus R_{\mathfrak{t}}(\bar{\ell}_{1,0}, \dots, \bar{\ell}_{j,0} - 1, \dots \bar{\ell}_{\nu,0})$ the inequality $\overline{w}(\bar{l}) \geq \overline{w}(\bar{l} - 1_j)$ holds. [Indeed, if $\Sigma\ell > \mathfrak{t}(\Sigma m + d) + a + 1$ then $\overline{w}(\bar{l}) = \overline{w}(\bar{l} - 1_j)$ from 11.5.9*(b)*. If $\Sigma\ell \leq \mathfrak{t}(\Sigma m + d) + a + 1$ then $\overline{w}(\bar{l}) = \overline{w}(\bar{l} - 1_j) + 1$ from 11.5.9*(a)*.] This defines a contraction from $R_{\mathfrak{t}}(\bar{\ell}_{1,0}, \dots, \bar{\ell}_{\nu,0})$ to $R_{\mathfrak{t}}(\bar{\ell}_{1,0}, \dots, \bar{\ell}_{j,0} - 1, \dots \bar{\ell}_{\nu,0})$.

Finally $R_{\mathfrak{t}}(\ell^\circ_{1,0}, \dots, \ell^\circ_{\nu,0})$ contracts onto the point $\mathfrak{t}\mathbf{m}$ using the same type of contraction (but now in an arbitrary order of the coordinates) via 11.5.9*(a)* (repeating the case $\Sigma\ell \leq \mathfrak{t}(\Sigma m + d) + a + 1$ and the proof of Lemma 11.5.4*(b)*.) □

Remark 11.5.11 The bound $\tilde{\mathfrak{t}}_0$ in Theorem 11.5.10 is optimal. This can be seen already at the $\mathbb{H}^0$-level (see the proof of the 'final reduction' for $q = 0$).

11.5.12 The Structure of $\mathbb{H}^{\geq 1}(R_{\mathfrak{t}}, \overline{w})$ The cohomology $\mathbb{H}^{\geq 1}(R_{\mathfrak{t}}, \overline{w})$ depends only on the $\overline{w}$-values on $\mathfrak{t}\mathbf{m}$, on $(\mathfrak{t}+1)\mathbf{m}$ and along $T^-_{\mathfrak{t}+1}$.

Indeed, for any $n \in \mathbb{Z}$ consider S_n as in 7.3.2. Then for $n < \min T^-_{\mathfrak{t}+1}$ the space $S_n \cap R_{\mathfrak{t}}$ has the same homotopy type as $S_n \cap \{\mathfrak{t}\mathbf{m}, (\mathfrak{t}+1)\mathbf{m}\}$; while for $n \geq \min T^-_{\mathfrak{t}+1}$ it has the homotopy type of the suspension of $S_n \cap T^-_{\mathfrak{t}+1}$. In particular, all the nontrivial homogeneous elements of $\mathbb{H}^{\geq 1}(R_{\mathfrak{t}}, \overline{w})$ have degree $\geq \min T^-_{\mathfrak{t}+1}$.

In fact, one can define on each $T^-_{\mathfrak{t}}$ a 'generalized lattice structure' and the corresponding lattice cohomology. This can be done in two different ways.

The first one is via the $\mathbb{S}^*$-representation: one defines

$$\mathbb{H}^*(T^-_{\mathfrak{t}}, \overline{w}) := \oplus_{n \geq \min T^-_{\mathfrak{t}}} H^*(S_n \cap T^-_{\mathfrak{t}}, \mathbb{Z}). \tag{11.5.16}$$

Equivalently, a lattice decomposition of $T^-_{\mathfrak{t}}$ (or, just of $T^-_{\mathfrak{t}} \cap R_{\mathfrak{t}-1}$) can be defined as follows: the 'generalized q-cubes' of $T^-_{\mathfrak{t}}$ are the intersections of the usual $\overline{L}$-latticial

$(q+1)$-cubes with $T_{\mathfrak{t}}^-$. Lemma 11.5.9 shows that if $\square_{q+1} \cap T_{\mathfrak{t}}^- \neq \emptyset$, then

$$\max\{w(v) \,:\, \text{vertex } v \text{ of } \square_{q+1}\} = \max\{w(v) \,:\, \text{vertex } v \text{ of } \square_{q+1} \cap T_{\mathfrak{t}}^-\}.$$

Therefore, the right hand side of this identity defines an intrinsic set of weight functions on the generalized cubes of $T_{\mathfrak{t}}^-$, hence a cochain complex determining the corresponding lattice cohomology as well.

With this 'generalized lattice structure' of $T_{\mathfrak{t}}^-$, one has the graded $\mathbb{Z}[U]$-module isomorphism

$$\mathbb{H}^q(R_{\mathfrak{t}}, \overline{w}) = \mathbb{H}^{q-1}_{red}(T^-_{\mathfrak{t}+1} \cap R_{\mathfrak{t}},\ \overline{w}) \quad \text{for } q > 0. \tag{11.5.17}$$

11.5.13 The Structure of $\mathbb{H}^*(T^-_{\mathfrak{t}+1} \cap R_{\mathfrak{t}}\,, \overline{w})$. The Modules $\mathbb{H}^*(\mathbb{T}_n^-, \overline{W})$ In most of the notations above, we have omitted the symbol a codifying the characteristic element k_r. In fact, for any $\mathfrak{t} \geq 0$ and $a \in \{0, \dots, d-1\}$, $T^-_{\mathfrak{t}+1} \cap R_{\mathfrak{t}}$ is

$$T^-_{\mathfrak{t}+1,a} := \{\bar{l} \,:\, \ell_i = \mathfrak{t}m_i + \ell_{i,0};\ \sum_i \ell_{i,0} = \mathfrak{t}d + a + 1\}.$$

Note that when $\mathfrak{t}$ runs over $\mathbb{Z}_{\geq 0}$ and $a \in \{0, \dots, d-1\}$, the integer $n = \mathfrak{t}d + a$ runs over $\mathbb{Z}_{\geq 0}$. This motivates to consider for any $n \in \mathbb{Z}_{\geq 0}$

$$\mathbb{T}_n := \{(\ell_{1,0}, \dots, \ell_{v,0}) \in [0, m_1] \times \cdots \times [0, m_v] \,:\, \sum_i \ell_{i,0} = n+1\}. \tag{11.5.18}$$

Then, for d and a fixed, $T^-_{\mathfrak{t}+1,a} = \mathbb{T}_{\mathfrak{t}d+a} + \mathfrak{t}\,\mathbf{m}$. If $\mathfrak{t} < \tilde{\mathfrak{t}}_0$ then $\mathfrak{t}d + a \leq \mu - 2$, hence the relevant index set of the $\mathbb{T}_n$'s is $0 \leq n \leq \mu - 2$ (this can be compared with the index set $\{\alpha_n\}_{n=0}^{\mu-2}$ of the coefficients of $Q(t)$). The expression $\mathbb{T}_{\mathfrak{t}d+a} + \mathfrak{t}\,\mathbf{m}$ shows also how they intersect the small rectangles: when we run a, an element of the set $\{\mathbb{T}_n + \lfloor n/d \rfloor \mathbf{m}\}_{0 \leq n \leq \mu-2}$ intersects $R_{\mathfrak{t}}$ if and only if $\lfloor n/d \rfloor = \mathfrak{t}$.

Up to the shift $\overline{w}(\mathfrak{t}\,\mathbf{m})$ (cf. 11.5.8), which is constant on each $\mathbb{T}_n$, but otherwise depends on $\mathfrak{t} = \lfloor n/d \rfloor$, the weights on $\mathbb{T}_n \cap \mathbb{Z}^v$ are given by the right hand side of (11.5.13). Or, up to a shift $\overline{w}((\mathfrak{t}+1)\,\mathbf{m})$, the weights are given by (11.5.14). Following this second version we set the following weights for any $\mathbb{T}_n$:

$$\overline{W}((\ell_{1,0}, \dots, \ell_{v,0})) = \sum_i \#\{s \notin \mathscr{S}_i \,:\, s \geq \ell_{i,0}\}. \tag{11.5.19}$$

That is, $\overline{W}|_{\mathbb{T}_n}(\bar{l} - \mathfrak{t}\,\mathbf{m}) = \overline{w}(\bar{l}) - \overline{w}((\mathfrak{t}+1)\,\mathbf{m})$, where $\mathfrak{t} = \lfloor n/d \rfloor$.

The weight function $\overline{W}$ restricted on all the level sets $\{\mathbb{T}_n\}_{n \geq 0}$ of $(\mathbb{Z}_{\geq 0})^v$ measures the very subtle distribution properties of the semigroups $\{\mathscr{S}_i\}_i$. Furthermore, up to a well-identified shift in degrees, the collection $(\mathbb{T}_n, \overline{W})$ provides all the lattice

cohomologies $\mathbb{H}^*(\Gamma(d), k_r)$ for all the possible values d and a. Here, and in the next discussion, we denote the dependence of Γ on d by $\Gamma(d)$.

More precisely, for any d and $a \in \{0, \dots, d-1\}$ and $q > 0$ one has:

$$\mathbb{H}^q(\Gamma(d), K + 2aE_+^*) = \bigoplus_{n \equiv a \,(\mathrm{mod}\, d),\ 0 \le n \le \mu - 2} \mathbb{H}^{q-1}_{red}(\mathbb{T}_n, \overline{W})[s_{n,d}], \qquad (11.5.20)$$

where $s_{n,d}$ is the value of the shift $2\overline{w}((\mathfrak{t}+1)\,\mathbf{m}) = 2(\mathfrak{t}+1)(1+a-\delta) + d(\mathfrak{t}+1)\mathfrak{t}$ with $\mathfrak{t} = \lfloor n/d \rfloor$. Moreover, the values $\{\min \overline{W}\,|\mathbb{T}_n\}_n$ and $s_{n,d}$ determine all the cohomology groups $\mathbb{H}^0(\Gamma(d), k_r)$ too, cf. 11.5.10*(c)*.

The second identity of 11.5.10*(d)* together with (11.5.14) reads as:

$$-m_{k_r} + \mathrm{rank}\ \mathbb{H}^0_{red}(\Gamma(d), K + 2aE_+^*) = \sum_{n \equiv a \,(\mathrm{mod}\, d),\ 0 \le n \le \mu - 2} \min\{\overline{W}|_{\mathbb{T}_n}\}. \qquad (11.5.21)$$

In particular, for any fixed $d > 0$ and $a \in \{0, \dots, d-1\}$ one has:

$$\begin{aligned} eu(\mathbb{H}^0(\Gamma(d), K + 2aE_+^*)) &= \sum_{n \equiv a \,(\mathrm{mod}\, d),\ 0 \le n \le \mu - 2} \min\{\overline{W}|_{\mathbb{T}_n}\}, \\ eu(\mathbb{H}^*(\Gamma(d), K + 2aE_+^*)) &= \sum_{n \equiv a \,(\mathrm{mod}\, d),\ 0 \le n \le \mu - 2} -eu(\mathbb{H}^*(\mathbb{T}_n, \overline{W})). \end{aligned} \qquad (11.5.22)$$

Example 11.5.14 For any $d > 0$ and $q > 0$ the summation of (11.5.20) over a gives

$$\mathbb{H}^q(\Gamma(d)) = \bigoplus_{a=0}^{d-1} \mathbb{H}^q(\Gamma(d), K + 2aE_+^*) = \bigoplus_{0 \le n \le \mu - 2} \mathbb{H}^{q-1}_{red}(\mathbb{T}_n, \overline{W})[s_{n,d}]. \qquad (11.5.23)$$

On the right hand side of (11.5.23) the numbers $s_{n,d}$ depend on d, but the rank of the right hand side is independent of d. In particular, up to shifts of different direct sum blocks, $\oplus_{q>0}\mathbb{H}^q(\Gamma(d))$ is independent of the choice of the integer d. (This can also be deduced from the surgery exact sequences from [483].)

Example 11.5.15

(a) Assume that for a certain d and a one gets $\tilde{\mathfrak{t}}_0 = 0$. Then $\mathbb{H}^*_{red}(\Gamma, k_r) = 0$, and $\mathbb{H}^0(\Gamma, k_r) = \mathscr{T}_0^+$.

(b) Assume that for a certain d and a one gets $\tilde{\mathfrak{t}}_0 = 1$. Then $\mathbb{H}^*(\Gamma, k_r) = \mathbb{H}^*(R_0, \overline{w})$, hence everything is determined by $T^-_{1,a}$. Indeed, from 11.5.8 and 11.5.9,

$$\min T^-_{1,a} = \min\Big\{ \sum_i \#\{s \in \mathscr{S}_i \,:\, s \le \ell_i - 1\},\ \text{where} \sum_i \ell_i = a+1 \Big\}$$
$$= \min\Big\{ \sum_i \#\{s \notin \mathscr{S}_i \,:\, s \ge \ell_i\},\ \text{where} \sum_i \ell_i = a+1 \Big\} + 1 + a - \delta,$$

$m_{k_r} = \min\{0, 1+a-\delta\}$, $\mathbb{H}^0_{red}(\Gamma, k_r)$ is generated by one element of degree $2\max\{0, 1+a-\delta\}$, $\operatorname{rank} \mathbb{H}^0_{red}(\Gamma, k_r) = \min T^-_{1,a} - \max\{0, 1+a-\delta\}$, and finally for $q > 0$ one has $\mathbb{H}^q(\Gamma, k_r) = \mathbb{H}^{q-1}_{red}(T^-_{1,a}, \overline{w}) = \mathbb{H}^{q-1}_{red}(\mathbb{T}_a, \overline{W})[2(1+a-\delta)]$, $(T^-_{1,a} = \mathbb{T}_a + \mathbf{m})$.

(c) If $d \ge \mu - 1$ then $\tilde{\mathfrak{t}}_0 = 1$ for $a < \mu - 1$, and $\tilde{\mathfrak{t}}_0 = 0$ for $a \ge \mu - 1$.

Remark 11.5.16 Assume that we know all the cohomology groups $\{\mathbb{H}^*(\Gamma(d), k_r)\}_{k_r}$ for some specific d with $d \ge \mu - 1$. Then using them, and also the values $\overline{w}(\mathbf{tm}) = \mathfrak{t}(1+a-\delta) + d\mathfrak{t}(\mathfrak{t}-1)/2$ for all $\mathfrak{t}, a$ and d, we can recover all the lattice cohomologies $\{\mathbb{H}^*(\Gamma(d), k_r)\}_{k_r}$ for any $d > 0$. [Use 11.5.13 and 11.5.15.]

Corollary 11.5.17 *For any $n \ge 0$ the coefficients of $Q(t) = \sum_n \alpha_n t^n$ satisfy*

$$\alpha_n = -eu\,(\mathbb{H}^*(\mathbb{T}_n, \overline{W})). \tag{11.5.24}$$

Proof Use the identities (11.5.5) and (11.5.22) for $d \gg 0$, cf. 11.5.16. □

This identity here is proved via properties of the Seiberg–Witten invariants of the plumbed 3-manifold. However, both sides of the identity depend only on the semigroups $\{\mathscr{S}_i\}_i$, and there is a direct combinatorial way to prove the identity, cf. [497, §7]. In that version even the semigroup properties are dropped: each $\mathscr{S}_i$ is merely a subset in $\mathbb{Z}_{\ge 0}$ with a finite complement.

Corollary 11.5.18 *The following objects are independent of the choice of $d > 0$:*

(a) $\sum_a eu\,(\mathbb{H}^*(\Gamma(d), k_r)) = \sum_{n \ge 0} \alpha_n$,
(b) $\sum_a eu\,(\mathbb{H}^0(\Gamma(d), k_r))$,
(c) $\sum_a m_{k_r} = -\delta(\delta-1)/2$,
(d) $\sum_a \operatorname{rank}\,(\mathbb{H}^q_{red}(\Gamma(d), k_r))$ *for any fixed $q \ge 0$,*
(e) $\oplus_a \mathbb{H}^q_{red}(\Gamma(d), k_r)$ *for any fixed $q \ge 0$, up to different shifts of different graded $\mathbb{Z}[U]$-module summands.*

Proof *(a)* follows from (11.5.5) and (11.5.22), cf. 11.5.17. Part *(b)* follows from *(a)* and (11.5.20). Part *(c)* can be proved as follows. We have to verify that

$$\sum_a \min_{t\in\mathbb{Z}_{\geq 0}} \{t(1+a-\delta)+dt(t-1)/2\} = -\delta(\delta-1)/2.$$

First we verify that for each a the minimum is realized for $t_a = \lceil(\delta-a-1)/d\rceil$, then the summation is elementary. *(d)* and *(e)* for $q > 0$ follows from (11.5.20). *(d)* for $q = 0$ follows from *(b)* and *(c)*. Finally, *(e)* can be proved by a direct combinatorial argument using the description from Theorem 11.5.10. □

The statements of the above Corollary 11.5.18, except for part (c), can be reproved and reinterpreted by the surgery exact sequence [231, 483] as well.

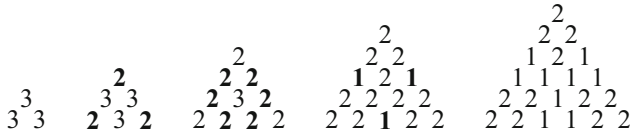
Example 11.5.19 Assume that $\nu = 2$ and both cusps are $(2, 3)$-cusps, hence $m_i = 6$ and $\mu_i = 2$. Then $\mathbb{T}_n = \{(i, j) \in R((0, 0), (6, 6)) : i+j = n+1\}, 0 \leq n \leq \mu-2 = 2$, and the $\overline{W}$-weights on the lattice points are 2,2 for $\mathbb{T}_0$, 1,2,1 for $\mathbb{T}_1$ and 1,1,1,1 for $\mathbb{T}_2$. In particular $-eu(\mathbb{H}^*(\mathbb{T}_n, \overline{W}))$ equals 2,0,1 for $n = 0, 1, 2$, a fact which is compatible via (11.5.24) with the coefficients of $Q(t) = 2 + t^2$.

The lattice cohomology of $\Gamma(d)$ for different d and a values are

		$d = 1$	$d = 2$	$d \geq 3$
$a = 0$	$\mathbb{H}^0$	$\mathscr{T}^+_{-2} \oplus \mathscr{T}_{-2}(1) \oplus \mathscr{T}_0(1)^2$	$\mathscr{T}^+_{-2} \oplus \mathscr{T}_0(1)^2$	$\mathscr{T}^+_{-2} \oplus \mathscr{T}_0(1)$
	$\mathbb{H}^1$	$\mathscr{T}_0(1)$	0	0
$a = 1$	$\mathbb{H}^0$		$\mathscr{T}^+_0 \oplus \mathscr{T}_0(1)$	$\mathscr{T}^+_0 \oplus \mathscr{T}_0(1)$
	$\mathbb{H}^1$		$\mathscr{T}_2(1)$	$\mathscr{T}_2(1)$
$a = 2$	$\mathbb{H}^0$			$\mathscr{T}^+_0 \oplus \mathscr{T}_2(1)$
	$\mathbb{H}^1$			0
$a \geq 3$	$\mathbb{H}^*$			$\mathscr{T}^+_0$

The reader is invited to verify the statements of Corollary 11.5.18 on this example.

Example 11.5.20 Assume that $\nu = 3$ and the cusps are $(2, 3)$-cusps. The next picture shows the weights of $\mathbb{T}_n$ for $0 \leq n \leq 4$.



Again, one can compare $-eu(\mathbb{H}^*(\mathbb{T}_n, \overline{W}))$ with the coefficients of $Q(t) = 3 + 3t^2 - t^3 + t^4$. Here we provide the lattice cohomology of $\Gamma(d)$ for $d = 1$: $\mathbb{H}^0 = \mathscr{T}^+_{-6} \oplus \mathscr{T}_{-6}(2) \oplus \mathscr{T}_{-4}(1)^2 \oplus \mathscr{T}_0(1)^2$, $\mathbb{H}^1 = \mathscr{T}_{-2}(1)^4$, $\mathbb{H}^2 = \mathscr{T}_{-2}(1)$. The reader is invited to determine the corresponding modules for any other d and a.

One can verify that for $d = 3$ and $a = 1$ the space S_1 is not connected (recall that S_1 is connected for for $h = 0$, cf. 7.3.13). Note also that the graph $\Gamma(d)$ is numerically Gorenstein if and only if $d \in \{1, 2, 4\}$. For $d = 3$ even for $a = 0$ we get an interesting non-symmetric graded root.

In both examples 11.5.19 and 11.5.20 one can see that $\mathbb{H}^{\nu-1} \neq 0$ (for any d and for some a). This happens in general, see the next statement.

Proposition 11.5.21 *Consider the lattice cohomology of $S^3_{-d}(\#_{i=1}^{\nu} K_i)$ of an arbitrary surgery manifold. Then $\mathbb{H}^{\geq \nu} = 0$ for any a, and $\mathbb{H}^{\nu-1}_{red} \neq 0$ for a certain a.*

Hence, the lattice cohomology of $S^3_{-d}(\#_{i=1}^{\nu} K_i)$ memorizes the number ν.

Proof If $\nu = 1$ then use Theorem 11.3.21, or the fact that due to the -1 vertex Γ cannot be rational, hence 11.1.28 applies.

Next assume that $\nu > 1$. In each $\mathscr{S}_i$ the Frobenius number $\max\{\gamma : \gamma \notin \mathscr{S}_i\}$ is $\mu_i - 1$. Note also that using the Gorenstein (or 'gap') symmetry, cf. 5.2.2, one has $\mu_i - 2 \in \mathscr{S}_i$ (†). Consider now the lattice point $\bar{l} = (\mu_1 - 1, \dots, \mu_\nu - 1) \in \mathbb{T}_n$, where $n + 1 = \sum_i (\mu_1 - 1) = \mu - \nu$. Then $\overline{W}(\bar{l}) = \nu$. But, via (†), the values of $\overline{W}$ on all the neighbor lattice points of $\bar{l}$ (in $\mathbb{T}_n$) are $\nu - 1$. Hence $\bar{l}$ is a local maximum and $\mathbb{H}^{\nu-2}_{red}(\mathbb{T}_n, \overline{W}) \neq 0$. The relevant spin^c structure is $a \equiv \mu - \nu - 1 \pmod{d}$. □

Remark 11.5.22 In the above discussion (e.g. in 11.5.12–11.5.13), the space $\mathbb{T}_n$—intersection of a simplex with a rectangle—can be replaced by the supporting simplex. Indeed, set

$$\Sigma_n := \{(\ell_{1,0}, \dots, \ell_{\nu,0}) \in (\mathbb{R}_{\geq 0})^\nu : \sum_i \ell_{i,0} = n + 1\}. \tag{11.5.25}$$

The simplex Σ_n has a natural 'generalized cube-decomposition' similarly as $\mathbb{T}_n$, cf. 11.5.12, where the 0-cubes are the lattice points $\Sigma_n \cap \mathbb{Z}^\nu$. Moreover, we take the same weight function on $\Sigma_n \cap \mathbb{Z}^\nu$ as for $\mathbb{T}_n \cap \mathbb{Z}^\nu$ given in (11.5.19).

First assume that $n+1 \leq \Sigma m$, i.e. $\mathbb{T}_n = \Sigma_n \cap [0, m_1] \times \cdots \times [0, m_\nu]$ is non-empty. (In fact, if $\mathfrak{t} < \tilde{\mathfrak{t}}_0$ as in the relevant cases above, then $n = \mathfrak{t}d + a < \mu - 1 < \Sigma m$.) We claim that the inclusion $(\mathbb{T}_n, \overline{W}) \hookrightarrow (\Sigma_n, \overline{W})$ induces an isomorphism of the corresponding lattice cohomologies $\mathbb{H}^*(\Sigma_n, \overline{W}) \to \mathbb{H}^*(\mathbb{T}_n, \overline{W})$.

This can be verified as follows. First we define a map $\varphi : \Sigma_n \cap \mathbb{Z}^\nu \to \Sigma_n \cap \mathbb{Z}^\nu$. If $\bar{l}_0 = (\ell_{1,0}, \dots, \ell_{\nu,0}) \in \mathbb{T}_n \cap \mathbb{Z}^\nu$ then $\varphi(\bar{l}_0) = \bar{l}_0$. Otherwise, let i be the smallest index for which $\ell_{i,0} > m_i$, and j the largest index for which $\ell_{j,0} < m_j$. Then $\varphi(\bar{l}_0) := (\ell_{1,0}, \dots, \ell_{i,0} - 1, \dots, \ell_{j,0} + 1, \dots, \ell_{\nu,0})$. One can check that $\overline{W}(\varphi(\bar{l}_0)) \leq \overline{W}(\bar{l}_0)$, repeated application of φ maps $\Sigma_n \cap \mathbb{Z}^\nu$ into $\mathbb{T}_n \cap \mathbb{Z}^\nu$, φ extends to the level of arbitrary dimensional cubes, hence induces the desired isomorphism at the level of lattice cohomology.

If $n + 1 \geq \mu$, then for a certain $\bar{l}_0$ with every $\ell_{i,0} \geq \mu_i$ one has $\overline{W}(\bar{l}_0) = 0$, hence $\min \overline{W} = 0$. Moreover, by similar retraction as above one can show that $\mathbb{H}^*_{red}(\Sigma_n, \overline{W}) = \mathbb{H}^*_{red}(\mathbb{T}_n, \overline{W}) = 0$.

In particular, in all the discussions above, $H^*_{red}(\mathbb{T}_n, \overline{W})$ can be replaced by $H^*_{red}(\Sigma_n, \overline{W})$ for every $n \geq 0$. Furthermore, if $n > \mu - 2$ then $H^*_{red}(\mathbb{T}_n, \overline{W}) = 0$ automatically, hence in several formulae above (e.g. in the summations from (11.5.20) and (11.5.23)) the restrictions $n \leq \mu - 2$ can be safely neglected.

11.5.A $\mathbb{H}^0\left(S^3_{-d}(\#_i K_i)\right)$ *and the Multiplicity Sequences*

11.5.23 In this section we present a 'stability' statement regarding $\mathbb{H}^0(S^3_{-d}(\#_i K_i))$, where the setting is as in Sect. 11.5, i.e. $\{K_i\}_i$ are algebraic knots, and d is an arbitrary positive integer. We prefer to fix d (and the corresponding spin^c structure $a \in \{0, \dots, d-1\}$), and also $\delta = \sum_i \delta_i$. However, we will vary the set $\{K_i\}_i$ under certain restriction, and we show that $\mathbb{H}^0(S^3_{-d}(\#_i K_i))$ stays stable. Recall that we already conjectured a stability property under certain analytic deformations in 11.3.39. The present case is very different, it is purely combinatorial.

In the previous subsection we reduced several computations to the weight function $\overline{W}|_{\mathbb{T}_n}$. It was connected with the weight function provided by the reduction formula via $\overline{W}|_{\mathbb{T}_n}(\bar{l} - \mathfrak{t}\mathbf{m}) = \overline{w}(\bar{l}) - \overline{w}((\mathfrak{t}+1)\mathbf{m})$, where $\mathfrak{t} = \lfloor n/d \rfloor$. Since each $\overline{w}(\mathfrak{t}\mathbf{m})$ is computable from d, a, δ, cf. 11.5.8, the lattice cohomology $\mathbb{H}^0(S^3_{-d}(\#_i K_i))$ is computable from d, a, δ and $\{\min \overline{W}|_{\mathbb{T}_n}\}_n$, cf. 11.5.10 *(c)*. Moreover, by (11.5.19)

$$\overline{W}((\ell_{1,0}, \dots, \ell_{v,0})) = \sum_i \#\{s \notin \mathscr{S}_i \,:\, s \geq \ell_{i,0}\}$$

$$= \sum_i \left(\delta_i - \#\{s_i \notin \mathscr{S}_i : s_i < \ell_{i,0}\}\right) = \sum_i (\delta_i - \ell_{i,0}) + \sum_i \#\{s_i \in \mathscr{S}_i : s_i < \ell_{i,0}\}.$$

Hence

$$\min\{\overline{W}|_{\mathbb{T}_n}\} = \delta - n - 1 + \min_{\sum_i \ell_{i,0} = n+1} \#\{s_i \in \mathscr{S}_i : s_i < \ell_{i,0}\}. \tag{11.5.26}$$

This motivates the replacement of the semigroup $\mathscr{S}_i$ with an equivalent object of it, with its 'counting function' $j \mapsto H_i(j)$,

$$H_i(j) := \#\{s \in \mathscr{S}_i \,:\, s < j\}. \tag{11.5.27}$$

From analytic point of view, $H_i(j)$ is the coefficient of t^j in the Hilbert function of the local singularity (C, p_i), associated with the filtration given by its normalization (for more see 8.1.26, especially (8.1.18)).

The above min-expression can be reformulated formally as follows. Consider any two functions H_1 and H_2 defined on integers and bounded from below. Then we define their *'minimum convolution'* (cf. [68, 5.3]), denoted by $H_1 \diamond H_2$ as

$$(H_1 \diamond H_2)(j) = \min_{j_1+j_2=j} \{H_1(j_1) + H_2(j_2)\}.$$

Then from the counting functions $\{H_i\}_{i=1}^{\nu}$ associated with $\{\mathscr{S}_i\}_{i=1}^{\nu}$ we consider

$$H := H_1 \diamond H_2 \diamond \cdots \diamond H_\nu. \tag{11.5.28}$$

Since the operator $\diamond$ is associative and commutative, the function H is well-defined.

From the above discussion $\mathbb{H}^0(S^3_{-d}(\#_i K_i))$ is computable from d, a, δ and H.

Next, we wish to modify the input data $\{H_i\}_{i=1}^{\nu}$ combinatorially under some restrictions (so that δ and $H_1 \diamond \cdots \diamond H_\nu$ stay stable). This will be done by a modification of the multiplicity sequences of the local singularities. Recall that the multiplicity sequence is a complete topological invariant of the local singularity type of the irreducible plane curve (similarly as $K_i \subset S^3$, or $\mathscr{S}_i$, or H_i), cf. 5.2.2.

Let $[m_1^{(1)}, \dots, m_{r_1}^{(1)}], \dots, [m_1^{(\nu)}, \dots, m_{r_\nu}^{(\nu)}]$ be the multiplicity sequences of the local singularities (with the convention $m_{r_i}^{(i)} > 1$, cf. 5.2.2). Recall also, cf. (5.2.1), that $\delta = \sum_{i=1}^{\nu} \sum_{j=1}^{r_i} m_j^{(i)}(m_j^{(i)} - 1)/2$. Hence under modification we wish to preserve this sum (at least).

We will show that the minimum convolution $H = H_1 \diamond \cdots \diamond H_\nu$ depends only on the *multiset* of multiplicities $\{\{m_1^{(1)}, \dots, m_{r_1}^{(1)}, \dots, m_1^{(\nu)}, \dots m_{r_\nu}^{(\nu)}\}\}$. By a multiset we mean a set, where the same element might be repeated and we keep track of the number of appearances (hence a multiset with integer entries is an element of $\mathbb{Z}[\mathbb{Z}]$).

Theorem 11.5.24 *[63] Take two collections of plane curve singularity types with their multiplicity sequences:*

$$[m_1^{(1)}, \dots, m_{r_1}^{(1)}], \dots, [m_1^{(\nu)}, \dots, m_{r_\nu}^{(\nu)}] \quad \text{and} \quad [\overline{m}_1^{(1)}, \dots, \overline{m}_{\overline{r}_1}^{(1)}], \dots, [\overline{m}_1^{(\overline{\nu})}, \dots, \overline{m}_{\overline{r}_{\overline{\nu}}}^{(\overline{\nu})}].$$

Denote the counting functions of their semigroups by $H_1, \dots, H_\nu$ and $\overline{H}_1, \dots, \overline{H}_{\overline{\nu}}$, respectively. If

$$\{\{m_1^{(1)}, \dots, m_{r_1}^{(1)}, \dots, m_1^{(\nu)}, \dots, m_{r_\nu}^{(\nu)}\}\} = \{\{\overline{m}_1^{(1)}, \dots, \overline{m}_{\overline{r}_1}^{(1)}, \dots, \overline{m}_1^{(\overline{\nu})}, \dots, \overline{m}_{\overline{r}_{\overline{\nu}}}^{(\overline{\nu})}\}\}$$

as multisets*, then*

$$H_1 \diamond \cdots \diamond H_\nu = \overline{H}_1 \diamond \cdots \diamond \overline{H}_{\overline{\nu}}.$$

Corollary 11.5.25 *Consider two collections of algebraic knots $\{K_1, \dots, K_\nu\}$ and $\{\overline{K}_1, \dots, \overline{K}_{\overline{\nu}}\}$. If the collections of numbers coming from multiplicity sequences cor-*

responding to $\{K_1, \dots, K_\nu\}$ *and* $\{\overline{K}_1, \dots, \overline{K}_{\overline{\nu}}\}$ *respectively are equal as multisets, then*

$$\mathbb{H}^0(S^3_{-d}(\#_i K_i), a) \cong \mathbb{H}^0(S^3_{-d}(\#_i \overline{K}_i), a)$$

for any $d > 0$ *and any spin*^{c} *structure* $a \in \{0, \dots, d-1\}$.

The same is true for $eu(\mathbb{H}^0)$ *as well. These facts are true independently of the algebraic realizability of the combinatorial data.*

Example 11.5.26 There exist cuspidal curves of degree 5 with the following multiplicity sequences: (A) $[3], [2_3]$ $(\nu = 2)$, (B) $[3, 2], [2_2]$ $(\nu = 2)$, (C) $[3], [2_2], [2]$ $(\nu = 3)$. Note that the multisets of the multiplicities are the same hence their $\mathbb{H}^0(S^3_{-d}(\#_i K_i))$ coincide too. On the other hand, one can consider the following candidate cusps as well: (D) $[3, 2], [2], [2]$ $(\nu = 3)$, (E) $[3], [2], [2], [2]$ $(\nu = 4)$. Their multiplicity multisets are the same as those from (A)-(C), though curves of degree 5 with these local cusp data do not exist. Nevertheless, their $\mathbb{H}^0(S^3_{-d}(\#_i K_i))$ is still the same.

11.5.27 Theorem 11.5.24 is the consequence of the following facts.

Let H be the counting function of the semigroup of a local irreducible singularity with multiplicity sequence $[m_1, m_2, \dots, m_r]$. For any $m \geq 2$, denote by $H_{[m]}$ the counting function of the semigroup $\langle m, m+1\rangle$ associated with a singularity type with multiplicity sequence $[m]$. Then we claim that

$$H = H_{[m_1]} \diamond H_{[m_2]} \diamond \cdots \diamond H_{[m_r]}. \tag{11.5.29}$$

For this it is enough to prove that $H_{[m_1, m_2, \dots, m_r]} = H_{[m_1]} \diamond H_{[m_2, \dots, m_r]}$. In order to prove this we analyse the behaviour of the counting function under the blow up.

Let $\mathscr{S}$ be the semigroup of an irreducible plane curve singularity, and let $\mathscr{S}'$ be the semigroup of the singularity obtained from the previous one by a single blow up. Denote by m the *multiplicity* of the original singularity, i.e. $m = \min\{s \in \mathscr{S} : 0 < s\}$.

The *Apéry set* of a numerical semigroup with respect to one of its elements is a standard invariant commonly used in semigroup theory. It consists of the smallest elements of the semigroup from each (nonempty) residue class of the given element. Let the Apéry set of $\mathscr{S}$ with respect to m be $\mathrm{Ap}(m, \mathscr{S}) = \{b_0, b_1, \dots, b_{m-1}\}$, where $0 = b_0 < b_1 < \cdots < b_{m-1}$. It is a complete residual system modulo m, and by the definition, for each i $(0 \leq i \leq m-1)$ we have $b_i \in \mathscr{S}$ but $b_i - m \notin \mathscr{S}$. The definition guarantees that $\mathscr{S} = \mathrm{Ap}(m, \mathscr{S}) + m \cdot \mathbb{Z}_{\geq 0}$.

Note that $m \in \mathscr{S}'$ as well. Indeed, the strict transform of the singular curve after the blow up and the reduced exceptional divisor have intersection multiplicity m. Let $\mathrm{Ap}(m, \mathscr{S}') = \{a_0, a_1, \dots, a_{m-1}\}$ be the Apéry set of $\mathscr{S}'$ associated with m.

Proposition 11.5.28 *[18, 137] The Apéry sets of $\mathscr{S}$ and $\mathscr{S}'$ (with respect to the multiplicity m of $\mathscr{S}$) are related by*

$$a_j + jm = b_j \quad (j = 0, \dots, m-1).$$

In [63] this was transformed into the following property.

Proposition 11.5.29 *If H and H' are the corresponding counting functions then $H = H_m \diamond H'$.*

For the proof see [63]. Note that 11.5.29 implies (11.5.29), hence 11.5.24 too. Indeed, if $H = H_{[m_1,m_2,\dots,m_r]}$ then $m = m_1$ and $H' = H_{[m_2,\dots,m_r]}$.

Remark 11.5.30 We wish to compare the above stability property with the Semigroup Distribution Property (SDP) 5.2.9.IV proved by Borodzik and Livingston in [68]. This says, via this new language, that for any rational cuspidal curve of degree d, the local singularity types and the degree d should satisfy $H(ld+1) = (l+1)(l+2)/2$ for any $l = 0, 1, \dots d-3$.

(a) Theorem 11.5.24 enlarges the applicability of SDP drastically: if we reorganize the multiplicity numbers of a rational cuspidal curve (by keeping the multiset), then the new combinatorial candidate satisfies the output of the SDP again regardless of its algebraic realizability. This shows that although in the proof of SDP the algebraic realizability is used, presumably some weaker assumption would suffice (e.g., only the smooth/symplectic realizability of the curve in $\mathbb{P}^2$).
(b) Since there exist rational cuspidal curves with arbitrarily long multiplicity sequences (see e.g. 5.2.8), the above corollary also shows that the SDP merely (or any invariant computable from H) cannot provide any restriction on the number of cusps of rational cuspidal curves. (This fact should be compared with the statement of 11.5.21, which recovers the number of cusps from $\mathbb{H}^*(S^3_{-d}(\#_{i=1}^{\nu} K_i))$.)
(c) The analogue statement of Corollary 11.5.25 is not true for $\mathbb{H}^q$ ($q \geq 1$), not even for the numerical value $eu(\mathbb{H}^*)$, see 11.5.46.

11.5.B Superisolated Singularities with More Cusps

In this subsection we consider a superisolated singularity associated with an irreducible rational cuspidal curve. For different notations and statements regarding the analytic and topological type see subsections 5.2, 8.6.B, 9.7.B, 11.3.B, 11.5.

Our goal is to discuss Conjectures 8.6.23 and 8.6.25 from the point of view of lattice cohomology. Let us recall the two statements. Set (cf. 8.6.21*(b)*)

$$N(t) = \sum_{l=0}^{d-3} \left(\alpha_{(d-3-l)d} - \frac{(l+1)(l+2)}{2} \right) t^{d-3-l}. \tag{11.5.30}$$

11.5.31 Conjecture 8.6.23: all the coefficients of $N(t)$ are non-positive. We will refer to this as '**Conjecture C**' ('Conjecture regarding the coefficients of $N(t)$').

11.5.32 Conjecture 8.6.25: $N(1)$ is non-positive. We will refer to this as the '**Conjecture I**' (we regard $N(1)$ as an 'index type invariant').

Clearly Conjecture C implies Conjecture I.

We will compare these statements with the Semigroup Distribution Property based on the properties of counting functions H_i of the semigroups and also on a subtle connection with lattice cohomology. In this subsection we follow [63].

We invite the reader to review the definition of the counting functions H_i of the semigroups $\mathscr{S}_i$ from (11.5.27). Let us define H as the minimum convolution of the functions $\{H_i\}_i$ as in (11.5.28). Recall also (cf. 11.5.30 and 5.2.9.IV) that the Semigroup Distribution Property (SDP) reads as $H(ld+1) = (l+1)(l+2)/2$ for any $l = 0, 1, \dots d-3$.

Example 11.5.33 [The Case $\nu = 1$] In this case $\alpha_j = \#\{s \notin \mathscr{S}_1 : s > j\}$, cf. (8.6.12). From the symmetry of the semigroup $\alpha_{2\delta-2-j} = H_1(j+1)$ for $j = 0, \dots, 2\delta-2$. Hence, the α-coefficient needed in (11.5.30) is $\alpha_{(d-3-l)d} = \#\{s \in \mathscr{S}_1 : s \le ld\} = H_1(ld+1)$, cf. (8.6.13). Recall that 5.2.9.III (Bézout's Theorem) implies $\alpha_{(d-3-l)d} = H_1(ld+1) \ge (l+1)(l+2)/2$. This inequality and (11.5.30) show that for $\nu = 1$ Conjecture C is equivalent to $N(t) \equiv 0$. But, they are also equivalent to Conjecture I, since if $N(1) \le 0$ then necessarily $N(t) \equiv 0$. Finally, the validity of all these statements follows from SDP.

However, for $\nu \ge 2$ the relationships (and statement validities) are more subtle. In the next parts we will prove the following.

Theorem 11.5.34 ([63]) *With the above notations one has:*

1. *If $\nu = 2$, then $\alpha_{2\delta-2-j} \le H(j+1)$ for any $j = 0, 1, \dots 2\delta-2$. Therefore, for bicuspidal curves the SDP implies Conjecture C (hence Conjecture I too).*
2. *If $\nu \ge 3$, then the inequality $\alpha_{2\delta-2-j} \le H(j+1)$ does not hold in general, not even for $j = ld$ ($l = 0, 1, \dots, d-3$), needed for Conjectures C and I.*

   Moreover, Conjecture C (that is, $\alpha_{2\delta-2-ld} \le H(ld+1)$ under the assumption of an analytic realization) is not true in general, and Conjecture I behaves independently from SDP. (Conjecture I remains as a conjecture, though its validity was verified directly for all 'known' curves.)

For the proof of part 1 see 11.5.45. For examples supporting part 2 see 11.5.37 and 11.5.38. For the verification of Conjecture I for the known curves see [63].

11.5.35 Combinatorial Reformulation [63] This subsection aims to make aware of the similarities and differences between the polynomial Q and the function H.

Let us start with ν semigroups $\{\mathscr{S}_i\}_{i=1}^{\nu}$ associated with local irreducible plane curve singularities. We wish to emphasize that in the next discussion/construction we will not require their realizability as singularities of a projective rational curve. However, regarding the realizability, we use the following terminology. If the sum δ

of delta-invariants of the local singularity types is of form $2\delta = (d-1)(d-2)$ for some integer d, then we say that these ν local topological types are *combinatorial candidates* for the ν singularities of a rational cuspidal plane curve of degree d. If such a curve really exists then (SDP) is valid for the corresponding local data and d.

The semigroups determine their counting (or Hilbert) functions H_i by (11.5.27) and the *minimal convolution* H of the functions $\{H_i\}_i$ by (11.5.28). For convenience, define also the sequences $\{h_j^{(i)}\}_{j=0}^{\infty}$ by $h_j^{(i)} := H_i(j+1)$ (notice the shift by one).

For any sequence $a = \{a_j\}_{j=0}^{\infty}$ denote by ∂a its *difference sequence*, i.e. $(\partial a)_j = a_j - a_{j-1}$ with the convention $a_{-1} = 0$. Similarly, we will denote by Σa the *sequence of partial sums*, i.e. $(\Sigma a)_j = a_0 + \cdots + a_j$. Of course, $\Sigma\partial a = a$ and $\partial\Sigma a = a$ for any sequence a.

By (11.5.27) and $\Delta_i(t) = (1-t)\cdot\sum_{s\in\mathscr{S}_i} t^s$ (cf. (5.2.2)) the coefficient $c_j^{(i)}$ of t^j in $\Delta_i(t)$ can be written as $c_j^{(i)} = (\partial\partial h^{(i)})_j$. The coefficient sequence of a polynomial product is the *usual convolution* of coefficient sequences of the factors. Hence, the coefficient c_j of t^j in $\Delta(t) = \prod_i \Delta_i(t)$ is $c_j = \sum_{j_1+\cdots+j_\nu=j} c_{j_1}^{(1)}\cdots c_{j_\nu}^{(\nu)}$. Denoting the convolution of two sequences $a = \{a_j\}_{j=0}^{\infty}$ and $b = \{b_j\}_{j=0}^{\infty}$ by $a * b$, i.e. $(a*b)_j = \sum_{k=0}^{j} a_k b_{j-k}$, we get $c_j = (\partial\partial h^{(1)} * \cdots * \partial\partial h^{(\nu)})_j$. Let us define:

$$F(j) := (\Sigma\Sigma(\partial\partial h^{(1)} * \cdots * \partial\partial h^{(\nu)}))_j. \tag{11.5.31}$$

Before we identify F, let us recall some symmetry properties. From the symmetry of $\Delta = 1 + (t-1)\delta + (t-1)^2 Q(t)$ (and from $\delta = \sum_i \delta_i$)

$$\alpha_{2\delta-2-j} = \alpha_j + j + 1 - \delta \quad \text{for} \quad 0 \le j \le 2\delta - 2. \tag{11.5.32}$$

This (or the symmetry of each semigroup) implies also $H_i(j_i) = H_i(2\delta_i - j_i) + j_i - \delta_i$, from which one also obtains

$$H(2\delta - 2 - j + 1) = H(j+1) - j - 1 + \delta \quad \text{for every } j \in \mathbb{Z}. \tag{11.5.33}$$

Next, if $A(t) = \sum_j a_j t^j$ and $B(t) = \sum_j b_j t^j$ satisfy $A(t) = A(1) + (t-1)B(t)$, then $(\Sigma a)_j = A(1) - b_j$. This applied twice for Δ gives $(\Sigma\Sigma c)_j = j + 1 - \delta + \alpha_j$. Hence, the definition of Q and (11.5.32) provide

$$\alpha_{2\delta-2-j} = (\Sigma\Sigma(\partial\partial h^{(1)} * \cdots * \partial\partial h^{(\nu)}))_j = F(j) \quad \text{for } 0 \le j \le 2\delta - 2. \tag{11.5.34}$$

In other words, the H-values are obtained from $\{h^{(i)}\}_i$ by minimal convolution (shifted by one), while the F-coefficients (or α-coefficient in opposite order) are obtained by the composition of $\partial\partial$, the usual convolution, and the $\Sigma\Sigma$ operator.

Then one has the following reinterpretations in terms of F and H.

Let $C \subset \mathbb{C}P^2$ be a rational cuspidal curve of degree d with ν cusps of given topological types (in particular, $d(d-3) = 2\delta - 2$. Set $F(j) := (\Sigma\Sigma(\partial\partial h^{(1)} * \cdots * \partial\partial h^{(\nu)}))_j$, where $h_j^{(i)} = H_i(j+1)$, and H_i is the semigroup counting function

(or the Hilbert function) of the i-th singularity. Set $H := H_1 \diamond \cdots \diamond H_\nu$. Then

$$\text{(Conjecture C)} \quad F(ld) \le \frac{(l+1)(l+2)}{2} \quad \text{for all } l = 0, 1, \dots, d-3. \tag{11.5.35}$$

$$\text{(Conjecture I)} \quad \sum_{l=0}^{d-3} F(ld) \le \sum_{l=0}^{d-3} \frac{(l+1)(l+2)}{2} = \frac{d(d-1)(d-2)}{6}. \tag{11.5.36}$$

$$\text{(SDP)} \quad H(ld+1) = \frac{(l+1)(l+2)}{2} \quad \text{for all } l = 0, 1, \dots, d-3. \tag{11.5.37}$$

Let us summarize the combinatorial situation. Starting from the semigroups of ν local singularities we define H and F.

If $\nu = 1$ (since $\Sigma\Sigma\partial\partial(h) = h$) then $F(j) = H(j+1)$ for each $j \in \mathbb{Z}_{\ge 0}$ (independently of realizability, hence not just for $j \in d \cdot \mathbb{Z}_{\ge 0}$).

On the other hand, for $\nu > 1$ the values $F(j)$ and $H(j+1)$ become different. Nevertheless, cf. Theorem 11.5.34.1 and paragraph 11.5.45, $F(j) \le H(j+1)$ remains true for $\nu = 2$ and every integer $j \ge 0$, again by combinatorial (lattice cohomology) argument (independently of realizability and d).

With these facts in mind, it is tempting to conjecture that maybe *the inequality $F(j) \le H(j+1)$ is always true — as a property of local singularity types —*, which would make Conjecture C a combinatorial corollary of SDP. But, for $\nu \ge 3$ there is no such relation between the local functions F and H, as we will show next.

Example 11.5.36 Take $\nu = 3$, and assume that all local singularities are (2,3)-cusps (with semigroup $\langle 2, 3\rangle$). Then the relevant terms for $0 \le j \le 2\delta - 2 = 4$ of $F(j)$ are $1, -1, 3, 0, 3$, while for and $H(j+1)$ are $1, 1, 2, 2, 3$. Hence for $j = 2$ the desired inequality $F(j) \le H(j+1)$ fails. Hence the inequality $F(j) \le H(j+1)$ cannot be true for *any integer* j. (In fact, this collection of local cusp types can be realized on a rational tricuspidal curve of degree four, cf. 8.6.22. Note that the inequality holds for $j = 4 = d$, hence one could still hope that $F(j) \le H(j+1)$ hold in the case of existing curves and for $j \in d \cdot \mathbb{Z}$, where d is the degree of the curve.)

Example 11.5.37 (Counterexample to Conjecture C) Consider the curve $C_{8,4} = C_{8,2}$ from 5.2.8.V. It has degree $d = 8$ and three cusps with multiplicity sequences $[6]$, $[2_4]$ and $[2_2]$. In this case $2\delta - 2 = 40$, and we will test $F(j)$ and $H(j+1)$ only for multiples of d. The next table shows these values:

j	0	...	8	...	16	...	24	...	32	...	40
$H(j+1)$	1	...	3	...	6	...	10	...	15	...	21
$F(j)$	1	...	4	...	5	...	9	...	16	...	21
$H(j+1) - F(j)$	0	...	−1	...	1	...	1	...	−1	...	0

For $l = 1$ and $l = 4$ the inequality $F(ld) \le (l+1)(l+2)/2 = H(ld+1)$ fails, hence for $v = 3$ we cannot expect $F(j) \le H(j+1)$ for any multiples of d neither. In particular, Conjecture C fails too. This also shows that the inequalities of Conjecture C are *not combinatorial consequences* of the equalities of SDP.

On the other hand, note that Conjecture I is not violated: the summation of the last row is zero.

Example 11.5.38 The following example shows that Conjecture I is *not a combinatorial consequence* of the equalities of SDP either. For this we will consider a combinatorial candidate, which admits a candidate d, which satisfies the SDP, but the inequality of Conjecture I fails. In particular, SDP does not obstruct the algebraic realization of the curve C, nevertheless, such a realization does not exist.

Consider three semigroups $\langle 3,5\rangle$, $\langle 2,3\rangle$, $\langle 2,3\rangle$. They are candidates for the singularity of a curve of degree $d = 5$. Since the complex projective quintics are classified (see e.g. [459] or 8.6.22 here), we know that such a curve does not exist. However, the SDP does not exclude the existence of such a curve, as the values $H(j+1)$ $j = 0, 5, 10$ are the right values $1, 3, 6$. On the other hand, by a computation, the values $H(j+1) - F(j)$ for $j = 0, 5, 10$ are $0, -1, 0$, hence Conjecture I fails. In this case the root of $S^3_{-5}(T_{2,3}\#T_{2,3}\#T_{3,5})$ (associated with $k_r = K$) coincides with the root of $\Sigma(5,5,6)$ (see 11.3.35), $\mathbb{H}^1 = 0$ and $\mathbb{H}^2 = \mathscr{T}_{-4}(1)$.

In particular, if Conjecture I would be proved—independently of the classification of projective curves—, it would provide *an independent tool for checking whether a given collection of local topological singularity types can be realized as the collection of cusp types of a rational cuspidal projective plane curve.*

11.5.39 Lattice Cohomological Reinterpretation Consider the combinatorial situation from 11.5.35. The semigroups $\mathscr{S}_i$ determine links $K_i \subset S^3$ of the corresponding (topological types) of plane curve singularities. Consider an arbitrary $d > 0$ and the surgery 3-manifold $S^3_{-d}(\#_i K_i)$ as in Sect. 11.5.

Though the above examples suggest that in general the function F and H behave rather differently and independently (for a large v), the next statements show their remarkable common feature.

Theorem 11.5.40 ([63]) *For any $d > 0$ and $0 \le a < d$ the following facts hold:*

$$\begin{aligned} eu\left(\mathbb{H}^0(S^3_{-d}(\#_i K_i), K + 2aE^*_+)\right) &= \sum_{\substack{j \equiv a \,(\mathrm{mod}\, d)\\ 0 \le j \le 2\delta-2}} \left(H(j+1) + \delta - 1 - j\right), \\ &= \sum_{\substack{j \equiv a \,(\mathrm{mod}\, d)\\ 0 \le j \le 2\delta-2}} H(2\delta - 2 - j + 1); \end{aligned} \tag{11.5.38}$$

$$\begin{aligned} eu\left(\mathbb{H}^*(S^3_{-d}(\#_i K_i), K + 2aE^*_+)\right) &= \sum_{\substack{j\equiv a (mod\ d)\\ 0\le j\le 2\delta-2}} (F(j)+\delta-1-j) \\ &= \sum_{\substack{j\equiv a (mod\ d)\\ 0\le j\le 2\delta-2}} F(2\delta-2-j). \end{aligned} \tag{11.5.39}$$

Proof We will use the identities from (11.5.22). In the first one, note that by (11.5.19), (11.5.18) and (11.5.26) $\min(\overline{W}|_{\mathbb{T}_j})$ is $\delta - j - 1 + H(j+1)$ and (11.5.38) follows (for its second identity use (11.5.33)).

For the second identity, note that $-eu(\mathbb{H}^*(\mathbb{T}_j, \overline{W})$ equals α_j by (11.5.24), which is $F(2\delta-2-j)$ by (11.5.34). Then use again the symmetry (11.5.32). □

Remark 11.5.41 In fact, by Theorem 11.5.10, the integer d, the sum of delta-invariants δ and the function H completely determine the whole $\mathbb{H}^0$ as a graded $\mathbb{Z}[U]$-module (and not just its Euler characteristic).

Corollary 11.5.42 *Assume that $d(d-3) = 2\delta - 2$ (that is, d and $\{\mathscr{S}_i\}_i$ constitute a package of combinatorial candidates for algebraic realizability). Then*

$$eu\left(\mathbb{H}^0(S^3_{-d}(\#_i K_i), K + 2aE^*_+)\right) = \sum_{\substack{j\equiv -a (mod\ d)\\ 0\le j\le 2\delta-2}} H(j+1),$$

$$eu\left(\mathbb{H}^*(S^3_{-d}(\#_i K_i), K + 2aE^*_+)\right) = \sum_{\substack{j\equiv -a (mod\ d)\\ 0\le j\le 2\delta-2}} F(j).$$

This for $a = 0$, for the canonical spin^c structure, reads as

$$eu\left(\mathbb{H}^0(S^3_{-d}(\#_i K_i), K)\right) = \sum_{0\le l\le d-3} H(ld+1),$$

$$eu\left(\mathbb{H}^*(S^3_{-d}(\#_i K_i), K)\right) = \sum_{0\le l\le d-3} F(ld).$$

Since by 5.2.9.III $H(ld+1) \ge (l+1)(l+2)/2$ for any $l = 0, \dots, d-3$, $\sum_{l=0}^{d-3} H(ld+1) = \sum_{l=0}^{d-3}(l+1)(l+2)/2$ is equivalent to SDP for every l (cf. (11.5.37)). In particular, in the presence of the algebraic realization, the valid SDP reads as:

$$\text{(SDP)} \quad eu\left(\mathbb{H}^0(S^3_{-d}(\#_i K_i), K)\right) = d(d-1)(d-2)/6. \tag{11.5.40}$$

Furthermore, under the same realizability assumption, Conjecture I reads as:

$$eu\left(\mathbb{H}^*(S^3_{-d}(\#_i K_i), K)\right) \le d(d-1)(d-2)/6. \tag{11.5.41}$$

They combined:

$$\text{(Conjecture I)} \quad eu\left(\mathbb{H}^*(S^3_{-d}(\#_i K_i), K)\right) \le eu\left(\mathbb{H}^0(S^3_{-d}(\#_i K_i), K)\right). \tag{11.5.42}$$

Remark 11.5.43 The algebraic realizability in (11.5.40) is essential. Indeed, cf. 11.3.35, $eu(\mathbb{H}^0(S^3_{-5}(T_{3,7}), K)) = 9 < 10$, and $eu(\mathbb{H}^0(S^3_{-6}(T_{2,21}), K)) = 22 > 20$. Hence the equality (11.5.40) valid in the algebraic cases might fail in both direction for non-realized combinatorial/topological situations.

11.5.44 Proof of Conjecture I for $\nu = 2$ (via SDP) First we recall from 11.5.21 that $\mathbb{H}^q(S^3_{-d}(\#_i K_i), k_r) = 0$ for any $q \ge \nu$ and any k_r. Then, for $\nu = 2$, one has $eu\left(\mathbb{H}^*(S^3_{-d}(\#_i K_i), K)\right) = eu\left(\mathbb{H}^0(S^3_{-d}(\#_i K_i), K)\right) - \text{rank}_{\mathbb{Z}}\mathbb{H}^1(S^3_{-d}(\#_i K_i), K)$, hence (11.5.42) follows.

For $\nu \ge 3$ the similar argument does not work. From this point of view, it is even more surprising that in all the known cases Conjecture I still holds, cf. 11.5.34.

11.5.45 Proof of Conjecture C and Part 1 of Theorem 11.5.34 for $\nu = 2$ We will show, essentially by the same argument as in 11.5.44, that in case of $\nu = 2$ the inequality $F(j) \le H(j+1)$ holds for any $0 \le j \le 2\delta - 2$. This inequality is purely combinatorial, completely independent of the parameter d, and it is also independent of the realizability of cusp types on an existing rational projective curve (and also on the validity of SDP).

Indeed, set $D > 2\delta - 2$. Then, we use 11.5.42 and $eu\left(\mathbb{H}^*(S^3_{-D}(\#_i K_i), k_r(j))\right) = eu\left(\mathbb{H}^0(S^3_{-D}(\#_i K_i), k_r(j))\right) - \text{rank}_{\mathbb{Z}}\mathbb{H}^1(S^3_{-D}(\#_i K_i), k_r(j))$, where $k_r(j) = K + 2jE^*_+$.

For a different, direct elementary proof, see [461].

Example 11.5.46 Finally, let us give an example promised in Remark 11.5.30(c), a pair of existing cuspidal curves, both of degree $d = 5$, with the same multiset of multiplicities but different F functions. The first curve C_1 has cusp types $[2_2]$, $[2_2]$, $[2_2]$, while the second one, C_2, has cusp types $[2_3]$, $[2]$, $[2]$, $[2]$.

The numerical values of functions F and H are as follows:

k	0	1	2	3	4	5	6	7	8	9	10
$H(k+1)$ for C_1 and C_2	1	1	2	2	3	3	4	4	5	5	6
$F(k)$ for C_1	1	−1	3	−3	6	−3	7	−1	6	3	6
$F(k)$ for C_2	1	−2	5	−5	8	−5	9	−3	8	2	6
$H(k+1)-F(k)$ for C_1	**0**	2	−1	5	−3	**6**	−3	5	−1	2	**0**
$H(k+1)-F(k)$ for C_2	**0**	3	−3	7	−5	**8**	−5	7	−3	3	**0**

Hence $eu(\mathbb{H}^0_{\text{can}}) - eu(\mathbb{H}^*_{\text{can}})$ for C_1 is $0+6+0=6$ while for C_2 it is $0+8+0=8$.

Remark 11.5.47 Above (see e.g. (11.5.37)) the SDP is formulated only for integers $j-1$, which are multiples of d, that is, only for the canonical spin^c structure, however several other invariants/statements/questions run for any integer j (any spin^c structures) as well. Next we try to explain some reasons for the lack of such extension. In the first approach SDP was motivated/found via the difference $Z_{0,\{v_+\}}(t) - P_{0,\{v_+\}}(t)$ (cf. 8.6.B), corresponding to $h=0$. Though its equivariant extension $Z_{\{v_+\}}(t) - P_{\{v_+\}}(t)$ is clearly present, we have very little knowledge (at least by the author) about $P_{\{v_+\}}(t)$ (contrary to $P_{0,\{v_+\}}(t)$, which is extremely simple). In general, the computation and identification of the analytic invariants of the universal abelian covering of a superisolated singularity is hard (for some details and examples see e.g. [665]). The second approach is the one used by Borodzik and Livingston in [68]. In this case the proof essentially uses the fact that $\mathbb{P}^2 \setminus C$ is a rational homology ball (hence its boundary has a vanishing Heegaard Floer d-invariant). However, the natural $\mathbb{Z}_d$-covering of $\mathbb{P}^2 \setminus C$ is the Milnor fiber $\{f=1\} \subset \mathbb{C}^3$ of the homogeneous hypersurface singularity f, where $f=0$ is the projective equation of C, and F is a rational homology ball only when C is a line, cf. [502, 24.3.3].

11.6 Path Lattice Cohomology

We reconsider the topological invariant $\text{Path}^i = \text{Path}$ introduced in 6.8.B (as a natural topological upper bound of p_g) in terms of lattice cohomology. In this new aspect, Path appears as the the minimal Euler characteristic of the lattice cohomologies associated with different (not necessarily increasing) paths. It turns out that $p_g \le \text{Path}^i = \text{Path} = \min_{\gamma\in\mathscr{P}} eu(\mathbb{H}^0(\gamma, (\mathbb{R}_{\ge 0})^s, K)) \le eu(\mathbb{H}^0(\Gamma, K))$. Both inequalities are equalities for superisolated germs. ♣

11.6.1 Fix $\mathbb{Z}^s$ as in 11.1 and fix also a compatible weight functions $\{w_q\}_q$ as in 11.1.4. Consider also a sequence $\gamma := \{l_i\}_{i=0}^t$ so that $l_0 = 0$, $l_i \ne l_j$ for $i \ne j$, and $l_{i+1} = l_i \pm E_{j(i)}$ for $0 \le i < t$. We write T for the union of 0-cubes marked

by the lattice points $\{l_i\}_i$ and of the segments (1–cubes) of type $[l_i, l_{i+1}]$. Then, by 11.1.15 we get a graded $\mathbb{Z}[U]$-module $\mathbb{H}^*(T, w)$, which will be called the *path lattice cohomology* associated with the 'path' γ and weights $\{w_q\}_{q=0,1}$. It is denoted by $\mathbb{H}^*(\gamma, w)$. It has an augmentation with $\mathscr{T}^+_{2m_\gamma}$, where $m_\gamma := \min_i\{w_0(l_i)\}$, and one also obtains the *reduced path lattice cohomology* $\mathbb{H}^0_{red}(\gamma, w)$ with

$$\mathbb{H}^0(\gamma, w) = \mathscr{T}^+_{2m_\gamma} \oplus \mathbb{H}^0_{red}(\gamma, w).$$

Similarly as in (11.1.3) we consider its 'Euler characteristic'

$$eu(\mathbb{H}^0(\gamma, w)) := -m_\gamma + \operatorname{rank}_{\mathbb{Z}}(\mathbb{H}^0_{red}(\gamma, w)). \tag{11.6.1}$$

Lemma 11.6.2 $\mathbb{H}^q(\gamma, w) = 0$ *for* $q \geq 1$, *and*

$$eu(\mathbb{H}^0(\gamma, w)) = -w_0(0) + \sum_{i=0}^{t-1} \big(w_1([l_i, l_{i+1}]) - w_0(l_{i+1})\big).$$

Proof Use induction comparing the paths $\{l_i\}_{i=0}^{n-1}$ and $\{l_i\}_{i=0}^{n}$ $(0 < n \leq t)$. □

If $w_1([l_i, l_{i+1}]) = \max\{w_0(l_i), w_0(l_{i+1})\}$ then

$$eu(\mathbb{H}^0(\gamma, w)) = -w_0(0) + \sum_{i=0}^{t-1} \max\{0, w_0(l_i) - w_0(l_{i+1})\}. \tag{11.6.2}$$

11.6.3 In the previous paragraphs of this section it was not necessary to assume that the terms of the path are effective (compare with the definition from 6.8.35, where this property was required). However, starting from the next paragraphs and in all the applications below we will need this property (note that in the proof of 6.8.38 it was crucial). For any such γ we will compare $\mathbb{H}^0((\mathbb{R}_{\geq 0})^s, w)$ with $\mathbb{H}^0(\gamma, w)$.

11.6.4 Weights Given By Γ In this section Γ denotes a connected negative definite graph such that $M(\Gamma)$ is a $\mathbb{Q}HS^3$. Then, similarly as in 11.1.A, in the presence of the Γ and a distinguished representative $k_r \in$ Char we consider the induced weight functions, see also 11.1.17. From 11.1.26 we know that the restriction $\mathbb{H}^*(\Gamma, k_r) = \mathbb{H}^*(\mathbb{R}^s, k_r) \to \mathbb{H}^*((\mathbb{R}_{\geq 0})^s, k_r)$ induced by the inclusion $(\mathbb{R}_{\geq 0})^s \hookrightarrow \mathbb{R}^s$ is an isomorphism of graded $\mathbb{Z}[U]$ modules. Accordingly, even for path lattice cohomology we will consider the starting pair $((\mathbb{R}_{\geq 0})^s, w)$, where the system w is associated with k_r, and all the terms of the path are effective. In such case the path lattice cohomology is denoted by $\mathbb{H}^0(\gamma, (\mathbb{R}_{\geq 0})^s, k_r)$. (Recall that $k_r = K + 2s_h$.)

For $k = K$ and $\gamma \in \mathscr{P}(Z)$ (for a certain $Z \geq 0$) $eu(\mathbb{H}^0(\gamma, (\mathbb{R}_{\geq 0})^s, K))$ equals $S(\gamma)$ from 6.8.37 (compare (6.8.15) with (11.6.2)), hence

$$\min_{\gamma \in \mathscr{P}} eu(\mathbb{H}^0(\gamma, (\mathbb{R}_{\geq 0})^s, K)) = \text{Path}. \tag{11.6.3}$$

See also 6.8.B for the definition of Path^i and 6.8.38 for the identity $\text{Path}^i = \text{Path}$.

Recall that in 11.3.10 we defined $eu(\mathbb{H}^0(\Gamma, k_r)) := -m_{k_r} + \text{rank}_{\mathbb{Z}}(\mathbb{H}^0_{red}(\Gamma, k_r))$. Then $m_{k_r} \leq m_\gamma = \min\{\chi_{k_r}(x_i)\}$. In the sequel we adopt the notations of 6.8.B too.

From analytic point of view we have the following inequality, the analogue of 11.3.8 (with similar proof).

Proposition 11.6.5 *Consider a normal surface singularity (X, o) with a resolution $\widetilde{X}$ and $\mathbb{Q}HS^3$ link. Fix also an increasing path $\gamma \in \mathscr{P}^i(Z)$ connecting 0 with Z. Then for any line bundle $\mathscr{L} \in Pic(\widetilde{X})$ with $c_1(\mathscr{L}) = -s_h$ one has*

$$h^1(Z, \mathscr{L}|_Z) \leq eu(\mathbb{H}^0(\gamma, (\mathbb{R}_{\geq 0})^s, k_r)). \tag{11.6.4}$$

11.6.6 Restriction map. Example The restriction map $r^0 : \mathbb{H}^0((\mathbb{R}_{\geq 0})^s, w) \to \mathbb{H}^0(\gamma, (\mathbb{R}_{\geq 0})^s, w)$ ($\gamma \in \mathscr{P}(Z)$), in general, is not onto. In order to see such an example, let us take w as in 11.6.4, $k = K$, and we will study different paths connecting $l_0 = 0$ with $l_t = Z_{min}$. For simplicity we assume that $\chi(Z_{min}) \leq 0$.

There is an 'optimal' way to find Z_{min}, given by Laufer's algorithm 6.6.7. If we take any path γ^{op} connecting 0 and Z_{min} provided by this algorithm (with $l_1 = E_1$), then $\chi(l_0) = 0$, $\chi(l_1) = \chi(E_1) = 1$, and after that χ is monotone non-increasing, so it will decrease to $\chi(Z_{min})$. Hence $\mathbb{H}^0(\gamma^{op}, (\mathbb{R}_{\geq 0})^s, K) = \mathscr{T}^+_{2\chi(Z_{min})} \oplus \mathscr{T}_0(1)$.

Assume next that the multiplicity of E_1 in Z_{min} is ≥ 2. Then one may take the 'non-optimal' increasing path γ^{nop} connecting 0 to Z_{min}, by taking $l_0 = 0$, $l_1 = E_1, l_2 = 2E_1$, and after that we proceed according to the generalized Laufer's algorithm 6.6.3. Then the maximum χ-value reached is $\chi(2E_1) = 2 - E_1^2 \geq 3$ and $\mathbb{H}^0(\gamma^{nop}, (\mathbb{R}_{\geq 0})^s, K) = \mathscr{T}^+_{2\chi(Z_{min})} \oplus \mathscr{T}_0(2 - E_1^2)$.

One sees that for γ^{op} the restriction r^0 is onto, while in the second case it is not. For the second statement it is enough to compare the 1-degree terms: by 7.3.13 $\text{rank}_{\mathbb{Z}}(\mathbb{H}^0((\mathbb{R}_{\geq 0})^s, K))_1 = 1$, while $\text{rank}_{\mathbb{Z}}(\mathbb{H}^0(\gamma^{nop}, (\mathbb{R}_{\geq 0})^s, K))_1 = 2$. E.g., if Γ is minimally elliptic, then one even has $eu(\mathbb{H}^0(\gamma^{nop}, (\mathbb{R}_{\geq 0})^s, K)) > eu(\mathbb{H}^0((\mathbb{R}_{\geq 0})^s, K))$.

Lemma 11.6.7 *Fix $Z \in L_{\geq 0}$ and consider all the paths $\mathscr{P}(Z)$ as in 11.6.1 with $l_0 = 0$ and $l_t = Z$ and $l_i \geq 0$. Then the following facts hold.*

(a) There exists $\gamma \in \mathscr{P}(Z)$ such that $r^0 : \mathbb{H}^0((\mathbb{R}_{\geq 0})^s, w) \to \mathbb{H}^0(\gamma, w)$ is onto.

(b) If for some $\gamma \in \mathscr{P}(Z)$ the restriction homomorphism r^0 is onto then the inequality $eu(\mathbb{H}^0(\gamma, w)) \leq eu(\mathbb{H}^0((\mathbb{R}^s_{\geq 0}, w))$ holds. In particular,

$$\min_{\gamma \in \mathscr{P}(Z)} eu(\mathbb{H}^0(\gamma, w)) \leq eu(\mathbb{H}^0((\mathbb{R}_{\geq 0})^s, w)).$$

Proof

(a) Take any γ from $\mathscr{P}(Z)$. If $n \gg 0$ then $S_n \cap \gamma$ contains all the vertices and segments of γ, hence it is contractible, $H^0(S_n \cap \gamma, \mathbb{Z}) = \mathbb{Z}$ and the restriction $r_n^0 : H^0(S_n, \mathbb{Z}) \to H^0(S_n \cap \gamma, \mathbb{Z})$ is onto. If r^0 is not onto, then let n be the largest integer for which r_n^0 is not onto. This means that there exists a connected component $S_{n,\alpha}$ of S_n such that $S_{n,\alpha} \cap \gamma$ has at least two connected components. In particular, when we travel along γ from $l_0 = 0$ to $l_t = Z$, there exists a certain term l_i of γ, with $l_i \in S_{n,\alpha}$, such that $l_{i+1}, \ldots, l_{j-1} \notin S_{n,\alpha}$, but $l_j \in S_{n,\alpha}$ again. It is possible that the intermediate path $l_{i+1}, \ldots, l_{j-1}$ intersects some other components $S_{n,\beta}$ of S_n. On the other hand, since $S_{n,\alpha}$ is connected, l_i and l_j can be connected by an injective path in $S_{n,\alpha}$, say $l_i = \bar{l}_0, \bar{l}_1, \ldots, \bar{l}_r = l_j$. Then in γ we replace the intermediate path $l_i, l_{i+1}, \ldots, l_j$ by $l_i = \bar{l}_0, \bar{l}_1, \ldots, \bar{l}_r = l_j$. Let this be $\bar{\gamma}$. In this way the number of components of $S_{n,\alpha} \cap \bar{\gamma}$ is strictly smaller than the number of components of $S_{n,\alpha} \cap \gamma$. If $l_{i+1}, \ldots, l_{j-1}$ intersects some other component $S_{n,\beta}$ then those components of $S_n \cap \gamma$ will disappear. Next we have to observe, that $\bar{\gamma}$ can be a non-injective path (in $S_{n,\alpha}$). Then we can eliminate from this (oriented) $\bar{\gamma}$ all its loops (oriented Euler cycles) similarly as in the proof of 6.8.38. The output is an injective path $\bar{\gamma}' \in \mathscr{P}(Z)$, which has strictly less connected components than γ in $S_{n,\alpha}$, and in any other component $S_{n,\beta}$ this number is not increasing. One also sees, that this modification step does not destroy the surjectivity of $r_{n'}^0$ for $n' > n$.

Repeating this procedure after finitely many steps we get the wished path.

(b) To the weight w and path γ one can associate in a natural way a graded root $R_{w,\gamma}$ as well. Since $H_0(S_n \cap \gamma, \mathbb{Z}) \to H_0(S_n, \mathbb{Z})$ is injective for any n, we get a natural graded root embedding $R_{w,\gamma} \hookrightarrow R_w$. Then use 11.2.11.

□

Corollary 11.6.8

$$p_g \leq Path^i = Path = \min_{\gamma \in \mathscr{P}} eu(\mathbb{H}^0(\gamma, \Gamma, K)) \leq eu(\mathbb{H}^0(\Gamma, K)).$$

Proof Combine (6.8.14), Lemma 6.8.38, (11.6.3) and Lemma 11.6.7 *(b)*. □

Example 11.6.9 As we have already pointed out, the inequality $p_g \leq \mathrm{Path}^i$ can be strict for certain analytic structures. E.g., in the elliptic case when the minimal resolution is good then $\mathrm{Path}^i = m + 1$ (cf. 7.2.66) and for the generic analytic structure $p_g = 1$ (cf. 7.2.98). Or, consider the discussion from Example 9.7.32.

Example 11.6.10 The inequality $\min_{\gamma \in \mathscr{P}} eu(\mathbb{H}^0(\gamma, \Gamma, K)) \leq eu(\mathbb{H}^0(\Gamma, K))$ for some graphs is an equality, but for certain graphs it can also be strict.

For example, in the situations of 11.6.11 below it is equality. Or, for a concrete example, take the graph from 9.7.32 where $eu(\mathbb{H}^0(\Gamma, K)) =$

$\min_{\gamma\in\mathscr{P}} eu(\mathbb{H}^0(\gamma, \Gamma, K)) = 10$ and $eu(\mathbb{H}^*(\Gamma, K)) = 8$ (see 9.7.32 and 11.4.12 or [348]).

On the other hand, if Γ is the graph from 11.4.11, then $eu(\mathbb{H}^0(\Gamma, K)) = 6$ while $eu(\mathbb{H}^*(\Gamma, K)) = \min_{\gamma\in\mathscr{P}} eu(\mathbb{H}^0(\gamma, \Gamma, K)) = 5$ (see e.g. [350]). It is instructive to chase the diagram from 11.4.11 to check $\min_{\gamma\in\mathscr{P}} eu(\mathbb{H}^0(\gamma, \Gamma, K)) = 5$.

Corollary 11.6.11 *The identities*

$$p_g = Path^i = \min_{\gamma\in\mathscr{P}} eu(\mathbb{H}^0(\gamma, \Gamma, K)) = eu(\mathbb{H}^0(\Gamma, K))$$

hold in the following cases:

(a) for superisolated singularities (with rational homology sphere link),
(b) if (X, o) satisfies SWIC for the canonical spin^c structure and $\mathbb{H}^{\geq 1}(\Gamma, K) = 0$. (Compare with 11.3.9.) This includes e.g. the weighted homogeneous germs and elliptic germs, cf. 10.6.17.

Proof Use 11.6.8 combined with (11.5.40) (and the fact that $p_g = d(d-1)(d-2)/6$) in the first case, and combined with 11.1.36 in the second case. □

Remark 11.6.12 The identities from 11.6.11 for case *(a)* show that for any superisolated singularity Conjecture I (equivalently, $eu(\mathbb{H}^*(\Gamma, K)) \leq eu(\mathbb{H}^0(\Gamma, K))$, cf. 11.5.42) is equivalent with $eu(\mathbb{H}^*(\Gamma, K)) \leq \text{Path}^i$.

11.6.A Lattice Cohomology of Newton Non-degenerate Germs

For a Newton non-degenerate hypersurface singularity with rational homology sphere link both identities $p_g = \min_{\gamma\in\mathscr{P}} eu(\mathbb{H}^0(\gamma, \Gamma, K))$ and $p_g = eu(\mathbb{H}^*(\Gamma, K)$ are valid. In fact, the first identity can be generalized to the $\mathbb{Q}$–Gorenstein–pointed Weil divisors. The path which realizes the identity is a special 'diagonal' computation sequence, an interplay between the combinatorial properties of Γ and $N\Gamma$. We follow [498, 499, 641]. ♣

11.6.13 Let $(X, 0)$ be a normal Newton non–degenerate Weil divisor given by a function f, with a rational homology sphere link. Moreover, assume that the polyhedron $N\Gamma_+(f)$ is $\mathbb{Q}$–Gorenstein–pointed as in 5.5.16. Hence $(X, 0)$ is $\mathbb{Q}$–Gorenstein, cf. 6.3.32.

In particular, for an isolated Newton non-degenerate hypersurface germs all these properties are satisfied. For a list of several numerical invariants and formulae for this classical case see 5.5.A.

In this section we discuss the following two topological characterizations of the geometric genus p_g.

Theorem 11.6.14

(a) If (X, o) normal Newton non–degenerate Weil divisor as in 11.6.13 then $p_g = Path^i = \min_{\gamma\in\mathscr{P}^i} eu(\mathbb{H}^0(\gamma, \Gamma, K))$.

(b) If (X, o) is an isolated Newton non-degenerate hypersurface germs in $(\mathbb{C}^3, 0)$ then $p_g = eu(\mathbb{H}^*(\Gamma, K))$, i.e. the Seiberg–Witten Invariant Conjecture holds for such germs and for the canonical Spin^c structure.

For hypersurfaces the proofs can be found in Sigurðsson's Thesis [641], part *(a)* was published in [498], for the more general toric case of *(a)* see [499]. The proofs of both parts *(a)* and *(b)* are technical. Here we will sketch the proof of *(a)*.

We fix the resolution given by Oka's algorithm 5.5.23 (what we will modify slightly, see below). At the beginning of the discussion we will also consider the following additional assumption:

$$\text{the } E_n\text{–coefficient of } Z_K \text{ for any node } n \in \mathscr{N} \text{ is } \geq 1. \tag{11.6.5}$$

This implies $\lfloor Z_K \rfloor > 0$, too. The assumption will be justified in 11.6.21.

We have to find a path $\gamma \in \mathscr{P}^i(Z)$ (for some $Z \geq \lfloor Z_K \rfloor$, cf. 6.8.40) such that $p_g = \min_{\gamma\in\mathscr{P}^i(Z)} S(\gamma)$. (For notations see 6.8.B.)

The construction of such a path is based on a remarkable combination of the combinatorics of the resolution graph Γ and of the toric pair $N\Gamma(f) \subset \Sigma^\vee$.

The geometric genus equals the cardinality of $(N\Gamma_+(f, \Sigma)^\circ \setminus N\Gamma_+(f)^\circ) \cap M$, cf. 5.5.21. Since the link is a rational homology sphere, in fact, we have $p_g = \#(N\Gamma_+(f, \Sigma)^\circ \setminus N\Gamma_+(f)) \cap M$, cf. 5.5.24. All these lattice points belong to $C(N\Gamma) \setminus N\Gamma(f)$, where

$$C(N\Gamma) \text{ is the convex closure of } N\Gamma(f) \text{ and } p, \tag{11.6.6}$$

the cone over $N\Gamma(f)$ with vertex p. Here the point p is that point where the polyhedron $N\Gamma_+(f)$ is $\mathbb{Q}$–Gorenstein–pointed, cf. 5.5.16.

We wish to filter this cone by a family of moving diagrams indexed by certain cycles $l' \in L'$ of Γ. They are defined as follows. If $l' = \sum_{v\in\mathscr{V}} l'_v E_v \in L'$, and n is a node of Γ, then we set the half–spaces $H_{n,\geq}(l') := \{q \in M_\mathbb{R} \cap \Sigma^\vee : \ell_n(q - p) \geq l'_n\}$. Here ℓ_n are the dual primitive vectors associated with the compact 2–faces $\{F_n\}_{n\in\mathscr{N}}$. We set $N\Gamma_+(l') = \cap_{n\in\mathscr{N}} H_{n,\geq}(l')$. Note that if $l'_1 \geq l'_2$ then $N\Gamma_+(l'_1) \subset N\Gamma_+(l'_2)$. Furthermore, by $Z_K - E = \mathrm{wt}(f) - \mathrm{wt}(p)$ (cf. (6.3.10)), $N\Gamma_+(Z_K - E)$ is exactly $N\Gamma_+(f)$.

This suggests that we should consider cycles $l' \in L'$ with $[l'] = [Z_K]$.

The idea is to consider a computation sequence $\{Z_i\}_i$ connecting $Z_K - \lfloor Z_K \rfloor$ with $Z_K - E$ and at any step compare $\chi(Z_{i+1}) - \chi(Z_i) = 1 - (Z_i, E_{v(i)})$ with the number of lattice points $N\Gamma_+(Z_i) \setminus N\Gamma_+(Z_{i+1})$. Note that if $p \in M$, i.e. if $(X, 0)$ is Gorenstein, then $Z_K - \lfloor Z_K \rfloor = 0$, and $p \in N\Gamma_+(0)$ contributes to p_g. In this case the computation sequence consists of integral cycles.

11.6.15 In the proof written in [499], motivated by some technical reasons, the resolution given by Oka's algorithm is modified as follows. Assume that the vertices n^{ext} and n of Γ^{ext} are associated with a non–compact 2–face $F_{n^{ext}}$ and with a compact 2–face F_n respectively. Assume that they are adjacent, and the determinant of the dual primitive vectors is one. In such a case in Oka's algorithm of Γ^{ext} we insert an edge (n^{ext}, n) (and then, in order to get Γ, we delete n^{ext} and this edge). In the modified version let us blow up this Γ at any such n, and this modified graph will be our 'new' Γ. In this way we create some non–intersecting (-1)–vertices. Let their collection be $\mathscr{U}$.

11.6.16 We will need the following version of the operator $\bar{l} \mapsto x(\bar{l})$ with respect to the nodes $\mathscr{N} \subset \mathscr{V}$, discussed in 7.3.23.

We claim that for any $l' \in L'$, there is a smallest cycle $x(l') \in l' + L$ satisfying

$$\begin{cases} \text{the } E_n\text{– coefficients of } x(l') \text{ and } l' \text{ agree for any } n \in \mathscr{N}, \\ (x(l'), E_v) \leq 0 \text{ for any } v \in \mathscr{V} \setminus \mathscr{N}. \end{cases} \tag{11.6.7}$$

The existence and uniqueness follows in a similar way as in 7.3.24. Note that $x(l')$ only depends on the E_n–coefficients of l' for $n \in \mathscr{N}$ and on the class $[l'] \in H = L'/L$.

The following properties hold for the operator x:

(a) *Monotonicity:* If $l'_1 \leq l'_2$ and $l'_1 - l'_2 \in L$ then $x(l'_1) \leq x(l'_2)$.
(b) *Idempotency:* We have $x(x(l')) = x(l')$ for any $l' \in L'$.
(c) *Generalized Laufer sequence:* Assume that $l' \leq x(l')$. Then there exists a generalized Laufer sequence which connects l' with $x(l')$ (as in 7.3.26).
(d) $x(Z_K - \lfloor Z_K \rfloor) \geq Z_K - \lfloor Z_K \rfloor$, and $x(Z_K) = Z_K + \sum_{u \in \mathscr{U}} E_u = Z_K + E_{\mathscr{U}}$.

(For the definition of $\mathscr{U}$ see 11.6.15.)

11.6.17 Next, we construct in two steps a computation sequence $\{Z_i\}_i$.

First, a *coarse 'diagonal' sequence* $\{\bar{Z}_i\}_{i=0}^{\bar{k}}$ is defined as follows. Start with $Z_0 = Z_K - \lfloor Z_K \rfloor$, and define $\bar{Z}_0 = x(Z_K - \lfloor Z_K \rfloor)$. Assuming $\bar{Z}_i$ $(i \geq 0)$ has been defined, and that $\bar{Z}_i|_{\mathscr{N}} < Z_K|_{\mathscr{N}}$, choose a $\bar{v}(i) \in \mathscr{N}$ minimizing the ratio

$$n \mapsto r(n) := \frac{E_n\text{–multiplicity of } \bar{Z}_i}{E_n\text{–multiplicity of } Z_K - E}, \quad n \in \mathscr{N}. \tag{11.6.8}$$

Then set $\bar{Z}_{i+1} = x(\bar{Z}_i + E_{\bar{v}(i)})$. If $\bar{Z}_i|_{\mathscr{N}} = (Z_K - E)|_{\mathscr{N}}$, then we record $\bar{k}' = i$. If $\bar{Z}_i|_{\mathscr{N}} = Z_K|_{\mathscr{N}}$, then we stop, and set $\bar{k} = i$.

We refine the above choice as follows. Choose some node $n_0 \in \mathscr{N}$ and define a partial order $\leq$ on the set $\mathscr{N}$: for $n, n' \in \mathscr{N}$, define $n \leq n'$ if n lies on the geodesic joining n' and n_0 (here we make use of the assumption that the link is a rational homology sphere, in particular, Γ is a tree). When choosing $\bar{v}(i)$, if several nodes minimizing $\{r(n)\}_n$ can be chosen, and $\min_n\{r(n)\} < 1$, then we choose $\bar{v}(i)$

minimal of those with respect to $(\mathscr{N}, \leq)$. If $\min_n\{r(n)\} = 1$, let $\mathscr{N}' \subset \mathscr{N}$ be the set of nodes n for which $r(n) = 1$. If $\mathscr{N}'$ has one element we have to chose that one. Otherwise, let Γ' be the minimal connected subgraph of Γ containing $\mathscr{N}'$, and we choose $\bar{v}(i)$ as an end–vertex of Γ'.

Next, we fill in this course sequence. Since $Z_0 = Z_K - \lfloor Z_K \rfloor \leq x(Z_K - \lfloor Z_K \rfloor) = \bar{Z}_0$ by 11.6.7(d), there exists a Laufer computation sequence connecting Z_0 with $\bar{Z}_0$. Furthermore, using the idempotency and monotonicity of the operator, we find

$$\bar{Z}_i + E_{\bar{v}(i)} = x(\bar{Z}_i) + E_{\bar{v}(i)} \leq x(\bar{Z}_i + E_{\bar{v}(i)}) = \bar{Z}_{i+1}.$$

Thus, we can join $\bar{Z}_i + E_{\bar{v}(i)}$ and $\bar{Z}_{i+1}$ by a Laufer computation sequence. Hence, we obtain a computation sequence $\{Z_i\}_i$, connecting $Z_K - \lfloor Z_K \rfloor$ with $x(Z_K) = Z_K + E_{\mathscr{U}}$.

11.6.18 In this part we relate the computation sequence to p_g. Note that the terms Z_i, in general, are not integral effective cycles, in particular, the statements of subsection 6.8.B cannot be applied directly. Here we present the dual situation. In this part $\widetilde{X}$ will denote the resolution of any normal surface singularity with rational homology sphere link and with $Z_K \geq 0$. We will consider an arbitrary computation sequence $\gamma = \{Z_i\}_{i=0}^k$, $Z_{i+1} = Z_i + E_{v(i)}$, such that $Z_0 = Z_K - \lfloor Z_K \rfloor$ and $Z_k - Z_K$ is the union of some reduced and non–intersecting (-1)-curves. For such γ we set $S^{\mathscr{L}}(\gamma) = \sum_{i=0}^{k-1} \max\{0, 1 - (Z_i, E_{v(i)})\}$, and $\mathrm{Path}^{\mathscr{L}} = \min_\gamma S^{\mathscr{L}}(\gamma)$.

This can be compared with p_g as follows. From (6.8.3) we have

$$p_g = \dim_{\mathbb{C}} H^0(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(K_{\widetilde{X}} + \lfloor Z_K \rfloor))/H^0(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(K_{\widetilde{X}})). \tag{11.6.9}$$

Furthermore, by (6.8.2) $H^0(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(K_{\widetilde{X}} + Z_K - Z_k)) \to H^0(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(K_{\widetilde{X}}))$ is an isomorphism. Therefore, if we define $\mathscr{L}_i := \mathscr{O}_{\widetilde{X}}(K_{\widetilde{X}} + Z_K - Z_i)$, then (similarly as in 6.8.B)

$$p_g = \sum_{i=0}^{k-1} \dim H^0(\widetilde{X}, \mathscr{L}_i)/H^0(\widetilde{X}, \mathscr{L}_{i+1}) \leq S^{\mathscr{L}}(\gamma), \tag{11.6.10}$$

with equality if and only if all the cohomological long exact sequences associated with $0 \to \mathscr{L}_i \to \mathscr{L}_{i+1} \to \mathscr{L}_{i+1}/\mathscr{L}_i \to 0$ split.

11.6.19 In [499] the following result is proved:

Theorem 11.6.20 *$p_g = \min_\gamma S^{\mathscr{L}}(\gamma)$ and the identity is realized for the computation sequence $\{Z_i\}_i$ constructed in 11.6.17.*

To sketch the proof, let us start with the following remarks.

There are two types of steps in the computation sequence $\{Z_i\}_{i=0}^k$, $\bar{Z}_i \rightsquigarrow \bar{Z}_i + E_{\bar{v}(i)}$, and those provided by the generalized Laufer algorithm connecting $\bar{Z}_i + E_{\bar{v}(i)}$

with $x(\bar{Z}_i + E_{\bar{v}(i)})$. Along the steps of the second type, $(Z_i, E_{v(i)}) > 0$, hence we have no contribution in the sum of $S^{\mathscr{L}}(\gamma)$.

Hence, we need to focus only on the steps of the first type. Well, one verifies that if $\bar{k}' \le i < \bar{k}$, then $(\bar{Z}_i, E_{\bar{v}(i)}) > 0$ as well. As a result, they also have zero contribution in the sum. Next, let us analyse the other steps $0 \le i \le \bar{k}'$.

For any $n \in \mathscr{N}$, corresponding to the face $F_n \subset \Gamma(f)$, denote by C_n the convex closure of the union of F_n and $\{p\}$. Set also

$$C_n^- = C_n \setminus \bigcup_{n' \ge n} C_{n'},$$

where we use the partial ordering $\le$ on $\mathscr{N}$ defined in 11.6.17. For $i = 0, \dots, \bar{k}' - 1$, let H_i be the hyperplane in $M_{\mathbb{R}}$ defined as the set of points $q \in M_{\mathbb{R}}$ satisfying $\ell_n(q-p) = m_{\bar{v}(i)}(\bar{Z}_i)$. Set also $F_i^- = C^-_{\bar{v}(i)} \cap H_i$. The sets C_n^- form a partition of the lattice points $C(N\Gamma) \setminus N\Gamma(f)$. These are the points 'under the Newton diagram', hence by 5.5.21 their number is p_g. It follows from the construction that the family $(F_i^- \cap M)_{i=0}^{\bar{k}'-1}$ forms a partition of these points. In particular,

$$p_g = \sum_{i=0}^{\bar{k}'-1} \#(F_i^- \cap M). \tag{11.6.11}$$

The key ingredient of the proof is the following geometric fact, which creates the bridge between Γ and $N\Gamma$: for any $0 \le i < \bar{k}'$ the following identity holds:

$$\#(F_i^- \cap M) = \max\{0, 1 - (\bar{Z}_i, E_{\bar{v}(i)})\}.$$

This means that along the computation sequence, the expressions $\max\{0, 1 - (\bar{Z}_i, E_{\bar{v}(i)})\}$ computed in Γ detect the lattice points under the Newton diagram partitioned by the sequence of moving diagrams $N\Gamma_+(\bar{Z}_i)$. The computation sequence is guided by the above 'ratio test': this guarantees that all the diagrams $N\Gamma_+(\bar{Z}_i)$ are as close to the homotheties of $N\Gamma(f)$ as possible. This is the reason why we called the sequence 'diagonal'. The amazing output of the construction is that this sequence of diagrams of $\Sigma^\vee$, together with the number of lattice points newly appearing at every step, are determined purely from the combinatorics of the resolution graph.

11.6.21 Finally, let us prove that $\mathrm{Path}^{\mathscr{L}} = \mathrm{Path}^i$. Consider a path $\{Z_i\}_{i=1}^k$ as in 11.6.18. We transform it into the *integral* increasing path $\{l_i\}_{i=0}^k$ via $l_i := Z_K - Z_{k-i}$. Note that $\chi(l_i) - \chi(l_{i+1}) = \chi(Z_{k-i}) - \chi(Z_{k-i-1})$, hence (for notation see 6.8.B)

$$S(\{l_i\}_i) = \sum_{i=0}^{k-1} \max\{0, \chi(l_i) - \chi(l_{i+1})\} = \sum_{i=0}^{k-1} \max\{0, 1 - (E_{v(i)}, Z_i)\} = S^{\mathscr{L}}(\{Z_i\}_i).$$

Since $\{Z_i\}_i$ connects $Z_K - \lfloor Z_K \rfloor$ with $Z_K + E_{\mathscr{U}}$, the path $\{l_i\}_i$ connects $-E_{\mathscr{U}}$ with $\lfloor Z_K \rfloor$. Let $\mathrm{Path}^i_{\{-1\}}$ be $\min_\gamma\{S(\gamma)\}$, where γ runs over increasing paths connecting a collection of non–intersection (-1)–curves with $\lfloor Z_K \rfloor$. Then by the above correspondence $\mathrm{Path}^i_{\{-1\}} = \mathrm{Path}^{\mathscr{L}}$. Next, using certain increasing and non–increasing paths we get $\mathrm{Path}^* \le \mathrm{Path}^i_{\{-1\}} \le \mathrm{Path}^i$, hence by 6.8.38, $\mathrm{Path}^i_{\{-1\}} = \mathrm{Path}^i$.

In particular, for any Newton non-degenerate $(X, 0)$ (under the assumption (11.6.5)) we obtain $p_g = \mathrm{Path}^i$, part *(a)* of 11.6.14.

11.6.22 Next, we discuss the assumption (11.6.5). First note that germs which do not satisfy this assumption really exist. Indeed, take e.g. the hypersurface singularity in $(\mathbb{C}^3, 0)$ with equation $x^3 + xy^3 + z^5 + y^{10}z$. It has two faces, let n be the node corresponding to the face with normal vector $(19, 2, 5)$. Then the E_n–multiplicity of Z_K is zero.

The point is that this face is 'removable' (in the sense of [499]). For any Newton non-degenerate Weil divisor $(X, 0)$ in $(Y, 0)$ (as in 5.5.B), which does not satisfy the assumption (11.6.5), in [499] another pair $(\bar{X}, 0) \subset (\bar{Y}, 0)$ is constructed in a canonical way, which satisfies the assumption. Additionally, the following facts also hold:

- $(\bar{X}, 0)$ is Newton non-degenerate.
- The singularities $(X, 0)$ and $(\bar{X}, 0)$ have diffeomorphic links.
- The singularities $(X, 0)$ and $(\bar{X}, 0)$ have the same geometric genera.
- If $(X, 0)$ is normal, then $(\bar{X}, 0)$ is normal. If $(X, 0)$ is Gorenstein, then $(\bar{X}, 0)$ is Gorenstein. In fact, if f is pointed at $p \in M_{\mathbb{Q}}$, then so is $\bar{f}$.

In particular, since $p_g = \mathrm{Path}^i$ holds for $(\bar{X}, 0)$ it must hold for $(X, 0)$ as well.

Remark 11.6.23 One of the consequences of Theorem 11.6.14 is that for isolated hupersurface Newton non-degenerate singularities

$$\min_\gamma \, eu(\mathbb{H}^0(\gamma, \Gamma, K)) = eu(\mathbb{H}^*(\Gamma, K)). \tag{11.6.12}$$

This is a purely topological identity satisfied by the link of a Newton non-degenerate germ, basically provided by the Newton diagram via Oka's algorithm.

It is even more remarkable that the above identity is not a consequence of certain vanishing statements of parts of the lattice cohomology. It holds despite of the fact that in general $\min_\gamma \, eu(\mathbb{H}^0(\gamma, \Gamma, K)) \neq eu(\mathbb{H}^0(\Gamma, K))$ and $\mathbb{H}^{\ge 1}(\Gamma, K) \neq 0$, see e.g. the Newton non-degenerate hypersurface germ 11.4.11.

On the other hand, the identity (11.6.12) does not hold for all topological types. For example, for the graph from 7.3.3 we have $\mathrm{Path}^i = \min_\gamma \, eu(\mathbb{H}^0(\gamma, \Gamma, K)) = 4$ and $eu(\mathbb{H}^*(\Gamma, K)) = 3$, cf. 11.2.19 and 6.8.44.

Similarly, one can consider any superisolated singularity (with rational homology sphere link). Then, by 11.6.12 we know that $\min_\gamma \, eu(\mathbb{H}^0(\gamma, \Gamma, K)) =$

$eu(\mathbb{H}^0(\Gamma, K)$. In particular, if $\nu = 2$ and $\mathbb{H}^1(\Gamma, K) \neq 0$ then $\min_\gamma eu(\mathbb{H}^0(\gamma, \Gamma, K)) \neq eu(\mathbb{H}^*(\Gamma, K)$.

11.7 Lattice Cohomology and Heegaard Floer Homology

The Seiberg–Witten invariant is the (normalized) Euler-characteristic of the Seiberg–Witten monopole Floer homology of Kronheimer–Mrowka, or equivalently, of the Heegaard Floer homology of Ozsváth and Szabó. These theories had an extreme influence on the modern mathematics, solving (or disproving) a long list of old conjectures (e.g. Thom Conjecture, or conjectures regarding classification of 4-manifolds, or famous old problems in knot theory); see the long list of distinguished articles of Kronheimer–Mrowka or Ozsváth–Szabó. See also a historical review in the prologue of Chapter 9 in the Introduction.
In [547] Ozsváth and Szabó provided a computation of the Heegaard Floer homology for some special plumbed 3-manifolds. This computation resonated incredibly with the theory of computation sequences used in Artin–Laufer program (see e.g. [363, 469, 476]). These two facts influenced considerably the definition of the lattice cohomology. ♣

11.7.1 Short Review of Heegaard Floer Homology $HF^+(M)$ We assume that M is an oriented rational homology 3–sphere, and we restrict ourselves to the $+$–theory of Ozsváth and Szabó. The Heegaard Floer homology $HF^+(M)$ is a $\mathbb{Z}[U]$–module with a $\mathbb{Q}$–grading compatible with the $\mathbb{Z}[U]$–action, where $\deg(U) = -2$. Additionally, $HF^+(M)$ has another $\mathbb{Z}_2$–grading; $HF^+(M)_{even}$, respectively $HF^+(M)_{odd}$ denote the graded parts. Moreover, $HF^+(M)$ has a natural direct sum decomposition of $\mathbb{Z}[U]$–modules (compatible with all the gradings): $HF^+(M) = \oplus_\sigma HF^+(M, \sigma)$ indexed by the spin^c structures σ of M. For any $\sigma \in \mathrm{Spin}^c(M)$ one has

$$HF^+(M, \sigma) = \mathscr{T}^+_{d(M,\sigma)} \oplus HF^+_{red}(M, \sigma),$$

a graded $\mathbb{Z}[U]$–module isomorphism, and $HF^+_{red}(M, \sigma)$ has finite $\mathbb{Z}$–rank and an induced $\mathbb{Z}_2$–grading. One also considers

$$\chi(HF^+(M, \sigma)) := \mathrm{rank}_{\mathbb{Z}}\, HF^+_{red,even}(M, \sigma) - \mathrm{rank}_{\mathbb{Z}}\, HF^+_{red,odd}(M, \sigma).$$

By changing the orientation we have $\chi(HF^+(M, \sigma)) = -\chi(HF^+(-M, \sigma))$ and $d(M, \sigma) = -d(-M, \sigma)$. By our convention of the Seiberg–Witten invariant (defined as in 9.6.1) it turns out that the Seiberg–Witten invariant $\mathfrak{sw}(-M, \sigma)$ equals $\chi(HF^+(M, \sigma)) - d(M, \sigma)/2$.

11.7.2 Relation Between the Lattice Cohomology and Heegaard Floer Theory

In [480] the author formulated the following

Conjecture 11.7.3 For any plumbed rational homology sphere associated with a connected negative definite graph Γ, and for any $k \in \mathrm{Char}$, one has

$$d(M,[k]) = \max_{k' \in [k]} \frac{(k')^2 + |\mathcal{V}|}{4} = \frac{k^2 + |\mathcal{V}|}{4} - 2 \cdot \min \chi_k.$$

Furthermore,

$$\begin{aligned} HF^+_{red,even}(-M,[k]) &= \bigoplus_{p \text{ even}} \mathbb{H}^p_{red}(\Gamma,[k])[-d], \quad \text{and} \\ HF^+_{red,odd}(-M,[k]) &= \bigoplus_{p \text{ odd}} \mathbb{H}^p_{red}(\Gamma,[k])[-d]. \end{aligned} \tag{11.7.1}$$

Both parts of the Conjecture were verified for almost rational graphs in [480], for two bad vertices in [551], see [480, 8.4] too.

A positive answer to this conjecture also shows that $\mathbb{H}^*$ has a richer structure: its q–filtration $\mathbb{H}^* = \oplus_q \mathbb{H}^q$ collapses at the level of HF^+ to a $\mathbb{Z}_2$ odd/even filtration.

The fact that both theories have the same Euler characteristic support the above conjecture as well. Another supporting evidence is the following fact.

11.7.4 Coincidence of the Vanishing of the Reduced Parts By 11.1.28 the graph Γ is rational if and only if $\mathbb{H}^*_{red}(\Gamma) = 0$. On the other hand, following Ozsváth and Szabó, by definition, M is an L–space if and only if $HF^+_{red} = 0$. Their equivalence is predicted by Conjecture 11.7.3. This 'tip of the iceberg' statement was proved in [485]:

Theorem 11.7.5 *The following facts are equivalent for a connected negative definite graph Γ:*

(i) Γ is a rational graph,
(ii) $M = M(\Gamma)$ is an L–space.

$(i) \Rightarrow (ii)$ follows from lattice cohomology theory [478, 480], while $(ii) \Rightarrow (i)$ uses partly the following equivalence $(ii) \Leftrightarrow (iii)$, where (iii) means that $\pi_1(M)$ is not a left-orderable group. The equivalence $(ii) \Leftrightarrow (iii)$ was proved in [255] for any graph–manifolds. For arbitrary 3–manifolds was conjectured by Boyer, Gordon and Watson [74], for different developments and other references see [255, 485].

A non trivial group G is said to be left-orderable if there exist a total order $<$ on G such that if $a < b$ then $ga < gb$ for every $g \in G$.

11.7.6 Having this characterization of rational graphs in terms of $\pi_1(M)$ it is natural to formulate the following:

Problem 11.7.7 Characterize elliptic singularities (or other non-rational families of singularities) by a certain property of the fundamental group of the link.

11.7.8 During the preparation of the last version of the present book a preprint of I. Zemke appeared [785], which claims the proof of the isomorphism of the lattice homology and HF^- as $\mathbb{Z}_2[[U]]$–modules compatibly with the decomposition over $\mathrm{Spin}^c(M)$ and relative grading.

11.7.9 For some more connection (e.g. applications of the graded roots in topology) see [132, 280, 296, 297].

11.8 Combinatorial Lattice Cohomology with Special Weight Functions

In this section we prove several combinatorial statements regarding the lattice cohomology associated with a weight function with certain combinatorial properties: the *Stability Property* and the *Combinatorial Duality Property*. The first one usually follows from the *matroid rank inequality* of certain filtrations, the second one usually is provided by geometric dualities. If they are satisfied then by a combinatorial argument a very mysterious fact happens: the coincidence of $eu(\mathbb{H}^*(R,w))$ and $eu(\mathbb{H}^*(\gamma,w))$ for any increasing path γ.
In this section we follow [8]. ♣

11.8.1 The Combinatorial Setup Fix $\mathbb{Z}^s$ with a fixed basis $\{E_v\}_{v\in\mathcal{V}}$, $|\mathcal{V}| = s$. Fix also an element $c \in \mathbb{Z}^s$, $c \geq 0$. Consider the real rectangles $R = R(0,c) := \{l \in \mathbb{R}^s : 0 \leq l \leq c\}$. Here we can consider the case $c = \infty$ too, in such a case $R = (\mathbb{R}_{\geq 0})^s$. Furthermore, assume that to each $l \in R \cap \mathbb{Z}^s$ we assign

(i) an integer $h(l)$ such that $h(0) = 0$ and $h(l+E_v) \geq h(l)$ for any $v, l, l+E_v \in R$,
(ii) an integer $h^\circ(l)$ such that $h^\circ(l+E_v) \leq h^\circ(l)$ for any v and $l, l+E_v \in R$.

Once h and $c < \infty$ are fixed with (i), a possible choice for h° is h^{sym}, where $h^{sym}(l) := h(c-l)$. Clearly $h^\circ = h^{sym}$ defined in this way depends on c.

We consider the set of cubes $\{\mathcal{Q}_q\}_{q\geq 0}$ of R as in 11.1.3 and the weight function

$$w_0 : \mathcal{Q}_0 \to \mathbb{Z} \text{ by } w_0(l) := h(l) + h^\circ(l) - h^\circ(0).$$

Clearly $w_0(0) = 0$. Moreover, similarly as in 11.1.17, we define $w_q : \mathcal{Q}_q \to \mathbb{Z}$ by $w_q(\square_q) = \max\{w_0(l) : l \text{ is a vertex of } \square_q\}$. We will use the symbol w for the system $\{w_q\}_q$. The compatible weight functions define the lattice cohomology $\mathbb{H}^*(R,w)$. Furthermore, for any increasing path γ connecting 0 and c we also have a path lattice cohomology $\mathbb{H}^0(\gamma,w)$ as in 11.6.1. Accordingly, we have the numerical Euler characteristics $eu(\mathbb{H}^*(R,w))$, $eu(\mathbb{H}^0(\gamma,w))$ and $\min_\gamma eu(\mathbb{H}^0(\gamma,w))$ too.

Lemma 11.8.2 *We have* $0 \le eu(\mathbb{H}^0(\gamma, w)) \le h^\circ(0) - h^\circ(c)$ *for any increasing path* γ *connecting 0 to c. The equality* $eu(\mathbb{H}^0(\gamma, w)) = h^\circ(0) - h^\circ(c)$ *holds if and only if for any i the differences* $h(x_{i+1}) - h(x_i)$ *and* $h^\circ(x_i) - h^\circ(x_{i+1})$ *simultaneously are not nonzero.*

Proof *(a)* Since $w_0(0) = 0$ we have $eu(\mathbb{H}^0(\gamma, w)) = -\min w_0 + \mathrm{rank}\mathbb{H}^0_{red}(\gamma, w) \ge 0$. Next, by 11.6.2 we have $eu(\mathbb{H}^0(\gamma, w)) = \sum_{i=0}^{t-1} \max\{0, w(x_i) - w(x_{i+1})\}$. On the other hand, $w(c) + \sum_i \max\{0, w(x_i) - w(x_{i+1})\} = -\sum_i \min\{0, w(x_i) - w(x_{i+1})\}$, hence $w(c) + 2 \cdot eu(\mathbb{H}^0(\gamma, w)) = \sum_i |w(x_i) - w(x_{i+1}| = \sum_i |h(x_i) + h^\circ(x_i) - h(x_{i+1}) - h^\circ(x_{i+1})| \le \sum_i (h(x_{i+1}) - h(x_i)) + \sum_i (h^\circ(x_i) - h^\circ(x_{i+1})) = h(c) + h^\circ(0) - h^\circ(c)$. □

Remark 11.8.3 The inequality $0 \le eu(\mathbb{H}^*(R, w))$ is not true in general. Take e.g. the following table ($s = 2$, $c = (2, 2)$) with $h^\circ = h^{sym}$:

$$h: \begin{matrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{matrix} \qquad w_0: \begin{matrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{matrix} \qquad \begin{matrix} \min w_0 = 0 \\ \mathbb{H}^0_{red} = 0,\ \mathbb{H}^1 = \mathbb{Z} \\ eu = -1 \end{matrix}$$

The inequality $eu(\mathbb{H}^*(R, w)) \le h^\circ(0) - h^\circ(c)$ (or $eu(\mathbb{H}^0(R, w)) \le h^\circ(0) - h^\circ(c)$) is not true either, see e.g.

$$h: \begin{matrix} 0 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{matrix} \qquad w_0: \begin{matrix} -2 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & -2 \end{matrix} \qquad (\text{with } h^\circ = h^{sym})$$

Then $eu(\mathbb{H}^0(\gamma, w)) = 2$, while $eu(\mathbb{H}^0(R, w)) = eu(\mathbb{H}^*(R, w)) = 4$.

11.8.4 We will focus on pairs (h, h°) which satisfy certain additional properties.

Definition 11.8.5 We say that h satisfies the *'matroid rank inequality'* if

$$h(l_1) + h(l_2) \ge h(\min\{l_1, l_2\}) + h(\max\{l_1, l_2\}), \quad l_1, l_2 \in R. \tag{11.8.1}$$

Note that the 'matroid rank inequality' implies the *'stability property'*

$$h(l) = h(l + E_v) \ \Rightarrow \ h(l + \bar{l}) = h(l + \bar{l} + E_v). \tag{11.8.2}$$

valid for any l and $\bar{l}$ such that $\bar{l} \ge 0$ and $|\bar{l}| \not\ni E_v$.

Example 11.8.6

(1) Fix $\mathbb{Z}^s$ and c as in 11.8.1. Let M be a finite dimensional vector space with a $\mathbb{Z}^s$-grading $\{M_{\mathbf{a}}\}_{\mathbf{a}}$ such that $M_{\mathbf{a}} = 0$ whenever either $\mathbf{a} \not\ge 0$ or $\mathbf{a} \ge c$. Let $\mathfrak{h}$ be the Hilbert function $\mathfrak{h}(l) = \sum_{\mathbf{a} \not\ge l} \dim M_{\mathbf{a}}$ and let h be its restriction to $R(0, c) \cap \mathbb{Z}^s$. Then $h(0) = 0$, $h(c) = \dim M$, and h satisfies the matroid rank inequality.

(2) Assume that M is a finite dimensional vector space endowed with a decreasing $\mathbb{Z}^s$-filtration such that $F(0) = M$ and $F(c) = 0$ and define $\mathfrak{h}(l) = \dim(M/F(l))$ for any $l \geq 0$. Again, define h as the restriction of $\mathfrak{h}$ to $R(0,c) \cap \mathbb{Z}^s$. Then usually the matroid rank inequality is not satisfied (see e.g. below in (3)).

(3) Let $\{\mathscr{F}(\mathbf{a})\}_{\mathbf{a}}$ be a $\mathbb{Z}^s$–filtration of a local ring $\mathscr{O}_{X,o}$ (cf. 8.1.22). If the filtration is provided by a collection of order functions (cf. 8.1.21) then $\{\mathscr{F}(\mathbf{a})\}_{\mathbf{a}}$ is a $\mathbb{Z}^s$–filtration of ideals and $\mathscr{F}(\mathbf{a}) \cap \mathscr{F}(\mathbf{b}) = \mathscr{F}(\max\{\mathbf{a},\mathbf{b}\})$. In particular, $\mathfrak{h}(\mathbf{a}) = \dim \mathscr{O}_{X,o}/\mathscr{F}(\mathbf{a})$ satisfies the matroid rank inequality. However, for more general filtrations the matroid rank inequality does not necessarily hold. E.g., for $\mathscr{F}^{pr}$ considered at the end of 8.1.22, $\mathfrak{p}((4,4)) = -1$, hence in the $(4,4)$–based 2-cube the matroid rank inequality fails.

(4) Let (X,o) be a normal surface singularity and $\widetilde{X}$ a fixed resolution. Then $l \in L_{\geq 0}$, $l \mapsto -h^1(\mathscr{O}_l)$ satisfies the matroid rank inequality (cf. 6.4.21(b)).

Definition 11.8.7 We say that the pair h and h° satisfies the 'Combinatorial Duality Property' (CDP) if $h(l+E_v) - h(l)$ and $h^\circ(l+E_v) - h^\circ(l)$ simultaneously cannot be nonzero for l, $l + E_v \in R \cap \mathbb{Z}^s$. Furthermore, we say that h satisfies the CDP if the pair (h, h^{sym}) satisfies it.

Example 11.8.8

(1) Let (X,o) be a normal surface singularity and let ϕ be a good resolution. Assume that $g_v = 0$ for every $v \in \mathscr{V}$. Let $l \mapsto h(l)$, $l \in L_{\geq 0}$ be the Hilbert function associated with the divisorial filtration and $h^\circ(l) = p_g - h^1(\mathscr{O}_l)$. Then the pair (h, h°) satisfies the CDP, cf. 11.9.15.

(2) The equivariant version is the following. Let (X,o) be as in (1) and assume that $H_1(L_X, \mathbb{Q}) = 0$. Let $Z_{coh,h}$ be the cohomological cycle of $\mathscr{O}_{\widetilde{X}}(-r_h)$, cf. 6.4.24.

Fix any $c \geq Z_{coh,h}$ and define $h(l) := \mathfrak{h}(l + r_h)$ and $h^\circ(l) := h^1(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(-r_h)) - h^1(\mathscr{O}_l(-r_h))$. Then (h, h°) satisfies the CDP. For the proof (based on Laufer duality) and details regarding this cohomology theory see [9].

(3) Let (X,o) be a normal surface singularity, ϕ a good resolution, and $c = Z_K$, where $Z_K \in L_{>0}$. Assume that $g_v = 0$ for every $v \in \mathscr{V}$. Let h be as in part (1) and $h^\circ(l) = h^{sym}(l) = h(Z_K - l)$. Then h satisfies the CDP. Indeed, otherwise there exist $s', s'' \in \mathscr{S}_{an}$ such that $\mathrm{div}_E(s') \geq l$, $s'_v = l_v$, $\mathrm{div}_E(s'') \geq Z_K - l - E_v$, $s''_v = (Z_K - l)_v$. Hence, $\mathrm{div}_E(s's'') \geq Z_K - E_v$ with equality at E_v–coordinate. But this contradicts $H^0(\mathscr{O}_{E_v}(-Z_K + E_v)) \simeq H^1(\mathscr{O}_{E_v})^* = 0$.

(4) Let h be the Hilbert function associated with a Gorenstein curve singularity (C,o) (via valuations given by the normalization). Let c be the conductor and define h^{sym} with respect to c. Then $h(l+E_v) - h(l) \in \{0,1\}$. Furthermore, $h(l+E_v) - h(l) = 1$ if and only if $h^{sym}(l) - h^{sym}(l+E_v) = 0$. This follows from the identity $h(c-l) - h(l) = \delta(C) - \sum_v l_v$, cf. [101]. Hence h satisfies the CDP.

For the general case of an isolated reduced curve singularity set h as before and $h^\circ(l) := h(l) - \sum_v l_v$. Then (h, h°) satisfies the CDP. For more on the cohomologies associated with (h, h°) see [11].

(5) For the combinatorial case associated with a Newton diagram, see 11.8.13.

Definition 11.8.9 We say that the pair (h, h°) satisfies the

(a) *'path eu-coincidence'* if $eu(\mathbb{H}^0(\gamma, w)) = h^\circ(0) - h^\circ(c)$ for any increasing path γ.

(b) *'eu-coincidence'* if $eu(\mathbb{H}^*(R, w)) = h^\circ(0) - h^\circ(c)$.

Remark 11.8.10 In 11.4.11 and 11.8.15 we present a w_0–table in the rectangle $R = R((0, 0), (14, 14))$, which is associated with a certain h and $h^\circ = h^{sym}$ (provided by a graded vector space). This diagram satisfies both the 'path eu-coincidence' and 'eu-coincidence' properties, and it shows the following two facts.

Even if h satisfies the path eu-coincidence (and $h^\circ = h^{sym}$), in general it is not true that $\mathbb{H}^0(\gamma, w)$ is independent of the choice of the increasing path. (This statement remains valid even if we consider only the symmetric increasing paths, where a path $\gamma = \{x_i\}_{i=0}^t$ is symmetric if $x_{t-l} = c - x_l$ for any l.)

Even if h satisfies both the path eu-coincidence and the eu-coincidence, in general it is not true that $\mathbb{H}^*(R, w)$ equals any of the path lattice cohomologies $\mathbb{H}^0(\gamma, w)$ associated with a certain increasing path. Indeed, in the case of 11.8.15 we have $\mathbb{H}^1(R, w) \neq 0$, a fact which does not hold for any path lattice cohomology. However, all the *Euler characteristics* agree.

Remark 11.8.11 In the next discussion assume that $h^\circ = h^{sym}$.

(a) The CDP of h implies the 'path eu-coincidence' of h, see 11.8.2. However, the CDP of h does not imply the 'eu-coincidence' of h. As an example consider the second case from 11.8.3. Note that in this case the matroid rank inequalities, or the stabilization properties, are not satisfied by h.

(b) On the other hand, an eu-coincidence type property cannot be hoped without (some type of) CDP. Indeed, consider the following case: $s = 2$, $c = (2, 2)$, and h is associated with a graded vector space of dimension 2 supported in $(0, 1)$ and $(1, 2)$, cf. 11.8.6(1). In this case $h(c) = 2$, $eu(\mathbb{H}^0(R, w)) = 0$ and for any symmetric increasing path $eu(\mathbb{H}^0(\gamma, w)) = 0$ too, and for any non-symmetric paths $eu(\mathbb{H}^0(\gamma, w)) = 1$.

Theorem 11.8.12

(a) Assume that h satisfies the stability property, and the pair (h, h°) satisfies the Combinatorial Duality Property. Then (h, h°) satisfies both the path eu- and the eu-coincidence properties. That is, for any increasing γ we have

$$eu(\mathbb{H}^*(\gamma, w)) = eu(\mathbb{H}^*(R, w)) = h^\circ(0) - h^\circ(c).$$

(b) Take $c = \infty$, that is, $R = (\mathbb{R}_{\geq 0})^s$. Then

$$\sum_{l\geq 0}\sum_{I}(-1)^{|I|+1}w((l,I))\,\mathbf{t}^l = \sum_{l\geq 0}\sum_{I}(-1)^{|I|+1}h(l+E_I)\,\mathbf{t}^l. \tag{11.8.3}$$

Proof

(a) The identity $eu(\mathbb{H}^*(\gamma, w)) = h^\circ(0) - h^\circ(c)$ follows from 11.8.2. Next we focus on the second identity. We claim that for any $I \subset \mathscr{V}$ we have

$$w((l,I)) - w(l) = h(l+E_I) - h(l). \tag{11.8.4}$$

We use induction over the cardinality $|I|$ of I. If $I = \{v\}$, then $w((l,I)) - w(l) = \max\{0, w(l+E_v) - w(l)\}$. But if $h(l+E_v) > h(l)$ then $h^\circ(l+E_v) = h^\circ(l)$ hence $w(l+E_v) - w(l) = h(l+E_v) - h(l)$. Otherwise $w(l+E_v) \leq w(l)$ and $w((l,I)) = w(l)$.

Next, assume that $|I| > 1$ and $h(l+E_v) = h(l)$ for every $v \in \mathscr{V}$. Then by iterated use of the stability property of h, $h(l+E_J) = h(l)$ for any $J \subset I$. Moreover, $w((l,I)) - w(l) = 0 = h(l+E_I) - h(l)$.

Finally, assume that $|I| > 1$ and $h(l+E_v) > h(l)$ for a certain $v \in \mathscr{V}$. This means that $w(l+E_v) > w(l)$, hence $w((l,I)) > w(l)$ (†). Assume that $w((l,I))$ is realized for a certain $J \subset I$. Let J be minimal by this property. By (†) we know that $J \neq \emptyset$.

By minimality of J, $w(l+E_{J\setminus u}) < w(l+E_J)$ for any $u \in J$. By CDP $h(l+E_{J\setminus u}) < h(l+E_J)$ too, hence by the stability of h we also have $h(l) < h(l+E_u)$.

In particular we found $u \in I$ such that $h(l) < h(l+E_u)$ (hence $h^\circ(l) = h^\circ(l+E_u)$) and $w((l,I)) = w((l+E_u, I^*))$, where $I^* := I \setminus u$. Now, we use induction applied for the cube $(l+E_u, I^*)$. In particular, $w((l,I)) - w(l) = w((l+E_u, I^*)) - w(l) = w(l+E_u) + h(l+E_I) - h(l+E_u) - w(l) = h(l+E_I) - h(l)$.

This ends the proof of the claim.

Let us denote by $\mathscr{Q}$ the set of cubes of R. For $eu(\mathbb{H}^*(R,w))$ we use the following formula (with identical proof as in (11.1.4))

$$eu(\mathbb{H}^*(R,w)) = \sum_{(l,I)\in\mathscr{Q}}(-1)^{|I|+1}w((l,I)).$$

We will subdivide the lattice points of R into the following disjoint subsets. For any $0 \leq t \leq s$ we denote by R_t the set of those points $l \in R \cap \mathbb{Z}^s$ for which the cardinality of $\{v \in \mathscr{V} : l_v = c_v\}$ is t. If $l \in R_t$ set $I(l) := \{v : l_v < c_v\}$. In particular, the set $\mathscr{Q}$ is also a disjoint union of the sets $\{(l,I) : l \in R_t,\ I \subset I(l)\}_t$. Then

$$eu(\mathbb{H}^*(R,w)) = \sum_{t}\sum_{l\in R_t}\sum_{I\subset I(l)}(-1)^{|I|+1}w((l,I)).$$

If $0 \le t < s$ and $l \in R_t$ then $I(l) \neq \emptyset$. In this case for any I-independent (but maybe l-dependent) 'constant' $a(l)$ we have $\sum_{I\subset I(l)}(-1)^{|I|+1}a(l) = 0$. Therefore, by (11.8.4), for any such l,

$$\sum_{I\subset I(l)} (-1)^{|I|+1} w((l, I)) = \sum_{I\subset I(l)} (-1)^{|I|+1} h(l + E_I). \tag{11.8.5}$$

On the other hand, if $t = s$, then $R_t = \{c\}$, $I(c) = \emptyset$, hence $\sum_{I\subset I(c)}(-1)^{|I|+1}$ $((w(c, I)) = -w(c)$. Hence, corresponding to $t = s$ (11.8.5) fails, i.e. it must be corrected by $-w(c) + h(c) = h^\circ(0) - h^\circ(c)$. Therefore,

$$eu(\mathbb{H}^*(R, w)) = h^\circ(0) - h^\circ(c) + \sum_{(l,I)\in\mathscr{Q}} (-1)^{|I|+1} h(l + E_I). \tag{11.8.6}$$

For any fixed $\widetilde{l} \in R$, consider the following summation over $\{(l, I) : l + E_I = \widetilde{l}\}$:

$$S(\widetilde{l}) := \sum (-1)^{|I|+1} h(l + E_I) = h(\widetilde{l}) \cdot \sum (-1)^{|I|+1}.$$

Then, whenever the cardinality of $\{(l, I) : l + E_I = \widetilde{l}\}$ is > 1, the above sum $\sum(-1)^{|I|+1} = 0$. Thus, $S(\widetilde{l}) = 0$ except when $\widetilde{l} = 0$. But, for $\widetilde{l} = 0$, $S(0) = -h(0) = 0$ too. Thus $eu(\mathbb{H}^*(R, w)) = h^\circ(0) - h^\circ(c) + \sum_{\widetilde{l}} S(\widetilde{l}) = h^\circ(0) - h^\circ(c)$.

(b) Use (11.8.4).

□

Example 11.8.13 (The Combinatorial Lattice Cohomology Associated with a Newton Diagram) Assume that $N\Gamma$ is a convenient Newton diagram in $\mathbb{R}^{n+1}_{\ge 0}$ (see 5.5.A or [498]). Let $\triangle^{(1)}$ denote the n-dimensional faces of $N\Gamma$, and for each $\sigma \in \triangle^{(1)}$ let ℓ_σ be the normal primitive integral vector of the corresponding face F_σ (with all entries positive). Then F_σ is on the affine n-plane $\langle \ell_\sigma, p\rangle = m_\sigma$. Let M be the $\mathbb{C}$–vector space generated by lattice points $p \in \mathbb{Z}^{n+1}_{>0}$, which sit either on or below $N\Gamma$. This means that $p \in \mathbb{Z}^{n+1}_{>0}$ should satisfy $\langle \ell_\sigma, p\rangle \le m_\sigma$ for at least one σ. Then $M \neq 0$ if and only if $\mathbf{1} = (1, \dots, 1)$ is such a point; in the sequel we will assume this fact.

We order $\triangle^{(1)}$ as $\{\sigma_1, \dots, \sigma_s\}$. We introduce a $\mathbb{Z}^s$-grading of M by

$$\deg(p) = (\, \langle \ell_{\sigma_1}, p - \mathbf{1}\rangle, \dots, \langle \ell_{\sigma_s}, p - \mathbf{1}\rangle\,) \in \mathbb{Z}^s.$$

Define $c = (c_1, \dots, c_s)$ by $c_i := m_{\sigma_i} + 1 - \langle \ell_{\sigma_i}, \mathbf{1}\rangle$.

Since $p \ge \mathbf{1}$ for any point (generator of M), we also have $\deg(p) \in \mathbb{Z}^s_{\ge 0}$. Moreover, for any p there exists σ_i with $\langle \ell_{\sigma_i}, p\rangle \le m_{\sigma_i}$, hence $\langle \ell_{\sigma_i}, p - \mathbf{1}\rangle < c_i$. Therefore, $\deg(p) \not\ge c$. In particular, M is a $\mathbb{Z}^s$–graded vector space such that the conditions from 11.8.6(1) are satisfied.

For any lattice point from $R(0,c)$ set $h(l)=\sum_{\mathbf{a}\not\geq l}\dim\, M_{\mathbf{a}}$ as in 11.8.6(1), hence it satisfies the stability property. Consider also $h^\circ = h^{sym}$ associated with the above c.

We claim that if *there is no face F_σ whose relative interior contains a lattice point p then the pair (h,h^{sym}) satisfies the CDP.* Indeed, assume that this is not the case. Then there exists l and $1\leq v\leq s$ such that $h(l)\neq h(l+E_v)$ and $h(c-l)\neq h(c-l-E_v)$. By the first fact there exists $p\in\mathbb{Z}^{n+1}_{>0}$ such that $\langle \ell_{\sigma_u}, p-\mathbf{1}\rangle \geq l_u$ for every u and $\langle \ell_{\sigma_v}, p-\mathbf{1}\rangle = l_v$, and by the second one there exists $q\in\mathbb{Z}^{n+1}_{>0}$ such that $\langle \ell_{\sigma_u}, q-\mathbf{1}\rangle \geq c_u-l_u$ for every $u\neq v$ and $\langle \ell_{\sigma_v}, q-\mathbf{1}\rangle = c_v-1-l_v$. In particular, $\langle \ell_{\sigma_u}, p+q-\mathbf{1}\rangle > m_{\sigma_u}$ for every $u\neq v$ and $\langle \ell_{\sigma_v}, p+q-\mathbf{1}\rangle = m_{\sigma_v}$. But then $p+q-\mathbf{1}$ is a relative interior lattice point of F_{σ_v}, a fact which contradicts the assumption.

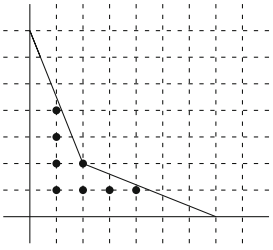
Let us call the lattice cohomology associated with the weight function $l\mapsto w_0(l)=h(l)+h^{sym}(l)-h^{sym}(0)$ the *combinatorial lattice cohomology associated with the Newton diagram $N\Gamma$*. It will be denoted by $\mathbb{H}^*_{N\Gamma}(R,w)$. Similarly, for any increasing path γ connecting 0 and c we consider the *combinatorial path lattice cohomology associated with the Newton diagram $N\Gamma$*, denoted by $\mathbb{H}^*_{N\Gamma}(\gamma,w)$.

Then, by the above discussion and by Theorem 11.8.12, if in the relative interior of the faces F_σ there are no lattice points, we have

$$eu(\mathbb{H}^*_{N\Gamma}(\gamma,w))=eu(\mathbb{H}^*_{N\Gamma}(R,w))=\dim\, M=\#\,N\Gamma_-\cap\mathbb{Z}^{n+1}_{>0}. \tag{11.8.7}$$

Example 11.8.14 Consider the situation of 11.8.13 with $n=1$ and assume that there is no 1–face of $N\Gamma$ whose relative interior contains a lattice point. Assume that $N\Gamma$ is the Newton boundary of an isolated singularity f which has a non-degenerate Newton principal part. Then the number of irreducible components s is exactly the number of 1–faces. Moreover, $\dim\, M$ equals the delta invariant of f. (This follows from the Milnor number formula from 5.5.4(a), Pick formula for the lattice poins and the delta invariant formula $\mu=2\delta-s+1$ from 5.2.2.) Furthermore, c defined in 11.8.13 is the conductor of f (use (8.1.20)). In particular, the combinatorial $\mathbb{H}^*_{N\Gamma}$ is a categorification of the delta invariant.

For example, let $N\Gamma$ be generated by the lattice points $(0,7)$, $(2,2)$, $(7,0)$. This is the Newton diagram of the Newton non-degenerate plane curve singularity $f=(x^2+y^5)(y^2+x^5)$. The normal vectors $(2,5)$ and $(5,2)$ of the two faces provide the degrees. The next diagram shows the points p, which generate M as base elements, and their degrees.



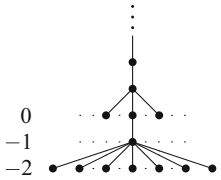
$(15,6)$
$(10,4)$
$(5,2)$ $(7,7)$
$(0,0)$ $(2,5)$ $(4,10)$ $(6,15)$

The conductor c is (8, 8), and $\dim M = \delta(f) = 8$. The next tables show the function h and the weight function w_0 on R.

8	6	6	6	6	6	7	7	8	8
7	5	5	5	5	5	6	6	7	8
6	4	4	4	4	4	5	5	6	7
5	3	3	3	4	4	5	5	6	7
4	2	2	2	3	3	4	4	5	6
3	2	2	2	3	3	4	4	5	6
2	1	1	1	2	2	3	4	5	6
1	1	1	1	2	2	3	4	5	6
0	0	1	1	2	2	3	4	5	6
	0	1	2	3	4	5	6	7	8

4	3	2	1	0	1	0	1	[0]
3	2	1	0	−1	0	−1	0	1
2	1	0	−1	[−2]	−1	[−2]	−1	0
1	0	−1	0	−1	(0)	−1	0	1
0	−1	[−2]	−1	[−2]	−1	[−2]	−1	0
1	0	−1	(0)	−1	0	−1	0	1
0	−1	[−2]	−1	[−2]	−1	0	1	2
1	0	−1	0	−1	0	1	2	3
[0]	1	0	1	0	1	2	3	4

The circled points show the generators of $\mathbb{H}^1(R, w)$, while the boxes the local minima of w_0. Hence $\mathbb{H}^1(R, w) = \mathscr{T}_{-2}(1)^2$ and $\mathbb{H}^0(R, w) = \mathscr{T}^+_{-4} \oplus \mathscr{T}_{-4}(1)^6 \oplus \mathscr{T}_0(1)^2$. The graded root is



$\min w_0 = -2$, $\operatorname{rank}(\mathbb{H}^0_{red}(R, w)) = 8$, $\operatorname{rank}(\mathbb{H}^1(R, w)) = 2$, and $eu(\mathbb{H}^*(R, w)) = 8$.

From analytic point of view, one can show that the above function h agrees with the Hilbert function associated with the filtration of valuations associated with the normalization. This can be verified e.g. by using the formula of the analytic Hilbert function via the multivariable Alexander polynomials, see 8.1.26 or [217].

(Nevertheless, we warn the reader that for a more general $N\Gamma$, several filtrations—projected, order, valuative—might all be different.)

Example 11.8.15 Consider the situation of 11.8.13 with $n = 2$ and assume that $N\Gamma$ is the Newton boundary of an isolated singularity f which has a non-degenerate Newton principal part. Then c is exactly the restriction of Z_K to the coordinates of the nodes (use 6.3.5). Moreover, $\dim M$ is the geometric genus of f, cf. 5.5.4(c). In particular, if the link is a $\mathbb{Q}HS^3$, the combinatorial $\mathbb{H}^*_{N\Gamma}$ is a categorification of p_g.

E.g., consider the hypersurface $x^{13}+y^{13}+x^2y^2+z^3$ with Newton nondegenerate principal part. For the resolution graph see 11.4.11.

Let $N\Gamma$ be its Newton diagram. It has two faces with normal vectors (6, 33, 26) and (33, 6, 26). The number of lattice point under the diagram is $p_g = 5$, they are (1, 1, 1), (2, 1, 1), (3, 1, 1), (1, 2, 1), (1, 3, 1). Furthermore, $R = R(0, Z_K|_{\text{nodes}}) = R((0, 0), (14, 14))$. Then a computation shows that the combinatorial weight function $w_0 : R \to \mathbb{Z}$ associated with $(M, \deg)$ from 11.8.13 is exactly the table obtained by the procedure of the (topological) Reduction Theorem applied for the

topological weight function χ, cf. 11.4.11. Hence we have the coincidence of the corresponding lattice cohomologies as well: $\mathbb{H}^*_{N\Gamma}(R, w) = \mathbb{H}^*_{top}(\Gamma, K)$.

11.9 Analytic Lattice Cohomology of Normal Surface Singularities

In this section we construct the *analytic lattice cohomology* $\mathbb{H}^*_{an}(X, o)$ of a normal surface singularity (X, o). We provide a detailed construction of $\mathbb{H}^*_{an,0}(X, o)$, the module associated with the canonical spin^c–structure of the link (or, with the structure sheaf $\mathscr{O}_{X,o}$). It is a graded $\mathbb{Z}[U]$ module. For the equivariant case see 11.9.19.

Then we prove that the pair $(\mathfrak{h}, \mathfrak{h}^\circ)$ satisfies the 'Combinatorial Duality Property' and in the Gorenstein case with the choice $c = Z_K$ (under the assumption $Z_K \geq 0$) we have $\mathfrak{h}^\circ = \mathfrak{h}^{sym}$. Moreover, we prove that the Euler characteristic of $\mathbb{H}_{an,0}(X, o)$ equals the geometric genus. We follow [8]. ♣

11.9.1 Let (X, o) be a normal surface singularity, and we fix a good resolution ϕ.

For any $c \in L$, $c \geq Z_{coh}$, we consider the rectangle $R(0, c)$. Here we might consider the $c = \infty$ case too, in this case $R(0, c) = (\mathbb{R}_{\geq 0})^s$. Then we consider the multivariable Hilbert function $\mathfrak{h} : R(0, c) \cap \mathbb{Z}^s \to \mathbb{Z}$, $\mathfrak{h}(l) = \dim H^0(\mathscr{O}_{\widetilde{X}})/H^0(\mathscr{O}_{\widetilde{X}}(-l))$ associated with the divisorial filtration of $\mathscr{O}_{X,o}$ and the resolution ϕ, cf. 8.2.6. Clearly $\mathfrak{h}$ is increasing (that is, $\mathfrak{h}(l_1) \geq \mathfrak{h}(l_2)$ whenever $l_1 \geq l_2$) and $\mathfrak{h}(0) = 0$.

Next, set $\mathfrak{h}^\circ(l) = p_g - h^1(\mathscr{O}_l)$ too (where, by definition, $h^1(\mathscr{O}_{l=0}) = 0$). Then $\mathfrak{h}^\circ$ is decreasing, $\mathfrak{h}^\circ(0) = p_g$ and $\mathfrak{h}^\circ(c) = 0$, cf. 6.4.7 and 6.4.A.

Finally we define the weight function

$$w_0 : \mathscr{Q}_0 \to \mathbb{Z}, \quad w_0(l) = \mathfrak{h}(l) + \mathfrak{h}^\circ(l) - \mathfrak{h}^\circ(0) = \mathfrak{h}(l) - h^1(\mathscr{O}_l). \tag{11.9.1}$$

Clearly, $w_0(0) = 0$. This weight function has several useful properties.

First of all, note that $0 \leq \mathfrak{h}^\circ(l) \leq p_g$ for every l, hence when $c = \infty$ then $\mathfrak{h}$ and w_0 have comparable asymptotic behaviour for $l \gg 0$. For any $l \in L$ let $s(l) \in L$ be the smallest element with $s(l) \geq l$ and $s(l) \in \mathscr{S}$. Then $\mathfrak{h}(l) = \mathfrak{h}(s(l))$, cf. (8.2.8). Using this fact, the monotonicity of $\mathfrak{h}$, and (6.4.2) a computation shows that w_0 satisfies the requirement 11.1.4(a), namely, $w_0^{-1}((\infty, n])$ is finite for any $n \in \mathbb{Z}$.

Since $\mathfrak{h}$ is induced by a filtration given by valuations, it satisfies the matroid rank inequality. On the other hand, h^1 satisfies the 'opposite' matroid rank inequality, see 6.4.21. Therefore, w_0 itself satisfies the matroid rank inequality. (Note that χ, the topological weight function, satisfies the matroid rank inequality too.)

We consider next the natural cube–decomposition of $R(0,c)$ as in 11.1.3, and, similarly as in 11.1.17, we define $w_q : \mathcal{Q}_q \to \mathbb{Z}$ by $w_q(\square_q) = \max\{w_0(l) : l$ is any vertex of $\square_q\}$. In the sequel we write w for the system $\{w_q\}_q$ if there is no confusion. This system defines the lattice cohomology $\mathbb{H}^*(R(0,c),w)$ and the graded root $R(R(0,c),\chi_{an})$, with grading χ_{an}, associated with $R(0,c)$ and w.

Lemma 11.9.2 *$\mathbb{H}^*(R(0,c),w)$ and $R(R(0,c),\chi_{an})$ are independent of the choice of $c \geq Z_{coh}$.*

Proof Fix some $c \geq Z_{coh}$ and choose $E_v \subset |c - Z_{coh}|$. Then for any $l \in R(0,c) \cap \mathbb{Z}^s$ with $l_v = c_v$ we have $\min\{l, Z_{coh}\} = \min\{l - E_v, Z_{coh}\}$. Therefore, by 6.4.21, $h^1(\mathscr{O}_{l-E_v}) = h^1(\mathscr{O}_l)$, thus $w_0(l - E_v) \leq w_0(l)$. Then for any $n \in \mathbb{Z}$, a strong deformation retraction in the direction E_v realizes a homotopy equivalence between the spaces $S_n \cap R(0,c)$ and $S_n \cap R(0, c - E_v)$. A retraction $r : S_n \cap R(0,c) \to S_n \cap R(0, c - E_v)$ can be defined as follows. If $\square = (l, I)$ belongs to $S_n \cap R(0, c - E_v)$ then r on $\square$ is defined as the identity. If $(l, I) \cap R(0, c - E_v) = \emptyset$, then $l_v = c_v$, and we set $r(x) = x - E_v$. Else, $\square = (l, I)$ satisfies $v \in I$ and $l_v = c_v - 1$. Then we retract (l, I) to $(l, I \setminus v)$ in the v–direction. The strong deformation retraction is defined similarly. □

Corollary 11.9.3 *Assume that $c \geq Z_{coh}$ or $c = \infty$ as above. Then*

(a) the graded root $R(R(0,c),\chi_{an})$ satisfies $|\chi_{an}^{-1}(n)| = 1$ for any $n \gg 0$;
(b) $\mathbb{H}^*_{red}(R(0,c),w)$ is a finitely generated $\mathbb{Z}$-module (for any finite or infinite $c \geq Z_{coh}$).

Proof For any $n \gg 0$ we have $R(0,c) = S_n$, hence S_n is contractible. □

Notation 11.9.4 We abridge $\mathbb{H}^*(R(0,c),w)$ as $\mathbb{H}^*_{an}(\phi)$ and $R(R(0,c),\chi_{an})$ as $R_{an}(\phi)$.

Theorem 11.9.5 ***(Independence of ϕ)*** *Assume that the resolution graph is a tree (a property independent of the resolution). Then $\mathbb{H}^*_{an}(\phi)$ and $R_{an}(\phi)$ are independent of the choice of the resolution ϕ.*

Proof Let us fix a resolution ϕ, and denote the blow up of a point of $E_{v_0} \setminus \cup_{w \neq v_0} E_w$ by π, and set $\phi' := \phi \circ \pi$. Let Γ and Γ' be the corresponding graphs, $L(\Gamma)$, $L(\Gamma')$ the lattices and $(\,,\,)$, $(\,,\,)'$ the intersection forms.

We denote the new (-1)-vertex of Γ' by E_{new}. We use the same notation for $E_v \in L$ and for its strict transform in L'. We have the natural morphisms: $\pi_* : L(\Gamma') \to L(\Gamma)$ defined by $\pi_*(\sum x_v E_v + x_{new} E_{new}) = \sum x_v E_v$, and $\pi^* : L(\Gamma) \to L(\Gamma')$ defined by $\pi^*(\sum x_v E_v) = \sum x_v E_v + x_{v_0} E_{new}$. Then $(\pi^* x, x')' = (x, \pi_* x')$. Thus $(\pi^* x, \pi^* y)' = (x, y)$ and $(\pi^* x, E_{new})' = 0$ for any $x, y \in L(\Gamma)$. Associated with ϕ, let $\mathfrak{h}$ be the Hilbert function, w_0 the analytic weight and $S_n(\phi) = \cup\{\square : w(\square) \leq n\}$. We use similar notations $\mathfrak{h}'$, w_0' and $S_n(\phi')$ for ϕ'.

Note that for any $x \in R(0,c) \cap \mathbb{Z}^s$, $H^0(\widetilde{X}', \mathscr{O}_{\widetilde{X}'}(-\pi^* x)) = H^0(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(-x))$. Even more, for $a \leq 0$ and x as above, $H^0(\widetilde{X}', \mathscr{O}_{\widetilde{X}'}(-\pi^* x - a E_{new})) =$

$H^0(\widetilde{X}', \mathscr{O}_{\widetilde{X}'}(-\pi^*x))$. Indeed, take the exact sequence of sheaves

$$0 \to \mathscr{O}_{\widetilde{X}'}(-\pi^*x) \to \mathscr{O}_{\widetilde{X}'}(-\pi^*x - aE_{new}) \to \mathscr{O}_{-aE_{new}}(-\pi^*x - aE_{new}) \to 0$$

and use that $h^0(\mathscr{O}_l(l) \otimes \mathscr{L}) = 0$ for any $l > 0$ and line bundle $\mathscr{L}$ with $(c_1\mathscr{L}, E_v) = 0$ for any $E_v \in |l|$, cf. 6.4.2. Therefore,

$$\mathfrak{h}'(\pi^*x + aE_{new}) \begin{cases} = \mathfrak{h}(x) \ \text{ for any } a \le 0 \\ \text{is increasing for } a \ge 0. \end{cases} \tag{11.9.2}$$

Using the exact sequence $0 \to \mathscr{O}_{aE_{new}}(-\pi^*x) \to \mathscr{O}_{\pi^*x+aE_{new}} \to \mathscr{O}_{\pi^*x} \to 0$ and Lipman's vanishing $h^1(\mathscr{O}_{aE_{new}}(-\pi^*x)) = 0$ from 6.4.10, we get that $h^1(\mathscr{O}_{\pi^*x+aE_{new}}) = h^1(\mathscr{O}_{\pi^*x})$ for any $a \ge 0$. Furthermore, from $0 \to \mathscr{O}_{E_{new}}(-\pi^*x + E_{new}) \to \mathscr{O}_{\pi^*x} \to \mathscr{O}_{\pi^*x-E_{new}} \to 0$ we get that $h^1(\mathscr{O}_{\pi^*x-E_{new}}) = h^1(\mathscr{O}_{\pi^*x})$ too. On the other hand (by Leray spectral sequence), $h^1(\widetilde{X}', \mathscr{O}_{\pi^*x}) = h^1(\widetilde{X}, \mathscr{O}_x)$. Therefore,

$$h^1(\mathscr{O}_{\pi^*x+aE_{new}}) \begin{cases} \text{is increasing for } a \le -1, \\ = h^1(\mathscr{O}_x) \ \text{ for any } a \ge -1. \end{cases} \tag{11.9.3}$$

These combined provide

$$a \mapsto w_0'(\pi^*x + aE_{new}) \begin{cases} \text{is decreasing for } a \le -1, \\ = w_0(x) \ \text{ for } a = -1 \text{ and } a = 0, \\ \text{is increasing for } a \ge 0. \end{cases} \tag{11.9.4}$$

Recall that we can compute $\mathbb{H}^*_{an}(\phi)$ using the cube $R(0, c)$ with $c \ge Z_{coh}(\phi)$, cf. 11.9.2. But then $\pi^*c \ge Z_{coh}(\phi')$ (use 6.4.22(c)). Hence $\mathbb{H}^*_{an}(\phi')$ can be computed in $R(0, \pi^*c)$, and π^* sends the lattice points of $R(0, c)$ into $R(0, \pi^*c)$.

Furthermore, if $w_0'(\pi^*x + aE_{new}) \le n$, then $w_0(x) \le n$ too. In particular, the projection $\pi_{\mathbb{R}}$ in the direction of E_{new} induces a well-defined map $\pi_{\mathbb{R}} : S_n(\phi') \to S_n(\phi)$. We claim that (whenever $S_n(\phi)$ is non-empty) $\pi_{\mathbb{R}}$ is a homotopy equivalence (with all fibers non-empty and contractible).

11.9.6 We proceed in two steps. First we prove that $\pi_{\mathbb{R}} : S_n(\phi') \to S_n(\phi)$ is onto. (Below we write $E_J := \sum_{v \in J} E_v$ for any subset $J \subset \mathscr{V}$, as usual.)

Consider a zero dimensional cube (i.e. lattice point) $x \in S_n(\phi)$. Then $w_0(x) \le n$. But then $w_0'(\pi^*x) = w_0(x) \le n$ too, hence $\pi^*(x) \in S_n(\phi')$ and $x = \pi_{\mathbb{R}}(\pi^*x) \in \mathrm{im}(\pi_{\mathbb{R}})$.

Next take a cube $(x, I) \subset S_n(\phi)$ ($I \subset \mathscr{V}$). This means that $w_0(x + E_{I'}) \le n$ for any $I' \subset I$. But then

$$\pi^*(x + E_{I'}) = \pi^*x + E_{I'} + \epsilon \cdot E_{new}, \tag{11.9.5}$$

where $\epsilon = 0$ if $v_0 \notin I'$ and $\epsilon = 1$ otherwise. Hence

$$w_0'(\pi^*x + E_{I'}) = w_0'(\pi^*(x + E_{I'}) - \epsilon E_{new}) \overset{(11.9.4)}{=} w_0(x + E_{I'}) \leq n. \qquad (11.9.6)$$

Therefore $(\pi^*x, I) \in S_n(\phi')$ and $\pi_{\mathbb{R}}$ projects (π^*x, I) isomorphically onto (x, I).

Next, we show that $\pi_{\mathbb{R}}$ is in fact a homotopy equivalence. In order to prove this fact it is enough to verify that if $\square \in S_n(\phi)$ and $\square^\circ$ denotes its relative interior, then $\pi_{\mathbb{R}}^{-1}(\square^\circ) \cap S_n(\phi')$ is contractible. Using the above facts, this follows similarly as in the proof of the parallel topological version 7.3.5.

11.9.7 The case when we blow up an intersection point $E_{v_0} \cap E_{v_1}$ starts very similarly, however at some point there is a major difference, hence we need an additional argument.

With very similar notations, in this case we define $\pi^*(\sum_v x_v E_v) = \sum_v x_v E_v + (x_{v_0} + x_{v_1})E_{new}$. Then all the statements till 11.9.6 remain valid (including the key (11.9.4)). However, the first part of 11.9.6 should be modified. The first difference is in (11.9.5). Indeed, in this case

$$\pi^*(x + E_{I'}) = \pi^*x + E_{I'} + \epsilon \cdot E_{new}, \qquad (11.9.7)$$

where ϵ is the cardinality of $I' \cap \{v_0, v_1\}$. This can be 0, 1 or 2. Therefore, if $\{v_0, v_1\} \not\subset I$, then $\epsilon \in \{0, 1\}$ for any I', hence for such cubes (x, I) all the arguments of 11.9.6 work.

Assume in the sequel that $\{v_0, v_1\} \subset I$. Write $J = I \setminus \{v_0, v_1\}$.

There are two cube candidates of $R(0, \pi^*c)$ which might cover the cube $(x, I) \in S_n(\phi)$. One of them is (π^*x, I) (as above). However, by (11.9.4) the lattice points $\pi^*(x + E_I) = \pi^*x + E_I + 2E_{new}$ and $\pi^*(x + E_I) - E_{new} = \pi^*x + E_I + E_{new}$ are in $S_n(\phi')$, but the vertex $\pi^*(x) + E_I$ of (π^*x, I) might not be in $S_n(\phi')$.

Another candidate is $(\pi^*x + E_{new}, I)$, but here again π^*x and $\pi^*x - E_{new}$ are in $S_n(\phi')$ but the vertex $\pi^*x + E_{new}$ of $(\pi^*x + E_{new}, I)$ might be not. So both cubes a priori are obstructed if we apply (11.9.4) only.

Next we analyze these obstructions in more detail and we show that one of the candidate cubes works.

11.9.8 Case 1 Assume that $w_0'(\pi^*x) = w_0'(\pi^*x + E_{new})$. Then by (11.9.2) and (11.9.3) we obtain that $\mathfrak{h}'(\pi^*x) = \mathfrak{h}'(\pi^*x + E_{new})$. By the matroid inequality of $\mathfrak{h}'$ we get that $\mathfrak{h}'(\pi^*x + E_{J'}) = \mathfrak{h}'(\pi^*x + E_{J'} + E_{new})$ for any $J' \subset J$. This again via (11.9.2) and (11.9.3) shows that $w_0'(\pi^*x + E_{J'}) = w_0'(\pi^*x + E_{J'} + E_{new})$. In particular,

$$w_0'(\pi^*x + E_{J'} + E_{new}) = w_0'(\pi^*x + E_{J'}) = w_0'(\pi^*(x + E_{J'})) = w_0(x + E_{J'}) \leq n.$$

That is, the vertices of type $\pi^*x + E_{J'} + E_{new}$ of $(\pi^*x + E_{new}, I)$ are in $S_n(\phi')$. For all other vertices we already know this fact (use (11.9.4)). Hence $(\pi^*x + E_{new}, I)$ is in $S_n(\phi')$ and it projects via $\pi_{\mathbb{R}}$ bijectively to (x, I). Furthermore,

$\pi_{\mathbb{R}}^{-1}(x, I)^\circ \cap S_n(\phi')$ admits a strong deformation retraction to $(\pi^*x + E_{new}, I)^\circ$, hence it is contractible.

11.9.9 Case 2 Assume that $w_0'(\pi^*x + E_I) = w_0'(\pi^*x + E_I + E_{new})$, or $w_0'(\pi^*(x + E_I) - 2E_{new}) = w_0'(\pi^*(x + E_I) - E_{new})$. Then by (11.9.2) and (11.9.3) we obtain that $h^1(\mathscr{O}_{\pi^*x+E_I}) = h^1(\mathscr{O}_{\pi^*x+E_I+E_{new}})$. By the opposite matroid inequality of h^1 and (11.9.2) and (11.9.3) again we obtain that $w_0'(\pi^*x + E_I - E_{J'}) = w_0'(\pi^*x + E_I - E_{J'} + E_{new})$. In particular,

$$w_0'(\pi^*x + E_I - E_{J'}) = w_0'(\pi^*x + E_I - E_{J'} + E_{new})$$

$$= w_0'(\pi^*(x + E_I - E_{J'}) - E_{new}) = w_0(x + E_I - E_{J'}) \leq n.$$

That is, the vertices of type $\pi^*x + E_I - E_{J'}$ of (π^*x, I) are in $S_n(\phi')$. For all other vertices we already know this fact (use (11.9.4)). Hence (π^*x, I) is in $S_n(\phi')$ and it projects via $\pi_{\mathbb{R}}$ bijectively to (x, I). Furthermore, $\pi_{\mathbb{R}}^{-1}(x, I)^\circ \cap S_n(\phi')$ admits a deformation retraction to $(\pi^*x, I)^\circ$, hence it is contractible.

11.9.10 Case 3 Assume that the assumptions from **Case 1** and **Case 2** do not hold. This means that

$$\begin{cases} \mathfrak{h}'(\pi^*x) < \mathfrak{h}'(\pi^*x + E_{new}), \text{ and} \\ h^1(\mathscr{O}_{\pi^*x+E_I}) < h^1(\mathscr{O}_{\pi^*x+E_I+E_{new}}). \end{cases}$$

This reads as follows, cf. (6.8.2),

$$\begin{cases} (a)\ \ H^0(\mathscr{O}_{\widetilde{X}'}(-\pi^*x - E_{new}) \subsetneq H^0(\mathscr{O}_{\widetilde{X}'}(-\pi^*x)), \text{ and} \\ (b)\ \ H^0(\widetilde{X}', \Omega^2_{\widetilde{X}'}(\pi^*x + E_I)) \subsetneq H^0(\widetilde{X}', \Omega^2_{\widetilde{X}'}(\pi^*x + E_I + E_{new})). \end{cases}$$

Part *(a)* means the following: there exists a function $f \in H^0(\widetilde{X}', \mathscr{O}_{\widetilde{X}'})$ such that $\mathrm{div}_{E'}(f) \geq \pi^*x$, and in this inequality the E_{new}–coordinate entries are equal. By part *(b)*, there exists a global 2–form ω such that $\mathrm{div}_{E'}(\omega) \geq -\pi^*x - E_I - E_{new}$ and the E_{new}–coordinate entries are equal.

Therefore, the form $f\omega \in H^0(\widetilde{X}' \setminus E', \Omega^2_{\widetilde{X}'})$ has the property that $\mathrm{div}_{E'}(f\omega) \geq -E_I - E_{new}$ with equality at the E_{new} coordinate. In particular, again by duality (6.8.2), we obtain that in $\widetilde{X}'$ the following strict inequality holds:

$$h^1(\mathscr{O}_{E_I+E_{new}}) > h^1(\mathscr{O}_{E_I})\ \ (\mathscr{V}' = \mathscr{V} \cup \{new\},\ I \subset \mathscr{V}). \tag{11.9.8}$$

But if the graph is a tree then this strict inequality cannot happen.

11.9.11 In particular, for any $I \subset \mathscr{V}$ either $\{v_0, v_1\} \not\subset I$, or in the opposite case either **Case 1** or **Case 2** applies. Hence, in any case, $\pi_{\mathbb{R}}^{-1}(x, I)^\circ \cap S_n(\phi')$

is contractible. Therefore, $S_n(\phi)$ and $S_n(\phi')$ have the same homotopy type by the argument from the end of 11.9.6.

□

Definition 11.9.12 Assume that Γ is a tree. Then $\mathbb{H}^*_{an}(\phi)$ and $R_{an}(\phi)$ are independent of ϕ. In the sequel we will use the notations $\mathbb{H}^*_{an,0}(X,o)$ and $R_{an,0}(X,o)$ for them. (The index 'zero' means 'canonical spin^c–structure.) They are called the *analytic lattice cohomology of* (X,o) and the *analytic graded root of* (X,o) (associated with the canonical spin^c–structure).

They are analytic invariants of the germ (X,o). By Corollary 11.9.3 $\mathbb{H}^*_{an,red,0}(X,o)$ has finite $\mathbb{Z}$–rank, hence the Euler characteristic $eu(\mathbb{H}^*_{an,0}(X,o))$ is well–defined.

We wish to emphasize that (similarly to the topological case, cf. 11.1.24), the above proof implies that the homotopy type of the tower $\{S_n\}_n$ is a resolution independent invariant of the analytic germ (X,o). It is a stronger invariant than the analytic lattice cohomology. We might call it the *analytic lattice homotopy type*.

In order to eliminate any misunderstanding between the two lattice cohomologies, the lattice cohomology associated in 11.1.A with the plumbing graph, RR weight function χ and the canonical spin^c structure will be denoted by $\mathbb{H}^*_{top,0}(M)$.

11.9.13 Next we list several properties of $\mathbb{H}^*_{an,0}(X,o)$.

The following reinterpretation of $\mathfrak{h}^\circ : R(0,c)\cap\mathbb{Z}^s \to \mathbb{Z}$ will be helpful.

By (6.8.2) $\dim H^0(\mathscr{O}_{\widetilde{X}}(K_{\widetilde{X}}+Z))/H^0(\mathscr{O}_{\widetilde{X}}(K_{\widetilde{X}})) = h^1(\mathscr{O}_Z)$ for any $Z>0$. On the other hand, by the opposite matroid rank inequality 6.4.21 we also have $h^1(\mathscr{O}_Z) = h^1(\mathscr{O}_{\min\{Z,\lfloor Z_K\rfloor_+\}})$. Hence, $\mathfrak{h}^\circ(l) = p_g - h^1(\mathscr{O}_l) = h^1(\mathscr{O}_c) - h^1(\mathscr{O}_l)$ appears as

$$\dim \frac{H^0(\widetilde{X},\mathscr{O}_{\widetilde{X}}(K_{\widetilde{X}}+c))}{H^0(\widetilde{X},\mathscr{O}_{\widetilde{X}}(K_{\widetilde{X}}+l))} = \dim \frac{H^0(\widetilde{X},\mathscr{O}_{\widetilde{X}}(K_{\widetilde{X}}+\lfloor Z_K\rfloor_+))}{H^0(\widetilde{X},\mathscr{O}_{\widetilde{X}}(K_{\widetilde{X}}+\min\{l,\lfloor Z_K\rfloor_+\}))}. \tag{11.9.9}$$

(In (11.9.9) $\lfloor Z_K\rfloor_+$ can be replaced by Z_{coh} as well.)

Example 11.9.14 (Gorenstein Germs) Assume that (X,o) is Gorenstein. Then $Z_K \in L$, and let us assume (for simplicity) that $Z_K \geq 0$ (this happens e.g. in the good minimal resolution, cf. 6.3.4). Then for any $l\in R(0,Z_K)\cap\mathbb{Z}^s$, by (11.9.9) we have $\mathfrak{h}^\circ(l) = \dim H^0(\mathscr{O}_{\widetilde{X}})/H^0(\mathscr{O}_{\widetilde{X}}(-Z_K+l)) = \mathfrak{h}(Z_K-l)$. Therefore, $w_0(l) = \mathfrak{h}(l)+\mathfrak{h}(Z_K-l)-p_g$ is obtained as the *symmetrization* of $\mathfrak{h}$. In particular $w_0(l) = w_0(Z_K-l)$. Note that this is true for the topological weight function too: $\chi(l) = \chi(Z_K-l)$. However, in the analytic case, the symmetry might fail for non-Gorenstein germs (even if we consider a numerically Gorenstein topological type).

The following property will be crucial in the Euler characteristic computation.

Lemma 11.9.15 (CDP) *Assume that $g_v = 0$ for any $v \in \mathcal{V}$. Then there exists no $l \in L_{\geq 0}$ and $v \in \mathcal{V}$ such that the differences $\mathfrak{h}(l+E_v)-\mathfrak{h}(l)$ and $\mathfrak{h}^\circ(l)-\mathfrak{h}^\circ(l+E_v)$ are simultaneously strictly positive.*

Proof If $\mathfrak{h}(l+E_v) > \mathfrak{h}(l)$ then there exists a global function $f \in H^0(\mathscr{O}_{\widetilde{X}})$ with $\mathrm{div}_E f \geq l$, where the E_v-coordinate is $(\mathrm{div}_E f)_v = l_v$. Similarly, via (11.9.9), if $\mathfrak{h}^\circ(l) > \mathfrak{h}^\circ(l+E_v)$ then there exists a global 2-form ω with possible poles along E, with $\mathrm{div}_E\omega \geq -l-E_v$ (i.e., the pole order is $\leq l+E_v$), and $(\mathrm{div}_E\omega)_v = -l_v - 1$. In particular, the form $f\omega$ satisfies $\mathrm{div}_E\, f\omega \geq -E_v$ and $(\mathrm{div}_E f\omega)_v = -1$. This implies $H^0(\Omega^2_{\widetilde{X}}(E_v))/H^0(\Omega^2_{\widetilde{X}}) \neq 0$, or, by (6.8.2), $h^1(\mathscr{O}_{E_v}) \neq 0$. □

11.9.16 Lemma 11.9.15 will allow us to determine the Euler characteristic of the analytic lattice cohomology $\mathbb{H}^*_{an,0}(X,o)$. Surprisingly, this Euler characteristic automatically equals the Euler characteristic of path cohomologies associated with any increasing path (this equality definitely does not hold in the topological version).

Theorem 11.9.17 (The Euler Characteristic $eu(\mathbb{H}^*_{an}(X,o))$**)** *Assume that the link is a $\mathbb{Q}HS^3$. Then $eu(\mathbb{H}^*_{an,0}(X,o)) = p_g(X,o)$. Furthermore, for any increasing path γ connecting 0 and c (where $c \geq Z_{coh}$) we also have $eu(\mathbb{H}^*_{an}(\gamma,w)) = p_g$.*

This in particular means that $\mathbb{H}^*_{an,0}(X,o)$ is a *categorification of the geometric genus*, that is, it is a graded cohomology module whose Euler characteristic is p_g.

Remark 11.9.18 The right hand side of (11.8.3) in the present case is the 0–component $P_0(\mathbf{t})$ of the multivariable Poincaré series, cf. (8.2.9). Hence, the identity (11.8.3) recovers $P_0(\mathbf{t})$ in terms of the analytic weighted cubes. This formula can be compared with its topological analogue (8.4.14), where $Z_0(\mathbf{t})$ is expressed in terms of topological weighted cubes by a similar expression.

11.9.19 The above facts can be extended to any spin^c–structure of the link whenever the link is a rational homology sphere. The pair (h,h°) in this equivalent case is given in 11.8.8(2). For details see [9].

11.9.A Analytic Reduction Theorem

In this section we prove the analytic analogue of the topological Reduction Theorem 11.4.3. By this result the rectangle $R = R(0,c)$ can be replaced by another rectangle sitting in a lattice of smaller rank.

Then, we compare the topological lattice cohomology $\mathbb{H}^*_{top,0}(M) = \mathbb{H}^*(\Gamma,K)$ of the link (introduced in 11.1.A) with the analytic lattice cohomology $\mathbb{H}^*_{an,0}(X,o)$. Under some conditions they are isomorphic. In this way, we can compute the analytic lattice cohomology for certain analytic types, see 11.9.28, including e.g. rational, Gorenstein elliptic, weighted homogeneous singularities, or superisolated singularities associated with rational unicuspidal curves. We also formulate

certain conjectures and problems regarding the structure of the canonical morphism $\mathbb{H}^*_{top,0}(M) \to \mathbb{H}^*_{an,0}(X,o)$ of graded $\mathbb{Z}[U]$ modules which connects them. ♣

11.9.20 Let us fix a resolution $\widetilde{X}$ with irreducible exceptional components $\{E_v\}_{v\in\mathcal{V}}$. In this section *we assume that the link is a rational homology sphere*.

In Reduction Theorems we decompose $\mathcal{V}$ as a disjoint union $\overline{\mathcal{V}} \sqcup \mathcal{V}^*$, where the vertices $\overline{\mathcal{V}}$ are the 'essential' ones, the ones which dominate the others, and the vertices $\mathcal{V}^*$ are those which 'can be eliminated'. In the topological context the possible choice of $\overline{\mathcal{V}}$ was dictated by combinatorial properties of χ with a special focus on the topological characterization of rational germs. In the present context we start with certain analytic properties of 2-forms (which reflects the dominance of $\overline{\mathcal{V}}$ over $\mathcal{V}^*$). (Recall that $p_g = 0$ if and only if $H^0(\widetilde{X}\setminus E, \Omega^2_{\widetilde{X}}) = H^0(\widetilde{X}, \Omega^2_{\widetilde{X}})$.)

Definition 11.9.21 We say that $\overline{\mathcal{V}}$ is an SR_{an}–set if it satisfies the following property: if some differential form $\omega \in H^0(\widetilde{X}\setminus E, \Omega^2_{\widetilde{X}})$ satisfies $(\mathrm{div}_E\omega)|_{\overline{\mathcal{V}}} \geq -E_{\overline{\mathcal{V}}}$ then necessarily $\omega \in H^0(\widetilde{X}, \Omega^2_{\widetilde{X}})$. By (6.8.2) this is equivalent with the vanishing $h^1(\mathcal{O}_Z) = 0$ for any $Z = E_{\overline{\mathcal{V}}} + l^*$, where $l^* \geq 0$ and it is supported on $\mathcal{V}^*$.

In the next proposition we prove that the SR_{an}–property is in fact topological.

Proposition 11.9.22 *$\overline{\mathcal{V}}$ is a SR_{an}–set if and only if it is a SR–set. (For the definition of SR–sets see 7.3.21.)*

Proof ($\mathrm{SR}\Rightarrow\mathrm{SR}_{an}$) Let Γ be the original graph, and let Γ' be that rational graph which is obtained from Γ be decreasing the numbers $\{E_v^2\}_{v\in\overline{\mathcal{V}}}$. Let $(\ ,\ )'$ be its intersection form and χ' the corresponding RR expression. Fix a cycle of type $Z = E_{\overline{\mathcal{V}}} + l^*$ with $|l^*| \subset \mathcal{V}^*$, $l^* \geq 0$. Then, regarding Γ', we claim the following: there exists a sequence $\{x_i\}_{i=0}^t$ such that

$$x_0 = 0,\ x_t = Z,\ x_{i+1} = x_i + E_{v(i)},\ (x_i, E_{v(i)})' \leq 1 \text{ at every step } i. \qquad (11.9.10)$$

We construct each x_i by decreasing order. Assume that x_i is already constructed. Then there exists at least one $E_u \subset |x_i|$ such that $\chi'(x_i - E_u) \leq \chi'(x_i)$, equivalently $(x_i - E_u, E_u)' \leq 1$. (Cf. the proof of 7.3.14(a) $\Rightarrow$ (c).) Then set $x_{i-1} := x_i - E_u$.

Now, since $E_{\overline{\mathcal{V}}}$ is reduced, and the self-intersections from the support of l^* are not modified, along this sequence $(x_i, E_{v(i)})' = (x_i, E_{v(i)})$. In particular, this sequence has the very same properties (11.9.10) in Γ too.

Then, using the exact sequences $0 \to \mathcal{O}_{E_{v(i)}}(-x_i) \to \mathcal{O}_{x_{i+1}} \to \mathcal{O}_{x_i} \to 0$ we get that $h^1(\mathcal{O}_{x_{i+1}}) = h^1(\mathcal{O}_{x_i})$. Since $h^1(\mathcal{O}_{x_0}) = 0$ by induction $h^1(\mathcal{O}_{x_t}) = 0$ too.

($\mathrm{SR}_{an}\Rightarrow\mathrm{SR}$) Consider the fixed resolution $\widetilde{X}$. For each $v \in \overline{\mathcal{V}}$ we blow up $\widetilde{X}$ in n_v different points of $E_v \setminus \cup_{u\neq v} E_u$. The obtained new resolution is denoted by $\widetilde{X}^b$. Let us denote the strict transform of any E_v ($v \in \mathcal{V}$) by E'_v and let $\widetilde{X}'$ be a small tubular neighbourhood of $\cup_{v\in\mathcal{V}} E'_v$ in $\widetilde{X}^b$. Using 6.3.1 and 2.2.5.5 one verifies that for any $v \in \overline{\mathcal{V}}$, if n_v is sufficiently large, then the E'_v–coefficient of $Z_K(\widetilde{X}')$ is < 2.

In particular, for any collection of $\{n_v\}$ with this property we have $\lfloor Z_K(\widetilde{X}')|_{\overline{\mathscr{V}}} \rfloor \leq E_{\overline{\mathscr{V}}}$. In particular, by 6.4.A, (i) $Z_{coh}(\widetilde{X}')|_{\overline{\mathscr{V}}} \leq E_{\overline{\mathscr{V}}}$. Similarly, one verifies that for $\{n_v\}$ sufficiently large, (ii) $Z_{min}(\widetilde{X}')|_{\overline{\mathscr{V}}} = E_{\overline{\mathscr{V}}}$. Let us fix $\widetilde{X}'$ (associated with a fixed collection $\{n_v\}$), which satisfies both these properties. Note that if (i) and (ii) are true for some collection $\{n_v\}$ then they are true for larger numbers too.

In particular, by Laufer Rationality Criterion 7.1.2, $\widetilde{X}'$ is rational if and only if the analogue of $\widetilde{X}'$ obtained by any larger collection of blow ups is rational.

Now, assume that $\overline{\mathscr{V}}$ is not SR. Then (by the previous paragraph) $\widetilde{X}'$ is not rational. Then via (i) there exists a form $\omega \in H^0(\widetilde{X}', \Omega^2_{\widetilde{X}'}(Z))$ with pole $Z \in L(\widetilde{X}')_{>0}$ such that $Z|_{\overline{\mathscr{V}}} \leq E'_{\overline{\mathscr{V}}}$.

Now consider the restriction map

$$H^0(\widetilde{X}^b, \Omega^2_{\widetilde{X}^b}(Z)/H^0(\widetilde{X}^b, \Omega^2_{\widetilde{X}^b}) \longrightarrow H^0(\widetilde{X}', \Omega^2_{\widetilde{X}'}(Z)/H^0(\widetilde{X}', \Omega^2_{\widetilde{X}'}).$$

It is injective, and both sides have dimension $h^1(\mathscr{O}_Z)$ (cf. (6.8.2)), hence this map is a bijection. In particular, there exists a form $\omega \in H^0(\widetilde{X}^b, \Omega^2_{\widetilde{X}^b}(Z))$ with pole $Z \in L(\widetilde{X}^b)_{>0}$ such that $Z|_{\overline{\mathscr{V}}} \leq E'_{\overline{\mathscr{V}}}$. But then the same property is true for the original $\widetilde{X}$ as well, a fact which contradicts the SR_{an} property of $\overline{\mathscr{V}}$. □

Example 11.9.23 By the above lemma, the set $\overline{\mathscr{V}} = \mathscr{N}$ of nodes is a SR_{an}–set. Moreover, if $\{\overline{v}\}$ is an SR–set of an AR graph, then it is an SR_{an}–set as well.

11.9.24 Associated with a disjoint decomposition $\mathscr{V} = \overline{\mathscr{V}} \sqcup \mathscr{V}^*$, we write any $l \in L$ as $\overline{l} + l^*$, or $(\overline{l}, l^*)$, where $\overline{l}$ and l^* are supported on $\overline{\mathscr{V}}$ and $\mathscr{V}^*$ respectively. We also write $\overline{R}$ for the rectangle $R(0, \overline{c})$, the $\overline{\mathscr{V}}$-projection of $R(0, c)$ with $c = Z_{coh}$.

For any $\overline{l} \in \overline{R}$ define the weight function

$$\overline{w}_0(\overline{l}) = \mathfrak{h}(\overline{l}) + \mathfrak{h}^\circ(\overline{l} + c^*) - p_g = \mathfrak{h}(\overline{l}) - h^1(\mathscr{O}_{\overline{l}+c^*}).$$

Consider all the cubes of $\overline{R}$ and the weight function $\overline{w}_q : \mathscr{Q}_q(\overline{R}) \to \mathbb{Z}$ by $\overline{w}_q(\square_q) = \max\{w_0(\overline{l}) \ : \ \overline{l} \text{ is any vertex of } \square_q\}$.

Theorem 11.9.25 (Reduction Theorem for the Analytic Lattice Cohomology) *If $\overline{\mathscr{V}}$ is an SR_{an}-set then*

$$\mathbb{H}^*_{an}(R, w) = \mathbb{H}^*_{an}(\overline{R}, \overline{w}).$$

Proof For any $\mathscr{I} \subset \mathscr{V}$ write $c_{\mathscr{I}}$ for the $\mathscr{I}$-projection of $c = Z_{coh}$. We proceed by induction, the proof will be given in $|\mathscr{V}^*|$ steps. For any $\overline{\mathscr{V}} \subset \mathscr{I} \subset \mathscr{V}$ we create the inductive setup. We write $\mathscr{I}^* = \mathscr{V} \setminus \mathscr{I}$, and according to the disjoint union

$\mathscr{I} \sqcup \mathscr{I}^* = \mathscr{V}$ we consider the coordinate decomposition $l = (l_{\mathscr{I}}, l_{\mathscr{I}^*})$. We also set $R_{\mathscr{I}} = R(0, c_{\mathscr{I}})$ and the weight function

$$w_{\mathscr{I}}(l_{\mathscr{I}}) = \mathfrak{h}(l_{\mathscr{I}}) + \mathfrak{h}^\circ(l_{\mathscr{I}} + c_{\mathscr{I}^*}) - p_g.$$

Then for $\overline{\mathscr{V}} \subset \mathscr{I} \subset \mathscr{J} \subset \mathscr{V}$, $\mathscr{J} = \mathscr{I} \cup \{v_0\}$ ($v_0 \notin \mathscr{I}$), we wish to prove that $\mathbb{H}^*_{an}(R_{\mathscr{I}}, w_{\mathscr{I}}) = \mathbb{H}^*_{an}(R_{\mathscr{J}}, w_{\mathscr{J}})$. For this consider the projection $\pi_{\mathbb{R}} : R_{\mathscr{J}} \to R_{\mathscr{I}}$.

For any fixed $y \in R_{\mathscr{I}} \cap \mathbb{Z}^{|\mathscr{I}|}$ consider the fiber $\{y + tE_{v_0}\}_{0 \le t \le c_{v_0},\ t \in \mathbb{Z}}$.

Note that $t \mapsto \mathfrak{h}(y + tE_{v_0})$ is increasing. Let $t_0 = t_0(y)$ be the smallest value t for which $\mathfrak{h}(y + tE_{v_0}) < \mathfrak{h}(y + (t+1)E_{v_0})$. If $t \mapsto \mathfrak{h}(y + tE_{v_0})$ is constant then we take $t_0 = c_{v_0}$. If $t_0 < c_{v_0}$, then the definition of t_0 implies the existence of a function

$$f \in H^0(\mathscr{O}_{\widetilde{X}}) \text{ with } (\mathrm{div}_E f)|_{\mathscr{I}} \ge y, \quad (\mathrm{div}_E f)_{v_0} = t_0. \tag{11.9.11}$$

Symmetrically, $t \mapsto \mathfrak{h}^\circ(y + c_{\mathscr{J}^*} + tE_{v_0})$ is decreasing. Let $t_0^\circ = t_0^\circ(y)$ be the smallest value t for which $\mathfrak{h}^\circ(y + c_{\mathscr{J}^*} + t'E_{v_0}) = \mathfrak{h}^\circ(y + c_{\mathscr{J}^*} + (t'+1)E_{v_0})$ for $t' \ge t$. From the definition of t_0° we obtain the existence of a form

$$\omega \in H^0(\widetilde{X} \setminus E, \Omega^2_{\widetilde{X}}) \text{ with } (\mathrm{div}_E \omega)|_{\mathscr{I}} \ge -y, \quad (\mathrm{div}_E \omega)_{v_0} = -t_0^\circ. \tag{11.9.12}$$

This shows that there exist a form $f\omega \in H^0(\widetilde{X} \setminus E, \Omega^2_{\widetilde{X}})$ such that $(\mathrm{div}_E f\omega)|_{\mathscr{I}} \ge 0$ and $(\mathrm{div}_E f\omega)_{v_0} = t_0 - t_0^\circ$. Since $\mathscr{I} \supset \overline{\mathscr{V}}$, by the SR_{an} property we necessarily must have $t_0 - t_0^\circ \ge 0$. Therefore, the weight $t \mapsto w_{\mathscr{J}}(y + tE_{v_0}) = \mathfrak{h}(y + tE_{v_0}) + \mathfrak{h}^\circ(y + tE_{v_0} + c_{\mathscr{J}^*}) - p_g$ is decreasing for $t \le t_0^\circ$, is increasing for $t \ge t_0$. Moreover, for $t_0^\circ \le t \le t_0$ it takes the constant value $\mathfrak{h}(y) + \mathfrak{h}^\circ(y + c_{v_0}E_{v_0} + c_{\mathscr{J}^*}) - p_g = w_{\mathscr{I}}(y)$.

Next we fix $y \in R_{\mathscr{I}} \cap \mathbb{Z}^{|\mathscr{I}|}$ and some $I \subset \mathscr{I}$ (hence a cube (y, I) in $R_{\mathscr{I}}$). We wish to compare the intervals $[t_0^\circ(y + E_{I'}), t_0(y + E_{I'})]$ for all subsets $I' \subset I$. We claim that they have at least one common element; in fact, it turns out that $t_0(y)$ works.

Note that $\mathfrak{h}(y + tE_{v_0}) = \mathfrak{h}(y + (t+1)E_{v_0})$ implies $\mathfrak{h}(y + tE_{v_0} + E_{I'}) = \mathfrak{h}(y + (t+1)E_{v_0} + E_{I'})$ for any I', hence $t_0(y) \le t_0(y + E_{I'})$. In particular, we need to prove that $t_0(y) \ge t_0^\circ(y + E_{I'})$. Similarly as above, the definition of $t_0^\circ(y + E_{I'})$ implies the existence of a form

$$\omega_{I'} \in H^0(\widetilde{X} \setminus E, \Omega^2_{\widetilde{X}}) \text{ with } (\mathrm{div}_E \omega_{I'})|_{\mathscr{I}} \ge -y - E_{I'}, \quad (\mathrm{div}_E \omega_{I'})_{v_0} = -t_0^\circ(y + E_{I'}).$$

Hence the from $f\omega_{I'} \in H^0(\widetilde{X} \setminus E, \Omega^2_{\widetilde{X}})$ satisfies $\mathrm{div}_E f\omega_{I'}|_{\mathscr{I}} \ge -E_{I'}$ and $(\mathrm{div}_E f\omega)_{v_0} = t_0(y) - t_0^\circ(y + E_{I'})$. By the SR_{an} property we must have $t_0(y) - t_0^\circ(y + E_{I'}) \ge 0$.

Set $S_{\mathscr{J},n}$ and $S_{\mathscr{I},n}$ for the lattice spaces defined by $w_{\mathscr{J}}$ and $w_{\mathscr{I}}$. If $y + tE_{v_0} \in S_{\mathscr{J},n}$ then $w_{\mathscr{J}}(y + tE_{v_0}) \le n$, hence by the above discussion $w_{\mathscr{I}}(y) \le n$ too. In particular, the projection $\pi_{\mathbb{R}} : R_{\mathscr{J}} \to R_{\mathscr{I}}$ induces a map $S_{\mathscr{J},n} \to S_{\mathscr{I},n}$. We claim

that it is a homotopy equivalence. The argument is similar to the proof from 11.9.5 via the above preparations. □

Corollary 11.9.26 *If (X, o) admits a resolution ϕ with a SR_{an}–set of cardinality $\overline{s}$, then $\mathbb{H}^{\geq \overline{s}}_{an,0}(X, o) = 0$. In particular, if Γ is almost rational (AR) then $\mathbb{H}^{\geq 1}_{an,0}(X, o) = 0$.*

Remark 11.9.27 Let $\overline{\mathscr{V}}$ be an SR (or, equivalently an SR_{an}) set. Then in the topological Reduction Theorem 11.4.3 (for the canonical spin^c structure) we used for the new weight function the expression $\overline{w}_{top}(\bar{l}) = \chi(x(\bar{l})) = w_{top}(x(\bar{l}))$, where $x(\bar{l})$ is the universal cycle defined in 7.3.24. On the other hand, in the analytic Reduction Theorem we consider $\overline{w}_{an}(\bar{l}) = \mathfrak{h}(\bar{l}) - h^1(\mathscr{O}_{\bar{l}+c^*})$. One might wonder regarding the apparent formal differences between them. The point is that they are not different at all: we claim that for any $c \geq Z_{coh}$,

$$\mathfrak{h}(\bar{l}) - h^1(\mathscr{O}_{\bar{l}+c^*}) = w_{an}(x(\bar{l})).$$

Indeed, let $s(\bar{l})$ be the smallest cycle in $\mathscr{S}$ with $\bar{l} \leq s(\bar{l})$ (cf. 6.6.3). Then by the universal properties $\bar{l} \leq x(\bar{l}) \leq s(\bar{l})$. Since $H^0(\mathscr{O}_{\widetilde{X}}(-\bar{l})) = H^0(\mathscr{O}_{\widetilde{X}}(-s(\bar{l}))$, we also have $\mathfrak{h}(\bar{l}) = \mathfrak{h}(x(\bar{l}))$. Next, for any $l^* \in L_{>0}$ with $|l^*| \in \mathscr{V}^*$ consider the exact sequence $0 \to \mathscr{O}_{l^*}(-x(\bar{l})) \to \mathscr{O}_{x(\bar{l})+l^*} \to \mathscr{O}_{x(\bar{l})} \to 0$. By the Lipman's vanishing 6.4.10 we have $h^1(\mathscr{O}_{l^*}(-x(\bar{l})) = 0$, hence

$$h^1(\mathscr{O}_{x(\bar{l})+l^*}) = h^1(\mathscr{O}_{x(\bar{l})}). \tag{11.9.13}$$

On the other hand, using the opposite matroid rank inequality from 6.4.21 and (11.9.13) applied for l^* sufficiently large, we obtain $h^1(\mathscr{O}_{x(\bar{l})+l^*}) = h^1(\mathscr{O}_{\min\{x(\bar{l})+l^*,c\}}) = h^1(\mathscr{O}_{\bar{l}+c^*})$. Thus $\overline{w}_{an}(\bar{l}) = \mathfrak{h}(\bar{l}) - h^1(\mathscr{O}_{\bar{l}+c^*}) = \mathfrak{h}(x(\bar{l})) - h^1(\mathscr{O}_{x(\bar{l})}) = w_{an}(x(\bar{l}))$.

Proposition 11.9.28 *Assume that for (X, o) and for its resolution ϕ with graph Γ the following facts hold: (i) Γ is almost rational, and (ii) $p_g = \min_\gamma eu(\mathbb{H}^0_{top}(\gamma, \Gamma, K))$. Then $\mathbb{H}^*_{an,0}(X, o) = \mathbb{H}^*_{top,0}(M)$ and $p_g = eu(\mathbb{H}^0_{top,0}(\Gamma, K))$.*

(Obviously, we have $\mathbb{H}^{\geq 1}_{an,0}(X, o) = \mathbb{H}^{\geq 1}_{top,0}(M) = 0$, cf. 11.4.4 and 11.9.26.)

Proof We can assume that ϕ is minimal good. Then $Z_K \geq 0$, cf. 6.3.4. Write $\mathscr{V}$ as $\{v_0\} \sqcup \mathscr{V}^*$, where $\{v_0\}$ is an SR-set. By 11.9.22 it is an SR_{an} set as well. In particular, $\mathbb{H}^*_{an,0}(X, o) = \mathbb{H}^*_{an}(\overline{R}, \overline{w})$, where $\overline{w}(\ell) = \mathfrak{h}(\ell E_{v_0}) - h^1(\mathscr{O}_{\ell E_{v_0}+\lfloor Z_K \rfloor^*})$, $0 \leq \ell \leq \lfloor Z_K \rfloor_{v_0}$. But, by 11.9.27, $\overline{w}(\ell) = w_{an}(x(\ell))$ too.

On the other hand, consider the exact sequence $0 \to \mathscr{O}_{\widetilde{X}}(-x(\ell)) \to \mathscr{O}_{\widetilde{X}} \to \mathscr{O}_{x(\ell)} \to 0$ and the morphism $r(\ell) : H^0(\mathscr{O}_{\widetilde{X}}) \to H^0(\mathscr{O}_{x(\ell)})$. Then $\mathfrak{h}(x(\ell)) + \dim \operatorname{coker}(r(\ell)) = h^0(\mathscr{O}_{x(\ell)})$, hence $\overline{w}(\ell) = \chi(x(\ell)) - \dim \operatorname{coker}(r(\ell))$.

The point in this identity is that in fact $r(\ell)$ is onto for all ℓ. Indeed, this follows from assumption *(ii)* and from the fact that $\min_\gamma eu(\mathbb{H}^0_{top}(\gamma, \Gamma, K))$ is realized by the concatenated computation sequence $\gamma = \{x_i\}_{i=0}^t$ having as intermediate terms the universal cycles $x(\ell)$ (defined in 7.3.D). This can be deduced from the proof of 11.3.5*(f)* and 11.3.9. In this case $H^0(\mathscr{O}_{x_{i+1}}) \to H^0(\mathscr{O}_{x_i})$ is necessarily onto, by the splitting of the cohomological exact sequence associated with consecutive members of the optimal path, see e.g. the proof of 11.3.8. Since $h^1(\mathscr{O}_{x_t}) = p_g = h^1(\mathscr{O}_{\widetilde{X}})$, we also conclude that $H^0(\mathscr{O}_{\widetilde{X}}) \to H^0(\mathscr{O}_{x_i})$ is also surjective for all x_i, in particular for the intermediate terms $x(\ell)$ as well.

In conclusion, $\overline{w}(\ell) = \chi(x(\ell))$. Then use the topological reduction theorem 11.4.3 for AR graphs. Thus $\mathbb{H}^*_{an,0}(X, o) = \mathbb{H}^*_{top,0}(M)$. Then apply the Euler characteristic formulae. □

Example 11.9.29 The two assumptions of Proposition 11.9.28 are satisfied in the following cases, cf. 6.8.43 and 11.6.11 (recall that we assume that the link is a rational homology sphere):

- rational singularities,
- Gorenstein elliptic singularities, and, more generally, (non-necessarily numerically Gorenstein) elliptic germs with $p_g =$ length of the elliptic sequence,
- AR singularities for which the SWIC for the canonical spin^c structure holds, (in particular, weighted homogeneous germs and superisolated germs associated with rational unicuspidal curves, or splice quotient singularities associated with an AR graph Γ).

Hence, for all these cases, $\mathbb{H}^*_{an,0}(X, o) = \mathbb{H}^*_{top,0}(M)$.

11.9.30 Comparison of $\mathbb{H}^*_{an,0}(X, o)$ with $\mathbb{H}^*_{top}(\Gamma, K)$ (Any (X, o) with $\mathbb{Q}HS^3$ Link) The modules $\mathbb{H}^*_{an,0}(X, o)$ with $\mathbb{H}^*_{top}(\Gamma, K)$ can be compared in any case, even without the assumptions of Proposition 11.9.28.

From the exact sequence $0 \to \mathscr{O}_{\widetilde{X}}(-l) \to \mathscr{O}_{\widetilde{X}} \to \mathscr{O}_l \to 0$ we obtain:

Lemma 11.9.31 *If w_{an} denotes the analytic weight function, then $w_{an}(l) = \chi(l) -$ dim coker $(r(l)) = w_{top}(l) -$ dim coker $(r(l))$, where $r(l) : H^0(\mathscr{O}_{\widetilde{X}}) \to H^0(\mathscr{O}_l)$ is the natural morphism (and $w_{top} = \chi$).*

11.9.32 Discussion Lemma 11.9.31 has several consequences.

(1) $w_{an} \le w_{top}$, hence min $w_{an} \le$ min $w_{top} =$ min χ.
(2) Set $S_{an,n} = \cup\{\square : w_{an}(\square) \le n\}$ and $S_{top,n} = \cup\{\square : w_{top}(\square) \le n\}$. Then $S_{top,n} \subset S_{an,n}$ for any $n \in \mathbb{Z}$.
(3) For any $q \ge 0$ there exists a *graded $\mathbb{Z}[U]$–module morphism*

$$\mathfrak{H}^q : \mathbb{H}^q_{an,0}(X, o) \to \mathbb{H}^q_{top,0}(\Gamma).$$

This is an isomorphism in the cases when Proposition 11.9.28 holds.

Similarly, the inclusion of the connected components induce a graded morphism of graded roots

$$R_{top,0}(\Gamma) \to R_{an,0}(X,o).$$

(4) The following (new) characterization of rational germs holds: (X,o) is rational if and only if $w_{an}|_{L_{>0}} > 0$. Indeed, if (X,o) is rational then $w_{an}(l) = \mathfrak{h}(l) > 0$ for $l > 0$. Conversely use $\chi \geq w_{an}$ and Artin's criterion 7.1.2.

(5) If $\min\, w_{an} = 0$, and (X,o) is not rational, then (X,o) is elliptic (use $\chi \geq w_{an}$). Conversely, if (X,o) is Gorenstein elliptic then $\min\, w_{an} = 0$ (use 11.9.29). Moreover, $\min\, w_{an} = 0$ holds for any elliptic singularity with generic analytic structure as well (see 11.9.37). However, in general $\min\, w_{an} = 0$ does not hold for any elliptic germ, see 11.9.45.

(6) In the next discussions it is convenient to assume that c is conveniently large or $c = \infty$. Let C be a component of $S_{an,n}$ such that for any $l \in S_{an,n}$ we have $w_{an}(l) = n$ (that is, the class of C represents a local minimum of the analytic weight in the analytic graded root). Assume that for some $s \in C$ we have $w_{an}(s + E_v) > w_{an}(s)$ for every $v \in \mathcal{V}$ (that is, s is a maximal element of C). Then $s \in \mathscr{S}_{an}$. (Indeed, $w_{an}(s + E_v) > w_{an}(s)$ implies $\mathfrak{h}(s + E_v) > \mathfrak{h}(s)$, cf. 11.9.15.) In fact, using the matroid rank inequality of w_{an} one can show that C has a unique maximal element. This shows that the 'ends' of the analytic graded root R_{an} represent elements of the analytic semigroup $\mathscr{S}_{an}$.

(7) Assume that $m \in L_{>0}$ satisfies $w_{an}(m - E_v) > w_{an}(m)$ for any $E_v \subset |m|$. (E.g., it is a minimal element of a component C as in (6).) By 11.9.15 $h^1(\mathscr{O}_{m-E_v}) < h^1(\mathscr{O}_m)$. This can happen only if $h^1(\mathscr{O}_{E_v}(-m+E_v)) = h^0(\mathscr{O}_{E_v}(m - Z_K)) \neq 0$, which implies $(Z_K - m, E_v) \leq 0$ for every $E_v \subset |m|$. In particular, $\chi(m) \leq 0$.

(8) Assume that $n \geq 0$. We claim that any connected component of $S_{an,n}$ contains at least one element of $S_{top,n}$. Indeed, assume the opposite, and let C_n be a component of $S_{an,n}$ such that $\chi(l) > n$ for any $l \in C_n$. Take a local w_{an}-minimum in C_n with value $k \leq n$, consider the component C of $S_{an,k}$ which contains it, and let m be a minimal element of C. Then, by (7), $\chi(m) \leq 0$. But this is a contradiction since $m \in C_n$, hence $\chi(m) > n \geq 0$.

(9) Recall that $S_{top,n}$ is connected for any $n \geq 1$, see Proposition 11.1.19*(b)*. Therefore, part (8) applied for $n \geq 1$ shows that $S_{an,n}$ is connected too. Furthermore, for $n = 0$ it implies that at the 0-graded level $\mathfrak{H}^0_{\deg=0} : \mathbb{H}^0_{an,0}(X,o)_{\deg=0} \to \mathbb{H}^0_{top,0}(\Gamma)_{\deg=0}$ is injective, or, at the level of degree zero vertices of roots, the map $R_{top,0}(\Gamma)_{\deg=0} \to R_{an,0}(X,o)_{\deg=0}$ is surjective.

Problem 11.9.33

(a) For a fixed topological type find all the possible graded $\mathbb{Z}[U]$–modules $\{\mathbb{H}^*_{an,0}\}_{an}$, associated with all the possible analytic structures supported on that topological type. (For such a concrete classification see Examples 11.9.39, 11.9.40 and 11.9.45.)

(b) For a fixed topological type (hence for a fixed $\mathbb{H}^*_{top,0}(\Gamma)$) and analytic type (X, o) supported on it find special properties of $\mathbb{H}^*_{an,0}(X, o)$ (and of the morphism $\mathbb{H}^*_{an,0}(X, o) \to \mathbb{H}^*_{top,0}(\Gamma)$), which might characterize the classification from part (a).

Several examples support the following conjecture (see e.g. 11.9.40), which makes Problem 11.9.33 more precise.

Conjecture 11.9.34 Fix a topological type. Then for any analytic type supported on it $\mathbb{H}^*_{an,0}(X, o) \to \mathbb{H}^*_{top,0}(\Gamma)$ is injective. At the graded roots level, the graded graph-morphism $R_{top,0}(\Gamma) \to R_{an,0}(X, o)$ is surjective (at the level of graded vertices and edges). In particular, Problem 11.9.33 reads as follows: characterize those graded $\mathbb{Z}[U]$–submodules of $\mathbb{H}^*_{top,0}(\Gamma)$ which appear as $\mathbb{H}^*_{an,0}(X, o)$.

Note that once the link is fixed, there are only finitely many graded $\mathbb{Z}[U]$–modules $\mathbb{H}^*_{an,0}(X, o)$ and graded roots $R_{an,0}(X, o)$ which satisfy this Conjecture.

11.9.B Examples

Example 11.9.35 We assume that the link is a rational homology sphere. We claim that the following facts are equivalent: (i) (X, o) **is rational**, (ii) $R_{an,0}(X, o) = R_{(0)}$, (iii) $\mathbb{H}^*_{an,0}(X, o) = \mathscr{T}_0^+$, (iv) $\mathbb{H}^*_{an,0}(X, o) = \mathscr{T}_{2m}^+$ for some m, (v) $S_{an,0}$ is nonempty and contractible.

Indeed, if (X, o) is rational then $h^1(\mathscr{O}_l) = 0$ for $l > 0$, hence w_{an} is increasing. In particular, any nonempty $S_{an,n}$ can be contracted to the origin, and the analytic root is $\mathfrak{R}_{(0)}$, and $\mathbb{H}^*_{an,0} = \mathscr{T}_0^+$. (ii)⇒(iii)⇒(iv)⇒(v) are clear. Assume that (v) holds. Since $w_{an}(E_v) = \mathfrak{h}(E_v) - h^1(\mathscr{O}_{E_v}) = 1$, the connectivity of $S_{an,0}$ implies that $w_{an}|_{L_{>0}} > 0$. Hence $\chi|_{L_{>0}} > 0$ too, and (X, o) is rational by Artin's criterion.

The above criterions are equivalent with the vanishing of the reduced cohomology $\mathbb{H}^*_{an,red,0}(X, o) = 0$ too.

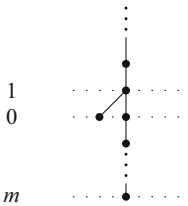
Example 11.9.36 Consider $\mathbb{H}^*$ of the hypersurface singularity with nondegenerate principal part from Example 11.8.15, computed in $R = R(0, Z_K|_{\text{nodes}})$ using the weights from 11.8.13, which correspond to the filtration $\mathscr{F}^{pr}$. Since $N\Gamma$ is bistellar $\mathscr{F}^{pr} = \mathscr{F}^o$ (cf. [384]). Furthermore, since f is a suspension Newton non-degenerate germ, these filtrations agree with the divisorial filtration associated with the nodes, cf. [643], see also 8.6.18. Hence, in this case, the symmetrized weight functions associated with the projected, order and divisorial filtrations agree. Furthermore, for this diagram, these weight functions agree with the topological weight function too restricted to R, which is given in 11.4.11.

In particular, for this germ, $\mathbb{H}^*_{an,0}(X, o) = \mathbb{H}^*_{top,0}(M)$.

This case shows that $\mathbb{H}^{>0}_{an,0}(X, o) \neq 0$ can happen in the analytic case as well.

Example 11.9.37 ($\mathbb{H}^*_{an,0}$ for the Generic Analytic Structure) Let us fix a *non–rational* resolution graph Γ with $M(\Gamma)$ a $\mathbb{Q}HS^3$. Assume that $\widetilde{X}$ is a resolution space of a singularity (X, o) with dual graph Γ and it has generic analytic structure in the sense of [360, 453]. In [453] the following facts are proved: (i) $p_g(X, o) = 1 - \min \chi$, (ii) $Z_{max} = \max\{l : \chi(l) = \min \chi\}$, (iii) $Z_{coh} = \min\{l : \chi(l) = \min \chi\}$. In particular, $Z_{coh} \leq Z_{max}$.

Since $\mathbb{H}^*_{an,0}$ can be computed in $R = R(0, Z_{coh})$ (cf. 11.9.2), and $\mathfrak{h}(0) = 0$ and $\mathfrak{h}(l) = 1$ for $0 < l \leq Z_{coh}$, $w_{an}(l) = 1 - h^1(\mathscr{O}_l)$ for any $l \in R \setminus \{0\}$. Note also that $h^1(\mathscr{O}_{Z_{coh}}) = p_g = 1 - \min \chi$. Therefore, $w_{an}(Z_{coh}) = 1 - p_g = \min \chi$, and $\mathbb{H}^0_{an,0}(X, o) \simeq \mathscr{T}^+_{2 \min \chi} \oplus \mathscr{T}_0(1)$ and $\mathbb{H}^{\geq 1}_{an,0}(X, o) = 0$. The analytic graded root is:



$(R_{gen}(m), \chi_{an})$, where $m = \min w_{an} = \min \chi$

Note that in general (for any analytic structure) $eu(\mathbb{H}^0_{an,0}) \geq 1 - \min w_{an}$ (use (11.6.1)). However, in the present case of the generic analytic structure we have the sharp equality: $eu(\mathbb{H}^0_{an,0}) = p_g = 1 - \min w_{an} = 1 - \min \chi = 1 - \min \chi_{an}$.

Definition 11.9.38 A graded root (R, χ_{an}) of the shape as in Example 11.9.37 is denoted by $R_{gen}(m)$, where $m = \min \chi_{an}$.

Example 11.9.39 Assume that Γ is star-shaped with central vertex -2, with five legs, each of them having one vertex with Euler number -4. Then the algorithm from 11.3.A (applied for the canonical spin^c–structure) shows that the topological graded root is $R_{gen}(-1)$ with $\min w_0 = -1$ (cf. Definition 11.9.38).

In fact, that algorithm also shows that that for *any* analytic structure $Z_{coh} = x(2) \leq x(3) = Z_{min}$, hence, the analytic graded root is $R_{gen}(-1)$ independently of the analytic structure.

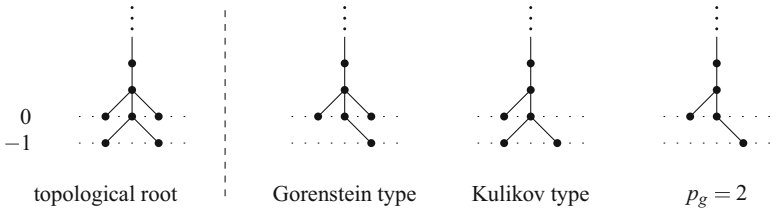
Note that $\mathbb{H}^*_{an,0}(X, o) = \mathbb{H}^*_{top,0}(M)$ can be obtained from 11.9.29 too. In particular, for this graph, $\mathbb{H}^*_{an,0} = \mathbb{H}^*_{top,0}$ for *any* analytic structure supported on Γ.

Example 11.9.40 Consider the topological type fixed by the graph Γ considered in 7.3.3 and 11.2.19. The link is an integral homology sphere and $\mathbb{H}^1_{top,0} \neq 0$.

The list of possible analytic structures supported on Γ is the following (see also 6.8.44):

(i) non-Gorenstein Kulikov analytic type with $p_g = 3$ and $Z_{max} = Z_{min}$,
(ii) Gorenstein splice (quotient) with $p_g = 3$ and $Z_{max} = 2Z_{min}$,
(iii) non-Gorenstein analytic structures with $p_g = 2$ and $Z_{coh} \leq Z_{min} < Z_{max}$.

In all these cases $\mathbb{H}^{\geq 1}_{an,0} = 0$, however the modules $\mathbb{H}^0_{an,0}$ are all different. Below we give the graded roots (and also the topological root, for comparison). The needed information regarding $\mathfrak{h}$ and $\mathfrak{h}^\circ$ can be deduced e.g. from [495].



The local minima of the topological graded root reflect elements of the topological semigroup $\mathscr{S}$. Indeed, the corresponding connected component C of $S_{top,n}$ has a unique maximal element l, $l \in \mathscr{S}$, and $C \cap R(0, Z_K) \neq \emptyset$. Similarly, for any fixed analytic type supported on Γ, the local minima of the analytic graded root reflect elements of the analytic semigroup $\mathscr{S}_{an}$: the corresponding connected component C of $S_{an,n}$ has a unique maximal element l, $l \in \mathscr{S}_{an}$, and $C \cap R(0, Z_{coh}) \neq \emptyset$ (cf. 11.9.32(6)). The above pictures indicate very intuitively which elements of $\mathscr{S}$ representing local minima of the topological root are not realized as divisors of functions (with the properties above) in different analytic structures: in the Goresntein case Z_{min}, in the non–Goresntein cases $s(Z_K) = 3Z_{min}$.

Remark 11.9.41 When we fix a topological type and we compare the possible analytic graded roots (and analytic lattice cohomologies) associated with different analytic structures then we realize that the 'simplest' object is provided by the generic analytic structure. However, it is very hard to identify those analytic structures which provide the 'most complicated' modules/roots. Of course, the candidate for the most complicated module/root is provided by $\mathbb{H}^*_{top,0}(M)$ and $R_{top,0}(M)$ respectively, cf. Conjecture 11.9.34. However, they are not always realized by a certain analytic structure, see Example 11.9.40.

Problem 11.9.42 Which $\mathbb{H}^*_{top,0}(M)$ and $R_{top,0}(M)$ is realized by a certain analytic structure as $\mathbb{H}^*_{an,0}(M)$ and $R_{an,0}(M)$? In the case of realization, for which (very special) analytic structure are they realized? Can this family be identified universally by certain analytic properties?

Remark 11.9.43 The Gorenstein case from Example 11.9.40 shows that even for **splice quotient** analytic types $\mathbb{H}^q_{an,0} \neq \mathbb{H}^q_{top,0}$ might happen. (In 11.9.40 they differ for both $q = 0, 1$.) However, the validity of SWIC guarantees that $eu(\mathbb{H}^q_{an,0}) = eu(\mathbb{H}^q_{top,0})$ for any splice quotient.

Problem 11.9.44 Consider the equisingular family of splice quotient singularities associated with a (convenient) graph Γ, cf. 5.4. Is it true that $\mathbb{H}^*_{an,0}$ is constant

along this equisingular family? If yes (hence $\mathbb{H}^*_{an,0}$ is determined by Γ) describe it combinatorially from Γ. (Note that usually it is not $\mathbb{H}^*_{top,0}(M(\Gamma))$, cf. 11.9.43.)

Example 11.9.45 Assume that (X, o) is a **numerically Gorenstein elliptic singularity** with rational homology sphere link (cf. 7.2.D).

In the Gorenstein case we already know (see 11.9.29) that $\mathbb{H}^*_{an,0}(X, o) = \mathbb{H}^*_{top,0}(M)$.

Next, consider a graph with $m = 2$. In the Gorenstein case $p_g = 3$, and $\mathbb{H}^*_{an,0}$ is determined above. Next, assume that $p_g = 2$, cf. 7.2.H. Using the notations from 7.2.H we have $\mathscr{A}_f = \mathscr{A}_g = \{1, 2\}$. Moreover, since (X_1, o_1) is Gorenstein and $h^1(\mathscr{O}_{C_1'}) = 2$, we have $Z_{coh} \leq C_1' = C + Z_{B_1}$. On the other hand, $H^0(\mathscr{O}_{\widetilde{X}}(-C_1)) = H^0(\mathscr{O}_{\widetilde{X}}(-Z_{min}))$, hence $Z_{max} \geq C_1 = Z_{min} + Z_{B_1}$. Therefore, $Z_{coh} < Z_{max}$, and $w_{an}(Z_{coh}) = 1 - p_g = -1$.

Since $eu = p_g = 2$, this shows that the analytic graded root is necessarily $R_{gen}(-1)$. The cohomology groups are $\mathbb{H}^{\geq 1}_{an,0} = 0$ and $\mathbb{H}^0_{an,0} = \mathscr{T}^+_{-2} \oplus \mathscr{T}_0(1)$.

Surprisingly, even if Γ is elliptic with $\min \chi = 0$, for this analytic structure $\min w_{an} = \min \chi_{an} < 0$.

Question 11.9.46 Let us fix a non-rational topological type Γ with $M(\Gamma)$ a $\mathbb{Q}HS^3$. What are the possible values of $\min w_{an}$ when we consider all the analytic types supported on Γ? (Note that $1 - p_g \leq \min w_{an} \leq \min \chi$.)

11.9.C $\mathbb{H}^*_{an,0}$ and p_g–Constant Deformations

11.9.47 First we formulate the following conjecture.

Conjecture 11.9.48 $\mathbb{H}^*_{an,0}$ is constant along flat p_g–constant deformations of normal surface singularities (with $\mathbb{Q}HS^3$ links).

Example 11.9.49 Consider the superisolated singularity (X, o) associated with a rational projective plane curve C degree 5 with two cusps, both with one Puiseux pair, namely (3,4) and (2,7) respectively (the multiplicity sequences are [3] and $[2_3]$, cf. 8.6.22). Denote the two singular points of C by p and q respectively. The curve C is projective equivalent with $y^3z^2 + x^4z + x^3yz - x^2y^2z/2 - x^5/4 + x^4y/16$ [770].

The resolution graph of (X, o) is given in 9.7.32, $\mathbb{H}^*_{top,0}$ is determined in 11.4.12: $\mathbb{H}^0_{top,0}(M) = \mathscr{T}^+_{-10} \oplus \mathscr{T}_{-10}(3) \oplus \mathscr{T}_0(1)^2$ and $\mathbb{H}^1_{top,0}(M) = \mathscr{T}_{-2}(1)^2$. $\mathfrak{R}_{top,0}$ agrees with the graded root of $\Sigma(5, 5, 6)$ pictured in 11.3.35.

The point is that there exists a K^2_{min}– and p_g–constant deformation from (X, o) to a singularity $(X_{5,5,6}, o)$ (equisingular to the Brieskorn hypersurface $x^5 + y^5 + z^6$), whose link is $\Sigma(5, 5, 6)$, cf. 11.3.38.

Since $(X_{5,5,6}, o)$ is AR and for it the SWIC holds, by 11.9.29 $\mathbb{H}^*_{an,0}(X_{5,5,6}, o) = \mathbb{H}^*_{top,0}(\Sigma(5,5,6))$. That is, $\mathbb{H}^{\geq 1}_{an,0}(X_{5,5,6}, o) = 0$ and the *analytic* graded root of $(X_{5,5,6}, o)$ agrees with the topological graded root of $\Sigma(5,5,6)$. On the other hand, the topological graded roots of $\Sigma(5,5,6)$ and of $M(\Gamma)$ agree (see above). Hence, the conjecture predicts that the same facts are valid for (X, o) too, just like for $(X_{5,5,6}, o)$. This, in particular means that $\mathbb{H}^0_{an,0}(X, o) = \mathbb{H}^0_{top,0}(M)$ and $\mathbb{H}^1_{an,0}(X, o) = 0$. This is what we will verify next.

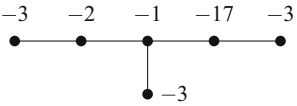
By the *analytic* Reduction Theorem 11.9.25 (applied for the two nodes) we have to complete the w_{an}–table on $R((0,0),(30,34))$. Since (X, o) is Gorenstein, $\mathfrak{h}^\circ(l) = \mathfrak{h}(Z_K - l)$, cf. 11.9.14, hence we can concentrate only on $\mathfrak{h}$. Note also that $Z_{max} = Z_{min}$, and the generic line has multiplicities $(12, 14)$ along the nodes. Therefore, $\mathfrak{h}(l) = 1$ for any $0 < l \leq (12, 14)$. Furthermore (by Laufer's algorithm [359]) $h^1(\mathscr{O}_{Z_{min}}) = 6$. Hence, basically it is enough to analyse the symmetric rectangle $R((12,14),(18,20))$, as in the topological case 11.4.12. The values of $\mathfrak{h}$ can be computed using computation sequences via the following principles: (a) if E_n is a node, $x(\ell) \in \mathscr{S}_{an}$ and $(x(\ell), E_n) = 0$ then (†) $\mathfrak{h}(x(\ell) + E_v) \geq \mathfrak{h}(x(\ell)) + 1$; (b) $\mathfrak{h}$ is constant along a computation sequence which connects $x(\ell) + E_n$ with $s(x(\ell) + E_n)$. In this rectangle we have the following elements of $\mathscr{S}_{an}$, all with the above properties: $x(12,14)$ being the divisor of the generic line, $x(15,14)$ of the generic line which contains p, $x(16,14)$ of the tangent line at p, $x(12,16)$ of the generic line which contains q, $x(15,16)$ of the line which contains both p and q, and finally, $x(12,18)$ of the tangent line at q. (Note also that $\mathfrak{h}((18,20)) = \mathfrak{h}(Z_K - Z_{min}) = \mathfrak{h}^\circ(Z_{min}) = p_g - h^1(\mathscr{O}_{Z_{min}}) = 4$, hence no other semigroup element can contribute and in (†) we always have equality.) These semigroup places are underlined in the next diagram from left, which shows the $\mathfrak{h}$ values in the rectangle $R((12,14),(18,20))$:

4	4	4	4	4	4	4
4	4	4	4	4	4	4
$\underline{3}$	4	4	4	4	4	4
3	4	4	4	4	4	4
$\underline{2}$	3	3	$\underline{3}$	4	4	4
2	3	3	3	4	4	4
$\underline{1}$	2	2	$\underline{2}$	$\underline{3}$	4	4

−2	−2	−3	−4	−4	−4	−5
−2	−2	−2	−3	−3	−3	−4
−3	−2	−2	−3	−3	−3	−4
−3	−2	−2	−2	−2	−2	−3
−4	−3	−3	−3	−2	−2	−3
−4	−3	−3	−3	−2	−2	−2
−5	−4	−4	−4	−3	−2	−2

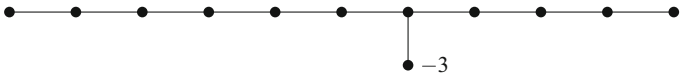
The w_{an}–table is given on the right diagram. Hence the wished statement can be read from it. It is instructive to compare the above analytic w_{an}–table with the $w_{top} = \chi$–table from 11.4.12 (the two (-1)–weights decrease).

Example 11.9.50 Let us consider the deformation of hypersurface singularities $zy^3 + x^5 + z^{11} + tx^2y^2 = 0$, where the deformation parameter t is small. Along the deformation the following objects stay constant: the (non–degenerate) Newton diagram, the Milnor number, the geometric genus, the link (and even the embedded topological type). The stable resolution graph is

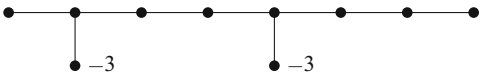


Since the graph is AR, and $p_g = \min_\gamma eu(\mathbb{H}^0_{top}(\gamma, \Gamma, K))$, cf. [498], by 11.9.29, $\mathbb{H}^*_{an,0} = \mathbb{H}^*_{top,0}$ for any t. Since $\mathbb{H}^*_{top,0}$ is t-stable, we obtain that $\mathbb{H}^*_{an,0}$ must stay t-stable as well. In this example the point is that the deformation is not μ^*–stable ($\mu_1^{t=0} = 11$ while $\mu_1^{t\neq 0} = 10$) (that is, the deformation does not admit a strong simultaneous resolution). However, $\mathbb{H}^*_{an,0}$ is still constant.

Example 11.9.51 Consider the deformation of hypersurface singularities $(X_t, o) = \{x^7 + y^5 + z^3 + tx^4y^2 = 0\}$, $t \in (\mathbb{C}, 0)$. If $t = 0$ then $(X_{t=0}, 0)$ is weighted homogeneous with the following minimal resolution graph, where the unmarked vertices have (-2) decorations.



For $t \neq 0$ the germ $(X_t, 0)$ has nondegenerate Newton principal part, its graph is



Note that the deformation modifies the Newton diagram, and also the topological type. However, along the deformation both $K^2_{\tilde{X}}$ and p_g remain constant: $K^2_{\tilde{X}} = -12$ and $p_g = 4$. (I.e., the deformation does not admit a weak simultaneous resolution, but it admits a very weak simultaneous resolution, cf. [365].) Note also that both graphs are AR. Since both analytic types are Newton nondegenerate, for both of them $p_g = \min_\gamma eu(\mathbb{H}^0_{top}(\gamma, \Gamma, K))$, cf. 11.6.14. Hence, for any fixed t, by Proposition 11.9.28 the analytic and topological lattice cohomologies agree. In fact, by AR property $\mathbb{H}^{\geq 1}_{an,0} = 0$ in all cases. Note also that by [641] $p_g = eu(\mathbb{H}^*_{top,0}(\Gamma, K))$ too, hence $p_g = \min_\gamma eu(\mathbb{H}^0_{top}(\gamma, \Gamma, K)) = eu(\mathbb{H}^0_{top,0}(\Gamma, K))$ as well.

On the other hand, for both $t = 0$ and $t \neq 0$, the topological graded roots can be computed by the AR–algorithm and it turns out that they agree. Hence $\mathbb{H}^0_{an,0}$ and the analytic graded root is independent of t as well. The (common) graded root agrees with the topological graded root from the Example 11.9.40.

In particular, since $\mathbb{H}^*_{an,0}$ is t–independent, the example supports the Conjecture, and it is also compatible with Conjecture 11.3.51, which predicts that along such deformations the topological graded root stays constant.

Problem 11.9.52 Consider the equisingular family of hypersurface singularities with Newton nondegenerate principal part associated with a Newton diagram $N\Gamma$. Is it true that $\mathbb{H}^*_{an,0}$ is constant along this family? If yes (hence $\mathbb{H}^*_{an,0}$ is determined by $N\Gamma$) describe it combinatorially from the Newton diagram $N\Gamma$.

Example 11.9.53 Let us consider the following Newton nondegenerate germs: $\{x^{13} + y^{13} + x^3y^2 + x^2y^3 + z^3 = 0\}$, or $\{x^{14} + y^{14} + x^2y^2 + z^3 = 0\}$, or $\{x^{13} + y^9z + x^2y^2 + z^3 = 0\}$. All of them are p_g-constant deformations of the germ $\{x^{13} + y^{13} + x^2y^2 + z^3 = 0\}$ considered in 11.8.15. Then for any of them, the corresponding w_0–table (determined by the Newton filtration) in the rectangle given by Z_K reduced to the nodes agree with the corresponding χ–table (though, the resolution graphs are rather different, and the size of $R(0, Z_K|_{\text{nodes}})$ is also changing). Furthermore, in all these cases, the identity $\mathbb{H}^*_{an,0}(X, o) = \mathbb{H}^*_{top,0}(M)$ holds, and this module stays stable under the corresponding deformations. (Compare with 8.6.18.)

Chapter 12
Appendix. Complex Analytic Spaces

12.1 Analytic Algebras

In Sects. 12.1 and 12.2 we follow the monographs [226–228, 230].

12.1.1 The Ring $\mathscr{O}_0^n$ Let $z_1, \dots, z_n$ denote the complex coordinates in $\mathbb{C}^n$. We denote by $\mathscr{O}_0^n = \mathbb{C}\{z_1, \dots, z_n\}$ the $\mathbb{C}$-algebra of convergent power series around $0 \in \mathbb{C}^n$. Its elements are called *germs of holomorphic functions at* 0. The ring $\mathscr{O}_0^n$ is Noetherian, factorial, and a local $\mathbb{C}$-algebra with maximal ideal $\mathfrak{m}$ generated by $z_1, \dots, z_n$.

12.1.2 Analytic Algebras A local $\mathbb{C}$-algebra is called a *local analytic algebra* if it is isomorphic to a residue class algebra $\mathscr{O}_0^n/\mathfrak{a}$, where $\mathfrak{a}$ is an ideal in $\mathscr{O}_0^n$ and $\mathfrak{a} \neq \mathscr{O}_0^n$. If $\mathfrak{m}_A$ is its maximal ideal then $A/\mathfrak{m}_A \approx \mathbb{C}$. Every analytic algebra A is Noetherian, Henselian and $\cap_{j\geq 1}\mathfrak{m}_A^j = \{0\}$.

An analytic algebra is called *regular* if it is isomorphic to $\mathscr{O}_0^d$ for some $d \in \mathbb{Z}_{\geq 0}$; it is called *reduced* (respectively *irreducible*) if it contains no non-zero nilpotent element (it is an integral domain, respectively). A is *Artinian* if it is a finite dimensional $\mathbb{C}$-vector space, or, equivalently, if $\mathfrak{m}_A^k = 0$ for a certain $k \geq 1$. In such a case $\mathfrak{m}_A$ is the only prime ideal of A.

$\mathbb{C}$-algebra homomorphisms between analytic algebras are called *analytic*; they are automatically *local*. Analytic algebras together with analytic homomorphisms form a category. An analytic homomorphism $\varphi : A \to B$ of analytic algebras is called *finite* if the A-module B (via φ) is finite.

Noether's Finiteness Lemma says that for every analytic algebra A there exists a finite analytic monomorphism $\varphi : \mathscr{O}_0^d \to A$. The integer d appearing in Noether's Lemma is independent of the choice of φ, it is called the *dimension* $\dim(A)$ *of* A. The geometric meaning of the lemma is that locally every complex space of dimension d is a ramified analytic covering of $\mathbb{C}^d$ (cf. 12.3.2). The integer $\dim(A)$

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also agrees with the *algebraic (Krull) dimension* of A, which, by definition, is the largest integer d for which a sequence $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_d$ of prime ideals of A exists.

12.1.3 Finite Extensions of Analytic Algebras. Normalization Finite $\mathbb{C}$-algebra extensions of analytic algebras are not necessarily analytic algebras. Nevertheless, if $A \to B$ is a finite $\mathbb{C}$-algebra homomorphism of an analytic algebra A into a $\mathbb{C}$-algebra B, then B is a unique finite sum $\oplus_j A_j$ of analytic algebras. In particular, if B is an integral domain then B is automatically an analytic algebra.

The normalization is such an extension. Consider a *reduced* analytic algebra A and its quotient ring $Q(A)$. The *integral closure* $\widehat{A}$ *of* A consists of all elements of $Q(A)$, which are integral over A, that is, they annihilate a monic polynomial with coefficients from A. The set $\widehat{A}$ is a $\mathbb{C}$-algebra extension of A, it is called the *normalization of* A. It is the maximal ring extension of A in $Q(A)$ which is a finite A-module. If $\widehat{A} = A$ then we say that A is *normal*. Factorial algebras are normal.

If the analytic algebra A is an integral domain then $\widehat{A}$ is an analytic algebra. In general, let $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ be the minimal prime ideals of A. Since A is reduced, one has $\cap_j \mathfrak{p}_j = 0$ and the residue maps $A \to A_j := A/\mathfrak{p}_j$ extend uniquely to epimorphisms $Q(A) \to Q(A_j)$ of the quotient rings inducing isomorphisms $Q(A) \to \oplus_j Q(A_j)$ and $\widehat{A} \to \oplus_j \widehat{A_j}$.

12.2 Complex Spaces

12.2.1 $\mathbb{C}$-Ringed Spaces A topological space $|X|$ together with a sheaf $\mathscr{A}_X$ of rings on $|X|$ is called a *ringed space*. We write $(X, \mathscr{A}_X)$, or X, and call $\mathscr{A}_X$ the *structure sheaf of* X (hence $|X|$ denotes the underlying topological space). We write $\mathbb{C}_X, \mathbb{Q}_X$, respectively $\mathbb{Z}_X$, for the constant sheaf of fields $\mathbb{C}$, $\mathbb{Q}$, respectively rings $\mathbb{Z}$, on $|X|$.

If $\mathscr{A}_X$ is a $\mathbb{C}$-module, then we call $\mathscr{A}_X$ a *sheaf of* $\mathbb{C}$*-algebras*. If every stalk $\mathscr{A}_x$ is a local ring with maximal ideal $\mathfrak{m}(\mathscr{A}_x)$ satisfying $\mathscr{A}_x/\mathfrak{m}(\mathscr{A}_x) \approx \mathbb{C}$, then we call $\mathscr{A}_X$ a *sheaf of local* $\mathbb{C}$-algebras. A ringed space X with a structure sheaf of local $\mathbb{C}$-algebras is called a $\mathbb{C}$*-ringed space*.

A morphism $(f, \tilde{f}) : (X, \mathscr{A}_X) \to (Y, \mathscr{A}_Y)$ *of* $\mathbb{C}$*-ringed spaces* consists of a continuous map $f : |X| \to |Y|$ together with a $\mathbb{C}$-algebra *lifting homomorphism* $\tilde{f} : \mathscr{A}_Y \to f_*(\mathscr{A}_X)$. We write $f : X \to Y$ for such morphism.

Every morphism $(f, \tilde{f})$ induces *stalk maps* $\tilde{f}_y : \mathscr{A}_{Y,y} \to f_*(\mathscr{A}_X)_y$ for every $y \in |Y|$. This composed with the natural restrictions $f_*(\mathscr{A}_X)_{f(x)} \to \mathscr{A}_{X,x}$ provides local $\mathbb{C}$-algebra homomorphism $\tilde{f}_x : \mathscr{A}_{Y,f(x)} \to \mathscr{A}_{X,x}$ for every $x \in |X|$. The collection $\{\tilde{f}_x\}_{x \in |X|}$ determines $\tilde{f}$ completely.

$\mathbb{C}$-ringed spaces form a category. $(f, \tilde{f})$ is an isomorphism in this category if f is a homeomorphism and $\tilde{f}$ a sheaf isomorphism.

12.2.2 Complex Model Spaces Let D be a domain of $\mathbb{C}^n$ and let $\mathscr{O}_D$ be its structure sheaf of holomorphic functions. Fix some holomorphic functions $h_1, \dots, h_m$ defined on D, and consider the ideal sheaf $\mathscr{I} \subset \mathscr{O}_D$ generated by them. They define a $\mathbb{C}$-ringed space $(V(\mathscr{I}), \mathscr{O}_{V(\mathscr{I})})$, where $|V(\mathscr{I})| = \{x \in D : h_1(x) = \dots = h_m(x) = 0\}$, and $\mathscr{O}_{V(\mathscr{I})}$ is the quotient sheaf $\mathscr{O}_D/\mathscr{I}$ with stalks the analytic algebras $\mathscr{O}_{D,x}/\mathscr{I}_x$.

The $\mathbb{C}$-ringed space $(V(\mathscr{I}), \mathscr{O}_{V(\mathscr{I})})$ is called the *complex model space defined by* $\mathscr{I}$ *in* D.

All analytic algebras appear as stalks of certain model spaces.

12.2.3 Complex Spaces A $\mathbb{C}$-ringed space $(X, \mathscr{O}_X)$ is called *complex space*, if $|X|$ is Hausdorff space and if every point of $|X|$ has an open neighborhood U such that the open $\mathbb{C}$-ringed subspace $(U, \mathscr{O}_U)$ of $(X, \mathscr{O}_X)$ is isomorphic to a complex model space. Complex spaces form a full subcategory of the category of $\mathbb{C}$-ringed spaces. Morphisms (isomorphisms) are called *holomorphic* (*biholomorphic*) maps, and $\mathscr{O}_X$-modules are called *analytic sheaves* on X.

Fix $x \in |X|$. We denote by (X, x) the *germ of* X *at* x, that is the space $(U, \mathscr{O}_U)$, where U is an arbitrarily small neighborhood of x. If (X, x) and (Y, y) are two germs of complex spaces, then $f : (X, x) \to (Y, y)$ denotes a holomorphic map defined on an arbitrary small neighborhood of x which satisfies $f(x) = y$.

A point $x \in X$ is called *smooth*, if $\mathscr{O}_x$ is regular. A space with only smooth points is a *complex manifold*. A point $x \in X$ is called *reduced* if $\mathscr{O}_x$ is reduced, *irreducible* if $\mathscr{O}_x$ is an integral domain, *normal* if $\mathscr{O}_x$ is normal. Normal points are irreducible, irreducible points are reduced. A space with reduced (irreducible, normal) points only is called *reduced* (*locally irreducible, normal*) space.

Non-regular points are called *singular*, non-irreducible points are called *reducible*. The locus of singular (or non-normal, or non-reduced) points is analytic in X (see 12.2.5 for the definition of analytic sets).

An *ideal subsheaf* $\mathscr{I} \subset \mathscr{O}_X$ *is finite* if every point of X has a neighborhood U with holomorphic functions $h_1, \dots, h_m \in \mathscr{O}_X(U)$ generating $\mathscr{I}_U$. Such a finite ideal subsheaf $\mathscr{I} \subset \mathscr{O}$ determines a *closed complex subspace* $(V(\mathscr{I}), \mathscr{O}_{V(\mathscr{I})})$ of $(X, \mathscr{O}_X)$. Here $V(\mathscr{I}) = \{x \in |X| : \mathscr{I}_x \neq \mathscr{O}_x\}$ and $\mathscr{O}_{V(\mathscr{I})} = (\mathscr{O}_X/\mathscr{I})|V(\mathscr{I})$.

A holomorphic map $f : X \to Y$ is called a (closed) *embedding* if f admits a factorization $f = \iota \circ g$ through a closed complex subspace $\iota : Z \hookrightarrow Y$ such that $g : X \to Z$ is biholomorphic. Fix $o \in X$. A holomorphic map $f : X \to Y$ is called an *embedding at* o *into* Y if there are neighborhoods U and V of o and $f(o)$ with $f(U) \subset V$ such that the restriction $f : U \to V$ is an embedding. For such local embedding we use the notation $f : (X, o) \hookrightarrow (Y, f(o))$. One has the following equivalent characterization: f is an embedding at o if and only if $\tilde{f}_o : \mathscr{O}_{Y,f(o)} \to \mathscr{O}_{X,o}$ is surjective.

Certainly, any (X, o) admits embedding into $(\mathbb{C}^n, 0)$ for some n. The smallest such n is the *embedding dimension* $\operatorname{emb}(\mathscr{O}_{X,o})$ of X at o.

12.2.4 Jacobi Criterion Fix a complex model space $X = V(\mathscr{I})$ in $D \subset \mathbb{C}^n$, where $\mathscr{I} = (h_1, \dots, h_m)$, cf. 12.2.2.

Then, for $x \in V(\mathscr{I})$, $\operatorname{emb}(\mathscr{O}_{X,x}) = n$ if and only if $\mathscr{I}_x \subset \mathfrak{m}^2_{\mathscr{O}^n_x}$.

More generally, if $\operatorname{rank}_x(h_1, \ldots, h_m)$ denotes the rank of the Jacobian matrix of $h_1, \ldots, h_m$ at x, then for all $x \in X$ one has

$$\operatorname{emb}(\mathscr{O}_{X,x}) + \operatorname{rank}_x(h_1, \ldots, h_m) = n. \tag{12.2.1}$$

Moreover,

$$\dim(\mathscr{O}_{X,x}) = n - \max_{y \in (X,x)} \operatorname{rank}_y(h_1, \ldots, h_m). \tag{12.2.2}$$

Therefore, $\operatorname{emb}(\mathscr{O}_{X,x}) \geq \dim(\mathscr{O}_{X,x})$, with equality if and only if x is a smooth point. X is called equidimensional if all the smooth points have the same dimension.

Furthermore, $\operatorname{emb}(\mathscr{O}_{X,x})$ is upper semi-continuous: for any integer k the set $X(k) := \{x \in X : \operatorname{emb}(\mathscr{O}_{X,x}) \geq k\}$ is analytic in X. If X is equidimensional of dimension d then $X(d) = X$, and the set of singular locus $\Sigma(X)$ equals $X(d+1)$.

(X, o) is called *a complete intersection* if in (12.2.2) $m = n - \dim(\mathscr{O}_{X,o})$. A complete intersection (X, o) is called *hypersurface* singularity if $m = 1$.

12.2.5 Analytic Sets Fix a $\mathbb{C}$-ringed space $(X, \mathscr{O}_X)$.

A subset $A \subset |X|$ is called *analytic at* $x \in |X|$ if there exists a neighborhood U of x in X, and a finite ideal $\mathscr{I} \subset \mathscr{O}_U$ such that $A \cap U$ is $|V(\mathscr{I})|$. For every analytic set A in X the family $\mathscr{I}_A := \{f \in \mathscr{O}(U) : \{f = 0\} \supset A \cap U\}_U$ is an ideal subsheaf of $\mathscr{O}_X$, called the *ideal of* A. Its stalks are reduced. Moreover, $|V(\mathscr{I}_A)| = A$ and $\mathscr{I}_A = \operatorname{rad}(\mathscr{I})$ (where $\mathscr{I} \subset \mathscr{O}$ is any ideal with zero set A, as in the definition of A). Recall that $\operatorname{rad}(\mathscr{I})$ of an ideal $\mathscr{I}$ is the ideal sheaf with stalks $\operatorname{rad}(\mathscr{I}_x) = \{f \in \mathscr{O}_x : f^m \in \mathscr{I}_x \text{ for large integer } m\}$.

Hence analytic sets can be identified with reduced complex spaces.

This fact is even more transparent if we consider the sheaf of continuous functions $\mathscr{C}_X$ on $|X|$, where X is a complex space. Every section $s \in \mathscr{O}(U)$ determines a $\mathbb{C}$-valued function $[s](x) = [s(x)] \in \mathbb{C} = \mathscr{O}_x/\mathfrak{m}(\mathscr{O}_x)$. Then $[s]$ is continuous, and the map $\mathscr{O}_X(U) \to \mathscr{C}_X(U)$, $s \mapsto [s]$, defines a $\mathbb{C}$-algebra homomorphism $\mathscr{O}_X \to \mathscr{C}_X$. Then, by *Rückert Nullstellensatz*, the nilradical $\mathfrak{N}_X$ of $\mathscr{O}_X$ is the kernel of $\mathscr{O}_X \to \mathscr{C}_X$.

If X is any complex space, $|X|$ with structure sheaf $\mathscr{O}_X/\mathfrak{N}_X$ (the *reduced structure of* X) is denoted by X_{red}.

An analytic set A is called *irreducible at* $a \in A$ if A as reduced complex space is irreducible at a; otherwise A is called *reducible at* a.

The analytic set A has an *irredundant decomposition* into irreducible components at every point $a \in A$. This means that for every $a \in A$ there exists a neighborhood U and a decomposition $A \cap U = \cup_j A_j$, where each A_j is analytic set in U and irreducible at a such that $A_j \cap V \not\subset A_k \cap V$ for all $j \neq k$ and all neighborhoods $V \subset U$ of a. The decomposition (up to a reordering) is unique.

The irredundant decomposition is provided by the *Lasker–Noether Theorem*, which guarantees that every reduced ideal $\mathfrak{a}$ in a Noetherian ring has a unique

decomposition into $\mathfrak{a} = \cap_j \mathfrak{p}_j$, where $\mathfrak{p}_j$'s are prime and $\mathfrak{p}_j \not\subset \mathfrak{p}_k$ for $j \neq k$. The decomposition of A at $a \in A$ is provided by the Lasker–Noether decomposition $\mathscr{I}_{A,a} = \cap_j \mathfrak{p}_j$ via $\mathscr{I}_{A_j,a} = \mathfrak{p}_j$.

12.2.6 Normal Reduced Complex Spaces Fix a *reduced* complex space X. Recall that X is normal at $x \in X$ if $\mathscr{O}_{X,x}$ is a normal ring. X is normal if it is normal at every point. This algebraic criterion is complemented by the following 'analytic' ones.

The following statements are equivalent:

1. X is normal.
2. The singular locus $\Sigma(X)$ has codimension ≥ 2 and for any open set $U \subset X$ the restriction $\mathscr{O}(U) \to \mathscr{O}(U \setminus \Sigma(X))$ is surjective.
3. For any open set $U \subset X$, every *bounded* holomorphic function in $U \setminus \Sigma(X)$ extends uniquely to a holomorphic function on U.

The above properties can be compared with the *Riemann Extension Theorem*, which says that if X is a (connected) manifold, A is an analytic subspace of X, $a \in A$, and f a holomorphic function on $X \setminus A$, then f has a unique holomorphic extension at a provided that either f is bounded near a, or the codimension of A in X at a is ≥ 2.

In particular, every connected normal space is equidimensional, locally irreducible, and the singular locus has codimension ≥ 2. Furthermore, if $A \subset X$ of dimension $< \dim(X)$, then $X \setminus A$ is connected too. Therefore, $X \setminus \Sigma(X)$ is connected.

Correspondingly, the (algebraic) normalization procedure has the following function theoretical reinterpretation. Fix a connected reduced complex space X.

By definition, a *weakly holomorphic function* on an open set $U \subset X$ is a holomorphic function on $U \setminus \Sigma(X)$ which is *bounded near* $\Sigma(X)$. The set $\mathscr{O}'(U)$ of all weakly holomorphic functions is an $\mathscr{O}(U)$-algebra, defining a sheaf $\mathscr{O}'$ of $\mathscr{O}$-algebras, and $\mathscr{O} \subset \mathscr{O}'$. Then the sheaf $\mathscr{O}'$ and the normalization $\widehat{\mathscr{O}}$ of $\mathscr{O}$ agree.

The inclusion $\mathscr{O} \subset \widehat{\mathscr{O}}$ induces the *normalization* $n : \widehat{X} \to X$. It is finite, surjective, and isomorphism over $X \setminus \Sigma(X)$. The normalization n dominates all holomorphic maps $\widetilde{X} \to X$ with these three properties. Furthermore, every surjective holomorphic map $Y \to X$ with Y normal has a unique factorization through n.

If X is locally irreducible then n is a homeomorphism. More generally, the cardinality of $n^{-1}(o)$ is the number of local irreducible components of (X, o). If (X, o) is irreducible at o then $\Sigma(X)$ is nowhere dense around o, and $U \setminus \Sigma(X)$ is connected for any small neighborhood U of o.

Usually, the fact that $\Sigma(X) \subset X$ has codimension ≥ 2 does not guarantee that X is normal. Nevertheless, if additional each $\mathscr{O}_{X,x}$ is Cohen–Macaulay (what happens e.g. in the case of complete intersections), the normality follows.

12.3 Analytic Coverings

In this section all the complex spaces are *reduced*. We relied on [103, 226–228, 230, 659].

12.3.1 Finite Maps Assume that X and Y are reduced complex spaces. A holomorphic map $f : X \to Y$ is called *finite* if it is proper and has finite fibers. This happens if and only if f is closed and has finite fibers. Moreover, for a holomorphic map $f : X \to Y$ the following statements are equivalent:

1. $x \in X$ is an isolated point of the fiber $f^{-1}(f(x))$ (that is, $\mathcal{O}_{f^{-1}(f(x)),x}$ is Artinian);
2. $\tilde{f}_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is finite.

Composition of two finite maps is finite again.

If X is closed subspace of Y then the inclusion $X \to Y$ is finite. In particular, finite maps, in general, are neither open, nor surjective.

Clearly, if $f : X \to Y$ is holomorphic, and A is an analytic subset of Y, then its inverse image $f^{-1}(A)$ is analytic in X. The converse is not true: images of analytic sets are not analytic in general. However, if f is finite, and A is analytic subset of X, then $f(A)$ is analytic in Y.

12.3.2 Coverings A *finite surjective* holomorphic map $p : X \to Y$ between reduced complex spaces is called an *analytic covering* (of Y), if there exists a nowhere dense analytic subset A of Y with the following properties:

1. the set $p^{-1}(A)$ is nowhere dense in X;
2. the induced map $X \setminus p^{-1}(A) \to Y \setminus A$ is locally biholomorphic.

The set of points $x \in X$ where p is not locally biholomorphic is called the *ramification locus* of p. Its image in Y is the *branch locus* of p, it is the minimal set A in Y for which the above definition holds. Usually it is denoted by B.

If $p : X \to Y$ and $\pi : Y \to Z$ are analytic coverings, then the composed map $\pi \circ p$ is an analytic covering. This has a partial converse: If $p : X \to Y$ is a finite holomorphic surjection and $\pi : Y \to Z$ and $\pi \circ p : X \to Z$ are analytic coverings, then p also is an analytic covering.

In general, analytic coverings are not open. (For example, project two disjoint lines onto two lines with one intersection point.) Nevertheless, if Y is *locally pure dimensional* then X is also *locally pure dimensional*. In addition, if Y is *locally irreducible*, then p is *open*. Furthermore, if $b(y)$ denotes the cardinality of $p^{-1}(y)$ ($y \in Y$), then $b(y)$ is locally constant in $Y \setminus B$. In particular, if $Y \setminus B$ is connected (e.g., whenever Y is normal and connected), $b(y)$ is constant on $Y \setminus B$.

Whenever $b(y)$ on $Y \setminus B$ is constant with value b, we say that p is *a b-sheeted analytic covering*. The integer b sometimes is also called the *degree* of the covering. For example, the normalization is a 1-sheeted covering, where B is the set of non-normal points of Y.

By the *Covering Theorem*, every open and finite holomorphic surjection is an analytic covering. If X and Y are irreducible then any finite surjective map $p : X \to Y$ is an analytic covering [228, p. 179].

The *Existence Theorem for open analytic coverings* says that if (X, x) is pure d-dimensional reduced complex space then there exist an open analytic covering $(X, x) \to (\mathbb{C}^d, 0)$ (compare with Noether's Finiteness Lemma).

12.3.3 Stein's Theorem [659] Next, we assume that X and Y are normal spaces. In this situation, if $p : X \to Y$ is an analytic covering, then above the smooth part of Y both loci, the ramification locus and the branch locus, are divisors.

In the next discussion we assume the local situation, namely the case of a germ of an analytic covering $p : (X, x) \to (Y, y)$, where both (X, x) and (Y, y) are normal. Recall, that in this case the degree of p is well-defined.

We say that two coverings $p : (X, x) \to (Y, y)$ and $p' : (X', x') \to (Y, y)$ are equivalent, if there exists an analytic isomorphism $\iota : (X, x) \to (X', x')$ with $p' \circ \iota = p$.

Let us fix a possible branch locus $(B, y) \subset (Y, y)$. By the duality between unramified coverings of $Y \setminus B$ and subgroups of the fundamental group $\pi_1(Y \setminus B)$, see e.g. [645], and the analytic realization provided by theorem of Stein, cf. [226, 659], there is a bijection between the following sets:

1. analytic coverings $p : (X, x) \to (Y, y)$ ramified over B, modulo equivalence,
2. subgroups K in $\pi_1(Y \setminus B)$ of finite index, modulo conjugation.

Under this correspondence, the index $[\pi_1(Y \setminus B) : K]$ of K equals the degree of p. The correspondence is *functorial*: an inclusion of subgroups $K_1 \subset K_2$ induces a covering $\iota : (X_{K_1}, x_1) \to (X_{K_2}, x_2)$ such that $p_{K_2} \circ \iota = p_{K_1}$, and conversely.

The above correspondence is valid even if B is not a divisor (that is, it is a smaller dimensional set). For example, if (Y, y) is a fixed normal surface singularity, then taking $B = \{y\}$, we have the bijection between the following sets:

1. analytic coverings $p : (X, x) \to (Y, y)$, with (X, x) normal and restriction $p : X \setminus \{x\} \to Y \setminus \{y\}$ *o*-ramified (i.e. locally biholomorphic), modulo equivalence,
2. subgroups K in $\pi_1(Y \setminus \{y\})$ of finite index, modulo conjugation.

For example, if $H_1(Y \setminus \{y\}, \mathbb{Z})$ is finite, then the commutator subgroup of $\pi_1(Y \setminus \{y\})$ is such a subgroup (in fact, it is even normal, being the kernel of the *abelianization* $\pi_1(Y \setminus \{y\}) \to H_1(Y \setminus \{y\}, \mathbb{Z})$). The corresponding covering is called the *universal abelian covering* of (Y, y).

12.3.4 Galois Coverings An *automorphism* of the covering $p : (X, x) \to (Y, y)$ is an analytic fiber-preserving isomorphism of (X, x) onto itself. The set of all automorphism form a group $G(p)$. The covering p is called *Galois*, if any of the following equivalent conditions below is satisfied:

(a) $\pi_1(X \setminus p^{-1}(B))$ is a normal subgroup in $\pi_1(Y \setminus B)$,
(b) $G(p) \simeq \pi_1(Y \setminus B)/\pi_1(X \setminus p^{-1}(B))$,

(c) $|G(p)| = \deg(p)$,
(d) $G(p)$ acts transitively on any fixed fiber $p^{-1}(y')$, $y' \notin B$.

In fact, if p is Galois, then by continuity, $G(p)$ acts transitively on any fiber $p^{-1}(y')$.

Hence, Galois coverings correspond to epimorphisms $\rho : \pi_1(Y \setminus B) \to G$ to finite groups G, which became the automorphism group of the covering.

For example, the universal abelian covering of (Y, y) is a Galois covering with $G(p) = H_1(Y \setminus \{y\}, \mathbb{Z})$ (provided that $H_1(Y \setminus \{y\}, \mathbb{Z})$ is finite). It dominates all Galois coverings with $G(p)$ abelian.

12.3.5 Cartan's Theorem [103] Starting from the pair $(B, y) \subset (Y, y)$ and an epimorphism $\rho : \pi_1(Y \setminus B) \to G$ to a finite group G, one constructs a Galois covering (X, x) with automorphism group G. In this way (Y, y) becomes an analytic realization of the topological quotient, or orbit space X/G. Conversely, one can also start with some X and the group G.

Indeed, fix a reduced complex space X and a subgroup G of the group of biholomorphic automorphisms of X. We assume that G acts *properly discontinuously* on X, that is, for all compact sets $\kappa \subset X$ the cardinality of $\{g \in G : g(\kappa) \cap \kappa\}$ is finite. This implies that for any fixed $x \in X$ the *isotropy* or *stabilizer* subgroup $G_x := \{g \in G : g(x) = x\}$ is also finite. Then the topological quotient space X/G is Hausdorff. Moreover, by a result of Cartan [103] (see also [395]), $Y = X/G$ admits a reduced complex space structure. The holomorphic quotient map $p : X \to Y$ is locally finite and surjective, and near any point $x \in X$ it is isomorphic to the quotient map $X \to X/G_x$. In particular, for any $y \in Y$, the analytic algebra $\mathcal{O}_{Y,y}$ can be identified with the invariant algebra $\mathcal{O}_{X,x}^{G_x}$ for any $x \in \rho^{-1}(y)$, and the canonical ring homomorphism $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is the inclusion $\mathcal{O}_{X,x}^{G_x} \subset \mathcal{O}_{X,x}$.

This can be complemented with the following facts at the level of analytic algebras [103, 227]. Let A be an analytic algebra and let G be a finite group of automorphism of A. Then the algebra of invariants A^G is an analytic algebra such that the inclusion $A^G \subset A$ is finite. Moreover, if A is reduced (irreducible, normal, Cohen–Macaulay), then so does A^G.

In particular, if X is smooth or normal, then X/G is normal.

12.3.6 The Case of Finite G. Linearization [103] Assume that (X, x) is a germ of a complex space and G a *finite* group of biholomorphic automorphisms of X. Then there exist (a) an open set U of X, $x \in U$, preserved by G, (b) a complex model space $(V(\mathcal{J}), \mathcal{O}_{V(\mathcal{J})})$ defined by the ideal sheaf $\mathcal{J} \subset \mathcal{O}_D$, $(D, 0)$ domain of $(\mathbb{C}^n, 0)$, and a linear action of G on $\mathbb{C}^n$, and finally (c) an isomorphism $(U, \mathcal{O}_U) \to (V(\mathcal{J}), \mathcal{O}_{V(\mathcal{J})})$ preserving the action of G. If G is abelian then we can even assume that G acts on $\mathbb{C}^n$ diagonally.

This basically follows from the following fact as well. Set $A := \mathcal{O}_{X,x}$ and consider the exact sequence $0 \to \mathfrak{m}_A^2 \to \mathfrak{m}_A \to \mathfrak{m}_A/\mathfrak{m}_A^2 \to 0$. This has an induced G action. Since G is finite, the exact sequence equivariantly splits, hence (X, x) embeds equivariantly into its tangent space m_A/m_A^2.

In particular, if (X, x) is regular, then $(X/G, x)$ is isomorphic to some $(\mathbb{C}^n/G, 0)$ for some finite subgroup of $GL_n(\mathbb{C})$.

12.3.7 Quotient Singularities $\mathbb{C}^n/G$ **[220, 221, 581]** An element $g \in GL_n(\mathbb{C}))$ of finite order is called *pseudo reflection* if it has eigenvalues 1 with multiplicity $n-1$ and another one $\xi \neq 1$ (equivalently, if its fixpoint set is a hyperplane). A group $G \subset GL_n(\mathbb{C}))$ is called *reflection group* if it is generated by pseudo reflections.

By [633] or [109] (see also [72, 647]), if $G \subset GL_n(\mathbb{C})$ is a finite reflection group then the invariant algebra $\mathbb{C}[z_1, \ldots, z_n]^G$ is generated by n algebraically independent homogeneous polynomials. In particular, $\mathbb{C}^n/G \simeq \mathbb{C}^n$ for such G.

A subgroup $G \subset GL_n(\mathbb{C}))$ is called *small*, if it does not contain any pseudo reflection. If $G \subset GL_n(\mathbb{C})$ is an arbitrary finite subgroup, then let $G_r \subset G$ be the subgroup of G generated by pseudo reflections of G. It is a normal subgroup of G and $\bar{G} := G/G_r$ is small. Since $\mathbb{C}^n/G = (\mathbb{C}^n/G_r)/(G/G_r) \simeq \mathbb{C}^n/\bar{G}$, any quotient singularity $\mathbb{C}^n/G$ can be written as $\mathbb{C}^n/\bar{G}$ with $\bar{G}$ small.

Moreover, if G_1 and G_2 are small subgroups of $GL_n(\mathbb{C})$, then $\mathbb{C}^n/G_1$ and $\mathbb{C}^n/G_2$ are isomorphic if and only if G_1 and G_2 are conjugate in $GL_n(\mathbb{C})$.

12.4 Complex Reduced Surfaces

12.4.1 Line Bundles and Divisors on Normal Spaces Fix an irreducible normal space X. With respect to the tensor product, holomorphic line bundles on X form a group. Their isomorphic classes is the *Picard group* $\mathrm{Pic}(X)$ of X. The cocycle interpretation of line bundles provides a natural isomorphism $\mathrm{Pic}(X) = H^1(X, \mathscr{O}_X^*)$, where $\mathscr{O}_X^*$ is the sheaf of invertible holomorphic functions. The *exponential exact sequence of sheaves*

$$0 \to \mathbb{Z}_X \hookrightarrow \mathscr{O}_X \xrightarrow{e} \mathscr{O}_X^* \to 0, \tag{12.4.1}$$

where $e(f) := \exp(2\pi i f)$, provides the *exponential exact cohomology sequences*

$$\cdots \to H^1(\mathbb{Z}_X) \to H^1(\mathscr{O}_X) \to H^1(\mathscr{O}_X^*) \xrightarrow{c_1} H^2(\mathbb{Z}_X) \to H^2(\mathscr{O}_X) \to \cdots \tag{12.4.2}$$

where $c_1(\mathscr{L})$ is the first Chern class of $\mathscr{L}$. Note that there is a similar exact sequence for topological (continuous) complex line bundles, with $\mathscr{C}_X$ instead of $\mathscr{O}_X$. In that case, since $\mathscr{C}_X$ is soft, hence $H^i(\mathscr{C}_X) = 0$ for $i > 0$. In particular, the classification of continuous line bundles is given by the isomorphism $c_1 : \mathrm{Pic}^{top}(X) \xrightarrow{\simeq} H^2(\mathbb{Z}_X)$. Hence $c_1 : \mathrm{Pic}(X) \to H^2(\mathbb{Z}_X) = H^2(X, \mathbb{Z})$ can also be identified with the 'forgetting morphism' $\mathrm{Pic}(X) \to \mathrm{Pic}^{top}(X)$.

Usually, the kernel of c_1 is denoted by $\mathrm{Pic}^0(X)$, the group of holomorphic line bundles, which are topologically trivial.

12.4.2 A **Weil divisor** on X is a formal sum $Z = \sum_i n_i C_i$, where $n_i \in \mathbb{Z}$ and $\{C_i\}_i$ is a locally finite family of irreducible one-codimensional analytic subsets of X. They form the group $\mathrm{Div}(X)$. The divisor Z is called *effective*, denoted by $Z \geq 0$, if all $n_i \geq 0$. A non-zero effective divisor is also called *positive*, denoted by $Z > 0$. If $\dim(X) = 2$, then the *positive divisors* are called *curves* as well.

Two divisors Z_1 and Z_2 are linearly equivalent, denoted by $Z_1 \sim Z_2$, if $Z_1 - Z_2$ is *principal*, that is, it is the divisor (the zero divisor minus the pole divisor) of a meromorphic function. The *divisor class group* is $\mathrm{Cl}(X) := \mathrm{Div}(X)/\sim$.

Locally principal divisors form the subgroup of *Cartier divisors* $\mathrm{CaDiv}(X)$ in $\mathrm{Div}(X)$. Their class group, $\mathrm{CaCl}(X) = \mathrm{CaDiv}(X)/\sim$, is a subgroup of $\mathrm{Cl}(X)$. If X is *smooth*, then the inclusions $\mathrm{CaDiv}(X) \hookrightarrow \mathrm{Div}(X)$ is an isomorphism.

For any Cartier divisor Z one considers the line bundle $\mathscr{O}_X(Z)$, having the property that the divisor of any meromorphic section of $\mathscr{O}_X(Z)$ is linearly equivalent to Z. The correspondence $Z \mapsto \mathscr{O}_X(Z)$ induces an injective homomorphism $\mathrm{CaCl}(X) \hookrightarrow \mathrm{Pic}(X)$. If a line bundle admits a global section then it is contained in the image. In fact, the image is generated by line bundles $\mathscr{L}$ with $H^0(X, \mathscr{L}) \neq 0$.

If X is projective then $\mathrm{CaCl}(X) \hookrightarrow \mathrm{Pic}(X)$ is an isomorphism.

The monomorphism $\mathrm{CaCl}(X) \hookrightarrow \mathrm{Pic}(X)$ extends to the Weil divisors as follows, see e.g. [607]. A coherent sheaf $\mathscr{F}$ on X is *reflexive* if $\mathscr{F}^{\vee\vee} = \mathscr{F}$, where $\mathscr{F}^{\vee}$ denotes the dual sheaf $\mathscr{H}om(\mathscr{F}, \mathscr{O}_X)$. A reflexive sheaf of rank one is called a *divisorial sheaf*. They form a group by the multiplication $\mathscr{F}_1, \mathscr{F}_2 \mapsto (\mathscr{F}_1 \otimes \mathscr{F}_2)^{\vee\vee}$. A rank one locally free sheaf (i.e. the sheaf of sections of a line bundles) is reflexive.

Let us denote by X^0 the regular part of X and $j : X^0 \to X$ the inclusion. A coherent sheaf $\mathscr{F}$ on X^0 is called *extendible* if it extends to a coherent sheaf on X. By [629] if $\mathscr{F}$ is extendible reflexive sheaf on X^0, then $j_*\mathscr{F}$ is a reflexive sheaf on X, and $j_*\mathscr{F}$ is unique as a reflexive extension of $\mathscr{F}$.

If Z is a divisor on X, then the invertible sheaf $\mathscr{O}(Z|_{X^0})$ on X^0 is extendible. In particular, $j_*(\mathscr{O}(Z|_{X^0}))$ is a divisorial sheaf on X, associated with Z. It is denoted by $\mathscr{O}_X(Z)$. Since $\mathscr{O}_X(Z_1 + Z_2) = (\mathscr{O}_X(Z_1) \otimes \mathscr{O}_S(Z_2))^{\vee\vee}$ we get a monomorphism from $\mathrm{Cl}(X)$ to the group of divisorial sheaves. If X is projective then this is an isomorphism.

If $\dim(X) = 2$ then the sheaf $\mathscr{O}_X(Z) = j_*(\mathscr{O}(Z|_{X^0}))$ has other realizations as well. Let $\phi : \widetilde{X} \to X$ be a resolution, and let Z^s be the strict transform of Z. Then $\mathscr{O}_X(Z) = \phi_*(\mathscr{O}_{\widetilde{X}}(Z^s))$ too. Even more, let us consider the (Mumford) pullback $\phi^*(Z)$ of Z [448], see also 12.4.9. It is defined as $Z^s + D$, where D is a rational divisor on $\widetilde{X}$ supported on the exceptional curves so that $(\phi^*(Z), E_v)_{\widetilde{X}} = 0$ for every irreducible exceptional divisor. Then $\mathscr{O}_X(Z) = \phi_*(\mathscr{O}_{\widetilde{X}}(\lfloor \phi^*(Z) \rfloor))$ too, cf. 6.4.5. [Note that $\phi_*(\mathscr{O}_{\widetilde{X}}(\lfloor \phi^*(Z) \rfloor)) = \phi_*(\mathscr{O}_{\widetilde{X}}(Z^s))$ since $\phi_*(\mathscr{O}_{\widetilde{X}}(\lfloor D \rfloor)) = \mathscr{O}_X$.]

12.4.3 If X is smooth and $\dim(X) = d$ then the global holomorphic d-forms on X, Ω^d_X, is a line bundle, called the *canonical line bundle* of X. If Ω^d_X admits a

divisor K_X, i.e. $\Omega_X^d = \mathscr{O}_X(K_X)$, then K_X is called a *canonical divisor*. We write $\omega_X := \Omega_X^d$.

If X is not smooth (but normal) and $\Omega_{X^0}^d$ admits a divisor, then we set $\omega_X := j_*(\Omega_{X^0}^d)$. It is a divisorial sheaf, it is called the *canonical divisorial sheaf*. It is associated with a Weil divisor, called the *canonical divisor*, still denoted by $K_X \in \mathrm{Cl}(X)$, well defined up to a linear equivalence.

12.4.4 In our situation, X is mostly a connected normal complex surface, not necessarily compact or smooth. However, in certain homological discussions it is convenient to assume that X is smooth. If X is smooth and non-compact, then usually, as a C^∞ 4-manifold, it is the set of interior points $\bar{X} \setminus \partial \bar{X}$ of a C^∞ manifold $\bar{X}$ with boundary $\partial \bar{X}$. We prefer not to make notational differences between X and $\bar{X}$.

Using this notation, any Weil divisor Z determines an element in $H_2(X, \partial X, \mathbb{Z})$, its Poincaré dual is $c_1(\mathscr{O}_X(Z)) \in H^2(X)$. If Z is compactly supported (it does not intersect ∂X), then its homology class, in fact, sits in $H_2(X, \mathbb{Z})$, and it is Poincaré dual to $c_1(\mathscr{O}_X(Z)) \in H_c^2(X)$.

Below, for any pair $(X, \mathscr{F})$ we write $\chi(\mathscr{F})$ for $\sum_q (-1)^q \cdot h^q(X, \mathscr{F})$.

12.4.5 Invariants of (Abstract) Curves If C is a (not necessarily embedded) *irreducible smooth compact curve*, then its *genus* $g(C)$ is by definition $h^1(\mathscr{O}_C) = 1 - \chi(\mathscr{O}_C) = 1 + \chi(\Omega_C^1)$, where Ω_C^1 is the sheaf of holomorphic 1-forms.

The isomorphism class of line bundles on C are given, as usual, by $\mathrm{Pic}(C) = H^1(\mathscr{O}_C^*)$. The exponential sequence $0 \to \mathbb{Z}_C \to \mathscr{O}_C \to \mathscr{O}_C^* \to 0$ provides $c_1 : Pic(C) \to H^2(C) = \mathbb{Z}$, denoted also by deg. For any locally free sheaf $\mathscr{F}$ of rank r one has the *Serre duality* $H^1(\mathscr{F}) = H^0(\mathscr{F}^\vee \otimes \Omega_C^1)^*$, and the *Riemann–Roch Theorem* $\chi(\mathscr{F}) = \deg(\det \mathscr{F}) + r \cdot \chi(\mathscr{O}_C)$.

If C is irreducible and compact (but not necessarily smooth), let $n : \widehat{C} \to C$ be its normalization. The *geometric genus* $g(C)$ of C is, by definition, the genus of $\widehat{C}$, while the *arithmetic genus* $p_a(C)$ is $h^1(\mathscr{O}_C) = 1 - \chi(\mathscr{O}_C)$.

In the *normalization exact sequence*

$$0 \to \mathscr{O}_C \to n_*\mathscr{O}_{\widehat{C}} \to n_*\mathscr{O}_{\widehat{C}}/\mathscr{O}_C \to 0 \tag{12.4.3}$$

the last sheaf is supported in the singular locus $\Sigma(C)$ of C, hence it is a finite dimensional vector space, its dimension is the *delta-invariant* $\delta(C)$ of C. For any $x \in \Sigma(C)$ we also write $\delta(C, x) = \dim(n_*\mathscr{O}_{\widehat{C}}/\mathscr{O}_C)_x$ for the *delta-invariant of the singular point* (C, x). In general $\delta(C) \geq 0$, and $\delta(C) = 0$ if and only if C is smooth.

From (12.4.3) one has

$$p_a(C) = g(C) + \delta(C). \tag{12.4.4}$$

The exponential cohomology sequence, Serre duality and Riemann–Roch Theorem extend to this case too, in fact, to the case of not necessarily irreducible, but reduced compact curves as well. The degree $\deg : \mathrm{Pic}(C) \to \mathbb{Z}$ appearing in Riemann–Roch is defined as follows. If $C = C_1 \cup \cdots \cup C_k$ is the irreducible decomposition of C then the target of $c_1 : \mathrm{Pic}(C) \to H^2(C, \mathbb{Z})$ is $H^2(C, \mathbb{Z}) = \mathbb{Z}^k$ whose components are the degrees of restrictions on C_i. Then $\deg : \mathrm{Pic}(C) \to \mathbb{Z}$ is defined as $\deg(\mathscr{L}) = \sum_i \deg(\mathscr{L}|C_i)$, the sum of all these components.

Moreover, one defines the *dualizing sheaf* ω_C, and, again, for $\mathscr{F}$ free of rank r:

$$H^1(\mathscr{F}) = H^0(\mathscr{F}^\vee \otimes \omega_C)^* \quad \text{(Serre duality)} \tag{12.4.5}$$

$$\chi(\mathscr{F}) = \deg(\det \mathscr{F}) + r \cdot \chi(\mathscr{O}_C) \quad \text{(Riemann–Roch)}. \tag{12.4.6}$$

If C is embedded into a smooth surface S then $\omega_C := \Omega^2_S \otimes \mathscr{O}_C(C)$.

12.4.6 At the end of this subsection we add some facts regarding the delta invariant of a reduced curve germ (C, o). We follow [90] and [661].

The delta invariant of a reduced curve can be determined inductively from the delta invariant of the components and the *Hironaka generalized intersection multiplicity*. Indeed, assume (C, o) is embedded in some $(\mathbb{C}^n, o)$, and assume that (C, o) is the union of two (not necessarily irreducible) germs (C'_1, o) and (C'_2, o) without common irreducible components. Assume that (C'_i, o) is defined by the ideal I_i in $\mathscr{O}_{(\mathbb{C}^n,o)}$ $(i = 1, 2)$. Then one can define *Hironaka's intersection multiplicity* by $(C'_1, C'_2)_{Hir} := \dim_{\mathbb{C}} (\mathscr{O}_{(\mathbb{C}^n,o)}/I_1 + I_2)$. Then, one has the following formula of Hironaka, see [264] or [661, 2.1] and [90],

$$\delta(C, o) = \delta(C'_1, o) + \delta(C'_2, o) + (C'_1, C'_2)_{Hir}. \tag{12.4.7}$$

In particular, if $(C, 0)$ has irreducible decomposition $\cup_{i=1}^r (C_i, o)$, and we set $(C^j, o) := \cup_{i=j+1}^r (C_i, o)$ then inductively $\delta(C, o) = \sum_{i=1}^r \delta(C_i, o) + \sum_{i=1}^{r-1} (C^i, C_i)_{Hir}$.

Assume that $(C, 0)$ is (analytically equivalent with) the union of the coordinate axes of $(\mathbb{C}^r, o)$ (called *ordinary r-tuple*). Then using any of the above formulae we get that $\delta(C, o) = r - 1$. Furthermore, for any (C, o) with r local irreducible components, (since $(C'_1, C'_2)_{Hir} \geq 1$ always) we have $\delta(C, o) \geq r - 1$. Conversely, if $\delta(C, o) = r - 1$ (is the sharp minimum) then (C, o) is necessarily an ordinary r-tuple.

For plane curve singularities $(C_1, C_2)_{Hir}$ coincides with the local intersection multiplicity $(C_1, C_2)_{\mathbb{C}^2,o}$ and $\delta(C, o) = \sum_i \delta(C_i, o) + \sum_{i<j} (C_i, C_j)_{\mathbb{C}^2,o}$. This is compatible with (12.4.7), since for plane curves $(C', C_1 \cup C_2)_{\mathbb{C}^2,o} = (C', C_1)_{\mathbb{C}^2,o} + (C', C_2)_{\mathbb{C}^2,o}$, a property which usually fails for non-plane germs in the general context of $(\ ,\ )_{Hir}$.

12.4.7 Invariants of Embedded Curves on Smooth Surfaces Below we will assume that S is a smooth surface. The facts from 12.4.5 can be generalized to any *embedded curve*, even non-reduced.

In this case, one defines $\mathscr{O}_C^*$ as the subsheaf of those germs f in $\mathscr{O}_C$ for which $f|C_{red}$ has no zero, or alternatively, $f = \tilde{f}|C$, where $\tilde{f} \in \mathscr{O}_S^*(U)$ on an open set $U \subset S$.

If C is compact with irreducible decomposition $C = \sum_i n_i C_i$, the degree map $\deg : \mathrm{Pic}(C) = H^1(\mathscr{O}_C^*) \to \mathbb{Z}$ is defined by $\deg(\mathscr{L}) = \sum_i n_i \deg(\mathscr{L}|C_i)$. Then Serre duality (12.4.5) and Riemann–Roch (12.4.6) work under unmodified form with $\omega_C := \Omega_S^2 \otimes \mathscr{O}_C(C)$. Moreover, $p_a(C)$ is still defined as $1 - \chi(\mathscr{O}_C) = 1 + \chi(\omega_C)$. These formulae applied for ω_C provide the *adjunction formula*:

$$\deg(\Omega_S^2 \otimes \mathscr{O}_C(C)) = \deg(\omega_C) \overset{RR}{=} \chi(\omega_C) - \chi(\mathscr{O}_C) = -2 + 2p_a(C). \qquad (12.4.8)$$

If C is smooth then $\Omega_C^1 = \Omega_S^2 \otimes \mathscr{O}_S(C)|_C$ (adjunction formula, smooth case), and the *normal bundle* $\mathscr{N}_{C/S}$ of C in S also equals $\mathscr{N}_{C/S} = \mathscr{O}_S(C)|_C$. The expression $\mathscr{N}_{C/S} := \mathscr{O}_S(C)|_C = \mathscr{O}_C(C)$ will serve as the *definition of the normal bundle* for any embedded, not necessarily reduced, curve C.

12.4.8 The cup product $H_c^2(S, \mathbb{Z}) \otimes H^2(S, \mathbb{Z}) \to H_c^4(S, \mathbb{Z}) = \mathbb{Z}$ defines a perfect pairing, denoted by (c, c'). Hence if Z_c is a divisor with compact support, Z any divisor, or $\mathscr{L}$ any line bundle, then one defines the *intersection numbers*

$$(Z_c, Z) := (c_1(\mathscr{O}_S(Z_c)), c_1(\mathscr{O}_S(Z))) \text{ and } (Z_c, \mathscr{L}) := (c_1(\mathscr{O}_S(Z_c)), c_1(\mathscr{L})).$$

They have the following reinterpretations. If Z_c is positive then $(Z_c, \mathscr{L}) = \deg(\mathscr{L}|Z_c)$. If both Z_c and Z are positive curves intersecting each other along finitely many points, then $(Z_c, Z) = \sum_{x \in Z_c \cap Z} (Z_c, Z)_{S,x}$, where $(,)_{S,x}$ denotes the intersection multiplicity at x (if f_c and f are the local equations of the divisors, then it is the codimension of the ideal (f_c, f) in $\mathscr{O}_{S,x}$). Z_c^2 abbreviates $(Z_c, Z_c) = \deg(\mathscr{N}_{Z_c/S})$.

The above cohomological definition of the pairing is Poincaré dual to the homological intersection perfect pairing $H_2(S, \mathbb{Z}) \otimes H_2(S, \partial S, \mathbb{Z}) \to \mathbb{Z}$. This, via $H_2(S, \mathbb{Z}) \to H_2(S, \partial S, \mathbb{Z})$ is compatible with the intersection pairing $H_2(S, \mathbb{Z}) \otimes H_2(S, \mathbb{Z}) \to \mathbb{Z}$.

12.4.9 If $\pi : S' \to S$ is a modification (e.g. blow up) of the smooth surface S, then for a divisor Z of S we define the divisor $\pi^*(Z)$ of S' (as the pull back of a Cartier divisor via pull back of the local equation, or, as the pull back of line bundles). Moreover, for a divisor Z' of S' we define the divisor $\pi_*(S')$ of S (as the homological image of Z'). Then the *projection formula* says that $(Z', \pi^*(Z))_{S'} = (\pi_*(Z'), Z)_S$.

E.g., if we blow up a point of a smooth curve $C \subset S$, then $\pi^*(C) = C^s + E$, where $C^s = \overline{\pi^{-1}(C \setminus p)}$ is the strict transform of C, and E is the exceptional curve. Then $(E, C^s + E) = (\pi_*(E), C) = (0, C) = 0$, hence $E^2 = -1$.

12.4.10 Let $\mathscr{L}$ be a line bundle on S and Z, Z_1, Z_2 positive divisors. We frequently will use the following exact sequence $0 \to \mathscr{L}(-Z) \to \mathscr{L} \to \mathscr{L}|_Z \to 0$, and the 'dévissage of $\mathscr{L}|_{Z_1+Z_2}$', namely, $0 \to \mathscr{L}(-Z_1)|_{Z_2} \to \mathscr{L}|_{Z_1+Z_2} \to \mathscr{L}|_{Z_1} \to 0$.

12.4.11 For more see also [41, 240, 241, 243, 246–249, 305, 358, 628].

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